Invent. math. 105, 473–545 (1991)

Inventiones mathematicae

The 3-manifold invariants of Witten and Reshetikhin-Turaev for sl(2, C)

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Oblatum 24-I-1991

0 Introduction

In the fall of 1988, Witten [W] gave the first intrinsically 3-dimensional interpretation of the Jones polynomial [J1, J2] of a link in the 3-sphere, using a quantum field theory with Chern-Simons action. In the process, he uncovered a family of new invariants for arbitrary closed framed 3-manifolds and for links in 3-manifolds.

Shortly afterwards, Reshetikhin and Turaev defined closely related invariants using the theory of quantum groups. In particular, starting with a simple Lie algebra $\mathfrak{g}$, they defined invariants for a framed link L in the 3-sphere using representations of associated quantum groups (which are Hopf algebra deformations of the universal enveloping algebra of $\mathfrak{g}$) [RT1]. For $\mathfrak{g}=\mathrm{sl}(2, \mathbf{C})$, these invariants $J_{L,\mathbf{k}}$ (which depend on a *coloring* $\mathbf{k}$ of L by representations) generalize the Jones polynomial. Their values at a fixed r^{th} root of unity $q=e^{2\pi i m/r}$ can be combined to produce a complex valued invariant of the oriented 3-manifold obtained by surgery on L [RT2], shown independent of L using [K1]. (Every 3-manifold can be obtained in this way [L1, Wa].) Presumably these 3-manifold invariants can be defined for any simple Lie algebra. What is needed is that the associated quantum groups have the structure of a *modular* Hopf algebra [RT2].

This paper gives a self-contained proof of the existence of the 3-manifold invariants for $\mathfrak{g}=\mathrm{sl}(2, \mathbf{C})$ and $q=e^{2\pi i/r}$ (§§ 1–5). It is similar in spirit to [RT2], but relies more on the elementary representation theory of the relevant Hopf algebra $\mathscr{A}$ and the topology of framed links, and less on the abstract theory of modular Hopf algebras.

There are several new features in the present treatment (described below). These yield manageable formulas for the 3-manifold invariants which can be interpreted in terms of familiar topological invariants for some small values of r (§§ 6–7). Perhaps more importantly, they lead to a family of new invariants underlying those of Witten and Reshetikhin-Turaev. These new invariants and some applications are discussed in the last section of the paper.

First, the 3-manifold invariants of Reshetikhin and Turaev are modified by splitting off a term involving the first Betti number. The resulting invariants

$\tau_r(M)$ conjugate under orientation reversal (that is $\tau_r(-M)=\overline{\tau_r(M)}$) and are thus useful in answering questions of amphicheirality. They also appear to be *exactly* the same as Witten's invariants of M with the canonical 2-framing of Atiyah [A] (normalized at 1 for the 3-sphere). This has been experimentally verified in many cases by Freed and Gompf [FG], and can in fact be proved for plumbed 3-manifolds using the formulas for Witten's invariants in [FG] together with the formulas for $\tau_r(M)$ in this paper (see Remark 1.9).

Second, a cabling formula (4.15) which reduces the generalized Jones polynomials $J_{L,\mathbf{k}}$ to the classical Jones polynomial for cables of L is derived. (Similar formulas have been obtained independently by Morton and Strickland [MS].) From this one obtains a formula for $\tau_r(M)$ in terms of classical Jones polynomials (4.17) (first announced in [KM1] with a proof sketched in [KM2]). Recently, Lickorish [L3, L4, L5] has found an elegant new proof of the invariance of such a formula, using the Temperley-Lieb algebra and linear skein theory and thereby avoiding the explicit use of quantum groups. It is clear however that the algebra $\mathscr{A}$ should not be deemphasized, for it encodes deep combinatorial properties of the Jones polynomial and appears to make calculations more accessible.

Using this formula, $\tau_r(M)$ is expressed in terms of familiar topological invariants for $r=3$ and 4, as the Jones polynomial has topological meaning at the corresponding roots of unity. (Note that $\tau_2(M)=1$ for all M. The Jones polynomial is also understood at the sixth root of unity (Appendix B), and so one would expect a similar formula for $\tau_6(M)$; see [KM2] for partial results.) For example,

$$\tau_3(M)=\exp(-2\pi i/8)^{\mu(M)+(|H,M|^2-1)/2}$$

if M is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere with μ-invariant $\mu(M)$ (6.5), and

$$\tau_4(M)=\sum_{\Theta}\exp(-6\pi i/16)^{\mu(M_\Theta)}$$

for general M, where $\mu(M_\Theta)$ is the μ-invariant of M with spin structure Θ, and the sum is over all spin structures (7.1). The general formula (6.3) for $\tau_3(M)$ depends on the (mod 2) first betti number and the Brown invariant [Br] of M, and is therefore a homotopy invariant, whereas $\tau_4(M)$ can distinguish homotopy equivalent manifolds. (The derivation of the formula for $\tau_4(M)$ suggested a new, elementary combinatorial proof of Rohlin's theorem, which appears in Appendix C.)

Finally, a Symmetry Principle (4.20) is proved which reduces greatly the number of steps required in calculating $\tau_r(M)$, and (coupled with the cabling formula) leads to an elementary proof of its invariance (see § 5). In particular, this proof avoids the difficult analysis of the structure of tensor products of irreducible representations of $\mathscr{A}$, which is central to the treatment of [RT2]. Note also the application to Jones polynomials of cables of a framed link at a root of unity (4.25).

The Symmetry Principle has many other interesting consequences. Some of these are discussed in § 8:

(1) For r odd,

$$\tau_r(M)=\begin{cases}\tau_3(M)\,\tau_r'(M) & \text{if } r\equiv 3 \pmod 4\\ \overline{\tau_3(M)}\,\tau_r'(M) & \text{if } r\equiv 1 \pmod 4\end{cases}$$

where $\tau_r'(M)$ is an invariant of M (see 8.10). In particular, $\tau_3(M)=0$ implies that $\tau_r(M)=0$ for all odd r. Note: it is shown in (6.3) that $\tau_3(M)=0$ if and only if there exists α in $H^1(M; \mathbf{Z}/2\mathbf{Z})$ with $\alpha\smile\alpha\smile\alpha\neq 0$.

(2) For r divisible by 4,

$$\tau_r(M)=\sum_{\Theta}\tau_r(M, \Theta)$$

where $\tau_r(M, \Theta)$ is an invariant of the manifold M with spin structure Θ, and the sum is over all spin structures on M (see 8.27). A similar statement holds for $r\equiv 2 \pmod 4$, with the spin structure replaced by an element of $H^1(M; \mathbf{Z}/2\mathbf{Z})$ (8.32). This result has been observed independently by Turaev [T3], also using the Symmetry Principle.

(3) The Casson invariant of a homology sphere M obtained by Dehn surgery on a knot is determined (mod 5) by $\tau_5(M)$ (8.20).

(4) The Jones polynomial of a knot K at the fifth root of unity is an invariant for integral surgery on K (8.14).

(5) If $K_{1/n}$ denotes the homology sphere obtained by $1/n$ surgery on the knot K, then $\tau_r(K_{1/n})$ is periodic in n with period r for odd r (8.15) and period $r/2$ for even r (8.26).

The paper is organized as follows: In §1 there is a general discussion of framed links, the K-move and 3-manifolds, the colored framed link invariants $J_{L,\mathbf{k}}$, and the definition of $\tau_r(M)$. The algebra $\mathscr{A}$ (2.7) is described in §2, along with its finite dimensional representations V^k (2.8) and W_k^r (2.16). Explicit formulas for the R-matrix (2.18 and 2.32) are derived. In §3, the $\mathscr{A}$-linear tangle operators F_T are defined (3.6), which specialize to $J_{L,\mathbf{k}}$ (3.25). Their behavior under direct sums, extensions and tensor products of colors (3.10) and under changes in orientation (3.18) is explored. The cabling formula (4.15) and symmetry principle (4.20) for $J_{L,\mathbf{k}}$ are established in §4, and §5 contains the proof of the invariance of τ_r under K-moves, and of its behavior under connected sums and orientation reversal. The evaluations of $\tau_r(M)$ for $r=3$ and 4 are found in §6 and §7. In §8, applications of the Symmetry Principle to the study of τ_r for odd r (8.7–21) and even r (8.23–33) are given, and the new invariants mentioned above are defined. Appendix A contains a combinatorial proof of the deepest identity in the algebra needed to define τ_r. Appendix B has a treatment of the Jones polynomial at $q=e^{2\pi i/6}$, and Appendix C deals with μ-invariants.

In a future paper [KM4] we will calculate $\tau_r(M)$ for lens spaces and Seifert fibered 3-manifolds, and give a Dehn surgery formula. The calculation of $\tau_r(M)$ for lens spaces led to a new definition of the Dedekind sum in terms of signatures, and new formulae for signature defects and the signature cocycle defining a central extension of SL(2, $\mathbf{Z}$) by $\mathbf{Z}$ [KM3].

Acknowledgements. We are especially grateful to Nicolai Reshetikhin for his lectures and conversations on his work, and to Vaughan Jones, Greg Kuperberg and Antony Wasserman for valuable insights into quantum groups. We also thank Andrew Casson, Peter Gilkey, Raymond Lickorish, Jim Milgram, Peter Sarnak, Frank Schmidt and Larry Taylor for helpful conversations. Part of this work was carried out while the second author was visiting Stanford University and the University of California at Berkeley. We thank both institutions for their hospitality.

1 The invariants of Reshetikhin and Turaev

Fix an integer $r>1$. In this section we describe in general terms the *3-manifold invariant* τ_r of Reshetikhin and Turaev [RT2], which assigns a complex number $\tau_r(M)$ to each oriented, closed connected 3-manifold M. It satisfies the following properties (see (5.9)):

(1) (multiplicativity) $\tau_r(M \# N)=\tau_r(M)\,\tau_r(N)$
(2) (orientation) $\tau_r(-M)=\overline{\tau_r(M)}$
(3) (normalization) $\tau_r(S^3)=1$.

In fact, τ_r is a slight modification of the invariant that appears in [RT2, § 3.3.2], which does not satisfy (2) (see (1.4) below).

$\tau_r(M)$ is defined as a linear combination of certain *colored framed link invariants* $J_{L,\mathbf{k}}$ (defined in [RT1, § 5]) of a framed link L associated with M. The $J_{L,\mathbf{k}}$ are generalizations of the Jones polynomial of L, and are described in more detail below and in § 3.

First we fix some notation to be used throughout the paper. Writing $e(a)$ for $\exp(2\pi i a)$, set

$$q=e\left(\frac{1}{r}\right) \qquad s=e\left(\frac{1}{2r}\right) \qquad t=e\left(\frac{1}{4r}\right)$$

so $q=s^2=t^4$. $\left(\text{Reshetikhin and Turaev consider other roots of unity } e\left(\frac{m}{r}\right),\right.$ $\left.\text{but we restrict to } m=1 \text{ for simplicity.}\right)$ For any integer k, define

(1.1) $$[k]=\frac{s^k-\bar{s}^k}{s-\bar{s}}=\frac{\sin\dfrac{\pi k}{r}}{\sin\dfrac{\pi}{r}}$$

(cf. $[k]_q$ in (2.28) below). Observe that $[k]$ depends on r and has symmetries $[k]=[r-k]=-[k+r]$. Finally, set

(1.2) $$b=\sqrt{\frac{2}{r}}\sin\frac{\pi}{r}$$
$$c=e\left(-\frac{3(r-2)}{8r}\right).$$

Framed links and 3-manifolds

Let L be a framed link in S^3. Recall that L determines a smooth oriented 4-manifold W_L obtained by adding 2-handles to the 4-ball B^4, oriented as the unit ball in $\mathbf{C}^2$, along the components L_i of L in $S^3=\partial B^4$ [K2]. If L is oriented then each L_i is identified with an element of $H=H_2(W_L;\mathbf{Z})$, also called L_i, formed from an oriented Seifert surface for L_i in S^3 and the core of the associated 2-handle. The L_i form a basis for H, and with respect to this basis the intersection form on H, denoted by $\cdot$, coincides with the linking matrix of L. That is, $L_i \cdot L_j = lk(L_i \cdot L_j)$ for $i \neq j$ and $L_i \cdot L_i$ is the framing on L_i. We write σ_L (or σ) for the *signature* of the linking matrix of L, or equivalently the index of W_L.

The 3-manifold $M_L = \partial W_L$, oriented using the "outward first" convention for boundaries, is the result of surgery on L in S^3. Any oriented 3-manifold M may be obtained in this way [L1, Wa], and if $M = M_L = M_{L'}$, then one can pass from L to L' by isotopy in S^3 and a combination of the following two moves [K1]:

Move 1 (blow up). *Add (or delete) a disjoint unknotted component with framing* ± 1.

Move 2 (handle slide). *For some $i \neq j$, replace L_i with $L_i' = L_i \# L_j$, a band connected sum of L_i with a push off of L_j (along the first vector in the framing), with framing $L_i' \cdot L_i' = (L_i + L_j) \cdot (L_i + L_j)$.*

In Move 1, *disjoint* means separated by a 2-sphere from the rest of the link. Move 2 corresponds to sliding the 2-handle for L_i over the 2-handle for L_j. These two moves can be combined into one [FR] which is more convenient for the work of Reshetikhin and Turaev [RT2]:

m-strand **K-move** (of type $\varepsilon = \pm 1$). *Locally, the following are interchangeable*

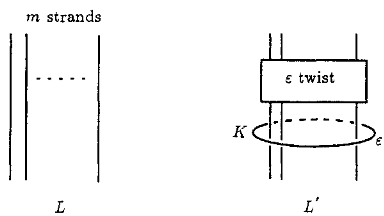


Fig. 1.3

where the framings on corresponding components J and J' of L and L' are related by $J' \cdot J' = J \cdot J + \varepsilon (K \cdot J')^2$.

The colored framed link invariants

At the heart of the 3-manifold invariant $\tau_r(M)$ are the colored framed link invariants $J_{L,\mathbf{k}}$. Here L is a framed link in S^3 with $M = M_L$ and $\mathbf{k}$ is a *coloring* of L, i.e. the assignment of an $\mathscr{A}$-module (or *color*) to each component of L, where $\mathscr{A}$ is a certain Hopf algebra over $\mathbf{C}$ (that depends on the fixed integer r) arising in the theory of *quantum groups*.

The algebra $\mathscr{A}$ will be described in detail in § 2. The reason for using a *Hopf* algebra is that the set of representations (i.e. $\mathscr{A}$-modules) is closed under taking tensor products and duals over the ground field $\mathbf{C}$. This is important for the construction of $J_{L,\mathbf{k}}$, given in § 3.

Here we will give a heuristic description of $J_{L,\mathbf{k}}$ in the language of topological quantum field theory (see e.g. [AHLS]). Orient L and represent it by a planar diagram D. After removing the extreme points (maxima and minima) of D,

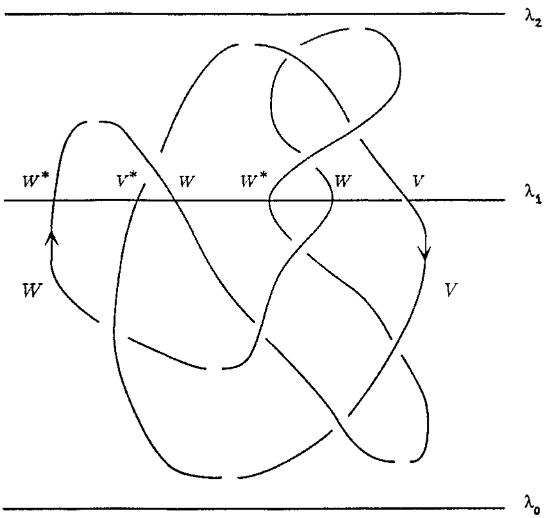


Fig. 1.4

assign the module V, or its dual V^*, to each V-colored strand of L (according to whether the strand is oriented down or up, see Fig. 1.4).

Any horizontal line λ which avoids crossings and extreme points hits D in a collection of points labeled by the colors and their duals. Associate to this line λ the module V_λ which is the tensor product of the labels in order. In Fig. 1.4, $V_{\lambda_0} = \mathbf{C} = V_{\lambda_2}$ and $V_{\lambda_1} = W^* \otimes V^* \otimes W \otimes W^* \otimes W \otimes V$. To each extreme point and each crossing, assign an $\mathscr{A}$-linear operator from the module just below to the one just above. The composition of these operators maps $\mathbf{C}$ to $\mathbf{C}$. Hence it is multiplication by a scalar which (after adjusting for framings) is defined to be $J_{L,\mathbf{k}}$.

This construction should be independent of the orientation on L and the chosen diagram D, so as to give an invariant of (unoriented) colored framed links. Suitable operators are provided in [RT1] using additional structure that exists on the algebra $\mathscr{A}$.

The 3-manifold invariant

Recall that r is a fixed integer >1. To define the 3-manifold invariant in terms of colored framed link invariants $J_{L,\mathbf{k}}$, it is necessary to restrict the colorings $\mathbf{k}$ to lie in a distinguished family $\mathscr{M}$ of $\mathscr{A}$-modules, consisting of one irreducible module V^k in each dimension $0<k<r$ (see § 2). We call these *$\mathscr{M}$-colorings* or write $\mathbf{k} \subset \mathscr{M}$, and denote the dimension of the color assigned to the component L_i of L by k_i, also called the *color* of L_i.

Now consider the following linear combination of colored framed link invariants.

(1.5) **Definition.** For any framed link L, define

$$\tau_L = \alpha_L \sum_{\mathbf{k} \subset \mathcal{M}} [\mathbf{k}] J_{L,\mathbf{k}}$$

where $\alpha_L = b^{n_L} c^{\sigma_L}$ and $[\mathbf{k}] = \prod_{i=1}^{n} [k_i]$. (Recall (1.2) that $b = \sqrt{\frac{2}{r}} \sin \frac{\pi}{r}$, $c = e\left(-\frac{3(r-2)}{8r}\right)$; $n = n_L$ is the number of components in L and σ_L is the signature of the linking matrix of L.) Note that since $\mathcal{M} = \{V^1, \ldots, V^{r-1}\}$, the sum $\sum_{\mathbf{k} \subset \mathcal{M}}$ may be written in multi-index notation as $\sum_{\mathbf{0} < \mathbf{k} < \mathbf{r}}$ (see § 4).

(1.6) **Theorem** [RT2] *τ_L is invariant under K-moves on L.*

Theorem 1.6 can be proved for 0 or 1-strand K-moves using three local properties of $J_{L,\mathbf{k}}$ and a standard Gauss sum (see 3.27, 5.1 and 5.4). Then the proof for m-strand K-moves is an easy inductive argument using the Symmetry Principle 4.20 (see (5.6)).

It follows from Theorem 1.6 and [K1, FR] that there is a well defined invariant for closed, oriented 3-manifolds:

(1.7) **Definition.** $\tau_r(M) = \tau_L$, where L is any framed link with $M = M_L$.

(1.8) *Remarks.* (1) The invariant that actually appears in [RT2] differs from $\tau_r(M)$ by a factor of c^ν, where ν is the nullity of the linking matrix of L (= first Betti number of M). The advantage of $\tau_r(M)$ is that it behaves nicely under orientation reversal.

As a convenience to the reader, here is a dictionary relating the notation of [RT2] with the notation in this paper: The algebra U_t of [RT2, § 8] (where $t = e(1/4r) = \exp(2\pi i/4r)$ as above) is our $\mathcal{A} = \mathcal{A}_r$, and the U_t-module V_i is our V^{i+1}. (Note that in U_t, the element uv^{-1} is just K^2 and $\dim V_i = [i+1]$). The link invariant $F(\Gamma(L, \omega, \lambda))$ of [RT2, § 3] (where ω is the orientation on L and λ is the coloring) corresponds to our $J_{L,\lambda}$. The constant C defined in [RT2, § 3.2 and 8.3.8] is our c^2, the coefficient d_i is equal to our $bc[i+1]$, and so the invariant $\{L\} = (bc)^n \sum_{\mathbf{k} \subset \mathcal{M}} [\mathbf{k}] J_{L,\mathbf{k}}$. Thus the 3-manifold invariant defined in [RT2, § 3.3.2] is

$$\mathcal{F}(M, L) = b^n c^{\sigma - \nu} \sum_{\mathbf{k} \subset \mathcal{M}} [\mathbf{k}] J_{L,\mathbf{k}} = c^{-\nu} \tau_r(M).$$

(2) It is often useful for calculations to write $\tau_r(M)$ (where $M = M_L$) in other ways. For example, let $a_i = L_i \cdot L_i$ be the framing on L_i, and write L_0 for the link L with all framings changed to zero. It is shown in Lemma 3.27b below that $J_{L,\mathbf{k}}$ changes by $t^{\pm(k_i^2 - 1)}$ if the framing of L_i is changed by ± 1. Thus $J_{L,\mathbf{k}} = t^{\Sigma a_i (k_i^2 - 1)} J_{L_0,\mathbf{k}}$. Now, since $c = t^3 e(-\frac{3}{8})$, (1.5) becomes

$$\tau_r(M) = b^n e(-\tfrac{3}{8})^\sigma t^{3\sigma - \Sigma a_i} \sum_{\mathbf{k} \subset \mathcal{M}} [\mathbf{k}] t^{\Sigma a_i k_i^2} J_{L_0,\mathbf{k}}. \tag{1.9}$$

The term $\varphi(L) = 3\sigma - \sum a_i$ has been identified in [FG] as the difference between the 2-framing on M determined naturally by L and the canonical 2-framing on M [A].

Furthermore, the Gauss sum $G=\sum_{k=1}^{4r} t^{k^2}$ (see [La] and (5.1)) equals $\sqrt{8r}\,e(\frac{1}{8})$, and so $b=\frac{G}{4r}(s-\bar{s})\,e(-\frac{3}{8})$ since $s-\bar{s}=2i\sin\left(\frac{\pi}{r}\right)$. Formula (1.9) becomes

$$\tau_r(M)=\left(G\frac{(s-\bar{s})}{4r}\right)^n e(-\tfrac{3}{8})^{n+\sigma}\, t^{3\sigma-\Sigma a_i} \sum_{\mathbf{k}\subset\mathcal{M}} [\mathbf{k}]\, t^{\Sigma a_i k_i^2} J_{L_0,\mathbf{k}}. \tag{1.10}$$

It follows that $\tau_r(M)$ belongs to the ring $\mathbf{Z}[t^{\pm\frac{1}{4}}, r^{-1}]$ since the $\frac{1}{4^n}$ disappears using results from §8. It is necessary to use $t^{\pm\frac{1}{4}}$ rather than t, since for example (see (5.11) and (6.3)) $\tau_3(S^1\times S^2)=\frac{1}{b}=\sqrt{2}=t^{3/2}-\bar{t}^{3/2}$. (Integer powers of t suffice if the nullity is put back into τ, see (1.8)). Question: is $\tau_r(M)$ always an element of $\mathbf{Z}[t^{\pm\frac{1}{4}}]$?

2 The quasitriangular Hopf algebra $\mathcal{A}$

In this section we shall define the Hopf algebra $\mathcal{A}$ and produce an R-matrix R in $\mathcal{A}\otimes\mathcal{A}$ making $\mathcal{A}$ a quasitriangular Hopf algebra. We also show that the associated operators $\check{R}$ in representations of $\mathcal{A}$ satisfy the Yang-Baxter equation.

Throughout this section, r will be a fixed integer greater than 1.

The algebra $\mathcal{A}$

As motivation, we first recall the definition of the Lie algebra sl(2, $\mathbf{C}$) and its representations: sl(2, $\mathbf{C}$) is a 3-dimensional complex vector space with preferred basis X, Y, H and Lie bracket given by $[H,X]=2X$, $[H,Y]=-2Y$ and $[X,Y]=H$. It has a unique (up to isomorphism) k-dimensional irreducible representation V^k for each positive integer k. Explicitly, sl(2, $\mathbf{C}$) acts on V^k (with preferred ordered basis $e_m, e_{m-1}, \ldots, e_{-m}$ where $k=2m+1$) by

$$\begin{aligned} Xe_j&=(m+j+1)\,e_{j+1}\\ Ye_j&=(m-j+1)\,e_{j-1}\\ He_j&=2j\,e_j. \end{aligned} \tag{2.1}$$

Note that the subscripts are integers if k is odd and half integers if k is even.

For example, the 1, 2 and 3-dimensional representations of sl(2, $\mathbf{C}$) are

(2.2) (1) $X=Y=H=0$

(2) $X=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $Y=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and $H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

(3) $X=\begin{pmatrix}0&2&0\\0&0&1\\0&0&0\end{pmatrix}$, $Y=\begin{pmatrix}0&0&0\\1&0&0\\0&2&0\end{pmatrix}$ and $H=\begin{pmatrix}2&0&0\\0&0&0\\0&0&-2\end{pmatrix}$

respectively.

We extend this discussion to the *universal enveloping algebra* $\mathbf{U} = U(\mathrm{sl}(2, \mathbf{C}))$, which is just the associative algebra over $\mathbf{C}$ with the same generators and relations as $\mathrm{sl}(2, \mathbf{C})$. (The bracket is interpreted as for matrices. Thus $HX - XH = 2X$, or equivalently $HX = X(H+2)$. Similarly $HY = Y(H-2)$ and $XY - YX = H$.) The representations above evidently extend to algebra representations of $\mathbf{U}$ and so there are unique irreducible $\mathbf{U}$-modules V^k in each dimension.

Note that $\mathbf{U}$ has a Hopf algebra structure (with comultiplication $\Delta: \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$, antipode $S: \mathbf{U} \to \mathbf{U}$ and counit $\varepsilon: \mathbf{U} \to \mathbf{C}$ given by $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$, $S(\alpha) = -\alpha$ and $\varepsilon(\alpha) = 0$ for all α in $\mathrm{sl}(2, \mathbf{C})$). This allows one to define $\mathbf{U}$-module structures on the duals $V^* = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ and tensor products $V \otimes W = V \otimes_{\mathbf{C}} W$ of $\mathbf{U}$-modules V and W. In particular, $(\alpha f)(v) = f(S(\alpha)v)$ and $\alpha(v \otimes w) = \Delta\alpha \cdot (v \otimes w)$ (where $\mathbf{U} \otimes \mathbf{U}$ acts diagonally on $V \otimes W$), for $\alpha \in \mathbf{U}$, $f \in V^*$, $v \in V$ and $w \in W$. (This is the reason for using *Hopf* algebras in the construction of the colored framed link invariants.)

Next we consider the *quantized universal enveloping algebra* $\mathbf{U}_h = U_h(\mathrm{sl}(2, \mathbf{C}))$, found by Kulish and Reshetikhin [KuR]. It can be defined as the algebra $\mathbf{U}[[h]]$ of *formal* power series in h (= Planck's constant for the physicists) with coefficients in $\mathbf{U}$, with the same relations as in $\mathbf{U}$ except that $[X, Y] = H$ is replaced with

$$[X, Y] = \frac{\sinh(hH/2)}{\sinh(h/2)} = H + \frac{H^3 - H}{24} h^2 + \dots.$$

Setting $q = e^h$ and (in analogy with the usual notation) $s = e^{h/2}$, $t = e^{h/4}$, $\bar{s} = s^{-1} = e^{-h/2}$ and

$$[H] = \frac{s^H - \bar{s}^H}{s - \bar{s}},$$

these relations may be written

(2.3)
$$HX = X(H+2)$$
$$HY = Y(H-2)$$
$$[X, Y] = [H].$$

If we introduce the element

$$K = t^H = e^{\frac{hH}{4}} = 1 + \frac{H}{4} h + \frac{H^2}{2!\,4^2} h^2 + \dots$$

(which will play an important role in the sequel), then we obtain the associated relations

(2.4)
$$KX = sXK$$
$$KY = \bar{s}YK$$
$$[X, Y] = [H] = \frac{K^2 - \bar{K}^2}{s - \bar{s}}$$

where $\bar{K} = K^{-1} = \bar{t}^H$.

