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# THE 3/2TH AND 2ND ORDER ASYMPTOTIC EFFICIENCY OF MAXIMUM PROBABILITY ESTIMATORS IN NON-REGULAR CASES*

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**Abstract.** In this paper we consider the estimation problem on independent and identically distributed observations from a location parameter family generated by a density which is positive and symmetric on a finite interval, with a jump and a nonnegative right differential coefficient at the left endpoint. It is shown that the maximum probability estimator (MPE) is 3/2th order two-sided asymptotically efficient at a point in the sense that it has the most concentration probability around the true parameter at the point in the class of 3/2th order asymptotically median unbiased (AMU) estimators only when the right differential coefficient vanishes at the left endpoint. The second order upper bound for the concentration probability of second order AMU estimators is also given. Further, it is shown that the MPE is second order two-sided asymptotically efficient at a point in the above case only.

*Key words and phrases*: Higher order two-sided asymptotic efficiency, maximum probability estimator, non-regular distributions, asymptotically median unbiased estimator, asymptotic concentration probability.

## 1. Introduction

In regular cases, it is known that the maximum likelihood estimator (MLE) is third order asymptotically efficient (e.g. see Pfanzagl and Wefelmeyer (1978), Ghosh *et al.* (1980), Akahira and Takeuchi (1981) and Akahira (1986)). However, in non-regular cases, the MLE is not asymptotically efficient. The maximum probability estimator (MPE) by Weiss and Wolfowitz (1967) is asymptotically equivalent to the maximum likelihood estimator in regular cases, but is not so in non-regular cases. In a truncated normal case, it was shown by Akahira and Takeuchi (1979, 1981) that the MPE is not asymptotically efficient in some sense. When considering the MPE, it should be noted that it depends on an interval $(-t, t)$ determined in advance. In higher order asymptotics of non-regular cases, higher order asymptotically efficient estimator may not generally exist. In fact, it

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is shown in Akahira (1988$a$) that the generalized Bayes estimator is not generally second order asymptotically efficient in non-regular cases. In such cases, it seems to be meaningful to consider the higher order asymptotic efficiency of an estimator at a certain point and then the MPE is one of the candidates.

In this paper, for a family of non-regular distributions, it is shown that the MPE is 3/2th order two-sided asymptotically efficient at a point in the sense that its asymptotic concentration probability (ACP) around the true parameter attains the 3/2th order upper bound for the ACP of 3/2th order AMU estimators at the point only when the nonnegative (nonpositive) right (left) differential coefficient at the left (right) endpoint vanishes. The second order upper bound for the ACP of second order AMU estimators is also obtained. Further, it is shown that the MPE is second order two-sided asymptotically efficient at the point among second order AMU estimators in the same case.

## 2. Preliminaries

Let $\mathcal{X}$ be an abstract sample space whose generic point is denoted by $x$, $\mathcal{B}$ a $\sigma$-field of subsets of $\mathcal{X}$ and $\{P_\theta: \theta \in \Theta\}$ a set of probability measures on $\mathcal{B}$, where $\Theta$ is called a parameter space. We assume that $\Theta$ is an open subset of Euclidean 1-space $R^1$. We denote by $(\mathcal{X}^{(n)}, \mathcal{B}^{(n)})$ the $n$-fold direct products of $(\mathcal{X}, \mathcal{B})$. Consider $n$-fold product measures $P_\theta^n$ of $P_\theta$. An estimator of $\theta$ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathcal{B}^{(n)}$-measurable functions $\hat{\theta}_n$ on $\mathcal{X}^{(n)}$ into $\Theta$. For simplicity we denote $\{\hat{\theta}_n\}$ by $\hat{\theta}_n$.

For an increasing sequence of positive numbers $\{c_n\}$ ($c_n$ tending to infinity) an estimator $\hat{\theta}_n$ is called $c_n$-consistent if for any $\eta \in \Theta$ there exists a sufficiently small positive number $\delta$ such that

$$\lim_{L\to\infty} \overline{\lim_{n\to\infty}} \sup_{\theta:|\theta-\eta|<\delta} P_\theta^n\{c_n|\hat{\theta}_n - \theta| \geq L\} = 0 \qquad \text{(Akahira (1975))}.$$

For any $k \geq 1$, a $c_n$-consistent estimator $\hat{\theta}_n$ is said to be $k$-th order asymptotically median unbiased ($k$-th order AMU for short) if for any $\eta \in \Theta$ there exists a positive number $\delta$ such that

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\eta|<\delta} c_n^{k-1}|P_\theta^n\{\hat{\theta}_n \leq \theta\} - (1/2)| = 0,$$

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\eta|<\delta} c_n^{k-1}|P_\theta^n\{\hat{\theta}_n \geq \theta\} - (1/2)| = 0.$$

We denote by $A_k$ a class of the all $k$-th order AMU estimators of $\theta$.

A $k$-th order AMU estimator $\hat{\theta}_n^*$ is called $k$-th order two-sided asymptotically efficient at a point $t$ if for any $k$-th order AMU estimator $\hat{\theta}_n$

$$\underline{\lim}_{n\to\infty} c_n^{k-1}[P_\theta^n\{c_n|\hat{\theta}_n^* - \theta| \leq t\} - P_\theta^n\{c_n|\hat{\theta}_n - \theta| \leq t\}] \geq 0.$$

This means that the estimator $\hat{\theta}_n^*$ has the most concentration probability at a point or the maximal probability in a fixed symmetric interval in the class $A_k$.

