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The 4-Choosability of Plane Graphs without 4-Cycles*

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A graph G is called k-choosable if k is a number such that if we give lists of k colors to each vertex of G there is a vertex coloring of G where each vertex receives a color from its own list no matter what the lists are. In this paper, it is shown that each plane graph without 4-cycles is 4-choosable. © 1999 Academic Press

Key Words: list coloring; choosability; plane graph; 4-cycle.

1. INTRODUCTION

In this paper, we consider only finite and simple graphs. Let G be a plane graph. $V(G)$, $E(G)$, and $F(G)$ shall denote the set of vertices, edges, and faces of G, respectively. Vertices u and v are *adjacent*, denoted by $uv \in E(G)$, if there is an edge in G joining them. $N_G(v)$, or $N(v)$ if there is no possibility of confusion, denotes the set of vertices adjacent to v in G, and ∂f denotes the set of vertices incident with the face f. The *degree* of u in G, written as $d_G(u)$, is the number of vertices in $N_G(v)$. A vertex u is called a *k-vertex* if $d_G(u) = k$. The minimum degree of G, $\min\{d_G(v) \mid v \in V(G)\}$, is denoted by $\delta(G)$. A face of a plane graph is said to be *incident* with all edges and vertices on its boundary. Two faces are *adjacent* if they have an edge in common. The degree of a face f of plane graph G, denoted by $d_G(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-face is a face of degree k. A *triangle* is synonymous with a 3-face.

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A *list coloring* of G is an assignment of colors to $V(G)$ such that each vertex v receives a color from a prescribed list $L(v)$ of colors and adjacent vertices receive distinct colors—see [8]. $L(G) = (L(v) \mid v \in V(G))$ is called a *color-list* of G. G is called *k-choosable* if G admits a list-coloring for all color-lists L with k colors in each list. The *choice number* of G, denoted by $\chi_l(G)$, is the minimum k such that G is k-choosable.

All 2-choosable graphs have been characterized by Erdős *et al.* [4]. N. Alon and M. Tarsi [1] proved that every plane bipartite graph is 3-choosable. Thomassen [6, 7] proved that every plane graph is 5-choosable and every plane graph of girth at least 5 is 3-choosable, where the *girth* of a graph G is the length of the shortest cycle in G. Also, every plane graph G without 3-cycles is 4-choosable because $\delta(G) \leqslant 3$. Examples of plane graphs which are not 4-choosable and plane graphs of girth 4 which are not 3-choosable were given by Voigt [9, 10]. Voigt and Wirth [11] also gave an example of a 3-colorable planar graph which is not 4-choosable.

In 1976, Steinberg (see [5, p. 229] or [2]) conjectured that every plane graph without 4- and 5-cycles is 3-colorable. In 1990, Erdős (also see [5, p. 229]) suggested the following relaxation of Steinberg's conjecture: Is there an integer $k \geqslant 5$ such that every plane graph without i-cycles, $4 \leqslant i \leqslant k$, is 3-colorable? In 1996, O. V. Borodin [2] proved that $k = 9$ is suitable. It would be significant to find an integer $k \geqslant 4$ such that every plane graph without i-cycles, $4 \leqslant i \leqslant k$, is 4-choosable. In this paper, we shall show that $k = 4$ is sufficient by proving the following theorem.

THEOREM 1. *Let G be a C_4-free plane graph. Then G is* 4-*choosable.*

In Section 2, we shall show that any C_4-free plane graph contains a subgraph with a special configuration, and this fact will be used to prove Theorem 2 in Section 3.

2. SPECIAL CONFIGURATION F_5^3

We use F_5^3 to denote a special C_4-free plane graph consisting of a 5-face with an exterior adjacent triangle. A subgraph H of G is called an *F_5^3-subgraph* if H is isomorphic to F_5^3 and $d_G(v) = 4$ for all $v \in V(H)$. We shall first prove that any C_4-free plane graph G with $\delta(G) = 4$ contains an *F_5^3-subgraph*.