There is a Hopf algebra structure on $\mathbf{U}_h$ as a module over the ring $\mathbf{C}[[h]]$ of formal power series, discovered by Sklyanin [Sk], with comultiplication Δ, antipode S and counit ε given by

(2.5)
$$\begin{aligned}\Delta(X)&=X\otimes K+\bar{K}\otimes X\\ \Delta(Y)&=Y\otimes K+\bar{K}\otimes Y\\ \Delta(H)&=H\otimes 1+1\otimes H\\ S(X)&=-sX\\ S(Y)&=-\bar{s}Y\\ S(H)&=-H\\ \varepsilon(X)&=\varepsilon(Y)=\varepsilon(H)=0.\end{aligned}$$

One may readily compute

(2.6)
$$\begin{aligned}\Delta(K)&=K\otimes K\\ S(K)&=\bar{K}\\ \varepsilon(K)&=1.\end{aligned}$$

We would like to specialize $\mathbf{U}_h$ at particular values of h, namely $h=\frac{2\pi i}{r}$ $\left(\text{so } q=e^h=e\left(\frac{1}{r}\right)\right)$, and then look for complex representations. This cannot be done using the full algebra $\mathbf{U}_h$, because of the presence of divergent series, and so we first restrict to the subalgebra, over the ring of *convergent* power series in h (i.e. entire functions), generated by X, Y, K and $\bar{K}$. Now, following Reshetikhin and Turaev, we define

$$\mathscr{A}=\mathscr{A}_r$$

(denoted $\mathbf{U}_t$ in [RT2]) to be the quotient of this subalgebra obtained by setting $h=\frac{2\pi i}{r}$, $X^r=0$, $Y^r=0$ and $K^{4r}=1$. (Omitting the last three relations yields the infinite dimensional algebra $\mathbf{U}_q$ of [Ji, RT1] known as the *q-analogue* of $\mathbf{U}$.) Thus $\mathscr{A}$ is a finite dimensional algebra over $\mathbf{C}$ with generators X, Y, K, $\bar{K}$ and relations

(2.7)
$$\begin{aligned}\bar{K}&=K^{-1}\\ KX&=sXK\\ KY&=\bar{s}YK\\ [X,Y]&=\frac{K^2-\bar{K}^2}{s-\bar{s}}\\ X^r&=Y^r=0\\ K^{4r}&=1\end{aligned}$$

where $s=e\left(\frac{1}{2r}\right)$, as usual. (We will retain the notation

$$[H]=\frac{s^H-\bar{s}^H}{s-\bar{s}}=\frac{K^2-\bar{K}^2}{s-\bar{s}}$$

even though H is no longer in our algebra.) $\mathscr{A}$ acquires a complex Hopf algebra structure from $\mathbf{U}_h$, and so tensor products and duals (over $\mathbf{C}$) of $\mathscr{A}$-modules are still $\mathscr{A}$-modules.

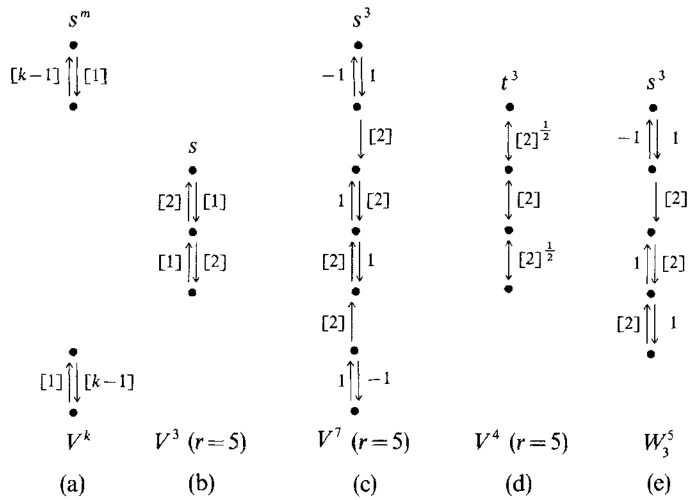


Fig. 2.11

Representations of $\mathscr{A}$

As with **U**, there are $\mathscr{A}$-modules V^k in each dimension $k>0$. In particular, $\mathscr{A}$ acts on V^k (with basis $e_m, \ldots, e_{-m}$, for $k=2m+1$) by

(2.8)
$$\begin{aligned} Xe_j &= [m+j+1]\, e_{j+1} \\ Ye_j &= [m-j+1]\, e_{j-1} \\ Ke_j &= s^j e_j = t^{2j} e_j \end{aligned}$$

(cf. (2.1), but note the brackets). The relation $[X, Y]=[H]$ follows from the identity $[a][b]-[a+1][b-1]=[a-b+1]$.

For example, the 1, 2 and 3 dimensional representations of $\mathscr{A}$ are

(2.9) (1) $X=0, Y=0$ and $K=1$

(2) $X=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $K=\begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix}$

(3) $X=\begin{pmatrix} 0 & [2] & 0 \\ 0 & 0 & [1] \\ 0 & 0 & 0 \end{pmatrix}$, $Y=\begin{pmatrix} 0 & 0 & 0 \\ [1] & 0 & 0 \\ 0 & [2] & 0 \end{pmatrix}$ and $K=\begin{pmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{s} \end{pmatrix}$

respectively.

(2.10) *Remark* It is useful to represent V^k by a graph in the plane with one vertex at height j for each basis vector e_j, and with oriented edges from e_j to $e_{j\pm 1}$ labeled by $[m\pm j+1]$ if $[m\pm j+1]\neq 0$ (recall $k=2m+1$), indicating the actions of X and Y. This graph, with the top vertex labeled by its *weight* (i.e. eigenvalue for K) s^m, is called the *diagram* of V^k with respect to $e_m, \ldots, e_{-m}$. (See Fig. 2.11a, with the special cases $k=3$ and 7 for $r=5$ shown in 2.11b and c, using the identities $[j]=[r-j]=-[r+j]$.) Similar diagrams can be used to describe other finite dimensional $\mathcal{A}$-modules with respect to bases of *weight vectors* (i.e. eigenvectors for K).

The diagram of a module is often simplified by rescaling the basis (i.e. changing the lengths of the basis vectors). In particular, if v is changed to av, then the label on each edge which starts (or ends) at the corresponding vertex is multiplied (or divided) by a.

For example for $k \leqq r$, the basis $e_m, \ldots, e_{-m}$ for V^k can be rescaled to a unique basis $b_m, \ldots, b_{-m}$ (up to a multiple), called a *balanced* basis, with

$$Xb_j = ([m+j+1][m-j])^{1/2} b_{j+1}$$
$$Yb_j = ([m-j+1][m+j])^{1/2} b_{j-1}$$
$$Kb_j = s^j b_j.$$

$\left(\text{Indeed } e_j = \begin{bmatrix} 2m \\ m-j \end{bmatrix}^{1/2} b_j, \text{ where } \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!} \text{ is the quantized binomial coefficient, cf. (2.29).}\right)$ Observe that in the corresponding diagram, any two oppositely oriented edges with the same end points have equal labels of the form $([i][k-i])^{1/2}$, and so can be combined into one doubly oriented edge. The case $r=5$, $k=4$ is shown in Fig. 2.11d.

It turns out that the representations V^k are irreducible if and only if $k \leqq r$ (see below), and so it is natural to define the distinguished family

(2.12) $$\mathcal{M} = \{V^1, \ldots, V^{r-1}\}$$

of $\mathcal{A}$-modules to be used in constructing the 3-manifold invariant τ_r. (V^r is excluded for technical reasons; see Lemma 3.29 below.)

The structure of the $\mathcal{A}$-modules V^k for $k \leqq r$, and their tensor products $V^k \otimes V^{k'}$ for $k+k' \leqq r+1$, is parallel to the classical case, and is summarized in the following well known result (see e.g. [Ji, Lu, RT2]).

(2.13) **Theorem** *If $k \leq r$, then the modules V^k are irreducible and self dual. In particular, the map $D: (V^k)^* \to V^k$ given by $D(b^j) = (-s)^j b_{-j}$, where b_j is a balanced basis with dual basis b^j, is an $\mathcal{A}$-linear isomorphism.* $\left(\textit{Equivalently, } D(e^j) = \begin{bmatrix} 2m \\ m-j \end{bmatrix}^{-1} (-s)^j e_{-j}.\right)$

Furthermore, if $k+k' \leq r+1$, then

$$V^k \otimes V^{k'} = \bigoplus_{p \in k \otimes k'} V^p$$

where $k \otimes k' = \{k+k'-1, k+k'-3, \ldots, |k-k'|+1\}$.

Proof. It is evident from its diagram that for $k \leqq r$, V^k contains no proper submodules generated by weight vectors e_j. But for $k < 2r$, *every* submodule of V^k is generated by weight vectors, since the weights of the e_j are distinct. Indeed, for any nonzero vector $v = \sum a_j e_j$, each e_j is a multiple of v by a suitable polynomial in K, and so the submodules generated by v and by the e_j with $a_j \neq 0$ coincide. It follows that V^k is irreducible for $k \leqq r$.

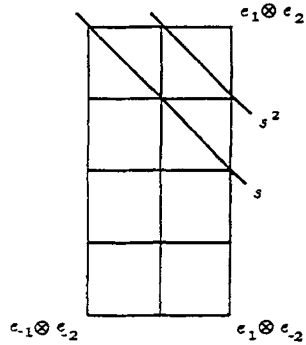


Fig. 2.14

For the second statement, note that a C-linear isomorphism D is $\mathscr{A}$-linear provided $D(\alpha b^j)=\alpha D(b^j)$ for $\alpha=K$, X and Y. For D of the form $D(b^j)=c_j b_{-j}$, this imposes no restriction on the c_j for $\alpha=K$, and the sole restriction $c_j=(-s)\,c_{j-1}$ for $\alpha=X$ or Y. To see this, one may compute the action of $\mathscr{A}$ on V^{k*}:

$$Xe^j=-s[m+j]\,e^{j-1}$$
$$Ye^j=-\bar{s}[m-j]\,e^{j+1}$$
$$Ke^j=\bar{s}^j e^j,$$

or in the balanced basis,

$$Xb^j=-s([m-j+1]\,[m+j])^{1/2}\,b^{j-1}$$
$$Yb^j=-\bar{s}([m+j+1]\,[m-j])^{1/2}\,b^{j+1}$$
$$Kb^j=\bar{s}^j\,b^j.$$

$\left(\text{Note that } b^j=\begin{bmatrix}2m\\ m-j\end{bmatrix}^{1/2} e^j.\right)$ The self duality result follows.

Finally, to verify the decomposition of $V^k\otimes V^{k'}$ observe that there are weight vectors v_p in $V^k\otimes V^{k'}$ of weight t^{p-1} with $Xv_p=0$ for each p in $k\otimes k'$, since the dimensions of the corresponding weight spaces decrease as p increases. (To see this it is useful to use the diagram for $V^k\otimes V^{k'}$ with respect to the weight basis $e_i\otimes e_j$ with vertices at (i,j), see Fig. 2.14 for the case $V^3\otimes V^5$.) It follows that the submodule generated by v_p is isomorphic to V^p. These subspaces are independent since any collection of equal weight vectors lying in distinct V^p are annihilated by distinct powers of X. A dimension count completes the argument. □

As a consequence of Theorem 2.13, each of the irreducible modules V^k can be expressed as a linear combination of powers of V^2 in the representation ring of $\mathscr{A}$ (where for example $2V+UW$ means $V\oplus V\oplus(U\otimes W)$, $(V)^2$ means $V\otimes V$ (not V^2), and $U=V-W$ means $U\oplus W=V$):

(2.15) **Corollary** *For $0 \leqq n < r$, the equality*

$$V^{n+1} = \sum_{j=0}^{n/2} (-1)^j \binom{n-j}{j} (V^2)^{n-2j}$$

holds in the representation ring, where the sum is over all integers j with $0 \leqq 2j \leqq n$.

Proof. Theorem 2.13 implies that $V^{n+1} = V^2 V^n - V^{n-1}$. Thus by induction (which starts trivially at $n=0$),

$$\begin{aligned} V^{n+1} &= V^2 \sum_j (-1)^j \binom{n-1-j}{j} (V^2)^{n-1-2j} - \sum_j (-1)^j \binom{n-2-j}{j} (V^2)^{n-2-2j} \\ &= \binom{n-1}{0} (V^2)^n - \binom{n-2}{1} (V^2)^{n-2} + \binom{n-3}{2} (V^2)^{n-4} - + \dots \\ &\quad - \binom{n-2}{0} (V^2)^{n-2} + \binom{n-3}{1} (V^2)^{n-4} - + \dots \\ &= \binom{n}{0} (V^2)^n - \binom{n-1}{1} (V^2)^{n-2} + \binom{n-2}{2} (V^2)^{n-4} - + \dots \\ &= \sum_j (-1)^j \binom{n-j}{j} (V^2)^{n-2j}. \quad \square \end{aligned}$$

It is amusing to note that the same identity holds with the bracket $[n]$ replacing V^n. The same proof works.

For small values of n, we have

$$\begin{aligned} V^3 &= V^2 \otimes V^2 - V^1 \\ V^4 &= V^2 \otimes V^2 \otimes V^2 - 2V^2 \\ V^5 &= V^2 \otimes V^2 \otimes V^2 \otimes V^2 - 3V^2 \otimes V^2 + V^1. \end{aligned}$$

Remark. The structure of the tensor product $V^k \otimes V^{k'}$ for $k + k' > r + 1$ is more complicated. This has been analyzed by Reshetikhin and Turaev [RT2] (and independently by A. Wasserman and J. Frölich-G. Keller) and is central to their proof of the invariance of the 3-manifold invariant. We will give a different proof of the invariance in §5 which depends on the Symmetry Principle 4.20. This in turn is based on the structure of certain r-dimensional $\mathscr{A}$-modules W_p^r discussed below (which arise as well in the general discussion of tensor products).

In contrast with the case of $\mathbf{U}$, the $\mathscr{A}$-modules V^k are reducible for $k > r$. In particular, the subspace V_k^r generated by e_j for $j > m - r$ is an r-dimensional submodule, since $Ye_{m-r+1} = [r]\, e_{m-r} = 0$. These modules are called *Verma modules.*

Observe that $V_r^r = V^r$, and (as is easily seen using Remark 2.10) V_k^r and V_{k+4r}^r are isomorphic. If $r \leqq k < 2r$, then V_k^r contains V^p, where $p = 2r - k$, as its *unique* proper submodule, which motivates the notation adopted in [RT2]

$$W_p^r = V_{2r-p}^r$$

for $0<p<r$. (See Fig. 2.11e for the diagram of $W_3^5=V_7^5$.) Indeed, it is clear from the diagram of W_p^r that V^p is the only submodule generated by weight vectors, but every submodule is of this form by the argument in the proof of Theorem 2.13.

In fact, the Verma modules W_p^r may be described as extensions of V^p by a twisted version $V^{r-p}(i)$ of V^{r-p}. In particular, observe that there are exactly four 1-dimensional $\mathscr{A}$-modules $V^1(\alpha)$, where $\alpha^4=1$, given by $K=\alpha$ and $X=Y=0$. (The value of K follows from the relation $[X, Y]=\dfrac{K^2-\bar{K}^2}{s-\bar{s}}$, which gives $K^4=1$ since $[X, Y]=0$ in $\mathbf{C}$, and the values of X and Y are immediate from the relations $KX=sXK$ and $KY=\bar{s}YK$.) For any $\mathscr{A}$-module V, put

$$V(\alpha)=V\otimes V^1(\alpha).$$

(A diagram for $V(\alpha)$ is obtained from one for V by multiplying the vertex weights by α.) It is now easy, using Remark 2.10, to establish the following result.

(2.16) **Lemma** [RT2, § 8.4] *There is a short exact sequence*

$$0\to V^p\to W_p^r\to V^{r-p}(i)\to 0$$

for $0<p<r$, where V^p is the unique proper submodule of W_p^r.

Similar considerations apply to the general Verma modules V_k^r, since V_{k+nr}^r is isomorphic to $V_k^r(i^n)$.

The R-matrix

The algebra $\mathscr{A}$ has the additional structure of a *quasi-triangular* Hopf algebra (see Drinfeld [D2]). That is, there exists an invertible element R in $\mathscr{A}\otimes\mathscr{A}$ satisfying the following properties:

(2.17) (a) $R\Delta(\alpha)R^{-1}=\check{\Delta}(\alpha)$ for all α in $\mathscr{A}$

(b) $(\Delta\otimes\mathrm{id})(R)=R_{13}R_{23}$

$(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12}$

where $\check{\Delta}(\alpha)=P(\Delta(\alpha))$, P is the permutation endomorphism of $\mathscr{A}$ given by P $P(\alpha\otimes\beta)=\beta\otimes\alpha$, $R_{12}=R\otimes 1$, $R_{23}=1\otimes R$ and $R_{13}=(P\otimes\mathrm{id})(R_{23})$. (Explicitly, if $R=\sum\alpha_i\otimes\beta_i$, then $R_{12}=\sum\alpha_i\otimes\beta_i\otimes 1$, $R_{23}=\sum 1\otimes\alpha_i\otimes\beta_i$ and $R_{13}=\sum\alpha_i\otimes 1\otimes\beta_i$.) Such an element R is called a *universal R-matrix* for $\mathscr{A}$, and is the central ingredient in the definition of the colored framed link invariants.

Historically, an R-matrix was first discovered in the algebra $\mathbf{U}_h$ by Drinfeld [D1] and independently by Jimbo [J] in the algebra $\mathbf{U}_q$. R-matrices in $\mathscr{A}$ have been written down by several authors, including Reshetikhin and Turaev [RT2] and A. Wasserman. We give a formula for R of the form $R=\sum c_{nab}X^nK^a\otimes Y^nK^b$, which is derived by recursively solving for the constants c_{nab} using the defining relation (2.17a). (This approach to finding R was suggested to us by A. Wasserman, who has previously carried out a similar calculation.)

(2.18) **Theorem** *The element*

$$R=\frac{1}{4r}\sum_{n,a,b}\frac{(s-\bar{s})^n}{[n]!}\,\bar{t}^{ab+(b-a)n+n}X^nK^a\otimes Y^nK^b,$$

where the sum is over all $0\leqq n<r$ *and* $0\leqq a,\ b<4r$, *is a universal R-matrix for* $\mathscr{A}$.

Proof. As mentioned above, we assume that there exists an R-matrix of the form

$$R=\sum_{n,a,b}c_{nab}X^nK^a\otimes Y^nK^b\qquad 0\leqq n<r,\ 0\leqq a,\ b<4r, \tag{2.19}$$

and then find c_{nab} by solving $R\Delta(X)=\breve{\Delta}(X)R$ and $R\Delta(Y)=\breve{\Delta}(Y)R$.

First note the following commutation relations in $\mathscr{A}$ (see 2.7):

$$\begin{gathered}KX=sXK\quad\text{and}\quad[H]X=X[H+2]\\ KY=\bar{s}YK\quad\text{and}\quad[H]Y=Y[H-2]\\ YX=XY-[H]\end{gathered}\tag{2.20}$$

where $[H+n]=\dfrac{s^{H+n}-\bar{s}^{H+n}}{s-\bar{s}}=\dfrac{s^nK^2-\bar{s}^n\bar{K}^2}{s-\bar{s}}$. By induction, using the identity

$$[a][H+c+b]+[b][H+c-a]=[a+b][H+c],$$

we can generalize the last relation in (2.20) to

$$\begin{aligned}Y^nX&=XY^n-[n][H+n-1]Y^{n-1}\\ YX^n&=X^nY-[n][H-n+1]X^{n-1}.\end{aligned}$$

Now we have (recalling from (2.5) that $\Delta(X)=X\otimes K+\bar{K}\otimes X$)

$$\begin{aligned}0&=R\Delta(X)-\breve{\Delta}(X)R\\ &=\sum_{n,a,b}c_{nab}(X^nK^aX\otimes Y^nK^{b+1}+X^nK^{a-1}\otimes Y^nK^bX\\ &\qquad-KX^nK^a\otimes XY^nK^b-X^{n+1}K^a\otimes\bar{K}Y^nK^b)\\ &=\sum_{n,a,b}c_{nab}(s^aX^{n+1}K^a\otimes Y^nK^{b+1}+s^bX^nK^{a-1}\otimes XY^nK^b\\ &\qquad-\frac{s^b[n]}{s-\bar{s}}(\bar{s}^{n-1}X^nK^{a-1}\otimes Y^{n-1}K^{b+2}-s^{n-1}X^nK^{a-1}\otimes Y^{n-1}K^{b-2})\\ &\qquad-s^nX^nK^{a+1}\otimes XY^nK^b-s^nX^{n+1}K^a\otimes Y^nK^{b-1}).\end{aligned}$$

Thus the coefficient of $X^nK^a\otimes XY^nK^b$ is $-s^nc_{n,a-1,b}+s^bc_{n,a+1,b}=0$ which implies that

$$c_{n,a+2,b}=s^{n-b}c_{nab}=\bar{s}^{b-n}c_{nab}. \tag{2.21}$$

Also, the coefficient of $X^n K^a \otimes Y^{n-1} K^b$ is

(2.22)
$$s^a c_{n-1,a,b-1} - \frac{s^{b-n-1}[n]}{s-\bar{s}} c_{n,a+1,b-2} + \frac{s^{b+n+1}[n]}{s-\bar{s}} c_{n,a+1,b+2} - s^{n-1} c_{n-1,a,b+1} = 0.$$

Similarly, using $R\Delta(Y) - \check{\Delta}(Y)R = 0$, we get

$$0 = \sum_{n,a,b} c_{nab}(\bar{s}^a X^n YK^a \otimes Y^n K^{b+1} + \bar{s}^b X^n K^{a-1} \otimes Y^{n+1} K^b - s^n X^a K^{a+1} \otimes Y^{n+1} K^b - s^n X^n YK^a \otimes Y^n K^{b-1} + \frac{s^n[n]}{s-\bar{s}}(s^{n-1} X^{n-1} K^{a+2} \otimes Y^n K^{b-1} - \bar{s}^{n-1} X^{n-1} K^{a-2} \otimes Y^n K^{b-1}),$$

and so the coefficient of $X^n YK^a \otimes Y^n K^b$ is $\bar{s}^a c_{n,a,b-1} - s^n c_{n,a,b+1} = 0$ which implies

(2.23)
$$c_{n,a,b+2} = \bar{s}^{a+n} c_{n,a,b}.$$

Also, the coefficient of $X^{n-1} K^a \otimes Y^n K^b$ is

(2.24)
$$\bar{s}^b c_{n-1,a+1,b} - s^{n-1} c_{n-1,a-1,b} + \frac{[n]}{s-\bar{s}} s^{2n-1} c_{n,a-2,b+1} - \frac{[n]}{s-\bar{s}} s c_{n,a+2,b+1} = 0.$$

Using (2.22) and (2.23) we obtain

(2.25)
$$c_{n,a,b} = \frac{s-\bar{s}}{[n]\, s^b} c_{n-1,a-1,b-1}$$

which can also be obtained from (2.21) and (2.24).

If we choose $c_{0,0,0} = 1$, then from (2.21) and (2.23) $c_{0,2a,2b} = \bar{s}^{2ab} = \bar{t}^{4ab}$. Thus it is a natural choice to let $c_{0ab} = \bar{t}^{ab}$ (which is consistent with (2.21) and (2.23)). It follows from (2.25) that for the values $c_{nab} = \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab+(b-a)n+n}$, the element R of (2.19) satisfies the first defining relation (2.17a) of the R-matrix. In fact, in order to satisfy the second defining relation as well, it is necessary to normalize by multiplying by $\frac{1}{4r}$. Thus, we put

(2.26)
$$c_{nab} = \frac{1}{4r} \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab+(b-a)n+n}$$

which gives the desired

$$R = \frac{1}{4r} \sum_{n,a,b} \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab+(b-a)n+n} X^n K^a \otimes Y^n K^b$$

satisfying (2.17 a), and it remains to verify that R is invertible and satisfies (2.17 b). We will check the axiom

$$(2.27) \qquad (\Delta \otimes \mathrm{id}) R = R_{13} R_{23}$$

and leave the rest as an exercise.

We need a quantized version of the binomial coefficients (this goes back to Gauss). If $BA = AB$, then $(A+B)^n = \sum_k \binom{n}{k} A^k B^{n-k}$ defines the binomial coefficient $\binom{n}{k}$. Similarly, if $BA = qAB$ (q arbitrary), then

$$(A+B)^n = \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_q A^k B^{n-k}$$

defines the binomial q-coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$. It can be verified by induction that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$$

where

$$(2.28) \qquad [n]_q = \frac{q^n - 1}{q - 1}.$$

This is just an unbalanced version of the $[n]$ used in this paper, i.e. $[n]_q = s^{n-1}[n]$. It follows that

$$(2.29) \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = s^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} \quad \text{where } \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]! [n-k]!}.$$

Returning to the axiom (2.27), the left hand side $(\Delta \otimes \mathrm{id}) R$ is

$$(2.30) \quad \frac{1}{4r} \sum_{n,a,b} \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab+(b-a)n+n} (X \otimes K + \bar{K} \otimes X)^n (K \otimes K)^a \otimes Y^n K^b$$

$$= \frac{1}{4r} \sum_{n,a,b} c_{nab} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\bar{q}} X^k K^{k-n+a} \otimes s^{k(n-k)} X^{n-k} K^{k+a} \otimes Y^n K^b,$$

since $(\bar{K} \otimes X)(X \otimes K) = \bar{q}(X \otimes K)(\bar{K} \otimes X)$, and the right hand side $R_{13} R_{23}$ is

$$(2.31) \quad \frac{1}{(4r)^2} \sum_{\substack{n', a', b' \\ n'', a'', b''}} c_{n'a'b'} c_{n''a''b''} X^{n'} K^{a'} \otimes X^{n''} K^{a''} \otimes \bar{s}^{b'n''} Y^{n'+n''} K^{b'+b''}.$$

We need to show equal the coefficients of

$$X^k K^{a'} \otimes X^{n-k} K^{a''} \otimes Y^n K^b$$

where $a' = k - n + a$ and $a'' = k + a$ in (2.30), and $k = n'$, $n'' = n - k$ and $b = b' + b''$ in (2.31). Furthermore, note that $a' = a'' - n$ in (2.30), but there are terms in (2.31) for which $a' \neq a'' - n$ so these terms must be shown zero.

Using (2.29) and (2.26), we compute this coefficient in (2.30) to be

$$\frac{1}{4r} c_{n,a,b} \begin{bmatrix} n \\ k \end{bmatrix} \bar{s}^{k(n-k)} s^{k(n-k)}$$
$$= \frac{1}{4r} \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab+(b-a)n+n} \cdot \frac{[n]!}{[k]![n-k]!}$$
$$= \frac{1}{4r} \frac{(s-\bar{s})^n}{[k]![n-k]!} \bar{t}^{ab+(b-a)n+n}.$$

Since b' and b'' can vary as long as $b' + b'' \equiv b (\mathrm{mod}\, 4r)$, the corresponding coefficient in (2.31) equals

$$\frac{1}{16r^2} \sum_{b'=0}^{4r-1} c_{ka'b'} c_{n-k,a'',b-b'} \bar{s}^{b'(n-k)}$$
$$= \frac{1}{16r^2} \sum_{b'=0}^{4r-1} \frac{(s-\bar{s})^k}{[k]!} \bar{t}^{a'b'+(b'-a')k+k}$$
$$\cdot \frac{(s-\bar{s})^{n-k}}{[n-k]!} \bar{t}^{a''(b-b')+(b-b'-a'')(n-k)+n-k} \cdot \bar{t}^{2b'(n-k)}$$
$$= \frac{1}{16r^2} \sum_{b'=0}^{4r-1} \frac{(s-\bar{s})^n}{[k]![n-k]!} \bar{t}^{b'(a'-a''+n)+(-a'k+a''b+bn-bk-a''n+a''k+n)}$$
$$= \frac{1}{16r^2} \frac{(s-\bar{s})^n}{[k]![n-k]!} \bar{t}^{(-a'k+a''b+bn-bk-a''n+a''k+n)} \sum_{b'=0}^{4r-1} \bar{t}^{b'(a'-a''+n)}.$$

When $a' - a'' + n \neq 0$, the sum on the right is zero, as it should be for there is no corresponding term in (2.30). If $a' - a'' + n = 0$, then the sum equals $4r$, and substituting $a' = k - n + a$ and $a'' = k + a$, we obtain

$$\frac{1}{4r} \frac{(s-\bar{s})^n}{[k]![n-k]!} \bar{t}^{ab+(b-a)n+n}$$

as the common value of this coefficient in (2.30) and (2.31). □

(2.32) **Corollary** (a) *R acts on the module $V^k \otimes V^{k'}$ by*

$$R(e_i \otimes e_j) = \sum_n \frac{(s-\bar{s})^n}{[n]!} \frac{[m+i+n]!}{[m+i]!} \frac{[m'-j+n]!}{[m'-j]!} t^{4ij-2n(i-j)-n(n+1)} e_{i+n} \otimes e_{j-n}$$

where $k = 2m+1$, $k' = 2m'+1$, $[n]! = [n][n-1]\dots[1]$ ($= 1$ for $n = 0$), and $\frac{[p]!}{[n]!} = [p][p-1]\dots[n+1]$. The sum is over all $n \geqq 0$ with $i + n \leqq m$ and $j - n \geqq -m'$. In particular, if $i = m$ or $j = -m'$, then $R(e_i \otimes e_j) = q^{ij} e_i \otimes e_j$.