## 3. Assumptions and maximum probability estimator

Let $\mathcal{X} = \Theta = R^1$, and we suppose that, for each $\theta \in \Theta$, $P_\theta$ is absolutely continuous with respect to the Lebesgue measure and constitutes a location parameter family. Then we denote the density $dP_\theta/dx$ by $g(x, \theta)$ and $g(x, \theta) = f(x-\theta)$. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independently and identically distributed real random variables with the density $f(x-\theta)$. We assume the following conditions:

(A.1)
$$f(x) > 0 \quad \text{for} \quad a < x < b,$$
$$f(x) = 0 \quad \text{for} \quad x \le a,\ x \ge b,$$

where both $a$ and $b$ are finite.

(A.2) $f(x)$ is twice continuously differentiable in the open interval $(a, b)$ and

$$\lim_{x \to a+0} f(x) = \lim_{x \to b-0} f(x) = c,$$
$$\lim_{x \to b-0} f'(x) = -\lim_{x \to a+0} f'(x) = h,$$

where $c$ is a positive constant and $h$ is a nonpositive constant.

(A.3) $f(x)$ is symmetric around $x = (a+b)/2$.

(A.4)
$$0 < I = \int_a^b \{f'(x)\}^2/f(x)dx < \infty.$$

For example, the following densities, $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ satisfy the conditions (A.1) to (A.4):

$$f_1(x) = \begin{cases} c_1 x^{\alpha-1}(1-x)^{\alpha-1} + c_1' & \text{for} \quad 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $2 < \alpha < \infty$ and $c_1$ and $c_1'$ are certain positive constants.

$$f_2(x) = \begin{cases} c_2 \exp\{(1-x^2)^\alpha\} & \text{for} \quad |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $1 < \alpha < \infty$ and $c_2$ is some constant.

$$f_3(x) = \begin{cases} c_3 \exp(-x^2/2) & \text{for} \quad |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $c_3$ is some constant.

$$f_4(x) = \begin{cases} c_4 \exp(-x^4 + 2ax^2) & \text{for} \quad |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < a < 1$ and $c_4$ is some constant. Then it follows that, for each $i = 1, 2, 3, 4$, the value of $h$ in the condition (A.2) on the density $f_i(x)$ is given by 0, 0, $-c_3 e^{-1/2}$ and $-4c_4(1-a)e^{-1+2a}$, respectively. In the regular case, the amount like $I$ in (A.4) is called the Fisher information on $f$. However, $I$ in (A.4) represents only the information on the inner part $(a, b)$ of the support of $f$, and so it does not involve that on the endpoints $a$ and $b$. Hence $I$ in (A.4) is different from the amount of Fisher information in the regular case. Further, in a non-regular case it does not necessarily hold that

$$\begin{aligned} &E_\theta[\{(\partial/\partial\theta)\log f(X-\theta)\}^2\chi_{(a,b)}(X-\theta)] \\ &\quad = -E[\{(\partial^2/\partial\theta^2)\log f(X-\theta)\}\chi_{(a,b)}(X-\theta)], \end{aligned}$$

where $\chi_A(x)$ denotes the indicator of a set $A$. For example, in the case when the density is given by $f_3(x)$, it is easily seen that the fact does not hold.

From the above conditions (A.1), (A.2) and (A.4) we have for the true parameter $\theta_0$,

$$E_{\theta_0}[\{(\partial^2/\partial\theta^2)\log f(X-\theta_0)\}\chi_{(a,b)}(X-\theta_0)] < 0. \tag{3.1}$$

Indeed, it follows from the conditions that

$$\begin{aligned} &E_{\theta_0}[\{(\partial^2/\partial\theta^2)\log f(X-\theta_0)\}\chi_{(a,b)}(X-\theta_0)] \\ &= E_{\theta_0}[\{\,(f_{\theta\theta}(X-\theta_0)/f(X-\theta_0)) \\ &\qquad - (f_\theta(X-\theta_0)/f(X-\theta_0))^2\}\chi_{(a,b)}(X-\theta_0)] \\ &= E_{\theta_0}[\{\,(f''(X-\theta_0)/f(X-\theta_0)) \\ &\qquad - (f'(X-\theta_0)/f(X-\theta_0))^2\}\chi_{(a,b)}(X-\theta_0)] \\ &= \int_a^b f''(x)dx - \int_a^b \{f'(x)\}^2/f(x)dx \\ &= f'(b-0) - f'(a+0) - I \\ &= 2h - I < 0, \end{aligned}$$

where $f_\theta(x-\theta) = (\partial/\partial\theta)f(x-\theta)$ and $f_{\theta\theta}(x-\theta) = (\partial^2/\partial\theta^2)f(x-\theta)$.

In the situation, it is known that the order $c_n$ of consistency is equal to $n$. It is easily seen that

$$\begin{aligned} &\prod_{i=1}^n f(x_i-\theta) > 0 \quad \text{for} \quad \underline{\theta} < \theta < \overline{\theta}, \\ &\prod_{i=1}^n f(x_i-\theta) = 0 \quad \text{otherwise}, \end{aligned}$$

where $\underline{\theta} = \max_{1\le i\le n} x_i - b$ and $\overline{\theta} = \min_{1\le i\le n} x_i - a$. We put $S = n(\underline{\theta}+\overline{\theta})/2$ and $T = n(\overline{\theta}-\underline{\theta})/2$. Let $R = (-t,\, t)$ for any fixed $t > 0$. The maximum probability estimator (MPE) $\hat{\theta}^t_{\mathrm{MP}}$ of $\theta$ at the point $t$ is defined as that value of $d$ maximizing

$$\int_{d-(t/n)}^{d+(t/n)} \prod_{i=1}^n f(x_i-\theta)d\theta$$

(Weiss and Wolfowitz (1967, 1974)). Then, in this case, the MPE $\hat{\theta}^t_{\mathrm{MP}}$ is shown to be

$$(3.2)\qquad n\hat{\theta}^t_{\mathrm{MP}} = \begin{cases} S & \text{for} \quad T \le t, \\ n\overline{\theta} - t & \text{for} \quad T > t,\ \hat{\theta}_0 \ge \overline{\theta} - (t/n), \\ n\underline{\theta} + t & \text{for} \quad T > t,\ \hat{\theta}_0 \le \underline{\theta} + (t/n), \\ n\hat{\theta}_0 & \text{for} \quad \underline{\theta} + (t/n) < \hat{\theta}_0 < \overline{\theta} - (t/n), \end{cases}$$

where $\hat{\theta}_0$ is the maximum likelihood estimator whose local uniqueness in the interval $(\underline{\theta}, \overline{\theta})$ is guaranteed from (3.1).

## 4. The 3/2th order two-sided asymptotic efficiency of the MPE

In this section, it will be shown that the MPE is 3/2th order two-sided asymptotically efficient at the point $t$ only when $h = 0$. First, the 3/2th order upper bound for the asymptotic concentration probability (ACP for short) of $\hat{\theta}_n$ ($\in A_{3/2}$) around the true parameter $\theta_0$, i.e., $P^n_{\theta_0}\{n|\hat{\theta}_n - \theta_0| \le t\}$ in the class $A_{3/2}$ of the all 3/2th order AMU estimators of $\theta_0$, up to the order $n^{-1/2}$, is established in Akahira (1988*a*).

THEOREM 4.1. *Assume that the conditions* (A.1) *to* (A.4) *hold. Then for any* $\hat{\theta}_n \in A_{3/2}$, *any* $\theta \in \Theta$ *and any* $z > 0$

$$P^n_\theta\{n|\hat{\theta}_n - \theta| \le z\} \le 1 - e^{-2cz} + \sqrt{2I/(\pi n)}\,z e^{-2cz} + o(1/\sqrt{n}).$$

The proof is omitted since the theorem is given as Theorem 4.1 in Akahira (1988*a*).

Next we shall obtain the ACP $P^n_\theta\{n|\hat{\theta}^t_{\mathrm{MP}} - \theta| \le z\}$ of the MPE $\hat{\theta}^t_{\mathrm{MP}}$ around $\theta$ up to the order $n^{-1/2}$.

THEOREM 4.2. *Assume that the conditions* (A.1) *to* (A.4) *hold. Then the ACP of the MPE* $\hat{\theta}^t_{\mathrm{MP}}$ *is given by*

$$P^n_\theta\{n|\hat{\theta}^t_{\mathrm{MP}} - \theta| \le z\} = \begin{cases} 1 - e^{-2cz} + \left(4hz/\sqrt{2\pi I n}\right) e^{-2ct} \\ \qquad + \sqrt{2I/(\pi n)}\{z - (c/2)(z-t)^2\}e^{-2ct} + o(1/\sqrt{n}) \\ \qquad\qquad \textit{for} \quad z \le t, \\ 1 - e^{-c(t+z)} + \left(2h/\sqrt{2\pi I n}\right)(t+z)e^{-c(t+z)} \\ \qquad + \sqrt{2I/(\pi n)}\,z e^{-c(t+z)} + o(1/\sqrt{n}) \\ \qquad\qquad \textit{for} \quad z > t. \end{cases}$$

*Further, if $h = 0$, then the MPE $\hat{\theta}^t_{\mathrm{MP}}$ is 3/2th order two-sided asymptotically efficient at $z = t$ in the class $A_{3/2}$ in the sense that the ACP of the MPE $\hat{\theta}^t_{\mathrm{MP}}$ attains the bound given by Theorem 4.1 at $z = t$ up to the order $n^{-1/2}$ in the class $A_{3/2}$.*

*Remark* 4.1. From Theorem 4.2 it is seen that the MPE $\hat{\theta}^t_{\mathrm{MP}}$ has the most ACP at the point $t$ up to the order $n^{-1/2}$ in the class $A_{3/2}$ only when $h = 0$. For example, if the density is given by $f_1(x)$ or $f_2(x)$ in Section 3, then it holds that $h = 0$. If $h < 0$, then it follows from Theorems 4.1 and 4.2 that the MPE $\hat{\theta}^t_{\mathrm{MP}}$ does not attain the bound at $z = t$, that is, it is not 3/2th order two-sided asymptotically efficient at the point $t$.

*Remark* 4.2. It is noted that the order $c_n$ of consistency is equal to $n$, but there exists a term of order $n^{-1/2}$ in the ACP of estimators in the class $A_{3/2}$. Hence, in the non-regular case, the order $k$ of asymptotic efficiency is equal to a fraction 3/2. This is quite different from the fact that the order $k$ takes only positive integer in regular cases.