(b) *If V is an $\mathscr{A}$-module and v is a vector in V of weight t^p (i.e. $Kv=t^p v$), then R maps $v\otimes e$ in $V\otimes V^1(\alpha)$ to $\alpha^p v\otimes e$, and $e\otimes v$ in $V^1(\alpha)\otimes V$ to $e\otimes\alpha^p v$. (For the definition of $V^1(\alpha)$, look above Lemma* 2.16.)

Proof. For the k-dimensional representation V^k with basis $e_m, e_{m-1}, \ldots, e_{-m}$, recall that $Ke_i=s^i e_i$, $Xe_i=[m+i+1]\, e_{i+1}$ and $Ye_i=[m-i+1]\, e_{i-1}$. It follows from the theorem that R acts on $V^k\otimes V^{k'}$ by

$$R(e_i\otimes e_j)=\frac{1}{4r}\sum_{n\geqq 0}\frac{(s-\bar{s})^n}{[n]!}\frac{[m+i+n]!}{[m+i]!}\frac{[m'-j+n]!}{[m'-j]!}\sum_{a,b}\bar{t}^{ab+(b-a)n+n-2ai-2bj}e_{i+n}\otimes e_{j-n}. \tag{2.33}$$

Observe that the exponent of $\bar{t}$ can be written as $(a+n-2j)(b-n-2i)+(n-2j)(n+2i)+n$. It is an elementary fact that

$$4r=\sum_{0\leqq a,b<4r}\bar{t}^{ab}=\sum_{0\leqq a,b<4r}\bar{t}^{(a+n-2j)(b-n-2i)}$$

so it follows that

$$\sum_{0\leqq a,b<4r}\bar{t}^{ab+(b-a)n+n-2ai-2bj}=4r\cdot\bar{t}^{n(n+1)-4ij+2n(i-j)}$$

and substitution in (2.33) gives

$$R(e_i\otimes e_j)=\sum_n\frac{(s-\bar{s})^n}{[n]!}\frac{[m+i+n]!}{[m+i]!}\frac{[m'-j+n]!}{[m'-j]!}\bar{t}^{n(n+1)+2n(i-j)-4ij}e_{i+n}\otimes e_{j-n}$$

which proves (a).

For (b), note that e is an element of the (twisted) 1-dimensional module $V^1(\alpha)$ with $\alpha=i^m=t^{rm}$ for some m (so $Ke=\alpha e=t^{rm}e$), and so we have

$$R(v\otimes e)=\frac{1}{4r}\sum_{a,b}\bar{t}^{ab-ap-brm}v\otimes e$$

$$=t^{rmp}\left(\frac{1}{4r}\sum_{a,b}\bar{t}^{(a-rm)(b-p)}\right)v\otimes e.$$

Thus $R(v\otimes e)=\alpha^p v\otimes e$, since the sum equals $4r$ as above. The other case follows in the same way. □

(2.34) *Remark* The action of the R-matrix in the modules $V^k\otimes V^{k'}$ given in the previous corollary can also be derived using Drinfeld's R-matrix in $\mathbf{U}_h$ [D2, p. 816],

$$R=\sum_{n=0}^{\infty}h^n Q_n(h)\, t^{H\otimes H+n(H\otimes 1-1\otimes H)}(X^+)^n\otimes(X^-)^n.$$

Here $X^{\pm}$ and H are generators for $\mathbf{U}_h$ with relations [D2, p. 807]

$$[H, X^{\pm}] = \pm 2X^{\pm}$$

$$[X^+, X^-] = \frac{2}{h}\sinh\left(\frac{hH}{2}\right)$$

$$= \frac{s-\bar{s}}{h}[H],$$

$$Q_n(h) = \bar{s}^n \prod_{k=1}^{n} \frac{q-1}{q^k-1} = \prod_{k=1}^{n} \frac{s-\bar{s}}{(s^k-\bar{s}^k)\, s^k}$$

$$= \prod_{k=1}^{n} \frac{1}{[k]} \bar{s}^k = \frac{1}{[n]!} \bar{t}^{n(n+1)}.$$

The generators X and Y correspond to $\left(\frac{h}{s-\bar{s}}\right)^{\frac{1}{2}} X^+$ and $\left(\frac{h}{s-\bar{s}}\right)^{\frac{1}{2}} Y^+$ (giving the relation $[X, Y] = [H]$), and substituting these in Drinfeld's R-matrix gives

$$R = \sum_{n=0}^{\infty} \frac{(s-\bar{s})^n}{[n]!} t^{H\otimes H + n(H\otimes 1 - 1\otimes H) - n(n+1)} X^n \otimes Y^n.$$

(Since $(s-\bar{s})^n \bar{t}^{n(n+1)} = (1-\bar{q})^n \bar{t}^{n(n-1)}$, it follows that there is a missing $\bar{t}^{n(n-1)}$ in the formula for the R-matrix in [RT1, §7.4], and a missing $t^{n(n-1)}$ in [KiR, §1.7].)

To compute the action of R on $e_i \otimes e_j$ in $V^k \otimes V^{k'}$, observe that $t^{H\otimes H} = 1\otimes 1 + \frac{h}{4} H\otimes H + \left(\frac{h}{4}\right)^2 \frac{1}{2!} H^2 \otimes H^2 + \ldots$ and $t^{n(H\otimes 1 - 1\otimes H)} = (K^n \otimes 1)(1\otimes \bar{K}^n) = K^n \otimes \bar{K}^n$, which gives

$$t^{H\otimes H} e_i \otimes e_j = t^{4ij} e_i \otimes e_j$$

$$t^{n(H\otimes 1 - 1\otimes H)} e_i \otimes e_j = t^{2n(i-j)} e_i \otimes e_j.$$

Thus

$$R(e_i \otimes e_j) = \sum_n \frac{(s-\bar{s})^n}{[n]!} t^{4(i+n)(j-n) + 2n((i+n)-(j-n)) - n(n+1)}$$

$$\cdot \frac{[m+i+n]!}{[m+i]!} \frac{[m'-j+n]!}{[m'-j]!} e_{i+n} \otimes e_{j-n},$$

which readily yields the formula given in Corollary 2.32.

(2.35) **Definition** The R-matrix, viewed as an operator on $V\otimes W$ for $\mathscr{A}$-modules V and W, can be composed with the permutation operator P to give an operator

$$\check{R} = P \circ R\colon\ V\otimes W \to W \otimes V$$

which we call the *$\check{R}$-matrix* (read "R flip matrix") on $V\otimes W$.

These are the operators associated with crossings in the definition of the colored framed link invariants.

(2.36) **Lemma.** *The $\check{R}$-matrices are $\mathscr{A}$-linear and satisfy the Yang-Baxter equation*

$$\check{R}_{23}\check{R}_{12}\check{R}_{23}=\check{R}_{12}\check{R}_{23}\check{R}_{12}$$

(as operators $U\otimes V\otimes W\to W\otimes V\otimes U$ for $\mathscr{A}$-modules U, V and W), where $\check{R}_{12}=\check{R}\otimes\mathrm{id}$ and $\check{R}_{23}=\mathrm{id}\otimes\check{R}$.

Proof. The first statement follows readily from the first defining property (2.17a) of R. Indeed, for X in $V\otimes W$, we have

$$\begin{aligned}\check{R}(\alpha X)&=P(R\Delta\alpha\cdot X) \qquad (\cdot = \text{diagonal action})\\ &=P(P(\Delta\alpha)R\cdot X) \qquad (\text{by (2.17a)})\\ &=\Delta\alpha P(R\cdot X)\\ &=\alpha\check{R}(X).\end{aligned}$$

The second also follows from the defining properties of R. First we derive the Yang-Baxter equation for the R-matrix, namely

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12},$$

as follows:

$$\begin{aligned}R_{12}R_{13}R_{23}&=R_{12}\cdot(\Delta\otimes\mathrm{id})(R) \qquad (\text{by (2.17b)})\\ &=(\check{\Delta}\otimes\mathrm{id})(R)\cdot R_{12} \qquad (\text{by (2.17a)})\\ &=(P\otimes\mathrm{id})(R_{13}R_{23})\cdot R_{12} \qquad (\text{by (2.17b)})\\ &=R_{23}R_{13}R_{12}.\end{aligned}$$

Now view this as an equation of operators $U\otimes V\otimes W\to U\otimes V\otimes W$ and multiply on the left by the operator $P_{23}P_{12}P_{23}=P_{12}P_{23}P_{12}$, where $P_{12}=P\otimes\mathrm{id}$ and $P_{23}=\mathrm{id}\otimes P$. Observing that $P_{ij}R_{ik}=R_{jk}P_{ij}$ for $j\neq k$, we obtain the Yang-Baxter equation for $\check{R}$. □

(2.37) **Examples** If the preferred weight basis $e_i\otimes e_j$ for $V^k\otimes V^{k'}$ is put in decreasing lexicographic order with respect to $(i+j, i, j)$, then the R-matrices given in Theorem 2.18 decompose into block sums, with constant $i+j$ in each block. For example, the R-matrix in $V^2\otimes V^2$ is given by

$$(t)\oplus\begin{pmatrix}\bar{t} & \bar{t}(s-\bar{s})\\ 0 & \bar{t}\end{pmatrix}\oplus(t)$$

(with respect to the basis $e_{1/2}\otimes e_{1/2}$, $e_{1/2}\otimes e_{-1/2}$, $e_{-1/2}\otimes e_{1/2}$, $e_{-1/2}\otimes e_{-1/2}$), and the corresponding $\check{R}$-matrix is

$$(t)\oplus\begin{pmatrix}0 & \bar{t}\\ \bar{t} & \bar{t}(s-\bar{s})\end{pmatrix}\oplus(t).$$

Similarly the $\check{R}$-matrix in $V^3 \otimes V^3$ is

$$(q) \oplus \begin{pmatrix} 0 & 1 \\ 1 & q-\bar{q} \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \bar{q} \\ 0 & 1 & 1-\bar{q} \\ \bar{q} & (q-\bar{q})(1+\bar{q}) & (q-\bar{q})(1-\bar{q}) \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & q-\bar{q} \end{pmatrix} \oplus (q),$$

and the $\check{R}$-matrix in $V^2 \otimes V^3$ is

$$(s) \oplus \begin{pmatrix} 0 & \bar{s} \\ 1 & \bar{t}(q-\bar{q}) \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ \bar{s} & \bar{t}(q-\bar{q})(s+\bar{s}) \end{pmatrix} \oplus (s).$$

3 Tangle operators and link invariants

The link invariants $J_{L,\mathbf{k}}$ (see Definition 3.25) are special cases of the more general *tangle operators* which we define first (in Theorem 3.6).

Tangles

Recall that a *tangle* T is a 1-manifold properly embedded (up to isotopy) in the unit cube I^3 in $\mathbf{R}^3 = S^3 - \infty$, with $\partial T \subset \frac{1}{2} \times I \times \partial I$. Define $\partial_- T = T \cap (I^2 \times 0)$ and $\partial_+ T = T \cap (I^2 \times 1)$, and call T an *(m, n)-tangle* if $m = |\partial_- T|$ and $n = |\partial_+ T|$. Thus a link is a $(0, 0)$-tangle, and a general tangle consists of a link together with a collection of proper arcs. All tangles will be assumed oriented and perpendicular to $I^2 \times \partial I$.

A *framed tangle* is a tangle T equipped with a framing of its normal bundle (up to isotopy rel ∂T) which is standard ($\mathbf{i}$, $\pm \mathbf{j}$) on ∂T (where the sign is chosen so that the frame followed by the oriented tangent to T is the standard frame on $\mathbf{R}^3$). Since we are working in S^3, there is a natural *0-framing* on each component of T, and so the framings may be specified by integers in the usual way. Alternatively, they may be specified by thickening the embeddings to ribbons in the direction of the second vector of the framing, as in the *homogeneous ribbon tangles* of [RT1].

As for links, one often studies tangles by their *diagrams* D in the square I^2 (obtained by regular projection onto $0 \times I^2$) with $\partial D \subset I \times \partial I$. Note that the 0-framing of a tangle T is in general different from the *blackboard framing* coming from a diagram D of T, in which the second vector is always parallel to $0 \times I^2$ (see Fig. 3.1).

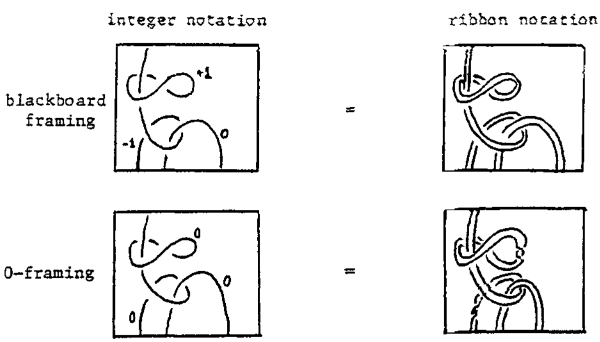


Fig. 3.1

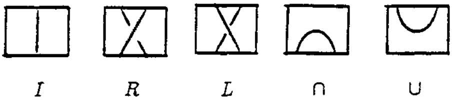


Fig. 3.2

Fig. 3.3

If the framing on T and the blackboard framing coincide, we call D a *good* diagram of T. Any diagram of T may be made into a good diagram by adding kinks. It is well known that every tangle diagram can be factored into the *elementary diagrams* I, R, L, $\cap$ and $\cup$ shown in Fig. 3.2 (with all possible orientations) using the *composition* $\circ$ (when defined) and the *tensor product* $\otimes$ of diagrams (see Fig. 3.3). Of course, distinct *factored diagrams* may represent the same tangle. For example $L \circ R = I \otimes I$ (with appropriate orientations), which may also be written as $\between = \|$. In fact, the following result follows easily from the work of Reidemeister [R].

(3.4) **Theorem** ([Ye, FY, T2, RT1]). *Any two factored good diagrams of a given framed tangle are related by a sequence of the following moves (with all possible orientations)*

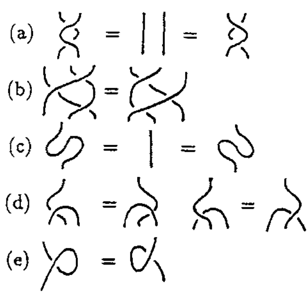


Fig. 3.4

together with the implicit associativity and identity relations and $(S \circ T) \otimes (S' \circ T') = (S \otimes S') \circ (T \otimes T')$ (*i.e. tangles are morphisms of a strict monoidal category, see e.g.* [FY]).

Remark. Moves (a)–(d) generate regular isotopy of tangle diagrams [Kf, Tr].

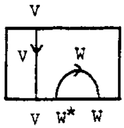


Fig. 3.5

Colored framed tangle operators F_T

Now fix a quasitriangular Hopf algebra (A, R) and define a *coloring* of a tangle T (or one of its diagrams) to be an assignment of an A-module to each component of T. This induces a coloring of ∂T as follows: if S is an arc of color V, then assign V to each endpoint of S where S is oriented down, and the dual module V^* to each endpoint where S is oriented up (see Fig. 3.5). Tensoring from left to right, this gives *boundary A-modules* $T_\pm$ assigned to $\partial_\pm T$ ($T_+ = V$ and $T_- = V \otimes W^* \otimes W$ in Fig. 3.5). By convention, the empty tensor product is $\mathbf{C}$, and so $T_\pm = \mathbf{C}$ if T is a link.

In the next result we show how to obtain tangle operators $T_- \to T_+$ for colored framed tangles T which behave well with respect to compositions and tensor products. This construction depends on some additional structure on the quasitriangular Hopf algebra (A, R) (with even more structure one obtains the *ribbon Hopf algebras* of Reshetikhin-Turaev, cf. Theorem 5.1 in [RT1] and Remark 3.16 below).

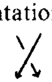
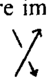
(3.6) **Theorem** *Let μ be an invertible element of a quasitriangular Hopf algebra $(A, R = \sum \alpha_i \otimes \beta_i)$ satisfying*
(a) $\mu \alpha \bar{\mu} = S^2(\alpha)$ *for all α in A*
(b) $\sum \alpha_i \bar{\mu} \beta_i = \sum \beta_i \mu \alpha_i$
where $\bar{\mu}$ denotes the inverse of μ. Then there exist unique A-linear operators

$$F_T = F_T^{A,R,\mu} \colon T_- \to T_+$$

assigned to each colored framed tangle T which satisfy $F_{T \circ T'} = F_T \circ F_{T'}$, $F_{T \otimes T'} = F_T \otimes F_{T'}$, and for the tangles given by the elementary diagrams (of Fig. 3.2) with the blackboard framing,

$$F_{|} = \mathrm{id}$$
$$F_{\mathrm{R}} = \check{R} \quad \textit{and} \quad F_{L} = \check{R}^{-1}$$
$$F_{\frown} = E \quad \textit{and} \quad F_{\frown} = E_\mu$$
$$F_{\smile} = N \quad \textit{and} \quad F_{\smile} = N_{\bar{\mu}}$$

where $E(f \otimes x) = f(x)$, $E_\mu(x \otimes f) = f(\mu x)$, $N(1) = \sum e_i \otimes e^i$ and $N_{\bar{\mu}}(1) = \sum e^i \otimes (\bar{\mu} e_i)$ (for any basis e_i). Note that for a link L (i.e. $(0, 0)$-tangle), $F_L \colon \mathbf{C} \to \mathbf{C}$ is just a scalar.

Proof. First assign operators $F_D \colon D_- \to D_+$ to each elementary diagram D, as in the theorem. (Observe that orientations are implicit in the definitions of the first three operators. For example  and  are assigned $\check{R}$-matrices on

$U \ V \quad U \ V$

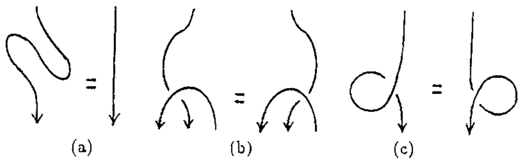


Fig. 3.7

$U \otimes V$ and $U^* \otimes V$, respectively.) A-linearity follows from Lemma 2.36 (for the crossings), and from the Hopf algebra axioms and property (3.6a) of μ (for the extrema). Indeed, for E: $V^* \otimes V \to \mathbf{C}$ we have, for all α in A

$$\begin{aligned} E(\alpha(f \otimes x)) &= f((m(S \otimes \mathrm{id})\Delta\alpha)\, x) \quad \text{where } m \text{ is multiplication} \\ &= f(\varepsilon(\alpha)\, x) \quad \text{by the antipode axiom} \\ &= \varepsilon(\alpha)\, f(x) \\ &= \alpha E(f \otimes x). \end{aligned}$$

Similarly N: $\mathbf{C} \to V \otimes V^*$ is A-linear. For E_μ: $V \otimes V^* \to \mathbf{C}$ we have

$$\begin{aligned} E_\mu(\alpha(x \otimes f)) &= E_\mu(\Delta\alpha(x \otimes f)) \\ &= E_\mu(\textstyle\sum (a_i x) \otimes (b_i f)) \quad \text{where } \Delta\alpha = \textstyle\sum a_i \otimes b_i \\ &= f(\textstyle\sum S(b_i)\, \mu a_i x) \\ &= f(\textstyle\sum S(b_i) S^2(a_i)\, \mu x) \quad \text{by (3.6a)} \\ &= f(S(m(S \otimes \mathrm{id})\Delta\alpha)\, \mu x) \\ &= f(S(\varepsilon(\alpha) 1)\, \mu x) \quad \text{by the antipode axiom} \\ &= f(\varepsilon(\alpha)\, \mu x) \\ &= \alpha E_\mu(x \otimes f). \end{aligned}$$

A similar argument shows that $N_{\bar{\mu}}$ is A-linear. Note that the role of μ (in the operators for the backward extrema) is essential, since the maps $x \otimes f \mapsto f(x)$ and $1 \mapsto \sum e^i \otimes e_i$ are not in general A-linear, due to the fact that the permutation map P is not in general A-linear.

Now extend the definition of F_D to arbitrary factored good diagrams D of colored framed tangles T by the rules $F_{D \circ D'} = F_D \circ F_{D'}$ and $F_{D \otimes D'} = F_D \otimes F_{D'}$. To show that these induce well defined operators F_T on tangles, it remains to show that F_D is invariant under the moves (3.4a–e).

Move (3.4a) follows from $\check{R}\check{R}^{-1} = I = \check{R}^{-1}\check{R}$, and (3.4b) follows from the Yang-Baxter equation of Lemma 2.36 (which holds, by the same proof, in any quasitriangular Hopf algebra). For moves (3.4c–e), we establish the cases shown in Fig. 3.7 and leave the rest as exercises.

For move (3.7a), which says $(E_\mu \otimes \mathrm{id})(\mathrm{id} \otimes N_{\bar\mu}) = \mathrm{id}$, we have

$$\begin{aligned}(E_\mu \otimes \mathrm{id})(\mathrm{id} \otimes N_{\bar\mu})(x) &= (E_\mu \otimes \mathrm{id})(\textstyle\sum x \otimes e^i \otimes \bar\mu e_i)\\ &= \textstyle\sum e^i(\mu x)\bar\mu e_i\\ &= \bar\mu \textstyle\sum e^i(\mu x) e_i\\ &= \bar\mu\mu x\\ &= x.\end{aligned}$$

For move (3.7b), which says $(\mathrm{id} \otimes E_\mu)(\check R \otimes \mathrm{id}) = (E_\mu \otimes \mathrm{id})(\mathrm{id} \otimes \check R^{-1})$, first note that the inverse of the R-matrix $R = \sum \alpha_i \otimes \beta_i$ can be computed easily using the antipode axiom as

$$R^{-1} = (S \otimes \mathrm{id})(R) = \textstyle\sum S(\alpha_i) \otimes \beta_i \tag{3.8}$$

(see e.g. [RT1, § 3.1.6]), and so

$$\begin{aligned}(\mathrm{id} \otimes E_\mu)(\check R \otimes \mathrm{id})(x \otimes y \otimes f) &= (\mathrm{id} \otimes E_\mu)(\textstyle\sum \beta_i y \otimes \alpha_i x \otimes f)\\ &= \textstyle\sum \beta_i y f(\mu\alpha_i x)\\ &= \textstyle\sum \beta_i y f(S^2(\alpha_i)\mu x) \qquad \text{by (3.6a)}\\ &= (E_\mu \otimes \mathrm{id})(\textstyle\sum x \otimes S(\alpha_i) f \otimes \beta_i y)\\ &= (E_\mu \otimes \mathrm{id})(\mathrm{id} \otimes \check R^{-1})(x \otimes y \otimes f).\end{aligned}$$

Finally, move (3.7c) says

$$(E \otimes \mathrm{id})(\mathrm{id} \otimes \check R)(N_{\bar\mu} \otimes \mathrm{id}) = (\mathrm{id} \otimes E_\mu)(\check R \otimes \mathrm{id})(\mathrm{id} \otimes N).$$

The value of the left and right hand sides on an element x are readily computed as $(\sum \alpha_i \bar\mu \beta_i)\, x$ and $(\sum \beta_i \mu \alpha_i)\, x$, respectively, and these are equal by (3.6b). □

Properties of tangle operators

Now we establish various properties of tangle operators. We shall always assume that we are in the setting of Theorem 3.6, so

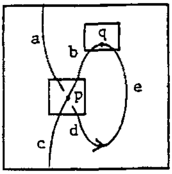
$$F_T = F_T^{A, R, \mu}$$

for some fixed quasitriangular Hopf algebra (A, R) and unit μ in A satisfying (3.6ab).

We begin with an elementary but useful fact about operators of (1, 1)-tangles.

(3.9) **Lemma** *Let T be a colored framed* (1, 1)*-tangle and V be the color of its (unique) arc component.*
(a) *If V is irreducible, then F_T is a scalar operator (i.e. a multiple of the identity).*
(b) *If V is reducible with a unique proper submodule, then F_T is the sum of a scalar operator and a nilpotent operator.*

Proof. Observe that any eigenspace for the operator F_T is a submodule of V (or V^* depending upon the orientation of the arc of T), since F_T is A-linear. The Lemma follows immediately by considering the Jordan canonical form of F_T. □



$a, b, c, d \in B_V$, $e \in B_V^*$

$\sigma_+(p) = a \otimes b$, $\sigma_-(p) = c \otimes d$

$\sigma_+(q) = 1$, $\sigma_-(q) = b \otimes e$

$\sigma_+ = a$, $\sigma_- = c$

Fig. 3.12

Next we consider how tangle operators behave under direct sums, extensions and tensor products of colors. Parts of this result were stated previously in §6.4 of [RT1].

(3.10) **Lemma** *Let T be a colored framed tangle and K be a component of color V. Write TX for the tangle obtained by changing the color on K to X.*
(a) *If $V = X \oplus Y$, or more generally V is an extension of Y by X (i.e. there is a short exact sequence $0 \to X \to V \to Y \to 0$ of A-modules), and K is closed, then*

$$F_T = F_{TX} + F_{TY}.$$

(see [RT1, §6.4.1] *for the case $V = X \oplus Y$).*
(b) *If $V = X \oplus Y$ and K is an arc between the bottom and top of the tangle, then*

$$F_T = F_{TX} \oplus F_{TY}$$

where the modules $T_\pm$ are naturally identified with $TX_\pm \oplus TY_\pm$.
(c) *If $V = X \otimes Y$, then*

$$F_T = F_{TXY}$$

where TXY is the tangle obtained by replacing K by two parallel pushoffs of itself (using the framing) colored X and Y, respectively. (*See* [RT1, §6.4.2].)

(3.11) *Remarks.* (1) Statements similar to (a) and (b) hold for arcs K joining the bottom or top of the tangle to itself.

(2) We will prove (a) using the following *states model* for F_T. Fix a factored good diagram D for T and preferred bases B_V for each color V. Let P be the set of critical points (i.e. extrema and double points) of D, and denote the elementary factor of D corresponding to a point p in P by Dp (i.e. Dp is the diagram in a small box about p).

A *state* σ of D is the assignment of a label $\sigma(S)$ to each component S of $D - P$ as follows: If S is V-colored, then $\sigma(S)$ is an element of B_V or B_V^* (the dual basis) according to whether S is oriented downward or upward. By taking tensor products, a state yields elements $\sigma_\pm(E)$ in the modules $E_\pm$ for any factor E of D. In particular, set $\sigma_\pm(p) = \sigma_\pm(Dp)$ and $\sigma_\pm = \sigma_\pm(D)$. (See Fig. 3.12).

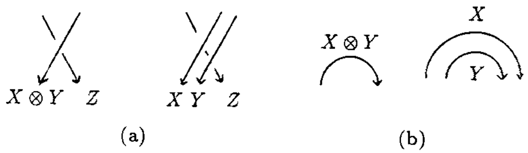


Fig. 3.14

Define the *weight* of a state σ to be the product

$$w(\sigma)=\prod_{p\in P} w_p(\sigma)$$

where $w_p(\sigma)$ is defined as the coefficient of $\sigma_+(p)$ in $F_{D_p}(\sigma_-(p))$.