PROOF. Without loss of generality, we assume that the true parameter $\theta_0$ is equal to 0. From (3.2) it follows that

$$
\begin{aligned}
(4.1)\qquad P_0\{n|\hat{\theta}^t_{\mathrm{MP}}| \le z\} &= P_0\{|S| \le z,\, T \le t\} \\
&\quad + P_0\{|n\overline{\theta} - t| \le z,\, T > t,\, n\hat{\theta}_0 \ge n\overline{\theta} - t\} \\
&\quad + P_0\{|n\underline{\theta} + t| \le z,\, T > t,\, n\hat{\theta}_0 \le n\underline{\theta} + t\} \\
&\quad + P_0\{n|\hat{\theta}_0| \le z,\, n\underline{\theta} + t < n\hat{\theta}_0 < n\overline{\theta} - t\} \\
&= p_1 + p_2 + p_3 + p_4 \qquad \text{(say)}.
\end{aligned}
$$

(i) $p_1$: Since the asymptotic density of $(S, T)$ is given by

$$
f_n(S, T) = \begin{cases} 2c^2 e^{-2cT} + O(1/n) & \text{for} \quad -T < S < T,\ T > 0, \\ 0 & \text{otherwise,} \end{cases}
$$

it follows that for $z \le t$

$$
\begin{aligned}
p_1 &= \int_0^z \int_{-T}^{T} 2c^2 e^{-2cT} dS dT + \int_z^t \int_{-z}^{z} 2c^2 e^{-2cT} dS dT + O(1/n) \\
&= -2cze^{-2ct} - e^{-2cz} + 1 + O(1/n),
\end{aligned}
$$

and for $z > t$

$$
\begin{aligned}
p_1 &= \int_0^t \int_{-T}^{T} 2c^2 e^{-2cT} dS dT + O(1/n) \\
&= -2cte^{-2ct} - e^{-2ct} + 1 + O(1/n).
\end{aligned}
$$

(ii) $p_2$: Putting $u = n\overline{\theta}$ and $v = n\underline{\theta}$, we have the asymptotic density of $(u, v)$

$$
g_n(u, v) = \begin{cases} c^2 e^{-c(u-v)} + O(1/n) & \text{for} \quad u > 0,\ v < 0, \\ 0 & \text{otherwise.} \end{cases}
$$

We also put $Z_1 = -(1/\sqrt{n})\sum_{i=1}^{n} l^{(1)}(X_i)$, where $l^{(1)}(x) = (d/dx)l(x)$ with $l(x) = \log f(x)$ for $a < x < b$. Then the asymptotic conditional cumulants of $Z_1$ given $\underline{\theta}$ and $\overline{\theta}$ on the set $\{\underline{\theta}, \overline{\theta} \mid u - v > 2t\}$, up to the third, are given by

$$E_0(Z_1 \mid \underline{\theta}, \overline{\theta}) = -h(n\underline{\theta} + n\overline{\theta})/\sqrt{n} + O_p\left(1/n\sqrt{n}\right),$$
$$V_0(Z_1 \mid \underline{\theta}, \overline{\theta}) = I + O_p(1/n),$$
$$\kappa_{3,0}(Z_1 \mid \underline{\theta}, \overline{\theta}) = O_p(1/n),$$

hence the Edgeworth expansion of the conditional distribution of $Z_1$ given $\underline{\theta}$ and $\overline{\theta}$ on the set is obtained by

$$(4.2)\quad P_0\{Z_1 \le z \mid \underline{\theta}, \overline{\theta}\} = \Phi\left(z/\sqrt{I}\right) + \left\{h(u+v)/\sqrt{In}\right\}\phi\left(z/\sqrt{I}\right) + o\left(1/\sqrt{n}\right),$$

where $\Phi(x) = \int_{-\infty}^{x}\phi(t)dt$ with $\phi(t) = \left(1/\sqrt{2\pi}\right)e^{-t^2/2}$. From (4.2) it follows that the conditional probability of the event $\{n\hat{\theta}_0 \ge u - t\}$ given $\underline{\theta}$ and $\overline{\theta}$ on the set $\{\underline{\theta}, \overline{\theta} \mid u - v > 2t\}$ is given by

$$\begin{aligned} P_0\{n\hat{\theta}_0 \ge u - t \mid \underline{\theta}, \overline{\theta}\} &= P_0\left\{Z_1 \ge I(u-t)/\sqrt{n} \mid \underline{\theta}, \overline{\theta}\right\} \\ &= 1 - \Phi\left(\sqrt{I/n}(u-t)\right) \\ &\quad - \left\{h(u+v)/\sqrt{In}\right\}\phi\left(\sqrt{I/n}(u-t)\right) + o(1/\sqrt{n}) \\ &= (1/2) - \sqrt{I/2\pi n}(u-t) \\ &\quad - h(u+v)/\sqrt{2\pi In} + o(1/\sqrt{n}). \end{aligned}$$

Hence we have for $z \le t$

$$\begin{aligned} p_2 &= P_0\{|u-t| \le z,\ (u-v)/2 > t,\ n\hat{\theta}_0 \ge u - t\} \\ &= \int_{t-z}^{t+z}\int_{-\infty}^{u-2t} c^2 e^{-c(u-v)}\left\{(1/2) - \sqrt{I/2\pi n}(u-t) - h(u+v)/\sqrt{2\pi In}\right\}dvdu \\ &\quad + o\left(1/\sqrt{n}\right) \\ &= cze^{-2ct} + \left(2hz/\sqrt{2\pi In}\right)e^{-2ct} + o\left(1/\sqrt{n}\right), \end{aligned}$$

and also for $z > t$

$$\begin{aligned} p_2 &= \left(\int_0^{2t}\int_{-\infty}^{u-2t} + \int_{2t}^{t+z}\int_{-\infty}^{0}\right)c^2e^{-c(u-v)} \\ &\quad \cdot\left\{(1/2) - \sqrt{I/2\pi n}(u-t) - h(u+v)/\sqrt{2\pi In}\right\}dvdu + o(1/\sqrt{n}) \\ &= cte^{-2ct} + 2hte^{-2ct}/\sqrt{2\pi In} \\ &\quad + \left\{(1/2) + \sqrt{I/2\pi n}(t - (1/c))\right\}(e^{-2ct} - e^{-c(t+z)}) \\ &\quad + \left\{\sqrt{I/2\pi n} + \left(h/\sqrt{2\pi In}\right)\right\}\{(t+z)e^{-c(t+z)} - 2te^{-2ct}\} + o(1/\sqrt{n}). \end{aligned}$$

(iii) $p_3$: From the condition (A.3) and (3.2) it follows that

$$\begin{aligned} p_3 &= P_0\{|n\underline{\theta}+t| \le z,\, T > t,\, n\hat{\theta}_0 \le n\underline{\theta}+t\} \\ &= P_0\{|n\overline{\theta}-t| \le z,\, T > t,\, n\hat{\theta}_0 \ge n\overline{\theta}-t\} = p_2, \end{aligned}$$

whose value is obtained in case (ii).

(iv) $p_4$: In this case we have

$$\begin{aligned} p_4 &= P_0\{-z \le n\hat{\theta}_0 \le z,\, v+t < n\hat{\theta}_0 < u-t\} \\ &= P_0\{\max(-z,\, v+t) \le n\hat{\theta}_0 \le \min(z,\, u-t)\} \\ &= \iint_{\substack{v<u-2t\\ v<0<u}} P_0\left\{\max(-z,\, v+t) \le \sqrt{n}Z_1/I \le \min(z,\, u-t) \mid \underline{\theta},\, \overline{\theta}\right\} \\ &\qquad \cdot c^2 e^{-c(u-v)} du dv + O(1/n) \\ &= \iint_{\substack{v<u-2t\\ v<0<u}} P_0\{(I/\sqrt{n})\max(-z,\, v+t) \le Z_1 \\ &\qquad \le (I/\sqrt{n})\min(z,\, u-t) \mid \underline{\theta},\, \overline{\theta}\} \\ &\qquad \cdot c^2 e^{-c(u-v)} du dv + O(1/n) \\ &= \iint_{\substack{v<u-2t\\ v<0<u}} \left[\sqrt{I/(2\pi n)}\{\min(z,\, u-t) - \max(-z,\, v+t)\}\right] \\ &\qquad \cdot c^2 e^{-c(u-v)} du dv + O(1/n). \end{aligned} \tag{4.3}$$

Then we obtain for $z \le t$

$$\begin{aligned} &\iint_{\substack{v<u-2t\\ v<0<u}} \{\min(z,\, u-t)\} c^2 e^{-c(u-v)} du dv \\ &\quad = \int_0^{z+t}\int_{-\infty}^{u-2t} (u-t)c^2 e^{-c(u-v)} dv du \\ &\qquad + \left(\int_{z+t}^{2t}\int_{-\infty}^{u-2t} + \int_{2t}^{\infty}\int_{-\infty}^{0}\right) z c^2 e^{-c(u-v)} dv du \\ &\quad = [z - \{c(z-t)^2/2\}]e^{-2ct}, \end{aligned}$$

and for $z > t$

$$\begin{aligned} &\iint_{\substack{v<u-2t\\ v<0<u}} \{\min(z,\, u-t)\} c^2 e^{-c(u-v)} du dv \\ &\quad = \left(\int_0^{2t}\int_{-\infty}^{u-2t} + \int_{2t}^{z+t}\int_{-\infty}^{0}\right) (u-t) c^2 e^{-c(u-v)} dv du \\ &\qquad + \int_{z+t}^{\infty}\int_{-\infty}^{0} z c^2 e^{-c(u-v)} dv du \\ &\quad = te^{-2ct} - (1/c)(e^{-c(z+t)} - e^{-2ct}). \end{aligned}$$

We also have for $z \leq t$

$$\iint_{\substack{v<u-2t\\ v<0<u}} \{\max(-z,\, v+t)\} c^2 e^{-c(u-v)} du dv$$
$$= \int_{-t-z}^{0} \int_{v+2t}^{\infty} (v+t) c^2 e^{-c(u-v)} du dv$$
$$+ \left( \int_{-2t}^{-z-t} \int_{v+2t}^{\infty} + \int_{-\infty}^{-2t} \int_{0}^{\infty} \right) (-z) c^2 e^{-c(u-v)} du dv$$
$$= (c/2)(t-z)^2 e^{-2ct} - z e^{-2ct},$$

and for $z > t$

$$\iint_{\substack{v<u-2t\\ v<0<u}} \{\max(-z,\, v+t)\} c^2 e^{-c(u-v)} du dv$$
$$= \left( \int_{-2t}^{0} \int_{v+2t}^{\infty} + \int_{-t-z}^{-2t} \int_{0}^{\infty} \right) (v+t) c^2 e^{-c(u-v)} du dv$$
$$+ \int_{-\infty}^{-t-z} \int_{0}^{\infty} (-z) c^2 e^{-c(u-v)} du dv$$
$$= -\{t + (1/c)\} e^{-2ct} + (1/c) e^{-c(t+z)}.$$