Now consider *basic* elements of $T_\pm$, i.e. tensor products of preferred basis elements. The operator F_T is determined by the coefficients $(F_T)^{x_+}_{x_-}$ of x_+ in $F_T(x_-)$, for all basic $x_\pm$ in $T_\pm$, and these evidently have the following *states formula*

$$(F_T)^{x_+}_{x_-}=\sum w(\sigma) \tag{3.13}$$

where the sum is over all states σ with $\sigma_\pm = x_\pm$.

Proof of 3.10 For (a) we adopt the notation of the previous remark, choosing preferred bases B_V, B_X and B_Y so that $B_X\subset B_V$ (viewing X as a subspace of V) and B_Y is the projection of $\tilde{B}_Y=B_V-B_X$. All state labels from B_X or B_X^* will be called *X-labels*, and those from $\tilde{B}_Y$ or $\tilde{B}_Y^*$ will be called *Y-labels*.

Observe that if σ is a state of D with non-zero weight, then the corresponding labels on the arcs of K must be either all X-labels (written $\sigma|K\subset X$) or all Y-labels (written $\sigma|K\subset Y$). This follows from the A-invariance of $X\subset V$ (and dually of $Y^*\subset V^*$), which shows that one cannot move from an X-label to a Y-label while traversing K in the direction opposite to its orientation.

Now for basic $x_\pm$ in $T_\pm$, the states formula (3.13) gives

$$\begin{aligned}(F_T)^{x_+}_{x_-}&=\sum w(\sigma)\\ &=\sum_{\sigma|K\subset X} w(\sigma)+\sum_{\sigma|K\subset Y} w(\sigma)\\ &=(F_{TX})^{x_+}_{x_-}+(F_{TY})^{x_+}_{x_-}\end{aligned}$$

and so $F_T=F_{TX}+F_{TY}$.

The proof of (b) is similar but easier, and is left to the reader.

Finally, (c) follows from definitions, including the second defining property (2.17b) of the R-matrix. In particular, the two operators corresponding to a crossing involving K in a diagram D for T, and the corresponding crossings in the associated diagram of TXY, are equal. We illustrate this with the right crossing shown in Fig. 3.14a, where both operators map $X\otimes Y\otimes Z$ to $Z\otimes X\otimes Y$.

Indeed, with the obvious notation

$$\begin{aligned}\check{R}_{X\otimes Y,Z} &= P_{X\otimes Y,Z}\circ R_{X\otimes Y,Z}\\ &= P_{X\otimes Y,Z}\circ(\Delta\otimes\mathrm{id})(R)_{(X\otimes Y)\otimes Z}\\ &= P_{X\otimes Y,Z}\circ(R_{13}R_{23})_{X\otimes Y\otimes Z}\\ &= \sum_{i,j}\beta_i\beta_j\otimes\alpha_i\otimes\alpha_j\\ &= (\check{R}_{X,Z}\otimes\mathrm{id}_Y)\circ(\mathrm{id}_X\otimes\check{R}_{Y,Z})\end{aligned}$$

where $R=\sum\alpha_i\otimes\beta_i$. An easier argument shows that the operators corresponding to an extreme point of D, for example $E_{X\otimes Y}$ and $E_X\circ(\mathrm{id}\otimes E_Y\otimes\mathrm{id})$ for the case shown in Fig. 3.14b, are equal. The result follows by the definition of tangle operators. □

Orientations

Finally we consider orientation questions for the tangle operators $F_T=F_T^{A,R,\mu}$. It turns out that to get a reasonable theory, one must assume that the element μ and its antipode $S(\mu)$ are inverses. Such an element μ, i.e. a unit in A satisfying

(3.15) (a) $\mu\alpha\bar{\mu}=S^2(\alpha)$ for all α in A

(b) $\sum\alpha_i\bar{\mu}\beta_i=\sum\beta_i\mu\alpha_i$ where $R=\sum\alpha_i\otimes\beta_i$

(c) $S(\mu)=\bar{\mu}$,

is said to be *charmed.*

(3.16) *Remark* If (A, R, v) is a *ribbon Hopf algebra* in the sense of [RT1], then $\mu=u\bar{v}$ is charmed, where $u=\sum S(\beta_i)\alpha_i$. We do not know if a charmed element μ in an arbitrary quasitriangular Hopf algebra (A, R) gives rise to a ribbon structure (with $v=u\bar{\mu}$).

Observe that any μ satisfying (3.15a) induces an A-linear isomorphism

(3.17) $$E_\mu\colon\ V\to V^{**}$$

for any A-module V, given by $E_\mu(x)=(\mu x)^{**}$ (=evaluation on μx), i.e. $E_\mu(x)(f)=f(\mu x)$. Indeed $E_\mu(\alpha x)=(\mu\alpha x)^{**}=(S^2(\alpha)\mu x)^{**}=\alpha(\mu x)^{**}=\alpha E_\mu(x)$. (Since this map is canonically identified with the map $E_\mu\colon V\otimes V^*\to\mathbf{C}$ in (3.6), we use the same notation.)

(3.18) **Lemma** *Let T be a colored framed tangle with a preferred component K of color V, and μ be charmed.*
(a) *If T^* is obtained from T by replacing K by $-K$ (opposite orientation) with color V^* (the dual module), then*

$$F_T=F_{T^*}$$

where (if K is not closed) $T_\pm$ and $T_\pm^*$ are identified by the isomorphism E_μ of (3.17) *between the V-colored endpoints of K and the V^{**}-colored endpoints of $-K$.*

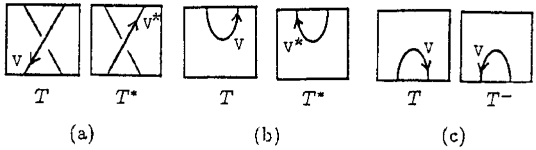
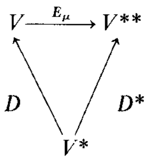


Fig. 3.19

(b) *If T^- is obtained from T by replacing K by $-K$ without changing the color, and if V is self dual by an isomorphism D: $V^* \to V$ for which $E_\mu = \pm D^* D^{-1}$, i.e.*



commutes up to sign, then

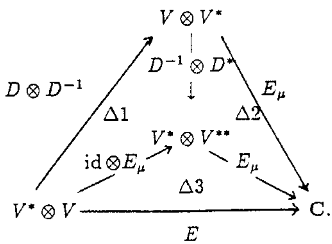
$$F_T = \varepsilon F_{T^-}$$

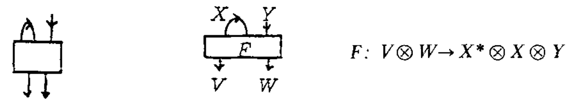
where (if K is not closed) $T_\pm$ and $T_\pm^-$ are identified by D between corresponding endpoints of K, and the sign $\varepsilon = \pm 1$ is -1 if and only if $E_\mu = -D^* D^{-1}$ and K is an arc joining one end of the tangle to itself. In particular $\varepsilon = +1$ if T is a link (i.e. (0, 0)-tangle).

Proof. It suffices to prove the lemma for elementary tangles, and for these it is straightfoward from definitions.

We illustrate the proof of (a) for the two (hardest) cases shown in Fig. 3.19ab. For (3.19a), we have $F_T(x \otimes y) = \sum \beta_i y \otimes \alpha_i x$ and $F_{T^*}(x \otimes y) = \sum \beta_i y \otimes E_\mu^{-1}(\alpha_i(\mu x)^{**})$. But these are equal by (3.15a), since $E_\mu(\alpha_i x) = (\mu \alpha_i x)^{**} = (S^2(\alpha_i) \mu x)^{**} = \alpha_i (\mu x)^{**}$. For (3.19b), we have $F_T(1) = \sum e_i \otimes e^i = \sum s(\mu) e_i \otimes \bar{\mu} e^i$ (note that $s(\mu) e_i$ and $\bar{\mu} e^i$ are dual bases) and $F_{T^*}(1) = \sum E_\mu^{-1}((e^i)^*) \otimes \bar{\mu} e^i$. These are equal, since $s(\mu) e_i = \bar{\mu} e_i$ and $E_\mu(\bar{\mu} e_i) = e_i^{**} = (e^i)^*$.

Note (b) follows from (a) by a diagram chase. For example, for the case shown in (3.19c) we must show $E = \varepsilon E_\mu (D \otimes D^{-1})$, where $E_\mu = \varepsilon D^* D^{-1}$, $\varepsilon = \pm 1$. Consider the diagram





(a) coupon (b) colored coupon

Fig. 3.21

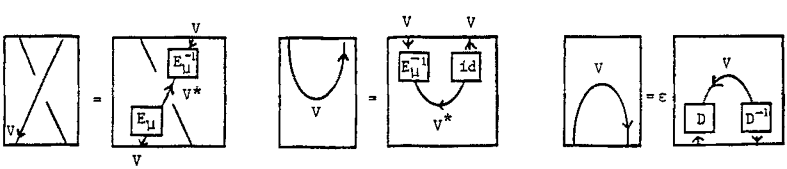
The small triangle $\Delta 1$ ε-commutes by hypothesis, $\Delta 2$ commutes by the A-linearity of D^{-1}, and $\Delta 3$ commutes since $S(\mu)=\bar{\mu}$. Thus the outer triangle ε-commutes, as desired. The remaining cases are left to the reader. □

(3.20) *Remark* It is convenient to rephrase the previous lemma using *framed* (or *ribbon*) *graphs*, introduced in [RT1], which are formed from compositions and tensor products of framed tangles and *coupons*. A *coupon* can be thought of as an empty tangle (with diagram □) which is permitted to compose with arbitrary tangles, as shown in Fig. 3.21a. The coupons are thought of as the vertices of the framed graph. *Coloring* the framed graph then consists of coloring the edges (and loops) of the graph by A-modules and the vertices (i.e. coupons) by appropriate A-linear operators, as indicated in Fig. 3.21b. As with colored framed tangles, there is an operator F_G associated with any colored framed graph G.

Now in Lemma 3.18, the identification of $T_\pm$ with $T_\pm^*$ in (a) and $T_\pm^-$ in (b) can be accomplished by inserting coupons colored with E_μ and D (and their inverses). For example, the tangle operator equalities (after suitable identifications), illustrated in Fig. 3.19 become exact graph operator equalities, as shown in Fig. 3.22.

Note that the operator equality of Fig. 3.22c shows that pushing a $D^{\pm 1}$-colored coupon over a maximum changes the associated operator by a factor of ε, and the same remark holds for minima. As a consequence we have the following corollary of Lemma 3.18.

(3.23) **Corollary** *Let K be a closed V-colored component of a colored framed tangle T, and assume that μ is charmed that there is an isomorphism $D: V^* \to V$ of A-modules with $E_\mu=(-1)^m D^* D^{-1}$ for some integer m. If G is a colored framed graph formed by introducing two $D^{\pm 1}$-colored coupons on K (changing orientations appropriately) at points separated by p extreme points of K (in some good diagram of T), then $F_G=(-1)^{mp} F_T$.*



(a) (b) (c)

Fig. 3.22

Tangle operators for $\mathscr{A}$

We now specialize to the quasitriangular Hopf algebras $\mathscr{A}=\mathscr{A}_r$ (quantized sl(2, **C**)) discussed in § 2, with R given in (2.18).

(3.24) **Theorem** *The element* $\mu=K^2$ *in* $\mathscr{A}$ *is charmed, i.e. satisfies* (3.15a–c).

The proof will be given below. The associated tangle operators F_T will be denoted by J_T, in honor of V.F.R. Jones (see § 4):

(3.25) **Definition** For any integer $r>1$ and colored framed tangle T, with coloring $\mathbf{k}$, define

$$J_T=J_{T,\mathbf{k}}=F_T^{\mathscr{A}_r,R,K^2}.$$

Note that the integer r, and sometimes the coloring $\mathbf{k}$, will be suppressed in this notation. If T is a link L, then $J_{L,\mathbf{k}}$ is a scalar which will be called the *colored framed link invariant* of $(L, \mathbf{k})$.

(3.26) *Remark* The tangle operators $J_{T,\mathbf{k}}$ are independent of the orientation on any closed component K of T whose color is one of the irreducible modules V^k $(1\leqq k\leqq r)$. Indeed, the isomorphism $D\colon (V^k)^*\to V^k$ given in Theorem 2.13 satisfies

$$D^*D^{-1}=(-1)^{k-1}E_{K^2}$$

(since $D^*D^{-1}(b^j)=(-s)^{2j}b_j^{**}=(-1)^{k-1}q^j b_j^{**}=(-1)^{k-1}E_{K^2}(b^j)$), and so the remark follows from Lemma 3.18b since K has an even number of extreme points. In fact, from Corollary 3.23, we obtain the more refined result that

$$F_G=(-1)^{(k-1)p}F_T$$

if G is the colored framed graph obtained by introducing two $D^{\pm 1}$-colored coupons on K separated by p extreme points.

Proof of 3.24 Properties (3.15a) and (c) are immediate from the definition of $\mathscr{A}$. In particular $KX=sXK$, $KY=\bar{s}YK$, $S(K)=\bar{K}$, $S(X)=-sX$ and $S(Y)=-\bar{s}Y$ imply $K^2X\bar{K}^2=qX=S^2(X)$, $K^2Y\bar{K}^2=\bar{q}Y=S^2(Y)$ and $S(K^2)=\bar{K}^2$. Property (3.15b) is deeper, and is proved in Appendix A. □

We conclude this section with some computations of specific tangle operators associated with the algebra $\mathscr{A}=\mathscr{A}_r$, using as colors the irreducible modules V^k (often identified by their dimensions $k\leqq r$) and the associated Verma modules W_k^r (for $0<k<r$), defined in § 2.

We begin with a basic result about local modifications of tangles.

(3.27) **Lemma** *Fix a colored framed tangle T and a preferred component J of color j (i.e. V^j with $j\leqq r$). Let T_0 be T with a disjoint k-colored unknot adjoined, let T_1 be T with the framing on J increased by* 1, *and let T_2 be T with a k-colored unknotted meridian to J adjoined. Then*

$$J_{T_i}=c_i\,J_T$$

where $c_0=[k]$, $c_1=t^{j^2-1}$ *and* $c_2=\dfrac{[jk]}{[j]}$ *(interpreted as $(-1)^{k-1}k$ when $j=r$). In pictures:*

(a) $J_{)\bigcirc_k}=[k]\,J_{)}$ (c) $J_{j\,\phi\,k}=\dfrac{[jk]}{[j]}\,J_{)j}$

(b) $J_{\wp j}=t^{j^2-1}\,J_{)j}$

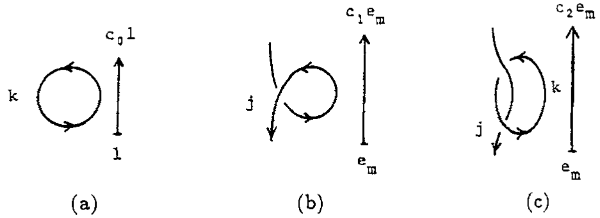


Fig. 3.28

Proof. It follows from Lemmas 3.9 a and 3.18 b that $J_{T_i} = c_i J_T$ for scalars c_i which are independent of orientations. Thus we may find the c_i by computing the values of the operators on 1 (for $i=0$) and e_m (for $i=1,2$), where $m=\frac{j-1}{2}$, as indicated in Fig. 3.28.

For (a), $1 \mapsto \sum_{i=-n}^{n} e_i \otimes e^i \mapsto \sum e^i(K^2 e_i) = \sum q^i = [k]$ $\left(\text{where } n=\frac{k-1}{2}\right)$, and so $c_0=[k]$.

For (b), $e_m \mapsto \sum_{i=-m}^{m} e_m \otimes e_i \otimes e^i \mapsto \sum q^{im} e_i \otimes e_m \otimes e^i$ (by Corollary 2.32) $\mapsto \sum q^{im} e^i(K^2 e_m)\, e_i = q^{m(m+1)} e_m = t^{j^2-1} e_m$, and so $c_1 = t^{j^2-1}$.

For (c), $e_m \mapsto \sum_{i=-n}^{n} e_m \otimes e_i \otimes e^i \mapsto \sum q^{mi} e_i \otimes e_m \otimes e^i \mapsto \sum q^{2mi} e_m \otimes e_i \otimes e^i$ (plus terms which will vanish at the next step) $\mapsto \sum q^{2mi} e^i(K^2 e_i)\, e_m = \sum q^{2mi+i} e_m = \sum_{i=-n}^{n} q^{ji} e_m = \frac{[jk]}{[j]} e_m$ for $j<r$ $\left(\text{where } n=\frac{k-1}{2}\right)$, and so $c_2 = \frac{[jk]}{[j]}$ for $j<r$. For $j=r$, we have $q^{ji} = (-1)^{k-1}$, and so $c_2 = \sum_{i=-n}^{n} (-1)^{k-1} = (-1)^{k-1} k$. □

We conclude with a global result about links. Recall from Lemma 2.15 that W_k^r contains V^k as a unique proper submodule. In particular, using the standard basis $e_{r-m-1}, \ldots, e_{-m}$ coming from the inclusion $W_k^r \subset V^{2r-k}$ $\left(\text{where } m=\frac{k-1}{2}\right)$, V^k is spanned by $e_m, \ldots, e_{-m}$.

(3.29) **Lemma** *If a colored framed link* $(L, \mathbf{k})$ *has a component of color* V^r *or* W_k^r, *then* $J_{L,\mathbf{k}} = 0$.

Proof. Write L as the closure of a (1, 1)-tangle T as shown in Fig. 3.30 with $V = V^r$ or W_k^r.

If $V = V^r$, then by Lemma 3.9 a, J_T is a scalar operator and so $J_{L,\mathbf{k}}$ is a scalar multiple of the invariant J_{O^r} of the r-colored unknot. But this is just $[r]=0$, by (3.27 a).

If $V = W_k^r$, then J_T is still a scalar operator (cf. Lemma 3.9 b). For, if λ is an eigenvalue for J_T, with eigenspace U, then U is a submodule and so $U \supset V^k$, i.e. $J_T(e_i) = \lambda e_i$ for $i \leqq m$. Thus we must show $U \neq V^k$, i.e. $J_T(e_i) = \lambda e_i$ for some

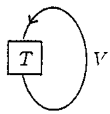


Fig. 3.30

$i>m$, for then $U=W_k^r$ and J_T is multiplication by λ. To see $J_T(e_{m+1})=\lambda e_{m+1}$, for example, linearity $J_T(\alpha e_{m+1})=\alpha J_T(e_{m+1})$ with $\alpha=Y$ shows that $J_T(e_{m+1}) =\lambda e_{m+1}+\nu e_{-m}$ for some ν. But then linearity with $\alpha=K$, together with the fact that e_{m+1} and e_{-m} have distinct eigenvalues for K, shows $\nu=0$.

The proof is completed as in the case $V=V^r$ with the observation that

$$\begin{aligned} J_{\bigcirc}\, W_k^r &= \sum_{i=-m}^{r-m-1} q^i \\ &= \bar{q}^m \sum_{i=0}^{r-1} q^i \\ &= 0. \quad \square \end{aligned}$$

4 Skein theory, cabling and the symmetry principle

The computation of the $\mathcal{M}$-colored framed link invariant $J_{L,\mathbf{k}}$ (see § 1 and (2.12)) directly from the R-matrices defined in § 2 becomes impractical as the crossing number of L and the colors k_i increase. If all $k_i=2$, however, then $J_{L,\mathbf{k}}$ $\left(\text{as a function of } q=e\left(\frac{1}{r}\right)\right)$ is just a variant of the Jones polynomial of L (Corollary 4.11) or the Kauffman bracket polynomial (Corollary 4.13) and can therefore be computed by the Conway skein calculus or (for $r=3$, 4 or 6) by topological means. Using this fact, we will give an expression for $J_{L,\mathbf{k}}$ as a linear combination of Jones polynomials of certain cables of L (Theorem 4.15). This will yield an alternative form for the 3-manifold invariant $\tau_r(M)$ (Theorem 4.17) which can be exploited for calculations.

At the end of this section we prove a symmetry principle (4.20) which describes the change in $J_{L,\mathbf{k}}$ when a color k is changed to $r-k$. This also leads to simplifications in the computation of $\tau_r(M)$, and appears to have interesting applications as well (including a simplified proof of the existence of $\tau_r(M)$; see § 5 below).

Skein theory, the Jones polynomial and the Kauffman bracket

(4.1) **Definition** Let L be a framed (unoriented) link. Define

$$J_L = J_{L,2}$$

where **2** denotes the constant 2-coloring (i.e. each component of L is colored with the untwisted 2-dimensional irreducible $\mathscr{A}$-module V^2).

The purpose of this subsection is to prove that J_L is equal to the value of the Jones polynomial at q, suitably normalized to account for the framing (Corollary 4.11), or equivalently the Kauffman bracket for a good diagram of L (Corollary 4.13).

Observe that for the zero framed m-component unlink $\bigcirc^m$, we have

(4.2) $$J_{\bigcirc^m} = [2]^m = (s+\bar{s})^m$$

by m applications of Lemma 3.27a.

Furthermore, J_L satisfies the following skein relations.

(4.3) **Theorem** (1) *(oriented skein relations) Let L_+, L_- and L_0 be oriented framed links with good diagrams (i.e. the framings are the blackboard framings) which are identical except in a disc where they are as shown in Fig.* 4.4a. *Then*

(a) $$tJ_{L_+} - \bar{t}J_{L_-} = (s-\bar{s})\,J_{L_0}.$$

If the framings are adjusted so that $L_+\cdot L_+ = L_-\cdot L_- = L_0\cdot L_0$, then (a) *becomes*

(b) $$qJ_{L_+} - \bar{q}J_{L_-} = (s-\bar{s})\,J_{L_0}.$$

(2) *(unoriented skein relations) Let R, V and H denote unoriented framed links with identical good diagrams except as shown in Fig.* 4.4*b. Then*

(a) $$J_R = tJ_V + \bar{t}J_H$$

if the two strands in the crossing come from different components of R, and

(b) $$J_R = \varepsilon(tJ_V - \bar{t}J_H)$$

if the two strands come from the same component of R, producing a crossing of sign $\varepsilon = \pm 1$ (i.e. appearing as in L_ε of Fig. 4.4a *if R is oriented).*

Proof. Recall from Example 2.37 that action of $\check{R}$ on $V^2\otimes V^2$ is given (in the preferred basis) by

$$\check{R} = (t)\oplus\begin{pmatrix}0 & \bar{t}\\ \bar{t} & \bar{t}(s-\bar{s})\end{pmatrix}\oplus(t).$$

We find (e.g. by computing the characteristic polynomial of $\check{R}$) that

(4.5) $$t\check{R} - \bar{t}\check{R}^{-1} = (s-\bar{s})\,I,$$

and (1a) follows. To adjust the framings to become equal we may add a left kink to L_+ and a right kink to L_- (see e.g. Fig. 3.28b), which changes $J_{L_\pm}$ by $t^{\mp 3}$ by Lemma 3.27b, and so the coefficients of $J_{L_\pm}$ become $t^{\pm 1}t^{\pm 3} = q^{\pm 1}$. This gives (1b).

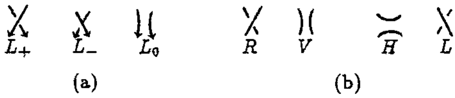


Fig. 4.4

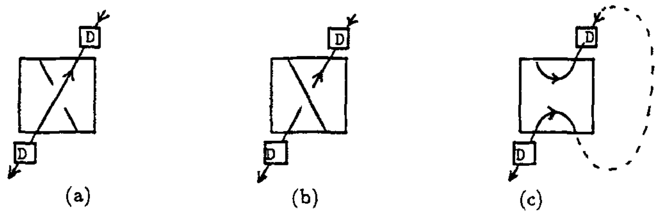


Fig. 4.8

For (2a), orient R so that the crossing looks like L_+, and then (1a) yields

$$tJ_R - \bar{t}J_L = (s-\bar{s})\,J_V \tag{4.6}$$

where L is as shown in Fig. 4.4b. Now reverse the orientation on one strand so that the same crossing looks like L_- when rotated by 90°, and so

$$-\bar{t}J_R + tJ_L = (s-\bar{s})\,J_H \tag{4.7}$$

by (1a) again. Multiplying (4.6) by t and (4.7) by $\bar{t}$ and adding gives (2a).

For (2b), first suppose $\varepsilon=1$, so (4.6) follows as above by orienting R. Now we may locally reorient one strand by introducing two coupons on R, giving the framed graph (see Remark 3.20) R' shown in Fig. 4.8a, and similarly construct L' and H' (Figs. 4.8bc). As in (4.7) above, we get $-\bar{t}J_{R'}+tJ_{L'}=(s-\bar{s})\,J_{H'}$. But by Remark 3.26 we have $J_{R'}=J_R$ and $J_{L'}=J_L$ (since our color 2 is odd and there are no extreme points in Figs. 4.8ab) and $J_{H'}=-J_H$ (since there are an odd number of extreme points between the coupons in Fig. 4.8c), and so (4.7) is replaced by

$$-\bar{t}J_R + tJ_L = -(s-\bar{s})\,J_H. \tag{4.9}$$

Multiplying (4.6) by t and (4.9) by $\bar{t}$ and adding now gives (2b) for $\varepsilon=1$.

If $\varepsilon=-1$, then the same argument establishes (4.7) and a revised (4.6), with the right hand side negated. This gives (2b) for $\varepsilon=-1$, as above. □

Remark. Theorem 4.3(2) can also be proved directly by computing the appropriate local tangle operators as in our proof of (1).

The skein relations in Theorem 4.3(1b) lead to a variant $\tilde{V}_L$ of the original Jones polynomial V_L of an *oriented* link L (in the variable q), characterized by

$$\tilde{V}_{\bigcirc} = 1 \tag{1}$$

$$q\tilde{V}_{L_+} - \bar{q}\tilde{V}_{L_-} = (s-\bar{s})\,\tilde{V}_{L_0} \tag{2}$$

where ○ denotes the unknot and L_+, L_- and L_0 are as in Fig. 4.2. (Note that for V_L one swaps q and $\bar{q}$ on the left side of (2).) In fact

(4.10) $$\tilde{V}_L = V_L(\bar{q})$$

where $\bar{q}^{1/2}$ must be chosen to be $-\bar{s}$ on the right (i.e., recalling that $V_L(q)$ is a polynomial in $q^{\pm 1/2}$, we substitute $\bar{q}$ for q, $-\bar{s}$ for $q^{1/2}$ and $-s$ for $q^{-1/2}$). Equivalently $\tilde{V}_L$ may be defined as the specialization of the Homfly polynomial at $(iq, i(\bar{s}-s))$ (see e.g. [L2]).

(4.11) **Corollary** *If L is a framed link, then*

$$J_L = [2]\, t^{3L\cdot L}\, \tilde{V}_L$$

for any orientation on L.

Proof. The right hand side is characterized by the same skein relations, (4.2) and (4.3(1a)), as J_L. □

(4.12) *Remarks* (1) The values of $\tilde{V}_L$ at certain roots of unity have topological significance, as they do for V_L [LM1, Lp, Mu]. In particular, the values at $q = e\left(\frac{1}{r}\right)$, for $r = 1, 2, 3, 4$ and 6, are as follows:

r	$\tilde{V}_L$	V_L
1	2^{n-1}	$(-2)^{n-1}$
2	$\det L$	$\det L$
3	1	$(-1)^{n-1}$
4	$a\sqrt{2}^{n-1}$	$a(-\sqrt{2})^{n-1}$
6	$\sqrt{3}^d(-i)^\omega$	$(-\sqrt{3})^d(-i)^\omega$

where n is the number of components of L, $\det L$ is the value at -1 of the (normalized) Alexander polynomial of L, a is $(-1)^{\mathrm{Arf}(L)}$ when L is proper (so the Arf invariant is defined) and 0 otherwise, d is the nullity of $Q(\mathrm{mod}\ 3)$ where Q is the quadratic form of L (represented by $S+S^t$ for any Seifert matrix S of L), and ω is the Witt class of $Q(\mathrm{mod}\ 3)$ in $W(\mathbf{Z}/3\mathbf{Z}) = \mathbf{Z}/4\mathbf{Z}$ (see Appendix B). It is well known that $|\det L| = |H_1(M)|$, where M is the 2-fold branched cover of S^3 along L, and $d = \dim H_1(M; \mathbf{Z}/3\mathbf{Z})$ (since any matrix representing Q is a presentation matrix for $H_1(M)$). Our expression for the value of V_L at $e(\frac{1}{6})$ may appear unfamiliar, although it can be shown to be equivalent to Lipson's.