From (4.3) it follows that

$$p_4 = \begin{cases} \sqrt{2I/(\pi n)}\{z - (c/2)(z-t)^2\} e^{-2ct} & \text{for} \quad z \leq t, \\ \sqrt{2I/(\pi n)}\{t e^{-2ct} - (1/c)(e^{-c(z+t)} - e^{-2ct})\} & \text{for} \quad z > t. \end{cases}$$

From (4.1) and cases (i) to (iv) we have

(4.4) $\quad P_0^n\{n|\hat{\theta}_{\mathrm{MP}}^t| \leq z\}$

$$= \begin{cases} 1 - e^{-2cz} + \left(4hz/\sqrt{2\pi I n}\right) e^{-2ct} \\ \qquad + \sqrt{2I/(\pi n)}\{z - (c/2)(z-t)^2\} e^{-2ct} + o(1/\sqrt{n}) \\ \qquad\qquad \text{for} \quad z \leq t, \\ 1 - e^{-c(t+z)} + \left(2h/\sqrt{2\pi I n}\right)(t+z) e^{-c(t+z)} \\ \qquad + \sqrt{2I/(\pi n)}\, z e^{-c(t+z)} + o(1/\sqrt{n}) \\ \qquad\qquad \text{for} \quad z > t. \end{cases}$$

If $h = 0$ and $z = t$, then

$$P_0\{n|\hat{\theta}_{\mathrm{MP}}^t| \leq t\} = 1 - e^{-2ct} + \sqrt{2I/(\pi n)}\, t e^{-2ct} + o(1/\sqrt{n}),$$

hence it follows that the ACP of the MPE $\hat{\theta}_{\mathrm{MP}}^t$ coincides with the bound given by Theorem 4.1 at point $t$; that is, the MPE is 3/2th order two-sided asymptotically efficient at $z = t$ in class $A_{3/2}$. Thus we complete the proof.

## 5. The second order two-sided asymptotic efficiency of the MPE

In the previous section the ACP of the MPE is obtained up to the order $n^{-1/2}$, and it is shown that the MPE attains the upper bound for the ACP of $\hat{\theta}_n$ ($\in A_{3/2}$) at point $t$ only when $h=0$. In this section, we shall consider the above up to the second order, i.e. order $n^{-1}$. First, in a similar way as in Akahira (1988*a*) we shall obtain the second order upper bound for the ACP of $\hat{\theta}_n$ ($\in A_2$) around the true parameter $\theta_0$, i.e. $P^n_{\theta_0}\{n|\hat{\theta}_n-\theta_0|\le z\}$ up to order $n^{-1}$ in class $A_2$. In order to do so, it is enough to get an upper bound for

$$P^n_{\theta_0-tn^{-1}}\{\hat{\theta}_n\le\theta_0\}-P^n_{\theta_0+tn^{-1}}\{\hat{\theta}_n\le\theta_0\} \tag{5.1}$$

in class $A_2$, up to order $n^{-1}$, since $\hat{\theta}_n$ is a second order AMU estimator. In a way similar to the fundamental lemma of Neyman and Pearson, it is shown in Akahira and Takeuchi ((1981), p. 76) that the $\hat{\theta}^*_n$ maximizing (5.1) is given by

$$\phi^*_n(\tilde{x}_n)=\begin{cases}1 & \text{for} \quad \prod_{i=1}^n f(x_i-\theta_0+tn^{-1})>\prod_{i=1}^n f(x_i-\theta_0-tn^{-1}),\\ 0 & \text{for} \quad \prod_{i=1}^n f(x_i-\theta_0+tn^{-1})<\prod_{i=1}^n f(x_i-\theta_0-tn^{-1}),\end{cases} \tag{5.2}$$

where $\tilde{x}_n=(x_1,\dots,x_n)$. Using (5.2) we have the following result.

THEOREM 5.1. *Assume that the conditions* (A.1) *to* (A.4) *hold. Then for any* $\hat{\theta}_n\in A_2$, *any* $\theta\in\Theta$ *and any* $z>0$

$$\begin{aligned}&P^n_\theta\{n|\hat{\theta}_n-\theta|\le z\}\\ &\qquad\le 1-e^{-2cz}+\sqrt{2I/(\pi n)}ze^{-2cz}+(2/n)(c^2-h)z^2e^{-2cz}+o(1/n).\end{aligned}$$

*Remark* 5.1. From Theorem 5.1 it is seen that the second order upper bound is affected by the endpoint $a$ or $b$ of the support $(a,b)$ of the density $f(x)$ through $c^2-h=f^2(a+0)+f'(a+0)=f^2(b-0)-f'(b-0)$ and the inner points of the interval through $I$.

PROOF. We put $A=\{\tilde{x}_n\mid \underline{\theta}<\theta_0-zn^{-1},\overline{\theta}<\theta_0+zn^{-1}\}$, $B=\{\tilde{x}_n\mid \underline{\theta}>\theta_0-zn^{-1},\overline{\theta}>\theta_0+zn^{-1}\}$, $C=\{\tilde{x}_n\mid \underline{\theta}<\theta_0-zn^{-1},\overline{\theta}>\theta_0+zn^{-1}\}$, $D=\{\tilde{x}_n\mid(1/\sqrt{n})Z_1\le\theta_0\}$ and $D'=\{\tilde{x}_n\mid(1/\sqrt{n})Z_1>\theta_0\}$. From (5.2) we have

$$\phi^*_n(\tilde{x}_n)=\begin{cases}1 & \text{for} \quad \tilde{x}_n\in A\cup(C\cap D),\\ 0 & \text{for} \quad \tilde{x}_n\in B\cup(C\cap D').\end{cases}$$