(2) It is often simpler to use J_L as the basic ingredient rather than $\tilde{V}_L$. For example, $J_\emptyset = 1$ is a better normalization than $\tilde{V}_\bigcirc = V_\bigcirc = 1$ since formulas are simpler (e.g. [2] disappears). Of course J_L does require a framing on L, but if one chooses a framing for which $L\cdot L = 0$, then $J_L = [2]\ \tilde{V}_L$ (cf. 4.3(1b)).

The skein relations in Theorem 4.3(2) remind one of Kauffman's bracket polynomial [Kf] (also see [L2]) in the variable t defined for a link diagram D. Our version $B_D(t)$ is normalized differently and is characterized by

(1) $$B_{\bigcirc^m} = (-[2])^m = (-s-\bar{s})^m \qquad (s = t^2)$$

(2) $$B_R = tB_V + \bar{t}B_H$$

where $\bigcirc^m$ is the standard diagram of the unlink of m components, and R, V and H are diagrams which are identical except as shown in Fig. 4.4b.

(4.13) **Corollary** *If L is a framed link, then*

$$J_L = (-i)^{L \cdot L} B_D(it)$$

for any good diagram D of L.

Proof. First note that $L \cdot L$ is only defined for oriented links, but $L \cdot L \pmod 4$ is independent of the choice of orientation (i.e. $(A+B)(A+B) \equiv (A-B)\cdot(A-B) \pmod 4$).

We prove the corollary by showing that the right side satisfies the same characterizing skein relations as J_L, namely (4.2) (obviously) and (4.3(2)). Observe that $R \cdot R \equiv V \cdot V + 1 \equiv H \cdot H - 1 \pmod 4$ if the two strands in the crossing belong to different components of R, whereas $R \cdot R \equiv V \cdot V + \varepsilon \equiv H \cdot H + \varepsilon$ if they belong to the same component. In the former case we compute

$$\begin{aligned}(-i)^{R \cdot R} B_R(it) &= (-i)^{V \cdot V + 1}(it) B_V(it) + (-i)^{H \cdot H - 1}(-i\bar{t}) B_H(it) \\ &= t((-i)^{V \cdot V} B_V(it)) + \bar{t}((-i)^{H \cdot H} B_H(it))\end{aligned}$$

and the latter case

$$\begin{aligned}(-i)^{R \cdot R} B_R(it) &= (-i)^{V \cdot V + \varepsilon}(it) B_V(it) + (-i)^{H \cdot H + \varepsilon}(-i\bar{t}) B_H(it) \\ &= \varepsilon(t(-i)^{V \cdot V} B_V(it) - \bar{t}(-i)^{H \cdot H} B_H(it)). \quad \square\end{aligned}$$

Remark. Alternatively, (4.13) can be proved directly from (4.10) and Corollary 4.11 by using the well known relation between the Jones polynomial and the bracket

$$[2]\, V_L(\bar{q}) = (-(it))^{-3L \cdot L} B_D(it)$$

(see e.g. [L2], and note that the [2] is there because of our normalization of the bracket, and the i's are there because of the choice of $q^{1/2}$ in (4.10)).

Cabling

The next result gives a formula for the general $\mathcal{M}$-colored framed link invariant $J_{L,\mathbf{k}}$ in terms of Jones polynomials $\left(\text{at } q = e\left(\frac{1}{r}\right)\right)$ of certain cables of L. (Recall that $\mathcal{M} = \{V^1, \ldots, V^{r-1}\}$.)

We will need the following lemma (which will correspond to zero cabling).

(4.14) **Lemma** *Let $(L, \mathbf{k})$ be a colored, framed link. If S is a sublink of L obtained by removing some 1-colored components, then*

$$J_{L,\mathbf{k}} = J_{S,\mathbf{k}|S}.$$

Proof. It was observed in (2.9) that K^2 acts by the identity on V^1, as does $\check{R}$ on $V^1 \otimes V^k$ and $V^k \otimes V^1$. Thus we may ignore 1-colored components of L when computing $J_{L,\mathbf{k}}$. $\square$

Now define a *cabling* $\mathbf{c}$ of a framed link L to be an assignment of nonnegative integers c_i to the components L_i of L. The associated *cable* of L, denoted $L^{\mathbf{c}}$, is obtained by replacing each L_i with c_i parallel pushoffs (using the framing). If $c_i=0$, simply delete L_i.

If L is oriented, then there is a natural choice of orientation on $L^{\mathbf{c}}$: for each component L_i of L, orient the pushoffs so that their sum is homologous in a tubular neighborhood of L_i to L_i or to 0 (depending upon whether c_i is odd or even). With this choice we say L and $L^{\mathbf{c}}$ are *compatibly oriented.*

We will use the multi-index notation $f(\mathbf{k})=\prod f(k_i)$, $\mathbf{k}<\mathbf{n}$ if $k_i<n_i$ for all i, etc. For example, $(-1)^{\mathbf{k}}=\prod(-1)^{k_i}=(-1)^{\Sigma k_i}$, $\binom{\mathbf{n}}{\mathbf{k}}=\prod\binom{n_i}{k_i}$, and $\sum_{\mathbf{k}=1}^{\mathbf{n}}$ is the sum over all $\mathbf{k}$ with $1\leqq k_i\leqq n_i$.

(4.15) **Theorem** *Let L be a framed link and $\mathbf{k}$ be an $\mathscr{M}$-coloring of L. Then setting $\mathbf{n}=\mathbf{k}-\mathbf{1}$,*

$$J_{L,\mathbf{k}}=\sum_{\mathbf{j}=0}^{\mathbf{n}/2}(-1)^{\mathbf{j}}\binom{\mathbf{n}-\mathbf{j}}{\mathbf{j}}J_{L^{\mathbf{n}-2\mathbf{j}}}$$
$$=[2]\sum_{\mathbf{j}=0}^{\mathbf{n}/2}(-1)^{\mathbf{j}}\binom{\mathbf{n}-\mathbf{j}}{\mathbf{j}}t^{3L^{\mathbf{n}-2\mathbf{j}}\cdot L^{\mathbf{n}-2\mathbf{j}}}\tilde{V}_{L^{\mathbf{n}-2\mathbf{j}}}$$

for any orientation on $L^{\mathbf{n}-2\mathbf{j}}$. In particular, if L and $L^{\mathbf{n}-2\mathbf{j}}$ are compatibly oriented for all $\mathbf{j}$, then

$$J_{L,\mathbf{k}}=[2]\,t^{3S\cdot S}\sum_{\mathbf{j}=0}^{\mathbf{n}/2}(-1)^{\mathbf{j}}\binom{\mathbf{n}-\mathbf{j}}{\mathbf{j}}\tilde{V}_{L^{\mathbf{n}-2\mathbf{j}}}$$

where S is the even colored sublink of L, consisting of all L_i with k_i even. (By convention $J_{L^0,2}=1$, $\tilde{V}_{L^0}=\dfrac{1}{[2]}$ and $L^0\cdot L^0=0$.)

Proof. The first equality is an immediate consequence of Corollary 2.15 and Lemmas 3.10 and 4.14. The second equality uses Corollary 4.11 and the last equality follows from the definition of compatibly oriented.

Remark. There is an analogous statement if $\mathbf{k}$ is only an $\mathscr{M}$-coloring on a sublink S of L,

$$J_{L,\mathbf{k}}=\sum_{\mathbf{j}=0}^{\mathbf{n}/2}(-1)^{\mathbf{j}}\binom{\mathbf{n}-\mathbf{j}}{\mathbf{j}}J_{S^{\mathbf{n}-2\mathbf{j}}\cup(L-S),\,\mathbf{2}\cup\mathbf{k}|(L-S)}$$

where $\mathbf{n}=\mathbf{k}|S-\mathbf{1}$.

(4.16) *Examples* If K is a framed knot, then

$$J_{K,k}=\sum_{j=0}^{n/2}(-1)^j\binom{n-j}{j}J_{K^{n-2j}}$$

where $n=k-1$. In particular

$$J_{K,3}=J_{K^2}-1$$
$$J_{K,4}=J_{K^3}-2J_K$$
$$J_{K,5}=J_{K^4}-3J_{K^2}+1.$$

As a consequence of Theorem 4.15, we obtain a formula for $\tau_r(M_L)$ in terms of Jones polynomials $\left(\text{at } q=e\left(\frac{1}{r}\right)\right)$ of cables of L. Recall from § 1 that

$$\tau_r(M_L)=\alpha_L \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{r}-\mathbf{1}} [\mathbf{k}]\, J_{L,\mathbf{k}}$$

where $\alpha_L=\left(\sqrt{\frac{2}{r}}\sin\frac{\pi}{r}\right)^{n_L}\left(e\left(\frac{-3(r-2)}{8r}\right)\right)^{\sigma_L}$.

(4.17) **Cabling Theorem** *For any framed link L,*

$$\tau_r(M_L)=\alpha_L \sum_{\mathbf{c}=0}^{\mathbf{r}-2} \langle \mathbf{c}\rangle\, J_{L^{\mathbf{c}}}$$

(i.e. sum over all cables $\mathbf{c}=(c_1, \ldots, c_n)$ with $0\leqq c_i\leqq r-2$), where

$$\langle \mathbf{c}\rangle = \sum_{\mathbf{j}=\mathbf{0}}^{(\mathbf{r}-\mathbf{c}-\mathbf{1})/2} (-1)^{\mathbf{j}}[\mathbf{c}+2\mathbf{j}+\mathbf{1}]\binom{\mathbf{c}+\mathbf{j}}{\mathbf{j}}$$

(i.e. sum over all $\mathbf{j}\geqq \mathbf{0}$ with $\mathbf{c}+2\mathbf{j}+\mathbf{1}<\mathbf{r}$).

The formula 4.17 can be rewritten in terms of the Jones polynomial variant $\tilde{V}$ by using the third equation in Theorem 4.15 and orienting cables compatibly:

$$\tau_r(M_L)=\alpha_L[2]\sum_{\mathbf{c}} t^{3L_{\mathbf{c}}\cdot L_{\mathbf{c}}}\langle \mathbf{c}\rangle\, \tilde{V}_{L^{\mathbf{c}}} \tag{4.18}$$

where $L_{\mathbf{c}}$ is the sublink of L consisting of components L_i with c_i odd, (thus $L^{\mathbf{c}-2\mathbf{j}}\cdot L^{\mathbf{c}-2\mathbf{j}}=L^{\mathbf{c}}\cdot L^{\mathbf{c}}=L_{\mathbf{c}}\cdot L_{\mathbf{c}}$).

(4.19) *Remark* It is often easier to calculate with cables by first changing all framings of L to zero (and adjusting by the appropriate power of t) and then taking cables using the 0-framing.

Symmetry Principle .

Finally we state the Symmetry Principle, which allows us to switch a color k to $r-k$. This cuts the number of terms in $\tau_r(M_L)$ from the order of $(r-1)^n$ to $\left(\frac{r}{2}\right)^n$, and makes possible an elementary proof of the invariance of τ under the m-strand k-move for $m>1$ (see § 5).

It is convenient to adopt the notation $L\cup K$ for a framed link with a distinguished component K. Colorings of $L\cup K$ will be written $\mathbf{l}\cup k$, where $\mathbf{l}$ is a coloring of L and k is the color of K. If the colors are selected from the modules V^1, V^2, ... defined in (2.8), then as above we identify these modules by their dimensions (also called colors) and so $\mathbf{l}$ is just a list of positive integers (i.e. l_i, or V^{l_i}, is the color of the component L_i).

(4.20) **Symmetry Principle** *Let $L\cup K$ be a framed link where K has framing a, and $\mathbf{l}=(l_1, \ldots, l_n)$ be a coloring of $L=L_1\cup\ldots\cup L_n$ by the modules defined in* (2.8). *If $0<k<r$, then*

$$J_{L\cup K,\mathbf{l}\cup(r-k)}=i^{(r-2k)a+2\lambda}\, J_{L\cup K,\mathbf{l}\cup k}$$

where $\lambda = \sum_{\text{even}\, l_i} K \cdot L_i$. *(Note that the exponent of i can also be written as $(r-2)K \cdot K + 2K \cdot E$ where E is the even colored sublink of $L \cup K$ for the coloring* $\mathbf{l} \cup k$.*)*

Before proving the Symmetry Principle, we illustrate its use in the context of 3-manifolds given by surgery on a knot. A more systematic study of its applications appears in § 8, including a form of the Symmetry Principle for τ_r (Theorem 8.5). Also see the end of this section for an application to Jones polynomials of cables.

(4.21) *Example* For $r=5$ and K a knot with framing a, we have

$$\begin{aligned}\tau_5(M_K) &= \alpha_K \sum_{k=1}^{4} [k]\, J_{K,k} \\ &= \alpha_K([1] + [2]\, J_K + [3]\, i^a\, J_K + [4]\, i^{3a}) \\ &= \alpha_K((1+i^{-a}) + [2](1+i^a)\, J_K).\end{aligned}$$

Thus if $a \equiv 2 \pmod 4$, then $\tau_5(M_K)=0$. If $a \not\equiv 2 \pmod 4$, then this shows that J_K (and so also the Jones polynomial of K at the fifth root of unity) is determined by $\tau_5(M_K)$, and thus is an invariant of M_K (cf. Theorem 8.22).

The proof of the Symmetry Principle needs

(4.22) **Lemma** *Let $L \cup K$ be a framed link, where K has framing zero, with colorings* $\mathbf{l}$ *on L and $V^1(i)$ on K (see above Lemma 2.16). Then*

$$J_{L \cup K, \mathbf{l} \cup V^1(i)} = (-1)^{1+\lambda}\, J_{L,\mathbf{l}}$$

where $\lambda = \sum_{\text{even}\, l_i} K \cdot L_i$.

Proof. Orient $L \cup K$ and draw it as a counterclockwise braid with the blackboard framing. We need to inspect three kinds of crossings as in Fig. 4.23.

In the first two cases, according to Corollary 2.32b, $\check{R}(e_j \otimes e) = i^{2j}\, e \otimes e_j$ and $\check{R}(e \otimes e_j) = i^{2j}\, e_j \otimes e$ since e_j has weight t^{2j} and $\alpha = i$. Similarly in the third case $\check{R}(e \otimes e) = i^r\, e \otimes e$ since e has weight $i = t^r$. On negative crossings we get the inverses $\check{R}^{-1}(e \otimes e_j) = i^{-2j}\, e_j \otimes e$, $\check{R}^{-1}(e_j \otimes e) = i^{-2j}\, e \otimes e_j$ and $\check{R}^{-1}(e \otimes e) = i^{-r}\, e \otimes e$ respectively.

Since K is 0-framed, K has an equal number of positive and negative crossings in the braid diagram and an odd number of maxima. Thus the self crossings of K together contribute nothing to $J_{L \cup K, \mathbf{l} \cup V^1(i)}$, and (since $K^2 = -1$ for $V^1(i)$) the maxima together contribute -1. That is, K contributes -1 (whether knotted or not). Hence we may change the self crossings of K so that K is unknotted, and then $L \cup K$ can be drawn as in Fig. 4.24. The crossings of K with L_i occur

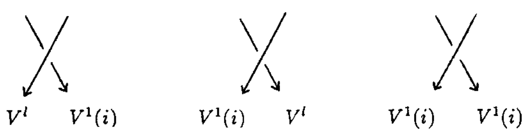


Fig. 4.23

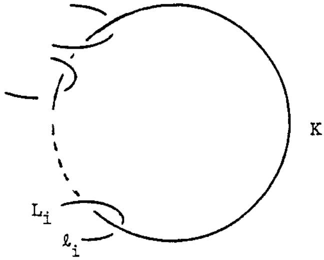


Fig. 4.24

in right or left handed pairs which algebraically sum to $K \cdot L_i$. So the contribution of pairs of crossings between K and L_i is $(-1)^{2jK \cdot L_i}$. When l_i is odd, then j is an integer so $(-1)^{2j}=1$ and there is no contribution. When l_i is even, then j is a half integer and $(-1)^{2j}=-1$, so we get a multiplicative contribution of $(-1)^{K \cdot L_i}$. □

Proof of the Symmetry Principle. Since we have the short exact sequence (2.16)

$$0 \to V^k \to W_k^r \to V^{r-k} \otimes V^1(i) \to 0$$

and since $J_{L,\mathbf{k}}$ is additive under extensions (3.10a) and $J_{L, W_k^r}=0$ (3.29), it follows that

$$0 = J_{L \cup K, \mathbf{1} \cup W_k^r} = J_{L \cup K, \mathbf{1} \cup k} + J_{L \cup K, \mathbf{1} \cup V^{r-k} \otimes V^1(i)}.$$

If the framing on K is zero, then replacing K with color $V^{r-k} \otimes V^1(i)$ by the 2-cable (using framing zero) of K with colors V^{r-k} and $V^1(i)$, it follows by Lemma 4.22 that we can eliminate the copy of K with color $V^1(i)$ by multiplying by $(-1)^{1+\lambda} = -i^{2\lambda}$ where $\lambda = \sum_{\text{even } l_i} K \cdot L_i$, and the result follows.

Finally, if the framing on K is a rather than zero, then we can change to framing zero by multiplying by $\bar{t}^{a(k^2-1)}$. Now apply the just proved zero framed case to switch the color on K from k to $r-k$, and then shift the framing back to a by multiplying by $t^{a((r-k)^2-1)}$. So the net change gives

$$\begin{aligned} J_{L \cup K, \mathbf{1} \cup (r-k)} &= t^{a((r-k)^2-1)} \cdot i^{2\lambda} \cdot \bar{t}^{a(k^2-1)} J_{L \cup K, \mathbf{1} \cup k} \\ &= i^{(r-2k)a+2\lambda} J_{L \cup K, \mathbf{1} \cup k}. \end{aligned}$$ □

We conclude this section with an application of the Symmetry Principle to the study of the values J_{L^c} of Jones polynomials for cables L^c of a framed link L at a *fixed* root of unity $q = e\left(\frac{1}{r}\right)$.

A cabling $\mathbf{m}$ of L will be called *minimal* if $\mathbf{m} \leqq \frac{r}{2} - 1$ (i.e. each component of L is replaced by at most $\frac{r}{2} - 1$ parallel copies). Now an easy inductive argument using the Symmetry Principle and the cabling formula (Theorem 4.15 and following remark) shows that for *any* cabling $\mathbf{c}$, J_{L^c} is an integer linear combination of the values $J_{L^\mathbf{m}}$ for minimal cablings $\mathbf{m}$ where the coefficients of the linear combination depend only on the linking matrix of L (and of course on $\mathbf{c}$ and r). That is, writing $J^\mathbf{c}(L)$ for J_{L^c}:

(4.25) **Corollary** *For each framed link L and cablings* **c** *and* **m** *with* **m** *minimal, there exist integers $a_{\mathbf{m}}^{\mathbf{c}}(L)$ such that*

$$J^{\mathbf{c}}(L)=\sum_{\text{minimal cablings } \mathbf{m}} a_{\mathbf{m}}^{\mathbf{c}}(L)\, J^{\mathbf{m}}(L),$$

and such that $a_{\mathbf{m}}^{\mathbf{c}}(L)=a_{\mathbf{m}}^{\mathbf{c}}(L')$ if L and L' have the same linking form.

In practice, the integers $a_{\mathbf{m}}^{\mathbf{c}}(L)$ can be computed easily.

(4.26) *Example* Let $r=5$. Consider 0-framed knots K and let v^c denote the value J_{K^c} of the Jones polynomial of the c-cable of K at $e(\frac{1}{5})$. By the Symmetry Principle $J_{K,2}=J_{K,3}$, and so by (4.16), $v^1=v^2-1$. Induction shows that $v^c=f_c v^1+f_{c-1}$, where f_c is the c^{th} term in the Fibonacci sequence 1, 1, 2, 3, 5, ...

5 The 3-manifold invariant $\tau_r(M)$

Proof that τ_r is a 3-manifold invariant

If M is described by surgery on a framed link L, then we have defined (1.5 and 1.7)

$$\tau_r(M)=\tau_L=\alpha_L \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{r}-\mathbf{1}} [\mathbf{k}]\, J_{L,\mathbf{k}}$$

where $\alpha_L=b^{n_L}c^{\sigma_L}$, $b=\sqrt{\dfrac{2}{r}}\sin\dfrac{\pi}{r}$, $c=e\left(\dfrac{-3(r-2)}{8r}\right)$, n_L is the number of components of L and σ_L is the signature of its linking matrix. For $\tau_r(M)$ to be well defined we must finally prove Theorem 1.6, that τ_L is invariant under K-moves on L, and hence $\tau_r(M)$ is independent of the choice of framed link used to describe M.

The proof of Theorem 1.6 depends on an elementary identity for Gauss sums. $\left(\text{Recall that } t=e\left(\dfrac{1}{4r}\right) \text{ and } s=e\left(\dfrac{1}{2r}\right).\right)$

(5.1) **Lemma** $\displaystyle\sum_{k=1}^{r-1} [jk]\,[kl]\, t^{j^2+k^2+l^2}=\frac{[jl]}{b}\, e\left(\frac{3}{8}\right).$

Proof. Note that

$$\begin{aligned}
&(s-\bar{s})^2 \sum_{k=1}^{4r} [jk]\,[kl]\, t^{j^2+k^2+l^2}\\
&=\sum_{k=1}^{4r} (t^{2jk}-\bar{t}^{2jk})(t^{2kl}-\bar{t}^{2kl})\, t^{j^2+k^2+l^2}\\
&=\sum_{k=1}^{4r} (\bar{t}^{2jl}(t^{(k+(j+l))^2}+t^{(k-(j+l))^2})-t^{2jl}(t^{(k+(j-l))^2}+t^{(k-(j-l))^2}))\\
&=2(\bar{t}^{2jl}-t^{2jl})\sum_{k=1}^{4r} t^{k^2}\\
&=2(\bar{s}^{jl}-s^{jl})\, 2\sqrt{2r}\, e(\tfrac{1}{8})
\end{aligned}$$

where the last equality is a standard Gauss sum (see [La]). Now $\sum_{k=1}^{r-1} = \frac{1}{4} \sum_{k=1}^{4r}$ by the symmetries of the bracket [ ], so

$$\sum_{k=1}^{r-1} [j\,k][k\,l]\, t^{j^2+k^2+l^2} = \frac{[j\,l]}{\bar{s}-s} \sqrt{2r}\, e\left(\frac{1}{8}\right)$$

$$= \frac{[j\,l]}{b}\, e\left(\frac{3}{8}\right)$$

since $\bar{s}-s = 2\sin\frac{\pi}{r}\, e(-\frac{1}{4}) = \sqrt{2r}\, b\, e(-\frac{1}{4})$. □

Now to prove Theorem 1.6 consider an m-strand K-move $L \leftrightarrow L^\varepsilon$ of type $\varepsilon = \pm 1$. Choose diagrams for L and L^ε which agree everywhere except for the tangles shown in Fig. 5.2.

For any $\mathcal{M}$-coloring $\mathbf{l}$ of L, let $\mathbf{l} \cup k$ denote the induced coloring of L^ε with the new component K colored $k<r$. Then $[\mathbf{l} \cup k] = [\mathbf{l}][k]$. Since $n_{L^\varepsilon} = n_L + 1$ and $\sigma_{L^\varepsilon} = \sigma_L + \varepsilon$, and so $\alpha_{L^\varepsilon} = b\, c^\varepsilon \alpha_L$, we have

$$\tau_L = \alpha_L \sum_{\mathbf{l}=\mathbf{1}}^{\mathbf{r-1}} [\mathbf{l}]\, J_{L,\mathbf{l}}$$

$$\tau_{L^\varepsilon} = \alpha_{L^\varepsilon} \sum_{\mathbf{l}=\mathbf{1}}^{\mathbf{r-1}} \left(\sum_{k=1}^{r-1} [\mathbf{l} \cup k]\, J_{L^\varepsilon, \mathbf{l} \cup k} \right)$$

$$= \alpha_L \sum_{\mathbf{l}=\mathbf{1}}^{\mathbf{r-1}} [\mathbf{l}] \left(b\, c^\varepsilon \sum_{k=1}^{r-1} [k]\, J_{L^\varepsilon, \mathbf{l} \cup k} \right).$$

Thus to prove $\tau_{L^\varepsilon} = \tau_L$ it suffices to establish the identity

$$b\, c^\varepsilon \sum_{k=1}^{r-1} [k]\, J_{L^\varepsilon, \mathbf{l} \cup k} = J_{L,\mathbf{l}} \tag{5.3}$$

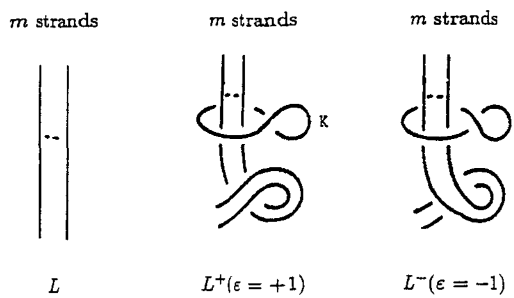


Fig. 5.2

for any fixed $\mathcal{M}$-coloring $\mathbf{l}$ on L. We will prove (5.3) by induction on the number of strands m, starting the induction for $m \leqq 1$ in the next result.

(5.4) **Lemma** *Identity* (5.3) *holds for* $m=0$ *and* 1, *and therefore* τ_L *is invariant under m-strand K-moves for* $m=0$ *and* 1.

Proof. The proof for $m=0$ is a special case of the proof for $m=1$ when the color j on the strand of L passing through K is 1, by Lemma 4.14. So we assume $m=1$. Then $J_{L^\varepsilon, \mathbf{l} \cup k} = \frac{[jk]}{[j]} t^{\varepsilon(j^2-1+k^2-1)} J_{L,\mathbf{l}}$ by Lemma 3.27, and so (5.3) reduces to

$$b c^\varepsilon \sum_{k=1}^{r-1} [k] \frac{[jk]}{[j]} t^{\varepsilon(j^2-1+k^2-1)} = 1.$$

Since these two identities (for $\varepsilon = \pm 1$) are conjugate, we need only consider the case $\varepsilon = +1$. But this identity follows from Lemma 5.1, with $l=1$, since $c = e(-\frac{3}{8}) t^3$. □

(5.5) *Remark* S^3 can be obtained by $+1$ surgery on the unknot, so

$$\tau_r(S^3) = b c \sum_{k=1}^{r-1} [k]^2 t^{k^2-1} = 1$$

by Lemma 3.27 and the identity established in the proof of (5.4) with $j=1$. Furthermore, it is not hard to show that τ_L is the only invariant of framed links under 1-strand K-moves of the form $a^\sigma \sum_{\mathbf{k}} \left(\prod_{i=1}^{n} d_{k_i} \right) J_{L,\mathbf{k}}$ with value 1 on the 1-framed unknot. (This is essentially how Reshetikhin and Turaev arrived at their formula.) Indeed, one readily shows as in the proof above that for any such invariant

$$a^\varepsilon \sum_{k=1}^{r-1} \left(\frac{[jk]}{[j]} t^{\varepsilon(j^2-1+k^2-1)} \right) d_k = 1$$

for all $0<j<r$ and $\varepsilon = \pm 1$. Solving for the d_k, using Lemma 5.1 and the fact that the matrix $(b[jk])$ is its own inverse $\left(\text{this is the well known orthogonality relation } b^2 \sum_{j=1}^{r-1} [ij][jk] = \delta_{ik}\right)$, gives $d_k = b c^\varepsilon a^{-\varepsilon} [k]$. Equating the values for $\varepsilon = \pm 1$ shows $a = \pm c$ and the case $a=-c$ is eliminated by the normalization on the unknot. Thus $a=c$ and $d_k = b[k]$, so the invariant is just τ_L.