Then we obtain for any $\hat{\theta}_n\in A_2$

$$\begin{aligned}&P^n_{\theta_0-zn^{-1}}\{\hat{\theta}_n\le\theta_0\}-P^n_{\theta_0+zn^{-1}}\{\hat{\theta}_n\le\theta_0\}\\ &\quad\le E^n_{\theta_0-zn^{-1}}(\phi^*_n)-E^n_{\theta_0+zn^{-1}}(\phi^*_n)\\ &\quad=P^n_{\theta_0-zn^{-1}}(A)-P^n_{\theta_0+zn^{-1}}(A)+P^n_{\theta_0-zn^{-1}}(C\cap D)-P^n_{\theta_0+zn^{-1}}(C\cap D).\end{aligned} \tag{5.3}$$

Putting $U = n(\bar{\theta} - \theta_0)$ and $V = n(\underline{\theta} - \theta_0)$, we have, as the joint density of $(U, V)$,

$$g_n(u, v) = \begin{cases} \{(n-1)/n\}\{F(b + (v/n)) - F(a + (u/n))\}^{n-2} \\ \qquad \cdot f(a + (u/n)) f(b + (v/n)) & \text{for} \quad u > 0,\ v < 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $F(x) = \int_{-\infty}^{x} f(t)dt$, hence, by its expansion,

$$g_n(u, v) = \begin{cases} c^2 e^{-c(u-v)}[1 + (1/n)\{-1 + 2c(u-v) + (h/4)((u+v)^2 + (u-v)^2) \\ \qquad - (c^2/2)(u-v)^2 - (h/c)(u-v)\}] + o(1/n) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad u < 0,\ v > 0, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Since, from the expansion of $g_n(u, v)$, the asymptotic density of $(S, T)$, up to the order $o(1/n)$, is given by

$$f_n(S, T) = \begin{cases} 2c^2 e^{-2cT}[1 + (1/n)\{-1 + 4cT + h(T^2 + S^2) \\ \qquad - 2c^2T^2 - (2h/c)T\}] + o(1/n) \\ \qquad\qquad\qquad\qquad\qquad \text{for} \quad -T < S < T,\ 0 < T, \\ 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

it follows that

$$\begin{aligned} (5.4)\ P^n_{\theta_0 - zn^{-1}}(A) &= P^n_{\theta_0 - zn^{-1}}\{\underline{\theta} < \theta_0 - zn^{-1},\ \bar{\theta} < \theta_0 + zn^{-1}\} \\ &= P^n_{\theta_0 - zn^{-1}}\{n(\underline{\theta} - (\theta_0 - zn^{-1})) < 0,\ n(\bar{\theta} - (\theta_0 - zn^{-1})) < 2z\} \\ &= P^n_0\{n\underline{\theta} < 0,\ n\bar{\theta} < 2z\} \\ &= P^n_0\{S - T < 0,\ S + T < 2z\} \\ &= \left(\int_0^z \int_S^{-S+2z} + \int_{-\infty}^0 \int_{-S}^{-S+2z}\right) f_n(T, S)dTdS \\ &= 1 - e^{-2cz} + (2/n)(c^2 - h)z^2 e^{-2cz} + o(1/n). \end{aligned}$$

Similarly we have

$$(5.5) \qquad P^n_{\theta_0 + zn^{-1}}(A) = o(1/n).$$

In a way similar to the proof of Theorem 4.2, it follows that the conditional probability of the event $\{Z_1/\sqrt{n} \le \theta_0\}$ given $S$ and $T$ on the set $C$, up to the order $n^{-1}$, is given by

$$P^n_{\theta_0 \mp zn^{-1}}\{Z_1/\sqrt{n} \le \theta_0 \mid S, T\} = (1/2) + (2hS \pm Iz)/\sqrt{2\pi I n} + o(1/n),$$

hence

$$(5.6)\qquad P^n_{\theta_0\mp zn^{-1}}(C\cap D)=\int_C P^n_{\theta_0\mp zn^{-1}}\{Z_1/\sqrt{n}\le\theta_0\mid S,T\}f_n(S,T)dSdT$$
$$=(1/2)e^{-2cz}\pm(1/2)\sqrt{2I/(\pi n)}ze^{-2cz}$$
$$+(1/n)[\{c+(h/2c)\}z-\{c^2-(h/2)\}z^2]e^{-2cz}$$
$$+o(1/n),$$

where the signs + and − should be read consistently. From (5.3) to (5.6) we have

$$P^n_{\theta_0}\{n|\hat{\theta}_n-\theta_0|\le z\}$$
$$\le 1-e^{-2cz}+\sqrt{2I/(\pi n)}ze^{-2cz}+(2/n)(c^2-h)z^2e^{-2cz}+o(1/n).$$

This completes the proof.

Next, we consider the second order two-sided asymptotic efficiency of the MPE.