Finally we prove the inductive step, completing the proof of Theorem 1.6.

(5.6) **Lemma** *Identity* (5.3) *holds for m-strand K-moves for* $m>1$ *provided it holds for n-strand K-moves for all* $n<m$. *Thus* τ_L *is invariant under K-moves.*

Proof. First suppose that L and L^ε are trivial outside of the tangles shown in Fig. 5.2, as shown in Fig. 5.7 (with blackboard framings).

Using the Symmetry Principle 4.20, we may assume that all colors j of components J of L satisfy $j \leqq \frac{r}{2}$. Indeed, if $j > \frac{r}{2}$, then change j to $r-j \leqq \frac{r}{2}$ on J (and

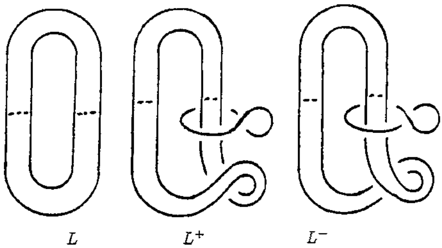


Fig. 5.7

on the corresponding component J^ε of L^ε). This changes the left side of (5.3) by

$$i^{\varepsilon(r-2j)+2(J^\varepsilon\cdot S+k-1)}$$

where S is the even-colored sublink of $L^\varepsilon-(J^\varepsilon\cup K)$, and leaves the right side unchanged. Next change k to $r-k$ on K. Then, using $[k]=[r-k]$, the left side of (5.3) changes by

$$i^{\varepsilon(r-2k)+2(J\cdot S+(r-j)-1)}$$

while the right side remains unchanged. Noting that $K\cdot S=J^\varepsilon\cdot S=|S|$, we see that the net change on the left side is

$$i^{2\varepsilon(r-j-k)+4(K\cdot S-1)+2(r-j+k)}=1$$

as it is on the right side.

Now by Lemma 3.10c, we may replace two components L_1 and L_2 of L, with colors l_1 and l_2, by a single component colored by $V^{l_1}\otimes V^{l_2}$ $=V^{l_1+l_2-1}\oplus\ldots\oplus V^{|l_1-l_2|+1}$ (using Theorem 2.13). Thus by Lemma 3.10a and distributivity, it is enough to establish (5.3) when L_1 and L_2 are replaced by a single j-colored component for $j<r$. But this is covered by the induction hypothesis.

Now consider the general case shown in Fig. 5.8, where T is an arbitrary tangle. We will reduce to the special case above (Fig. 5.7) using cabling and skein theory.

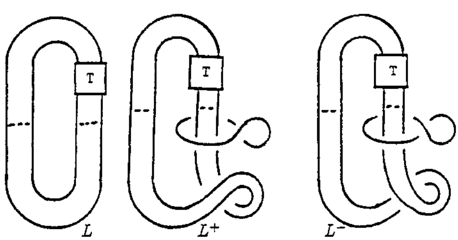


Fig. 5.8

First suppose that $\mathbf{l}=2$, the constant 2-coloring. Then we prove (5.3) by induction on the number of crossings in T. The induction begins with zero crossings, which is covered by the special case. (Note that T may have maxima and minima, as well as vertical strands, which simply pull through K, reducing m.) In general, we may smooth a crossing of T in two ways in both L and L^{ε}, and (5.3) follows by induction for each smoothing, using Theorem 4.3(2).

Finally, for general $\mathbf{l}$, $J_{L,\mathbf{l}}$ and $J_{L^{\varepsilon},\mathbf{l}\cup k}$ can be computed using the cabling formula of Theorem 4.15 (and the subsequent remark) applied to L and $L^{\varepsilon}-K$, respectively. This reduces the proof of (5.3) to the case $\mathbf{l}=\mathbf{2}$ proved above. □

(5.9) **Theorem** τ_r *satisfies the following three properties*
(1) $\tau_r(M \# N)=\tau_r(M)\,\tau_r(N)$
(2) $\tau_r(-M)=\bar{\tau}_r(M)$
(3) $\tau_r(S^3)=1$.

Proof. For (1), choose framed links L and L' with $M_L=M$, $M_{L'}=N$, and so $M_{L\cup L'}=M \# N$ where $L\cup L'$ denotes disjoint, (L and L' are separated by a 2-sphere). Note that

$$J_{L\cup L',\mathbf{k}\cup\mathbf{k}'}=J_{L,\mathbf{k}}\,J_{L',\mathbf{k}'}$$

(this is immediate from the definition of the colored framed link invariants, see the proof of Theorem 3.6), and so (1) follows from the definition of τ_r and distributivity.

For (2), observe that $(-M_L)=M_{\bar{L}}$, while $\bar{L}$ is the mirror image (obverse) of L. Now (2) follows from the Cabling Theorem 4.17 since $J_{\bar{L}}=\bar{J}_L$ and $\sigma_{\bar{L}}=-\sigma_L$.

Finally, (3) was shown in Remark 5.5. (Alternatively (3) follows from (1) once it is known that τ_r is nontrivial, i.e. $\tau_r(M)\neq 0$ for some M.) □

(5.10) *Remark* Observe that $\tau_2(M)=1$ for all M. Indeed $\tau_2(M_L)=J_{L,\mathbf{1}}=1$. In the subsequent sections we will give formulas for $\tau_r(M)$ for small values of $r>2$.

Examples

Computations similar to the one made for S^3 in Remark 5.5 can be made for $S^2\times S^1$ and the lens spaces $L(p,1)$, obtained by surgeries on the unknot with framings 0 and p, respectively. For example

$$\tau_r(S^2\times S^1)=b\sum_{k=1}^{r-1}[k]^2=\frac{1}{b}=\sqrt{\frac{r}{2}}\csc\left(\frac{\pi}{r}\right) \tag{5.11}$$

(which approaches ∞ like $r^{3/2}$ as $r\to\infty$), and for *even* r

$$\tau_r(\mathbf{R}P^3)=b\,c\sum_{k=1}^{r-1}[k]^2\,t^{2(k^2-1)}=\frac{1}{\sqrt{2}}\sec\left(\frac{\pi}{2r}\right) \tag{5.12}$$

(which approaches $1/\sqrt{2}$ as $r\to\infty$)) where the last equality is derived as in the proof of Lemma 5.1 by expanding, completing the square, and using a Gauss sum. (Note that $\mathbf{R}P^3=L(2,1)$.) A similar argument shows that for r odd, $\tau_r(\mathbf{R}P^3)=0$ (cf. (8.9)). In [KM4], we will give general formulas for τ_r (lens space).

6 The case $r=3$

In this section we give two formulas for $\tau_3(M)$. The first (6.1) depends only on the linking matrix of L, where $M=M_L$. It follows that $\tau_3(M)$ (unlike $\tau_4(M)$, see §7) is a homotopy invariant (Remark 6.2). The second (Theorem 6.3) expresses $\tau_3(M)$ in terms of "classical" invariants of M.

Let L be a framed link with n components and signature σ. If $\mathbf{k}$ is a coloring of L with all colors 1 or 2, and S is the 2-colored sublink of L, then by Corollary 4.11 and Lemma 4.14, $J_{L,\mathbf{k}}=J_{S,\mathbf{2}}=[2]\,t^{3S\cdot S}\,\tilde{V}_S$ (for any orientation on S). In particular for $r=3$

$$J_{L,\mathbf{k}}=i^{S\cdot S}$$

since $\tilde{V}_S=1$ (Remark 4.12), $t^3=i$ and $[2]=1$. Thus the 3-manifold invariant (1.7) reduces to

$$\tau_3(M_L)=\frac{1}{\sqrt{2^n}}\,c^{\sigma}\sum_{S<L} i^{S\cdot S} \tag{6.1}$$

where $c=e\left(-\frac{1}{8}\right)=\frac{1-i}{\sqrt{2}}$. Here $<$ denotes sublink and $\emptyset\cdot\emptyset=0$ by convention. (Alternatively, (6.1) follows from the cabling formulas in Theorem 4.15 or from the Symmetry Principle 4.20.)

(6.2) *Remark* Evidently Formula (6.1) depends only on the linking matrix A of L. Since it is a 3-manifold invariant, it must be invariant under change of orientation on L, and under blowups and handle slides, that is under *stable equivalence* of A. It follows that $\tau_3(M_L)$ is a homotopy invariant, determined in fact by the first Betti number of M_L and the linking pairing on $\operatorname{Tor} H_1(M_L)$ (for it is known that these determine the stable equivalence class of A [KP, Du, Wk]).

Note that there is an easy direct proof of the invariance of Formula (6.1) under stable equivalence of A (giving an elementary proof, using the two moves in [K1], that τ_3 is a 3-manifold invariant). First observe that the formula is multiplicative under block sums of matrices (since σ is additive). Invariance under blowing up (summing with (± 1)) is now evident since $\frac{1}{\sqrt{2}}\,c^{\pm 1}(1+i^{\pm 1})=1$. Reversing the orientation on a component of L (multiplying a row and corresponding column by -1) leaves σ unchanged and alters $S\cdot S$ by a multiple of 4 for all sublinks S, and thus leaves the formula invariant. Finally consider handle slides. Let L' be obtained from L by sliding component L_i over L_j and then, for convenience, reversing the orientation on L_j (i.e. replace L_i by $L'_i=L_i+L_j$ and L_j by $L'_j=-L_j$). Each sublink S of L corresponds to a sublink S' of L' with $S\cdot S=S'\cdot S'$, namely

$$S'=\begin{cases} S & \text{if } S \text{ does not contain } L_i \\ S-(L_i+L_j)+L'_i & \text{if } S \text{ contains } L_i \text{ and } L_j \\ S-L_i+(L'_i+L'_j) & \text{if } S \text{ contains } L_i \text{ but not } L_j. \end{cases}$$

(In fact $S=S'$ as homology classes in W_L.) This correspondence is one-to-one, and so Formula (6.1) is invariant under handle slides.

The (cumbersome) sum in Formula (6.1) can be eliminated by using Ed Brown's $\mathbf{Z}/8\mathbf{Z}$ invariant λ of the linking matrix A of L, defined as follows ([Br, Ma]): View A as the matrix of a $\mathbf{Z}/4\mathbf{Z}$-valued quadratic form on a $\mathbf{Z}/2\mathbf{Z}$-vector space by reducing mod 4 along the diagonal and mod 2 off the diagonal. Two matrices are *Witt equivalent* if they represent the same form after possibly block summing with copies of $(1)\oplus(-1)$. It is easy to show that A is Witt equivalent to a diagonal matrix (mod 2). Let n_j denote the number of diagonal entries congruent to j (mod 4). Assume $n_2=0$. Then the Brown invariant is defined by

$$\lambda = n_1 - n_3 \pmod 8.$$

(If $n_2 \neq 0$, then the form is classified up to Witt equivalence by its nullity over $\mathbf{Z}/2\mathbf{Z}$, and the Brown invariant is not defined.)

Observe that if $n_2=0$ (which is equivalent to the topological statement that there exists α in $H^1(M;\mathbf{Z}/2\mathbf{Z})$ with $\alpha\smile\alpha\smile\alpha\neq 0$, see Theorem 6.3) then

$$\beta(M) = \sigma - \lambda \pmod 8$$

is an invariant of the 3-manifold $M=M_L$ by [K1], since σ and λ change equally under blowing up and remain unchanged under handle slides. This will be called the *Brown invariant* of M. We can now state:

(6.3) **Theorem** *Let M be a closed, oriented 3-manifold. Then $\tau_3(M)=0$ if and only if any one of the following equivalent conditions holds:*
(1) *M has two spin structures with distinct μ-invariants* mod 4 *(see Appendix C)*
(2) *M contains an embedded closed surface of odd euler characteristic*
(3) *there exists α in $H^1(M;\mathbf{Z}/2\mathbf{Z})$ with $\alpha\smile\alpha\smile\alpha\neq 0$.*
Otherwise,

$$\tau_3(M) = \sqrt{2}^{\,b(M)}\, c^{\beta(M)}$$

where $b(M)=\mathrm{rk}\, H^1(M;\mathbf{Z}/2\mathbf{Z})$, $c=e(-1/8)$ and $\beta(M)$ is the Brown invariant (defined above).

Proof. First we show the equivalence of the three conditions. Let Θ_1 and Θ_2 be two spin structures on M. Following [KT, Theorem 4.11] or [T1], we have $\mu(\Theta_1)-\mu(\Theta_2)=2\beta(F)$ where F is a surface which is Poincaré dual to the class in $H^1(M;\mathbf{Z}/2\mathbf{Z})$ which measures the difference between Θ_2 and Θ_1; F gets a Pin^- structure from Θ_1 and $\beta(F)$ is its Pin^- bordism class in $\Omega_2^{\mathrm{Pin}^-}=\mathbf{Z}/8\mathbf{Z}$. Now the odd classes in $\Omega_2^{\mathrm{Pin}^-}$ are represented by odd multiples of $\mathbf{R}P^2$, and the even by even, and so the equivalence of (1) and (2) follows. The equivalence of (2) and (3) is well known, and follows from an elementary geometric argument (see for example [KT]).

Now choose a framed link L of n components and signature σ with $M=M_L$. Orient L and let A be the associated linking matrix. As above, we may assume that A is diagonal (mod 2) with n_j diagonal entries congruent to j (mod 4).

Observe that

$$b(M) \equiv n_0 + n_2 \pmod 4$$

since A is a presentation matrix for $H_1(M)$. By Formula (6.1),

$$\tau_3(M) = c^{\sigma} w$$

where

$$w = \frac{1}{\sqrt{2^n}} \sum_{S<L} i^{S \cdot S}.$$

Now, since w (as a function of A) is multiplicative under block sum and depends only on A mod 4 on the diagonal and mod 2 off the diagonal, we have

$$w = \prod_{j=0}^{3} w_j^{n_j}$$

where $w_0 = \sqrt{2}$, $w_1 = \bar{c}$, $w_2 = 0$ and $w_3 = c$. Thus

$$w = \begin{cases} \sqrt{2}^{n_0} c^{n_3 - n_1} = \sqrt{2}^{b(M)} c^{-\lambda} & \text{if } n_2 = 0 \\ 0 & \text{if } n_2 > 0 \end{cases}$$

and so

$$\tau_3(M) = \begin{cases} \sqrt{2}^{b(M)} c^{\beta(M)} & \text{if } n_2 = 0 \\ 0 & \text{if } n_2 > 0. \end{cases}$$

It remains to show that $n_2 = 0$ if and only if all the μ-invariants of M are congruent (mod 4). It is known that the spin structures on M are in one-to-one correspondence with the *characteristic* sublinks C of L (i.e. $C \cdot L_i \equiv L_i \cdot L_i \pmod 2$ for all components L_i of L), and their μ-invariants are given by

$$\mu_C = \sigma - C \cdot C + 8\,\mathrm{Arf}(C) \pmod{16}$$

(see Appendix C).

It is evident that C is characteristic if and only if it contains all L_i with $L_i \cdot L_i$ odd (since $L_i \cdot L_j$ is even for $i \neq j$ by assumption). Now if $n_2 = 0$, then (working mod 4) $\mu_C \equiv \sigma - \lambda = \beta(M)$ for all characteristic C (since $C \cdot C \equiv \lambda$). If $n_2 > 0$, however, then $\mu_C \equiv \beta(M)$ if C contains an even number of L_i with $L_i \cdot L_i \equiv 2$, and $\mu_C = \beta(M) + 2$ otherwise. □

(6.4) **Corollary** *If M is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere, then*

$$\tau_3(M) = \pm c^{\mu(M)}$$

where $c = e(-\frac{1}{8})$, $\mu(M)$ is the μ-invariant of M, and the sign is chosen according to whether $|H_1(M)| \equiv \pm 1$ or $\pm 3 \pmod 8$.

Proof. Since $b(M) = \mathrm{rk}\, H_1(M; \mathbf{Z}/2\mathbf{Z}) = 0$, we must show

$$\beta(M) \equiv \mu(M) + \delta(M) \pmod 8$$

where $\delta(M) = 0$ if $|H_1(M)| \equiv \pm 1 \pmod 8$, and $\delta(M) = 4$ otherwise.

First note that $n_0 = n_2 = 0$ (in the notation of the proof of Theorem 6.3). In addition, after a change of basis we may assume that A is diagonal (mod 4) with m_j diagonal entries congruent to $j \pmod 8$.

Working mod 8 we have

$$\lambda \equiv m_1 - m_3 + m_5 - m_7$$

$$|H_1(M)| \equiv 3^{m_3} 5^{m_5} 7^{m_7} \equiv \begin{cases} \pm 1 & \text{if } m_3 + m_5 \text{ is even} \\ \pm 3 & \text{otherwise.} \end{cases}$$

$$C \cdot C \equiv m_1 + 3m_3 + 5m_5 + 7m_7$$

where C is the (unique) characteristic sublink of L. Thus

$$\begin{aligned} \beta(M) &\equiv \sigma - \lambda \equiv \mu(M) + C \cdot C - \lambda \\ &\equiv \mu(M) + 4(m_3 + m_5) \equiv \mu(M) + \delta(M). \quad \square \end{aligned}$$

(6.5) *Remarks* (1) $\tau_3(M)$ is not in general determined by $H_1(M)$ and the μ-invariants of M (whereas $\tau_4(M)$ is, see §7). For example, for $M = L(4, 1) \# L(8, 1)$ one readily computes $\beta(\pm M) = \pm 2$ whence $\tau_3(\pm M) = \pm 2i$. Yet M and $-M$ have the same homology and μ-invariants.

(2) Let $\nu(M) = \operatorname{rk} H_1(M)$. Then the modified invariant $c^{\nu(M)} \tau_3(M)$ (see Remark 1.8) is always a Gaussian integer. This follows from Theorem 6.3 and the elementary observation that $b(M) \equiv \beta(M) + \nu(M)$ (mod 2).

7 The case $r = 4$

In this section we give a formula for $\tau_4(M)$ in terms of the μ-invariants of spin structures on M (Theorem 7.1). It is derived using Rohlin's Theorem on the signature of spin 4-manifolds from a related formula (7.2) which involves the Arf invariants of sublinks of a framed link L with $M_L = M$. It turns out that Formula (7.2) can be shown directly to be an invariant of M using only elementary properties of the Arf invariant and [K1], and this in turn yields a new short proof of Rohlin's theorem (see Appendix C).

(7.1) **Theorem** *Let M be a closed, oriented 3-manifold. Then*

$$\tau_4(M) = \sum_{\Theta} c^{\mu(M_\Theta)}$$

where $c = e(-\frac{3}{16})$ and $\mu(M_\Theta)$ is the μ-invariant of the spin structure Θ on M (the sum is taken over all spin structures).

Proof. Choose a framed link L of n components and signature σ with $M_L = M$. By Theorem 4.17,

$$\tau_4(M) = \sqrt{2}^{\,1-2n} c^{\sigma} \sum_{\mathbf{c}=\mathbf{0}}^{\mathbf{2}} \langle \mathbf{c} \rangle J_{L^{\mathbf{c}}}$$

where the doubly cabled components of $L^{\mathbf{c}}$ are oppositely oriented, and

$$\langle \mathbf{c} \rangle = \bar{c}^{S \cdot S} \sum [\mathbf{c} + 2\mathbf{j} + \mathbf{1}](-1)^{\mathbf{j}} \binom{\mathbf{c}+\mathbf{j}}{\mathbf{j}}.$$

Here S is the sublink of L which is cabled once (S depends on $\mathbf{c}$), and the sum is over all $\mathbf{j} \geqq \mathbf{0}$ with $\mathbf{c}+2\mathbf{j}+\mathbf{1}<\mathbf{4}$.

First we show that $\langle \mathbf{c} \rangle = 0$ for any $\mathbf{c}$ in which some $c_p = 0$. Indeed, each $\mathbf{j}$ with $j_p = 0$ in the sum can be paired with $\mathbf{j}'$, identical to $\mathbf{j}$ except that $j'_p = 1$. The corresponding terms in the sum differ only in the p^{th} position, where we have $[0+0+1](-1)^0 \binom{0}{0} = 1$ for $\mathbf{j}$ and $[0+2+1](-1)^1 \binom{1}{1} = -1$ for $\mathbf{j}'$, and therefore cancel.

Now if $\mathbf{c}$ has all $c_i = 1$ or 2, then

$$\langle \mathbf{c} \rangle = \bar{c}^{S \cdot S} \sqrt{2}^{n_S}$$

where S (as above) is the sublink of all L_i with $c_i = 1$, and n_S is the number of components of S. Furthermore, recall from Remark 4.12 that $J_{L^c} = a\sqrt{2}^{2n - n_S - 1}$, where $a = (-1)^{\operatorname{Arf}(L^c)}$ or 0, depending upon whether L^c is proper or not (see Appendix C). But L^c is proper if and only if S is characteristic, since the components of $L - S$ are doubled. Hence

$$J_{L^c} = (-1)^{\operatorname{Arf}(S)} \sqrt{2}^{2n - n_S - 1}$$

if S is characteristic, and 0 otherwise.

Putting these calculations together gives

$$\tau_4(M) = \sqrt{2}^{1-2n} c^{\sigma} \sum_C \bar{c}^{C \cdot C} \sqrt{2}^{n_C} (-1)^{\operatorname{Arf}(C)} \sqrt{2}^{2n - n_C - 1} \tag{7.2}$$
$$= \sum_C c^{\sigma - C \cdot C + 8 \operatorname{Arf}(C)}$$

where the sum is over all characteristic sublinks C of L. It is shown in Appendix C that characteristic sublinks C of L naturally correspond to spin structures Θ on M, and the associated μ-invariants $\mu(M_\Theta)$ are given by $\sigma - C \cdot C + 8 \operatorname{Arf}(C)$ (mod 16) (see Eq. (C.3)). Thus Formula (7.2) may be written as in the statement of the theorem.

(7.3) *Remark* There is, of course, a quicker proof of Theorem 7.1 using the Symmetry Principle 4.20, which we leave to the reader.

8 Applications of the Symmetry Principle

In this section, the Symmetry Principle is used to simplify the formulas for $\tau_r(M)$, and to split $\tau_r(M)$ into finer invariants. Several applications are given. We begin by reformulating the Symmetry Principle.

The function $\phi_{\mathbf{k}}$ and the sum $m_{L,\mathbf{k}}$

Let L be a framed link and $\mathbf{k}$ be an $\mathscr{M}$-coloring of L. Denote the corresponding even colored sublink of L by $E_{\mathbf{k}}$. The Symmetry Principle (4.20) describes how the invariant $J_{L,\mathbf{k}}$ changes if the color k on some component K is switched to $r-k$: it is multiplied by

$$i^{(r-2)K\cdot K+2E_{\mathbf{k}}\cdot K}.$$

Now we give a formula when several colors are switched in this way, and show how to apply this to the study of the 3-manifold invariant τ_r.

For each sublink S of L, let $\mathbf{k}_S$ denote the coloring of L obtained from $\mathbf{k}$ by switching the color as above on *each* component of S. Then applying the Symmetry Principle repeatedly, we have

(8.1) $$J_{L,\mathbf{k}_S}=i^{\phi_{\mathbf{k}}(S)}J_{L,\mathbf{k}}$$

for some $\mathbf{Z}/4\mathbf{Z}$-valued function $\phi_{\mathbf{k}}$ on the sublinks of L, where

(8.2) $$\phi_{\mathbf{k}}(K)=(r-2)\,K\cdot K+2\,E_{\mathbf{k}}\cdot K \pmod 4$$

for a single component K of color k. Noting that there is a one to one correspondence between the sublinks of L and the elements of $H_2(W_L;\mathbf{Z}/2\mathbf{Z})$, where W_L is the 4-manifold defined by L (see § 1), $\phi_{\mathbf{k}}$ may be viewed as a function

$$\phi_{\mathbf{k}}\colon\ H_2(W_L;\mathbf{Z}/2\mathbf{Z})\to\mathbf{Z}/4\mathbf{Z}.$$

(8.3) **Lemma** *If r is odd, then $\phi_{\mathbf{k}}$ is a quadratic enhancement of the $\mathbf{Z}/2\mathbf{Z}$ intersection form $\cdot$ on $H_2(W_L;\mathbf{Z}/2\mathbf{Z})$. If r is even, then $\phi_{\mathbf{k}}$ is linear.*

Proof. For r odd, we must show

$$\phi_{\mathbf{k}}(S+T)=\phi_{\mathbf{k}}(S)+\phi_{\mathbf{k}}(T)+2(S\cdot T) \pmod 4$$

for any (sublinks) S and T in $H_2(W_L;\mathbf{Z}/2\mathbf{Z})$. Note that by (8.2), $\phi_{\mathbf{k}}(K)+K\cdot K$ is even (since r is odd), and so this is immediate for $S=T=K$. The general case now reduces easily to the case when $T=K$ and S is an arbitrary sublink of L not containing K.

In this case, consider $S_0=E_{\mathbf{k}}\cap S$ (the even colored sublink of S) and $S_1=S-S_0$ (the odd colored sublink of S). Now compute $\phi_{\mathbf{k}}(S+K)$ by first switching colors on the components of S (this gives $\phi_{\mathbf{k}}(S)$), and then switching the color on K (which adds $(r-2)\ K\cdot K+2(E_{\mathbf{k}}+S_1-S_0)\cdot K\equiv\phi_{\mathbf{k}}(K)+2(S_1-S_0)\cdot K \pmod 4$, since the parities of all the colors on the components of S have changed). Hence,

$$\phi_{\mathbf{k}}(S+K)=\phi_{\mathbf{k}}(S)+\phi_{\mathbf{k}}(K)+2(S\cdot K) \pmod 4$$

since $S\cdot K=(S_1+S_0)\cdot K\equiv(S_1-S_0)\cdot K \pmod 2$.

When r is even, $\phi_{\mathbf{k}}(K)$ is even by (8.2). The argument now proceeds as above except that the parities of the colors on S remain unchanged, and so switching the color on K simply adds $\phi_{\mathbf{k}}(K)$. Thus $\phi_{\mathbf{k}}(S+K)=\phi_{\mathbf{k}}(S)+\phi_{\mathbf{k}}(K) \pmod 4$. □

For any function $\phi\colon V\to\mathbf{Z}/4\mathbf{Z}$ on an n-dimensional $\mathbf{Z}/2\mathbf{Z}$-vector space V, consider the *Monsky sum* [Br]

$$m_\phi=\sum_{v\in V} i^{\phi(v)}.$$

If ϕ is linear, then m_ϕ vanishes unless ϕ is identically zero, in which case $m_\phi = 2^n$ [Br, Lemma 3.1]. If ϕ is quadratic (i.e. $\phi(v+w) = \phi(v) + \phi(w) + 2v \cdot w$ for some symmetric bilinear form $\cdot$ on V) then m_ϕ is either 0 or of the form $\sqrt{2}^d e(1/8)^\beta$, where $d = n + \text{nullity}_{\mathbf{Z}/2\mathbf{Z}} \phi$ and β is the Brown invariant of ϕ (see §6, or [Br, Theorem 1.20] for the nonsingular case).

Now, for any coloring $\mathbf{k}$ of L, let

(8.4) $$m_{L,\mathbf{k}} = m_{\phi_\mathbf{k} | T_\mathbf{k}} = \sum_{S<L} i^{\phi_\mathbf{k}(S)}$$

where $T_\mathbf{k}$ is the sublink of L consisting of all components with colors unequal to $r/2$ (and of course $m_{L,\mathbf{k}} = 1$ if $T_\mathbf{k}$ is empty). Note that $m_{L,\mathbf{k}} = m_{\phi_\mathbf{k}}$ for r odd, since $T_\mathbf{k} = L$ in this case.