THEOREM 5.2. *Assume that the conditions* (A.1) *to* (A.4) *hold. If* $h=0$, *then the ACP of the MPE* $\hat{\theta}^t_{\mathrm{MP}}$ *is given by*

$$P^n_\theta\{n|\hat{\theta}^t_{\mathrm{MP}}-\theta|\le z\}$$
$$=\begin{cases} 1-e^{-2cz}+\sqrt{2I/(\pi n)}\{z-(c/2)(z-t)^2\}e^{-2ct} \\ \quad +(2c^2/n)t^2e^{-2ct}+o(1/n) & \text{for}\quad z\le t, \\ 1-e^{-c(t+z)}+\sqrt{2I/(\pi n)}ze^{-c(t+z)} \\ \quad +(1/n)[2c^2t^2e^{-2ct}+\{(c^2/2)(t+z)^2 \\ \qquad +(2c^2-3c)(t+z)/2+c-(3/2)\}e^{-c(t+z)} \\ \qquad -\{2c^2t^2+(2c^2-3c)t+c-(3/2)\}e^{-2ct}]+o(1/n) \\ & \text{for}\quad z>t, \end{cases}$$

*and so the MPE* $\hat{\theta}^t_{\mathrm{MP}}$ *is second order two-sided asymptotically efficient at* $z=t$ *in the class* $A_2$.

The proof is similar to that of Theorem 4.2.

OUTLINE OF THE PROOF. Without loss of generality, we assume that $\theta$ is equal to 0. We also have (4.1) in the proof of Theorem 4.2. It is noted that the asymptotic density of $(U, V)$, up to the order $o(1/n)$, is given by

$$(5.7)\qquad f_n(u, v) = \begin{cases} c^2 e^{-c(u-v)}[1 + (1/n)\{-1 + 2c(u-v) \\ \qquad - (c^2/2)(u-v)^2\}] + o(1/n) & \text{for} \quad v < 0 < u, \\ 0 & \text{otherwise,} \end{cases}$$

and the asymptotic conditional probability of the event $\{Z_1/\sqrt{I} \le a\}$ given $\underline{\theta}$ and $\overline{\theta}$ is given by

$$(5.8)\qquad P_0^n\{Z_1/\sqrt{I} \le a \mid \underline{\theta}, \overline{\theta}\} = \Phi(a) + O(1/n).$$

Then $p_i$ $(i = 1, 2, 3, 4)$ defined in (4.1) are given as follows.

$$p_1 = \begin{cases} 1 - e^{-2cz} - 2cze^{-2ct} + (z/n)\{4c^2t(ct-1)e^{-2ct} + 2c^2ze^{-2cz}\} + o(1/n) & \text{for} \quad z \le t, \\ 1 - e^{-2ct} - 2cte^{-2ct} + (2/n)c^2t^2(2ct-1)e^{-2ct} & \text{for} \quad z > t. \end{cases}$$

$$p_2 = p_3 = \begin{cases} cze^{-2ct} + (2/n)c^2tz(1-ct)e^{-2ct} + o(1/n) & \text{for} \quad z \le t, \\ cte^{-2ct} + (1/2)(e^{-2ct} - e^{-c(t+z)}) \\ \quad -\sqrt{I/(2\pi n)}\{(t + (1/c))e^{-2ct} - (z + (1/c))e^{-c(t+z)}\} \\ \quad +(1/n)[2c^2t^2(1-ct)e^{-2ct} + \{(c^2/4)(t+z)^2 \\ \qquad + ((c^2/2) - (3c/4))(t+z) + (c/2) - (3/4)\}e^{-c(t+z)} \\ \qquad - \{c^2t^2 + (c^2 - (3c/2))t + (c/2) - (3/4)\}e^{-2ct}] \\ \quad +o(1/n) & \text{for} \quad z > t. \end{cases}$$

$$p_4 = \begin{cases} \sqrt{2I/(\pi n)}\{z - (c/2)(z-t)^2\}e^{-2ct} + o(1/n) & \text{for} \quad z \le t, \\ \sqrt{2I/(\pi n)}\{(t + (1/c))e^{-2ct} - (1/c)e^{-c(t+z)}\} + o(1/n) & \text{for} \quad z > t. \end{cases}$$

Hence, by (4.1)

$$P_0^n\{n|\hat{\theta}_{\mathrm{MP}}^t| \le z\} = p_1 + p_2 + p_3 + p_4$$

$$= \begin{cases} 1 - e^{-2cz} + \sqrt{2I/(\pi n)}\{z - (c/2)(z-t)^2\}e^{-2ct} \\ \quad +(2c^2/n)t^2 e^{-2ct} + o(1/n) & \text{for} \quad z \le t, \\ 1 - e^{-c(t+z)} + \sqrt{2I/(\pi n)} z e^{-c(t+z)} \\ \quad +(1/n)[2c^2t^2e^{-2ct} + \{(c^2/2)(t+z)^2 \\ \qquad + (2c^2-3c)(t+z)/2 + c - (3/2)\}e^{-c(t+z)} \\ \qquad - \{2c^2t^2 + (2c^2-3c)t + c - (3/2)\}e^{-2ct}] \\ \quad +o(1/n) & \text{for} \quad z > t. \end{cases}$$

If $z = t$, then

$$P_0^n\{n|\hat{\theta}_{\mathrm{MP}}^t| \le t\} = 1 - e^{-2ct} + \sqrt{2I/(\pi n)}te^{-2ct} + (2c^2/n)t^2e^{-2ct} + o(1/n),$$

hence it follows that the ACP of the MPE $\hat{\theta}_{\mathrm{MP}}^t$ at point $t$ coincides with the bound given in Theorem 5.1; that is, the MPE is second order two-sided asymptotically efficient at $z = t$ in class $A_2$. Thus we complete the proof.

*Remark* 5.2. In the above discussion we assume that $I > 0$ in the condition (A.4). If $I = 0$, the density $f(x)$ coincides with a uniform density on the interval $(a, b)$. Then it is shown in Akahira (1988*a*, 1988*b*) that the generalized Bayes estimator is 3/2th and second order asymptotically efficient.

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