In view of (8.1), these sums may be used to compute the 3-manifold invariants $\tau_r(M)$, where $M = M_L$. In particular, define two colorings $\mathbf{k}$ and $\mathbf{k}'$ of L to be *equivalent* if on each component, the corresponding colors are either equal or add up to r. Note that each equivalence class contains $2^{|T_\mathbf{k}|}$ elements, and exactly one of these is minimal (where $\mathbf{k}$ is called *minimal* if no color exceeds $r/2$, also written $\mathbf{k} \leqq \mathbf{r}/2$ in multi-index notation). Now we may group the colorings into equivalence classes and rewrite $\tau_r(M) = \alpha_L \sum_{0<\mathbf{k} \leqq \mathbf{r}} [\mathbf{k}] J_{L,\mathbf{k}}$ from Definition 1.5 (using $[k] = [r-k]$):

(8.5) **Theorem** $\tau_r(M) = \alpha_L \sum_{0<\mathbf{k} \leqq \mathbf{r}/2} m_{L,\mathbf{k}} [\mathbf{k}] J_{L,\mathbf{k}}$.

This reduces the number of terms by roughly a factor of 2^n where n is the number of components in L. Thus it is of interest to evaluate the sums $m_{L,\mathbf{k}}$.

(8.6) **Lemma** *Let L be a framed link and* $\mathbf{k}$ *be a coloring of L. If r is even, then*

$$m_{L,\mathbf{k}} = \begin{cases} 2^{|T_\mathbf{k}|} & \text{if } \phi_\mathbf{k} | T_\mathbf{k} \text{ is identically zero} \\ 0 & \text{otherwise} \end{cases}$$

where $T_\mathbf{k}$ *is the sublink of L consisting of components with colors strictly less than r/2 in the coloring* $\mathbf{k}$.

For r odd,

$$m_{L,\mathbf{k}} = i^{r E_\mathbf{k} \cdot E_\mathbf{k}} m_{L,\mathbf{1}},$$

where $\mathbf{1}$ *is the constant* 1-*coloring and* $E_\mathbf{k}$ *is the even colored sublink of L for the coloring* $\mathbf{k}$.

Proof. The formula for even r follows immediately from the fact that $\phi_\mathbf{k}$ is linear (Lemma 8.3) and the remarks above. For r odd, it suffices to establish the following:

Assertion. *If* $\mathbf{k}$ *and* $\mathbf{k}'$ *are colorings of L which differ on only one component K, with colors k and k′ respectively, then*

$$m_{L,\mathbf{k}} = \begin{cases} m_{L,\mathbf{k}'} & \text{if } k \equiv k' \pmod 2 \\ i^{\phi_\mathbf{k}(K)} m_{L,\mathbf{k}'} & \text{if } k \not\equiv k' \pmod 2. \end{cases}$$

The formula in the Lemma then follows by induction on the number of components in $E_\mathbf{k}$ (since for k even, $E_\mathbf{k} = E_{\mathbf{k}'} \cup K$, and so as r is odd, $r E_\mathbf{k} \cdot E_\mathbf{k}$

$=rE_{\mathbf{k}'}\cdot E_{\mathbf{k}'}+2rE_{\mathbf{k}'}\cdot K+rK\cdot K\equiv rE_{\mathbf{k}'}\cdot E_{\mathbf{k}'}+2E_{\mathbf{k}}\cdot K+(r-2)K\cdot K\equiv rE_{\mathbf{k}'}\cdot E_{\mathbf{k}'}+\phi_{\mathbf{k}}(K)$ (mod 4)).

The assertion is obvious for $k\equiv k'(\text{mod } 2)$, since $\phi_{\mathbf{k}}=\phi_{\mathbf{k}'}$ by (8.2), and so we assume that k and k' have opposite parities.

First some notation. We shall write ϕ for $\phi_{\mathbf{k}}$ and ϕ' for $\phi_{\mathbf{k}'}$; these are both quadratic forms on $V=H_2(W_L;\mathbf{Z}/2\mathbf{Z})$ by Lemma 8.3. Observe that $E_{\mathbf{k}'}=E_{\mathbf{k}}+K$ (in V) and so

$$\phi'(S)\equiv\phi(S)+2(S\cdot K)\,(\text{mod } 4)$$

for any S in V (e.g. $\phi'(K)\equiv-\phi(K)(\text{mod } 4)$). Also, let V_1 be the 1-dimensional $\mathbf{Z}/2\mathbf{Z}$-vector space generated by N with $N\cdot N=1$, and $\phi^{\pm}$ be the quadratic forms on V_1 given by $\phi^{\pm}(N)=\pm1(\text{mod } 4)$. Note that

$$m_{\phi^{\pm}}=1\pm i.$$

Now to prove the assertion, it remains to show that $m_\phi=i^{\phi(K)}m_{\phi'}$. There are three cases.

Case 1 $\phi(K)\equiv0\,(\text{mod } 4)$. Then ϕ and ϕ' are equivalent, i.e. there is an isometry T on $(V,\cdot)$ with $\phi'=\phi\circ T$. Indeed, define $T(S)=S+(S\cdot K)K$ for any S in V. (In the language of the calculus of framed links, we slide over K each component of $L-K$ which links K oddly.) It follows that $m_\phi=m_{\phi'}=i^{\phi(K)}m_{\phi'}$.

Case 2 $\phi(K)\equiv\pm1\,(\text{mod } 4)$. Then $\phi\oplus\phi^+$ and $\phi'\oplus\phi^{\pm}$ are equivalent (as forms on $V\oplus V_1$). Indeed, an explicit isometry T is given by $T(S)=S+(S\cdot K)(K+N)$ for S in V (e.g. $T(K)=N$) and $T(N)=K$. (As framed links, we slide off K and over N each component of $L-K$ which links K oddly.) It follows that $(1+i)m_\phi=(1\pm i)m_{\phi'}$, since Monsky sums multiply under direct sums, and so $m_\phi=i^{\pm1}m_{\phi'}=i^{\phi(K)}m_{\phi'}$.

Case 3 $\phi(K)\equiv2\,(\text{mod } 4)$. Then $\phi\oplus\phi^+\oplus\phi^+$ and $\phi'\oplus\phi^-\oplus\phi^-$ are equivalent (as forms on $V\oplus V_1\oplus V_1$). To see this, consider $\psi=\phi\oplus\phi^+$ and $\psi'=\phi'\oplus\phi^-$. Note that $\psi(K+N)=-1$ and $\psi'(K+N)=1$, and so we are in the situation of case 2 (with ϕ replaced by ψ and K replaced by $K+N$). Thus $\psi\oplus\phi^+$ and $\psi'\oplus\phi^-$ are equivalent. It follows that $(1+i)^2m_\phi=(1-i)^2m_{\phi'}$, and so $m_\phi=-m_{\phi'}=i^{\phi(K)}m_{\phi'}$. □

The case of odd r

Combining Theorem 8.5 and Lemma 8.6 we obtain the following formula for $\tau_r(M)$ when r is odd (where $M=M_L$ as usual).

(8.7) **Theorem** *If r is odd, then*

$$\tau_r(M)=m_{L,\mathbf{1}}\,\alpha_L\sum_{0<\mathbf{k}<\mathbf{r}/2}i^{rE_{\mathbf{k}}\cdot E_{\mathbf{k}}}[\mathbf{k}]\,J_{L,\mathbf{k}},$$

where $E_{\mathbf{k}}$ is the even colored sublink of L for the coloring **k**.

We now derive some consequences of this formula.

Splitting for odd r

Observe that there is a natural quadratic form ϕ_L on $(H_2(W_L;\mathbf{Z}/2\mathbf{Z}),\cdot)$ given by

$$\phi_L(S)=S\cdot S\ (\mathrm{mod}\ 4)$$

with associated Monsky sum

$$m_L=m_{\phi_L}=\sum_{S<L} i^{S\cdot S}=\sqrt{2}^n\, e\left(\frac{\sigma}{8}\right)\tau_3(M), \tag{8.8}$$

where the last equality follows from the formula for $\tau_3(M)$ in (6.1). It is readily verified that

$$m_{L,1}=\begin{cases} m_L & \text{if } r\equiv 3\ (\mathrm{mod}\ 4)\\ \overline{m_L} & \text{if } r\equiv 1\ (\mathrm{mod}\ 4)\end{cases}$$

and so we deduce from Theorem 8.7 that τ_r splits as a product for odd r:

(8.9) **Corollary** *If r is odd, set*

$$\tau_r'(L)=\sqrt{2}^n\, e\left(\frac{\pm\sigma}{8}\right)\alpha_L \sum_{0<\mathbf{k}<\mathbf{r}/2} i^{rE_{\mathbf{k}}\cdot E_{\mathbf{k}}}[\mathbf{k}]\, J_{L,\mathbf{k}}$$

with the + or − sign chosen according to whether $r\equiv 3$ *or* $r\equiv 1$ (mod 4). *(Here n is the number of components in L and σ is the signature of the linking matrix of L.) Then*

$$\tau_r(M)=\begin{cases} \tau_3(M)\,\tau_r'(L) & \text{if } r\equiv 3\ (\mathrm{mod}\ 4)\\ \overline{\tau_3(M)}\,\tau_r'(L) & \text{if } r\equiv 1\ (\mathrm{mod}\ 4).\end{cases}$$

In particular $\tau_r(M)=0$ *whenever* $\tau_3(M)=0$.

The Corollary suggests that $\tau_r'(L)$, if invariant under the moves of the calculus of framed links (K-moves), would be a more useful invariant of $M=M_L$ than $\tau_r(M)$ because it would not vanish for "trivial" reasons. It is evident that it is invariant when $\tau_3(M)\neq 0$, since $\tau_3(M)$ and $\tau_r(M)$ are, and in fact it is invariant in general:

(8.10) **Theorem** *If L is a framed link and r is odd, then* $\tau_r'(L)$ *(defined in* (8.9)*) is invariant under K-moves on L, and hence defines an invariant* $\tau_r'(M)$ *of the associated* 3*-manifold* $M=M_L$.

Proof. We adopt the notation of the proof in § 5 of Theorem 1.6 (which established the invariance of $\tau_r(M)$). In particular we have a K-move $L\leftrightarrow L^\varepsilon$ of type $\varepsilon=\pm 1$, a fixed coloring $\mathbf{l}$ of L, and induced colorings $\mathbf{l}\cup k$ of L^ε for each k (the color of the new component K). Note that $K\cdot K=\varepsilon$.

Let E denote the even colored sublink of $L^\varepsilon-K$ for the coloring $\mathbf{l}\cup k$ (this is independent of k), and as above $E_{\mathbf{l}}$ and $E^\varepsilon_{\mathbf{l}\cup k}$ denote the even colored sublinks of L and L^ε for the colorings $\mathbf{l}$ and $\mathbf{l}\cup k$, respectively. Observe that

$$E^\varepsilon_{\mathbf{l}\cup k}=\begin{cases} E\cup K & \text{if } k \text{ is even}\\ E & \text{if } k \text{ is odd.}\end{cases} \tag{8.11}$$

It follows that

$$E^{\varepsilon}_{1\cup k}\cdot E^{\varepsilon}_{1\cup k}\equiv E_1\cdot E_1+\varepsilon\,\delta_k \pmod 4$$

where

$$\delta_k=\begin{cases}1 & \text{if } k\equiv E\cdot K \pmod 2\\ 0 & \text{if } k\not\equiv E\cdot K \pmod 2),\end{cases}$$

since $E\cdot E=E_1\cdot E_1+\varepsilon(E\cdot K)^2$ (by the way framings change under K-moves, see §1). Using this, the proof of the Theorem reduces (as in the proof (1.6)) to the identity

$$(8.12)\qquad (1-i^{\varepsilon r})\,b\,c^{\varepsilon}\sum_{0<k<r/2} i^{\varepsilon\delta_k r}[k]\,J_{L^{\varepsilon},1\cup k}=J_{L,1},$$

which is the analogue of (5.3).

To prove (8.12), consider the contribution $s_k=(1-i^{\varepsilon r})\,b\,c^{\varepsilon}\,i^{\varepsilon\delta_k r}[k]\,J_{L^{\varepsilon},1\cup k}$ of each color k to left hand side. Using the Symmetry Principle (4.20) we have

$$(8.13)\qquad s_k=b\,c^{\varepsilon}([k]\,J_{L^{\varepsilon},1\cup k}+[r-k]\,J_{L^{\varepsilon},1\cup(r-k)}).$$

Indeed, $[r-k]\,J_{L^{\varepsilon},1\cup(r-k)}=i^{\varepsilon(r-2)+2(E^{\varepsilon}_{1\cup k}\cdot K)}[k]\,J_{L^{\varepsilon},1\cup k}=i^{\varepsilon(r-2)+2\delta_k}[k]\,J_{L^{\varepsilon},1\cup k}$ (since $E^{\varepsilon}_{1\cup k}\cdot K\equiv\delta_k \pmod 2$ by (8.11)). Thus to prove (8.13), it suffices to show $(1-i^{\varepsilon r})\,i^{\varepsilon\delta_k r}=1+i^{\varepsilon(r-2)+2\delta_k}$, which is readily verified.

Now the left hand side of (8.12) can be rewritten using (8.13) as

$$b\,c^{\varepsilon}\sum_{0<k<r}[k]\,J_{L^{\varepsilon},1\cup k}.$$

But this is just the left hand side of (5.3), and so (8.12) follows. □

As an application, we illustrate the use of Theorem 8.10 in studying the manifolds K_a obtained by surgery on K with integer framing a. Recall from Example 4.21 that the value of the Jones polynomial of K at the fifth root of unity is an invariant of K_a provided the framing $a\not\equiv 2 \pmod 4$. This is in fact true for all a. In particular, by Theorem 8.10

$$\tau'_5(K_a)=\sqrt{2}\,e\left(\frac{-\sigma}{8}\right)\alpha_K(1+q^{2a}[2]^2\,\overline{V_K(q)})$$

is an invariant of K_a (where $q=e(1/5)$, and the right hand side is obtained using (4.10) and Corollary 4.11). It follows that $V_K(q)$ is as well. We have proved:

(8.14) **Theorem** *Let K and K' be knots in S^3 whose Jones polynomials have distinct values at the fifth root of unity $e(1/5)$. Then the 3-manifolds K_a and K'_a (obtained by surgery with framing a) are distinct for each integer a.*

Homology spheres and the Casson invariant

For another application of Theorem 8.7, consider the 3-manifold $K_{p/q}$ obtained by p/q Dehn surgery on a knot K in S^3. Then we have:

(8.15) **Corollary** (periodicity for homology spheres for odd r) *If r is odd, then $\tau_r(K_{1/n})=\tau_r(K_{1/(n+r)})$ for every integer n. The same statement holds for the invariants τ_r'.*

Proof. $K_{1/n}$ may be obtained by surgery on a two component link L consisting of K with the zero framing together with a meridian J (unknotted) of K with framing $-n$.

Observe that the linking matrix of L has zero signature. (Note that it follows immediately that $\tau_r(K_{1/n})=\tau_r(K_{1/(n+4r)})$, by the Definition 1.5 of τ_r and the way that the invariants $J_{L,\mathbf{k}}$ change under change of framing (3.27b).) One readily computes $m_{L,1}=2$ and so by Theorem 8.7

$$\tau_r(K_{1/n})=2\,b^2 \sum_{0<j,k<r/2} [j]\,[k]\,i^{rE_{j\cup k}\cdot E_{j\cup k}} J_{L,j\cup k}$$

where j and k are the colors on J and K respectively, and $E_{j\cup k}$ is the associated even colored sublink of L as usual.

Evidently $E_{j\cup k}\cdot E_{j\cup k}$ is 0 if j is odd, and is $-n$ or $2-n$ if j is even (depending upon whether k is odd or even), that is

$$E_{j\cup k}\cdot E_{j\cup k}\equiv 2(j-1)(k-1)+n(j^2-1)\ (\mathrm{mod}\ 4).$$

Also we have

$$J_{L,j\cup k}=\frac{[j\,k]}{[k]}\,t^{-n(j^2-1)}$$

by Lemma 3.27c. Thus

$$\begin{aligned}\tau_r(K_{1/n})&=2\,b^2 \sum_{0<j,k<r/2} [j]\,[j\,k]\,i^{r(2(j-1)(k-1)+n(j^2-1))}\,t^{-n(j^2-1)}\,J_{K,k}\\ &=2\,b^2 \sum_{0<j,k<r/2} [j]\,[j\,k](-1)^{(j-1)(k-1)}\,q^{n(j^2-1)(r^2-1)/4}\,J_{K,k}\end{aligned}$$

since $i=t^r$ and $q=t^4$. (Note that $(r^2-1)/4$ is an integer since r is odd.) It is now evident that changing n to $n+r$ does not change τ_r, since $q^r=1$.

The analogous result for τ_r' follows immediately from Corollary 8.9 since $\tau_3(K_{1/n})=1$ by Corollary 6.5, whence $\tau_r'(K_{1/n})=\tau_r(K_{1/n})$. □

(8.16) *Example* Let $r=5$. Then $b^2=\frac{2}{5}\sin^2\left(\frac{\pi}{5}\right)=\frac{1}{10}(2-q-q^4)$, and a straightforward calculation from the last formula in the proof of the previous proposition yields

$$\tau_5(K_{1/n})=\tfrac{1}{5}(2-q-q^4)((1+[2]^2\,q^{3n})+[2]\,(1-q^{3n})\,J_K)),$$

where $J_K=J_{K,2}$ as usual.

It is illuminating to write this formula in terms of the *reduced Jones polynomial* W_K of K, defined by $V_K(x)=1-P(x)\,W_K(x)$, where $P(x)=(1-x)(1-x^3)$. (That V_K can be so written follows from the evaluations (4.12) $V_K(1)=V_K(\omega)=1$, where $\omega=e(1/3)$, and $V_K'(1)=0$. Note that $P(1)=P'(1)=P(\omega)=0$. The polynomials W_K are tabulated in Jones' original papers [J1, J2].) Now, since $J_K=[2]\,(1-P(\bar{q})\,W(\bar{q}))$ by (4.10) and Corollary 4.11, we compute

(8.17) $$\tau_5(K_{1/n}) = 1 - (1+q)(1-q^{3n})\, W_K(\bar{q}).$$

$\left(\text{Recall that } \sum_{j=0}^{4} q^j = 0.\right)$

There is an interesting consequence of this formula, relating the Casson invariant $\lambda = \lambda(K_{1/n})$ (see [AM]) with $\tau = \tau_5(K_{1/n})$. (This relationship was first observed experimentally using data generated in Mathematica [Wo].)

First recall from Casson's surgery formula that $\lambda = \frac{n}{2}\, \Delta_K''(1)$, where Δ_K denotes the normalized Alexander polynomial of K. But $V_K''(1) = -3\Delta_K''(1)$ by a well-known skein computation [J2, p. 369], and so

(8.18) $$\lambda = n\, W_K(1)$$

since $P(1) = P'(1) = 0$ and $P''(1) = 6$.

Next observe from (8.17) that τ is an element of the ring $\mathbf{Z}[q]$ of cyclotomic integers in the cyclotomic field $\mathbf{Q}[q]$ (where $q = e(1/5)$). Consider the map

$$T\colon\ \mathbf{Z}[q] \to \mathbf{Z}/5\mathbf{Z}$$

given by $T(\sum \lambda_j q^j) = \sum \lambda_j (\bmod 5)$ $(= -\mathrm{tr}(\sum \lambda_j q^j)(\bmod 5))$. Observe that T is both an additive and a multiplicative homomorphism (i.e. $T(\alpha+\beta) = T(\alpha) + T(\beta)$ and $T(\alpha\beta) = T(\alpha)\, T(\beta)$). Evidently $T(\tau - 1) = 0$ (or equivalently $\mathrm{tr}(\tau - 1) \equiv 0 (\bmod 5)$).

Noting that $T(W_K(\bar{q})) = W_K(1)(\bmod 5)$, it follows from (8.17) and (8.18) that

$$\lambda \equiv n\, T\left(\frac{1-\tau}{(1+q)(1-q^{3n})}\right) (\bmod 5)$$

for $n \not\equiv 0 (\bmod 5)$. (If $n \equiv 0 (\bmod 5)$, then $\lambda \equiv 0 (\bmod 5)$.)

The last expression is in fact independent of n. To see this, write

$$\frac{1-\tau}{(1+q)(1-q^{3n})} = \frac{1-\tau}{1-q}\, u_2\, u_{3n}$$

where $u_j = \frac{1-q}{1-q^j}$. An easy computation $\left(\text{using the fact that } \prod_{j=1}^{4} (1-q^j) = 5\right)$ shows that $T(u_j) \equiv j (\bmod 5)$ for $j \equiv \pm 1 (\bmod 5)$ and $T(u_j) \equiv -j (\bmod 5)$ for $j \equiv \pm 2 (\bmod 5)$. It follows readily that

(8.19) $$\lambda \equiv T\left(\frac{1-\tau}{1-q}\right) (\bmod 5).$$

(Note that this holds even if $n \equiv 0 (\bmod 5)$, for then $\tau = 1$ by (8.17), and so the right hand side is 0 as expected.) Thus the mod 5 Casson invariant is determined by τ. In summary, we have

(8.20) **Theorem** *Let M be a homology sphere obtained by Dehn surgery on a knot in S^3, and let q be the fifth root of unity $e(1/5)$. Then $\tau_5(M)$ is a cyclotomic integer (i.e. an element of the ring $\mathbf{Z}[q]$). Furthermore, the element $\alpha(M)$*

$=\tau_5(M)-1=\tau_5(M)-\tau_5(S^3)$ *has zero trace, is divisible by* $1-q$, *and satisfies*

$$\lambda(M)\equiv \operatorname{tr}\left(\frac{\alpha(M)}{1-q}\right) \pmod 5$$

where λ *denotes the Casson invariant.*

(8.21) *Remark* The theorem holds equally well for connected sums of homology spheres, each of which is obtained by Dehn surgery on a knot in S^3.

The case of even r

From Theorem 8.5 and Lemma 8.6, we obtain the formula

$$\tau_r(M)=\alpha_L \sum_{\substack{0<\mathbf{k}\leqq r/2 \\ \phi_{\mathbf{k}}|T_{\mathbf{k}}=0}} 2^{|T_{\mathbf{k}}|}[\mathbf{k}]\, J_{L,\mathbf{k}} \tag{8.23}$$

for even r, where as usual $M=M_L$ and $T_{\mathbf{k}}$ is the sublink of components whose colors are less than $r/2$ for the coloring $\mathbf{k}$.

Observe that the condition $\phi_{\mathbf{k}}|T_{\mathbf{k}}=0$ can be replaced by the more restrictive condition $\phi_{\mathbf{k}}=0$. For if $\phi_{\mathbf{k}}\neq 0$ on some $r/2$-colored component K, then the Symmetry Principle applied to K yields $J_{L,\mathbf{k}}=-J_{L,\mathbf{k}}$ (since $\phi_{\mathbf{k}}$ is even valued for r even), and so $J_{L,\mathbf{k}}=0$.

Furthermore, the condition $\phi_{\mathbf{k}}=0$ holds if and only if

$$\begin{aligned} &E_{\mathbf{k}}\cdot K\equiv K\cdot K \pmod 2 \quad &\text{if } r\equiv 0 \pmod 4\\ &E_{\mathbf{k}}\cdot K\equiv 0 \pmod 2 \quad &\text{if } r\equiv 2 \pmod 4\end{aligned} \tag{8.24}$$

for each component K of L, where $E_{\mathbf{k}}$ is the even colored sublink for $\mathbf{k}$. A coloring $\mathbf{k}$ satisfying (8.24) for all K will be called a *characteristic* coloring (since for r divisible by 4 this is just the condition that $E_{\mathbf{k}}$ be a characteristic sublink of L, see Appendix C). Thus we have

(8.25) **Theorem** *If r is even, then*

$$\tau_r(M)=\alpha_L \sum_{0<\text{characteristic}\,\mathbf{k}\leqq r/2} 2^{|T_{\mathbf{k}}|}[\mathbf{k}]\, J_{L,\mathbf{k}}$$

where $T_{\mathbf{k}}$ *is the sublink of L consisting of components with colors strictly less than* $r/2$ *in the coloring* $\mathbf{k}$.

Homology spheres

For an application, consider once again the homology spheres $K_{1/n}$ obtained by $1/n$ Dehn surgery on a knot K in S^3 (cf. Proposition 8.15).

(8.26) **Corollary** (periodicity for homology spheres for even r) *If r is even, then* $\tau_r(K_{1/n})=\tau_r(K_{1/(n+(r/2))})$ *for every integer n.*

Proof. The argument is analogous to the proof of Corollary 8.15, and we adopt the notation used there. Using Theorem 8.25 in place of Theorem 8.7, we obtain

$$\tau_r(K_{1/n}) = b^2 \sum_{0<\text{characteristic } j\cup k \leq r/2} 2^{|T_{j\cup k}|} [j][jk]\, t^{-n(j^2-1)} J_{K,k}.$$

Now if $r \equiv 0 \pmod 4$, then $j \cup k$ characteristic means that j and $k+n$ are odd. If $r \equiv 2 \pmod 4$, then it means that j and k are odd. In either case, j is always odd and so j^2-1 is divisible by 8. Thus the term $t^{n(j^2-1)}$ (which is the only term that depends on n) can be rewritten as $(q^2)^{mn}$ for some integer m. Since q^2 has order $r/2$, the Corollary follows. □

Splittings for even r

Recall from Corollary 8.9 and Theorem 8.10 that for odd r, the invariant $\tau_r(M)$ can be written as a *product* of two other invariants of M. It turns out that for even r, it can be written as a *sum* of invariants.

For r divisible by 4, these are invariants of spin structures on M. That this should be so is suggested by the fact that the only terms which contribute to $\tau_r(M)$ come from colorings whose even colored sublinks are *characteristic*, and characteristic sublinks correspond to spin structures.

(8.27) **Theorem** *Let M be a 3-manifold and Θ be a spin structure on M. Choose a framed link L for which $M = M_L$, and let C be the characteristic sublink corresponding to Θ (see Lemma C.1). If $r \equiv 0 \pmod 4$, then*

$$\tau_r(M, \Theta) = \alpha_L \sum_{0<\mathbf{k}\leq r/2 \text{ with } E_{\mathbf{k}}=C} 2^{|T_{\mathbf{k}}|} [\mathbf{k}]\, J_{L,\mathbf{k}}$$

is an invariant of the spin manifold M_Θ. Furthermore,

$$\tau_r(M) = \sum_{\Theta} \tau_r(M, \Theta)$$

where the sum is over all spin structures on M.

Remark. $\tau_r(M, \Theta)$ can equally well be written as

$$\tau_r(M, \Theta) = \alpha_L \sum_{0<\mathbf{k}<r \text{ with } E_{\mathbf{k}}=C} [\mathbf{k}]\, J_{L,\mathbf{k}} \tag{8.28}$$

using Lemma 8.6 and the fact that k and $r-k$ have the same parity (for even r).

Proof of Theorem 8.27 We must show that the right hand side of (8.28), denoted $\tau_r(L, C)$, is invariant under K-moves $(L, C) \leftrightarrow (L^\varepsilon, C^\varepsilon)$ of characteristic pairs (see Appendix C). Here

$$C^\varepsilon = \begin{cases} C+K & \text{if } C\cdot K \text{ is even} \\ C & \text{if } C\cdot K \text{ is odd} \end{cases}$$

is the characteristic sublink of L^ε (Remark C.2), where C denotes both the characteristic sublink of L and the corresponding sublink of L^ε and K is the new component of L^ε (as usual).

Proceeding as in the proof of Theorem 1.6 in §5, or Theorem 8.10 above, the proof of the invariance of $\tau_r(L, C)$ reduces to the identity

$$(8.29) \qquad b\, c^{\varepsilon} \sum_{\substack{0<k<r \\ k \equiv C\cdot K \pmod 2}} [k]\, J_{L^\varepsilon, \mathbf{l} \cup k} = J_{L, \mathbf{l}}$$

for any coloring $\mathbf{l}$ of L with $E_\mathbf{l} = C$.

To prove this identity, first assume that $C \cdot K$ is even. Then for *odd* k we have $\phi_{\mathbf{l}\cup k}(K) \equiv (r-2)\,\varepsilon + 2E_{\mathbf{l}\cup k} \cdot K \equiv 2 + 2C\cdot K \equiv 2 \pmod 4$, and so by the Symmetry Principle (4.20),

$$(8.30) \qquad [r-k]\, J_{L^\varepsilon, \mathbf{l}\cup(r-k)} = -[k]\, J_{L^\varepsilon, \mathbf{l}\cup k}.$$

It follows that the condition $k \equiv C\cdot K \pmod 2$ may be omitted in the sum in (8.29), as the additional terms cancel in pairs, and so (8.29) reduces to the identity (5.3).

If $C\cdot K$ is odd, then for *even* k we have $\phi_{\mathbf{l}\cup k}(K) \equiv 2 + 2(C+K)\cdot K \equiv 2 \pmod 4$, and (8.30) follows. Thus (8.29) holds in this case as well, and so the first statement in the Theorem is proved.

The last statement in the Theorem follows immediately from Theorem 8.25. □

(8.31) *Example* If $r=4$, then $\tau_4(M, \Theta) = c^{\mu(M_\Theta)}$, where $c = e(-3/16)$. Indeed, since $b = 1/2$ and $[2] = \sqrt{2}$, we compute $\tau_4(M, \Theta) = c^{\sigma} J_{C,2}/\sqrt{2}^{|C|}$, where σ is the signature of a framed link L with $M = M_L$ and C is the characteristic sublink corresponding to Θ. Using Corollary 4.11 and Remark 4.12, it follows that $\tau_4(M, \Theta) = c^{\sigma - C\cdot C + \operatorname{Arf}(C)} = c^{\mu(M_\Theta)}$. (Note that this yields a proof of Theorem 7.1 without cabling, cf. Remark 7.3.)

For $r \equiv 2 \pmod 4$, the invariant $\tau_r(M)$ splits as a sum of invariants, one for each element in $H^1(M; \mathbf{Z}/2\mathbf{Z})$. Indeed, the only terms which contribute to the computation of $\tau_r(M)$ come from colorings whose even colored sublinks E intersect *each* component of L evenly (where $M = M_L$). Such sublinks E are in one-to-one correspondence with elements α of $H^1(M; \mathbf{Z}/2\mathbf{Z})$. In particular, α is the unique class which is one on meridians of E and zero on meridians of $L - E$.

(8.32) **Theorem** *Let M be a 3-manifold and α be an element of $H^1(M; \mathbf{Z}/2\mathbf{Z})$. Choose a framed link L for which $M = M_L$, and let E be the sublink corresponding to α (see above). If $r \equiv 2 \pmod 4$, then*

$$\tau_r(M, \alpha) = \alpha_L \sum_{0 < \mathbf{k} \le \mathbf{r}/2 \text{ with } E_\mathbf{k} = E} 2^{|T_\mathbf{k}|}\, [\mathbf{k}]\, J_{L,\mathbf{k}}$$

is an invariant of (M, α). Furthermore,

$$\tau_r(M) = \sum_{\alpha} \tau_r(M, \alpha)$$

where the sum is over all elements α in $H^1(M; \mathbf{Z}/2\mathbf{Z})$.

The proof is similar to the proof of Theorem 8.27 and is left to the reader. Note that one uses a calculus for pairs (L, E), with $E \cdot K$ even for all K, where the K-move replaces E by E' given by

$$E' = \begin{cases} E & \text{if } E \cdot K \text{ is even} \\ E + K & \text{if } E \cdot K \text{ is odd.} \end{cases}$$

(8.33) *Example* If $\alpha = 0$, then the corresponding sublink E is empty, and so to compute $\tau_r(M, 0)$ we only consider odd colorings of L (where $M = M_L$). It follows readily that

$$\tau_6(M, 0) = \frac{i^{-\sigma}}{\sqrt{3^n}} \sum_{S < L} J_{S,3}$$

since $b = 1/\sqrt{12}$, $c = i^{-1}$ and $[3] = 2$.

Appendix A. Identities in $\mathscr{A}$

As noted after (2.20), the relation $YX = XY - [H]$ may be generalized to $Y^n X = XY^n - [n][H+n-1]\, Y^{n-1}$. A more general formula is

$$\text{(A.1)} \quad Y^n X^k = \sum_{0 \leq i \leq \min(n,k)} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} [i]! \begin{bmatrix} k \\ i \end{bmatrix} [i]! \begin{bmatrix} H+n-k-1+i \\ i \end{bmatrix} X^{k-i} Y^{n-i}$$

$\left(\text{where} \begin{bmatrix} H+m \\ j \end{bmatrix} = [H+m] \ldots [H+m-j+1]/[j]!\right)$, which follows by induction on k using the identity $[a][H+c+b] + [b][H+c-a] = [a+b][H+c]$.

In particular, we will need (A.1) when $k = n$:

$$\text{(A.2)} \qquad Y^n X^n = \sum_{0 \leq i \leq n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}^2 [i]!^2 \begin{bmatrix} H-1+i \\ i \end{bmatrix} X^{n-i} Y^{n-i}.$$

For our purposes, it is convenient to expand the term

$$\text{(A.3)} \quad \begin{bmatrix} H-1+i \\ i \end{bmatrix} = \frac{1}{[i]!(s-\bar{s})^i} (K^2 - \bar{K}^2)(sK^2 - \bar{s}K^2) \ldots (s^{i-1}K^2 - \bar{s}^{i-1}\bar{K}^2)$$

in powers of K:

(A.4) **Lemma**

$$(K^2 - \bar{K}^2)(sK^2 - \bar{s}\bar{K}^2) \ldots (s^{i-1}K^2 - \bar{s}^{i-1}\bar{K}^2) = \sum_{0 \leq j \leq i} (-1)^{i-j} c_{ij} K^{4j-2i}$$

where $c_{ij} = t^{(2j-i)(i-1)} \begin{bmatrix} i \\ j \end{bmatrix}$. *(The exponent of* K *arises by choosing* j K^2*'s and* $(i-j)$ $\bar{K}^2$*'s.)*

Proof. We use double induction starting with the cases $j=i$ (where $c_{ii} = s^{0+\dots+(i-1)} = t^{i(i-1)}$ is the coefficient of K^{2i}) and $j=0$ (where $c_{i0} = \bar{s}^{0+\dots+(i-1)} = \bar{t}^{i(i-1)}$ is $(-1)^i$ times the coefficient of $\bar{K}^{2i}$).

Recursively, $c_{ij} = s^{i-1} c_{i-1,j-1} - \bar{s}^{i-1} c_{i-1,j}$ for $j<i$, so c_{ij} is determined by c_{kk} and c_{k0} for $k \leqq j$. But $t^{(2j-i)(i-1)} \begin{bmatrix} i \\ j \end{bmatrix}$ satisfies the same recursive formula, using the quantized Pascal relation

$$s^j \begin{bmatrix} i \\ j \end{bmatrix} = s^i \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} + \begin{bmatrix} i-1 \\ j \end{bmatrix}. \quad \square \tag{A.5}$$

Using (A.3–4) we may rewrite (A.2) as

$$Y^n X^n = \sum_{0 \leqq j \leqq i \leqq n} (-1)^j \begin{bmatrix} n \\ i \end{bmatrix}^2 [i]! \frac{t^{(2j-i)(i-1)}}{(s-\bar{s})^i} \begin{bmatrix} i \\ j \end{bmatrix} X^{n-i} Y^{n-i} K^{4j-2i}. \tag{A.6}$$

We are now in a position to complete the proof of Theorem 3.20.

(A.7) **Theorem** *The identity*

$$\sum \alpha_i \bar{K}^2 \beta_i = \sum \beta_i K^2 \alpha_i$$

is satisfied in $\mathscr{A}$, where $R = \sum \alpha_i \otimes \beta_i$ is the R-matrix given in Theorem 2.18.

Proof. Using Theorem 2.18, the left hand side is

$$\sum c_{nab} X^n K^a \bar{K}^2 Y^n K^b = \sum c_{nab} \bar{s}^{n(a-2)} X^n Y^n K^{a+b-2}$$

and the right hand side is

$$\sum c_{nab} Y^n K^b K^2 X^n K^a = \sum c_{nab} s^{n(b+2)} Y^n X^n K^{a+b+2},$$

where the sums are over all $0 \leqq n < r$ and $0 \leqq a,\ b < 4r$. Multiplying by $4r\bar{K}^2$ and substituting for c_{nab}, the left hand side becomes

$$\sum \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab+(b+a)n-3n} X^n Y^n K^{a+b-4} = \sum \lambda_{np} X^n Y^n K^p \tag{A.8}$$

and the right hand side becomes

$$\sum \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab-(b+a)n-3n} X^n Y^n K^{a+b} = \sum \rho_{np} X^n Y^n K^p \tag{A.9}$$

summed over all $0 \leqq n < r$ and $0 \leqq p < 4r$. It remains to show that coefficients λ_{np} and ρ_{np} of $X^n Y^n K^p$ are equal for all n and p.

From (A.8) we compute

$$\lambda_{np} = \sum_{a+b-4 \equiv p(\mathrm{mod}\, 4r)} \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab+(b+a)n-3n}$$

$$= \sum_{0 \leqq a < 4r} \frac{(s-\bar{s})^n}{[n]!} t^{a^2-(p+4)a-(p+1)n}$$

$$= \sum_{0 \leqq a < 4r} \frac{(s-\bar{s})^n}{[n]!} t^{a^2-pa-2p-4-(p+1)n}$$

where the last equality is derived by replacing a by $a+2$, and noting that the sums $\sum\limits_{0 \leqq a < 4r}$ and $\sum\limits_{-2 \leqq a < 4r-2}$ are equal.

To compute ρ_{np}, first use (A.6) to move Y^n past X^n on the left side of (A.9)

$$\sum_{\substack{0 \leqq n < r \\ 0 \leqq a,b < 4r \\ 0 \leqq j \leqq i \leqq n}} \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab-(b+a)n-3n} (-1)^j \begin{bmatrix} n \\ i \end{bmatrix}^2 [i]! \frac{t^{(2j-i)(i-1)}}{(s-\bar{s})^i} \begin{bmatrix} i \\ j \end{bmatrix} X^{n-i} Y^{n-i} K^{a+b+4j-2i}.$$

It follows as above that ρ_{np} is

$$\sum_{\substack{0 \leqq a < 4r \\ 0 \leqq j \leqq i < r-n}} \frac{(s-\bar{s})^{n+i}}{[n+i]!} t^{a^2-a(p-4j+2i)+(n+i)(p-4j+2i+3)+(2j-i)(i-1)}$$

$$(-1)^j \begin{bmatrix} n+i \\ i \end{bmatrix}^2 \frac{[i]!}{(s-\bar{s})^i} \begin{bmatrix} i \\ j \end{bmatrix}$$

$$= \sum_{\substack{0 \leqq a < 4r \\ 0 \leqq j \leqq i < r-n}} \frac{(s-\bar{s})^n}{[n]!} t^{a^2-pa+(pn+3n+2jp-4j^2-4jn-2j)+i(2j+2n+4)} \begin{bmatrix} n+i \\ i \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

where the second equality follows by cancellation and substitution of $a-2j+i$ for a.

Since n and p are fixed, λ_{np} and ρ_{np} have a common factor

$$C = \frac{(s-\bar{s})^n}{[n]!} \sum_{0 \leqq a < 4r} t^{a^2-pa}.$$

If p is odd, then $C=0$. Indeed, $\frac{p-1}{2}$ is an integer in this case, and so we can replace a by $a+\frac{p-1}{2}$ in the sum:

$$\sum_{0 \leqq a < 4r} t^{a^2-pa} = t^{\frac{1-p^2}{4}} \sum_{0 \leqq a < 4r} t^{a^2-a}.$$

But $t^{(2r-a+1)^2-(2r-a+1)} = t^{-2r+a^2-a} = -t^{a^2-a}$, since $t^{4r}=1$, and so the terms in the sum on the right cancel in pairs. Thus $\lambda_{np} = \rho_{np} = 0$.

For p even, we divide λ_{np} and ρ_{np} by C, and it remains to show that

$$t^{-2p-4-(p+1)n} = \sum_{0\leq j\leq i<r-n} (-1)^j t^{m+i(2j+2n+4)} \begin{bmatrix} n+i \\ i \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} \tag{A.10}$$

where $m = pn + 3n + 2jp - 4j^2 - 4jn - 2j$ is independent of i.

It is straightforward to check that the term on the right hand side of (A.10) corresponding to $i=j=r-n-1$ is equal to the left hand side. $\Bigl($Note that p is even, $(-1)^i = t^{2r(r-n-1)}$, and $\begin{bmatrix} n+i \\ i \end{bmatrix} = \begin{bmatrix} i \\ j \end{bmatrix} = 1$ in this case.$\Bigr)$

We now finish by showing that for fixed $j<r-n-1$, the sum over i on the right of (A.10) is zero. Factoring out $(-1)^j t^m$, it suffices to show

$$\sum_{j\leq i<r-n} t^{i(2j+2n+4)} \begin{bmatrix} n+i \\ i \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} = 0. \tag{A.11}$$

It is convenient at this point to use the binomial q-coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q = s^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}$ (see (2.29)). The left hand side of (A.11) then becomes

$$\sum_{j\leq i<r-n} t^{2j^2+4i} \begin{bmatrix} n+i \\ i \end{bmatrix}_q \begin{bmatrix} i \\ j \end{bmatrix}_q = \frac{t^{2j^2}}{[n]_q! [j]_q!} \sum_{j\leq i<r-n} q^i \frac{[i+n]_q!}{[i-j]_q!}\,.$$

The last sum is just $\sum_{j\leq i<r-n} q^i [i+n]_q \dots [i-j+1]_q$. To see that this is zero, note that the product of brackets ranges from a high of $[r-1]_q$ when $i=r-n-1$ to a low of $[1]$ when $i=j$. Thus we may extend the range of i to $0\leq i<r$, as that only adds terms containing a factor of either $[r]_q=0$ or $[0]_q=0$. Hence the left hand side of (A.11) is a multiple of

$$\begin{aligned}
&\sum_{i=0}^{r-1} q^i [i+n]_q \dots [i-j+1]_q \\
&= \frac{1}{(q-1)^{n+j}} \sum_i q^i (q^{i+n}-1) \dots (q^{i-j+1}-1) \\
&= \frac{1}{(q-1)^{n+j}} \sum_i q^i \left(\sum_{k=1}^{2n+j} \pm q^{a_k i + b_k} \right) \quad \text{(for suitable } a_k, b_k) \\
&= \frac{1}{(q-1)^{n+j}} \sum_k \pm (\sum_i q^{(a_k+1)i+b_k}) \\
&= \frac{1}{(q-1)^{n+j}} \sum_k \pm \left(\sum_{i=0}^{r-1} q^i \right) \\
&= 0
\end{aligned}$$

since the sum of the r^{th}-roots of unity is zero. □

Appendix B. The Jones polynomial at the sixth root of unity

In this appendix we derive an expression for the value of the variant $\tilde{V}_L$ of the Jones polynomial at $q=e(\frac{1}{6})$ (see Remark 4.12). There are of course similarities between our derivation and that of Lipson [Lp] for the Jones polynomial V_L.

Let A be a symmetric matrix over $\mathbf{Z}/3\mathbf{Z}$. The corresponding quadratic form is classified up to Witt equivalence by its *nullity* d_A and *Witt class* ω_A in the Witt group $W(\mathbf{Z}/3\mathbf{Z})=\mathbf{Z}/4\mathbf{Z}$ (see e.g. [MH]). These invariants may be computed as follows: diagonalize A (over $\mathbf{Z}/3\mathbf{Z}$) writing all entries as 0 or ± 1. Then d_A is the number of diagonal 0's, and ω_A is the trace (or signature) viewed as an integer mod 4.

Set

$$\lambda_A=\sqrt{3}^{d_A}(-i)^{\omega_A}.$$

Note that λ multiplies under block sum. If A is the mod 3 reduction of a matrix representing the quadratic form of a link L, write $\lambda_L=\lambda_A$.

(B.1) **Theorem** *$\tilde{V}_L=\lambda_L$ at $q=e(\frac{1}{6})$.*

Proof. Let L_ε ($\varepsilon=+,-,0$) be as in Fig. 4.4a. Choose corresponding connected Seifert surfaces F_ε which locally appear as in Fig. B.2 and coincide otherwise.

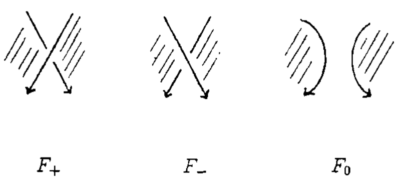


Fig. B.2

With respect to suitable bases of $H_1(F_\varepsilon)$, the associated symmetrized Seifert matrices A_ε satisfy

$$A_+=\begin{pmatrix} a & * \\ * & A_0\end{pmatrix} \quad\text{and}\quad A_-=\begin{pmatrix} a+2 & * \\ * & A_0\end{pmatrix}.$$

By a change of basis (first diagonalizing A_0 mod 3) we may arrange that either

$$A_+\equiv(a)\oplus B,\qquad A_-\equiv(a-1)\oplus B,\qquad A_0\equiv B\ (\mathrm{mod}\ 3)$$

(for A_0 nonsingular), or

$$A_+\equiv\begin{pmatrix} a & b \\ b & 0\end{pmatrix}\oplus B,\qquad A_-\equiv\begin{pmatrix} a-1 & b \\ b & 0\end{pmatrix}\oplus B,\qquad A_0\equiv(0)\oplus B\ (\mathrm{mod}\ 3)$$

for some matrix B. Here $\oplus$ denotes block sum.

Now set

$$d_\varepsilon=d_{A_\varepsilon}-d_B$$

$$\omega_\varepsilon=\omega_{A_\varepsilon}-\omega_B$$

$$\lambda_\varepsilon=\lambda_{A_\varepsilon}/\lambda_B=\sqrt{3}^{d_\varepsilon}(-i)^{\omega_\varepsilon}.$$

If A_0 is nonsingular, or A_0 is singular with $b \neq 0$, we have

	d_+	d_-	d_0	ω_+	ω_-	ω_0	λ_+	λ_-	λ_0
$a \equiv 0$ (mod 3)	1	0	0	0	-1	0	$\sqrt{3}$	i	1
$a \equiv 1$ (mod 3)	0	1	0	1	0	0	$-i$	$\sqrt{3}$	1
$a \equiv 2$ (mod 3)	0	0	0	-1	1	0	i	$-i$	1
$b \neq 0$	0	0	1	0	0	0	1	1	$\sqrt{3}$

For A_0 singular with $b=0$, the d_ε are 1 more than the corresponding d_ε in the nonsingular case, and the ω_ε remain the same, and so the λ_ε go up by a factor of $\sqrt{3}$. Now for $q=e\left(\frac{1}{6}\right)=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $s-\bar{s}=e\left(\frac{1}{12}\right)-e\left(-\frac{1}{12}\right)=i$, and one readily verifies that

$$q\,\lambda_+ - \bar{q}\,\lambda_- = (s-\bar{s})\,\lambda_0$$

in all cases. Multiplying by λ_B gives

$$q\,\lambda_{L_+} - \bar{q}\,\lambda_{L_-} = (s-\bar{s})\,\lambda_{L_0}$$

since $\lambda_{L_\varepsilon} = \lambda_{A_\varepsilon}$ by definition. It is evident that $\lambda_{\text{unknot}}=1$, and so $\tilde{V}_L = \lambda_L$ as desired. $\square$

Appendix C. μ-invariants

Let M be an oriented, closed connected 3-manifold. A *spin structure* Θ on M can be viewed as a homotopy class of trivializations of the tangent bundle τ_M over M-point [K2]. It is well known that M has a spin structure, since $w_2(\tau_M)=0$. It follows by obstruction theory that the number of distinct spin structures is equal to the number of elements in $H^1(M;\mathbf{Z}/2\mathbf{Z}) = H_1(M;\mathbf{Z}/2\mathbf{Z})$.

Recall that the *μ-invariant* of M_Θ is defined to be the signature (mod 16) of any smooth, compact spin 4-manifold W with spin boundary M_Θ,

$$\mu(M_\Theta) = \sigma(W) \pmod{16}.$$

This is well defined by Rohlin's theorem, which states that the signature of a smooth, closed spin 4-manifold is divisible by 16.

Now suppose that M is described by a framed link L, so $M=M_L=\partial W_L$ (see §1). A sublink C of L is *characteristic* if $C \cdot L_i \equiv L_i \cdot L_i \pmod 2$ for all components L_i of L, and the pair (L, C) is then called a *characteristic pair.*

(C.1) **Lemma** *There is a natural one-to-one correspondence between the spin structures on M_L and characteristic sublinks of L.*

Proof. Assign to any spin structure Θ on M_L the sublink C of L consisting of all components L_i such that Θ does not extend across the 2-handle in W_L attached to L_i. An elementary geometric argument shows that C must be characteristic (see e.g. §3 of [MK]).

The map carrying Θ to C is one-to-one. Indeed, if C is assigned to some other spin structure Θ', then Θ and Θ' agree on the link L' in M_L which is the core of the surgery on L. Since $H=H_1(M_L;\mathbf{Z}/2\mathbf{Z})$ is carried by L', we have $\Theta=\Theta'$ by obstruction theory.

It remains to show that there are exactly $|H|$ characteristic sublinks (as this is the number of spin structures). To see this, recall that the mod 2 linking matrix A of L is a presentation matrix for H, and so $|H|=|\ker A|$. But C (viewed as a column vector of 0's and 1's) is characteristic if and only if $AC=D$, where D is the main diagonal of A, and so the number of characteristic sublinks is $|\ker A|$ as well. □

If θ is a spin structure on M_L and C is the corresponding characteristic sublink of L, then the spin manifold $(M_L)_\Theta$ will be denoted by $M_{L,C}$.

(C.2) *Remark* The argument in [K1] on the calculus of framed links for oriented 3-manifolds yields a calculus of characteristic pairs for spin 3-manifolds. In particular, $M_{L,C}=M_{L',C'}$ as spin manifolds (i.e. there is a diffeomorphism between them which preserves spin structures) if and only if one can pass from (L, C) to (L', C') by isotopy in S^3 and a combination of the following two *moves of characteristic pairs* (cf. §1):

Move 1 (blow up) *Add (or delete) a disjoint unknotted component with framing* ± 1 *and replace* C *by* $C'=C+K$.

Move 2 (handle slide) *For some* $i \neq j$, *slide* L_i *over* L_j *to get* $L'_i=L_i+L_j$ *and replace* C *by*

$$C'=\begin{cases} C & \text{if } C \text{ does not contain } L_i \\ C-(L_i+L_j)+L'_i & \text{if } C \text{ contains } L_i \text{ and } L_j \\ C-L_i+(L_j+L'_i) & \text{if } C \text{ contains } L_i \text{ but not } L_j \end{cases}$$

(cf. Remark 6.2). As in [FR], these moves may be combined into one, the *K-move*, defined as in §1 (Fig. 1.3) with

$$\mathrm{C}'=\begin{cases} \mathrm{C}+\mathrm{K} & \text{if } \mathrm{C}\cdot\mathrm{K} \text{ is even} \\ \mathrm{C} & \text{if } \mathrm{C}\cdot\mathrm{K} \text{ is odd.} \end{cases}$$

Here, C denotes both the characteristic sublink of L and the corresponding sublink of L′. Note that it can happen that $M_{L,C}=M_{L,C'}$ for some $C \neq C'$.

Now let (L, C) be a characteristic pair. Note that C is a *proper* link (i.e. characteristic as a sublink of itself, or equivalently $L_i\cdot(C-L_i)$ is even for all components L_i of C), and therefore has a well defined *Arf invariant* Arf(C). If L is oriented, then define the *μ-invariant* of (L, C) to be

(C.3) $$\mu(L, C)=\sigma-C\cdot C+8\,\mathrm{Arf}(C)\ (\mathrm{mod}\ 16)$$

where $\sigma=\sigma(W_L)$ is the signature of the linking matrix of L.

(C.4) **Theorem** *$\mu(L, C)$ is an invariant of the spin manifold $M_{L,C}$.*

Proof. By the previous remark, it suffices to show that $\mu(L, C)$ is independent of the orientation on L and is invariant under moves 1 and 2 of characteristic pairs.

First suppose that the orientation on a component K of L is reversed, giving (L', C'). Evidently σ remains unchanged. If C does not contain K, then $C \cdot C$ and $\mathrm{Arf}(C)$ are unchanged as well, and so μ is unchanged. If C contains K, then homologically $C' = C - 2K$, and so $C' \cdot C' = C \cdot C - 4K \cdot (C - K)$. Thus the invariance of μ follows from the fact that $\mathrm{Arf}(C') \equiv \mathrm{Arf}(C) + \frac{1}{2} K \cdot (C - K) (\mathrm{mod}\ 2)$, or equivalently

$$(-1)^{\mathrm{Arf}(C') - \mathrm{Arf}(C)} = (-1)^{\frac{1}{2} K \cdot (C - K)}.$$

(Note that $K \cdot (C - K)$ is even since C is characteristic.) This fact can be proved by elementary methods using Seifert surfaces, but it is quicker to use the formula for the Jones polynomial at i (Remark 4.12), which gives $(-1)^{\mathrm{Arf}(C') - \mathrm{Arf}(C)} = V_{C'}(i)(V_C(i))^{-1}$, and the Jones reversing result (see Proposition 4.3 of [L2]), which gives $V_{C'}(i) = (-1)^{\frac{1}{2} K \cdot (C - K)} V_C(i)$. (Note: Jones' reversing result can be proved using Remark 3.26 and Corollary 4.11.)

If L is changed by Move 1, then σ and $C \cdot C$ change equally (by ± 1) and $\mathrm{Arf}(C)$ remains unchanged, and so μ is unchanged.

If L is changed by Move 2, sliding L_i over L_j to get $L'_i = L_i + L_j$, then σ is unchanged, and $C \cdot C$ and $\mathrm{Arf}(C)$ change only if C contains L_i. In that case, since μ is independent of orientation, we can change the orientation of L_j after the handle slide ($L'_j = -L_j$). Then homologically $C' = C$, and so $C' \cdot C' = C \cdot C$. Also $\mathrm{Arf}(C') = \mathrm{Arf}(C)$, since the Arf invariant does not change under orientation preserving band connected sum, and evidently we can get C from C' by summing L'_i and L'_j appropriately. Thus μ is unchanged. □

(C.5) **Corollary** $\mu(M_{L,C}) = \mu(L, C)$.

Proof. It is known that (L, C) may be changed by moves of characteristic pairs to (L', C') with $C' = \emptyset$, and so $W_{L'}$ is a spin 4-manifold bounded by $M_{L,C}$ [Ka]. Thus $\mu(M_{L,C}) = \sigma(W_{L'}) (\mathrm{mod}\ 16) = \mu(L', C') = \mu(L, C)$. □

Remark. Normally Rohlin's theorem is used to show that the μ-invariant is a $\mathbf{Z}/16\mathbf{Z}$ invariant (rather than a $\mathbf{Z}/8\mathbf{Z}$ invariant as the algebra of intersection forms predicts). One may, however, reverse the order of things by showing the invariance of μ from the calculus of framed links (Lemma C.4), and then deduce Rohlin's theorem:

(C.6) **Corollary** (Rohlin's Theorem) *If W is a smooth, closed spin 4-manifold, then $\sigma(W) \equiv 0 (\mathrm{mod}\ 16)$.*

Proof. Using smoothness, decompose W as a handlebody with one 0 and one 4-handle. We may assume that W has no 1 or 3-handles, since they may be changed into 2-handles by surgery (preserving the spin structure and signature σ). Now $W_0 = W - (\text{4-handle}) = W_L$, with $M_L = S^3$, for some framed link L. Since W_L is spin, the characteristic sublink corresponding to the (unique) spin structure on S^3 is empty, and so $\mu(L, \emptyset) = \sigma(W_L) (\mathrm{mod}\ 16)$. But $S^3 = M_\emptyset$ as well, and so $\mu(L, \emptyset) = \mu(\emptyset, \emptyset) = 0$ by (C.4). Thus $\sigma(W) = \sigma(W_L) \equiv 0 (\mathrm{mod}\ 16)$. □

Whether this proof is really easier depends on ones view of the calculus of framed links whose proof ([K1]) uses Cerf theory.

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