Let G be a C_4-free plane graph with $\delta(G) \geqslant 4$. The set of all 5-faces adjacent to exactly four triangles and the set of all 5-faces adjacent to five triangles are denoted by F_4 and F_5 respectively. The subset of F_4 (respectively of F_5) consisting only of faces incident with five 4-vertices is denoted by $\bar{F}_4$ (respectively $\bar{F}_5$).

Let $f \in \bar{F}_5 \cup \bar{F}_4$, and let T be a triangle adjacent to f. The vertex $u \in \partial(T) \backslash \partial(f)$ is called an *M^*-vertex* of f. Moreover, if $d_G(u) = 5$, then u is called an *M-vertex* of f. Since a 5-vertex u can be incident with at most two triangles, u can be an M-vertex of at most two faces in $\bar{F}_5 \cup \bar{F}_4$. If u is an M-vertex of a face in $\bar{F}_5 \cup \bar{F}_4$, we denote by $m^*(u)$ the number of faces in $\bar{F}_5 \cup \bar{F}_4$ for which u is an M-vertex. It follows that $m^*(u) = 1$ or 2. A face f' is called an *M^*-face* of f at the vertex u_0 if $\partial(f) \cap \partial(f') = \{u_0\}$ and there exist two triangles which are adjacent to both f and f'. If in addition, $d_G(f') = 5$, then f' is called an *M-face* of f. If f' is an M-face of f at u_0, then $d_G(u_0) = 4$. If G contains no *F_5^3-subgraph* and $N(u_0) \cap \partial f' = \{u_1, u_2\}$, then $d_G(u_i) \geqslant 5$ for $i = 1$ and 2. Therefore a 5-face f' may be the M-face of at most two faces $f_1, f_2 \in \bar{F}_5 \cup \bar{F}_4$. If f' is an M-face of a face in $\bar{F}_5 \cup \bar{F}_4$, then we denote by $m^*(f')$ the number of faces in $\bar{F}_5 \cup \bar{F}_4$ for which f' is an M-face. It also follows that $m^*(f) = 1$ or 2. Moreover, if $m^*(f') = 2$, then f' is incident with three vertices of degree five or higher.

In [2], Borodin proved that $\delta(G) \leqslant 4$ for each plane graph without adjacent triangles. The following lemma is proved using the method of Borodin. Note that if a graph is C_4-free, then it has no 4-face and no adjacent triangles.

Lemma 1. *Let G be a plane graph without 4-faces and without adjacent triangles. If $\delta(G) = 4$, then G contains an F_5^3-subgraph.*

Proof. Suppose there exists a plane graph satisfying all assumptions of the Lemma and containing no F_5^3-subgraph. Let G be a graph of minimum order among such graphs.

Claim 1. *Each $f \in \bar{F}_5$ has at least two M-vertices. If f has exactly two M-vertices which are incident with the same M^*-face f' of f, then all M^*-faces of f, with the possible exception of f', are M-faces.*

Proof of Claim 1. Let $f \in \bar{F}_5$ be surrounded by triangles as in Fig. 1a. Because G contains no F_5^3-subgraph, we have $d_G(v_i) \geqslant 5$ for $i = 1, ..., 5$.

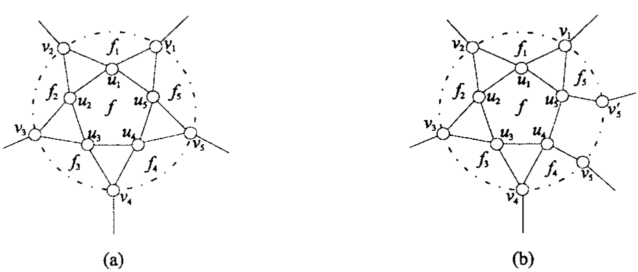

FIGURE 1

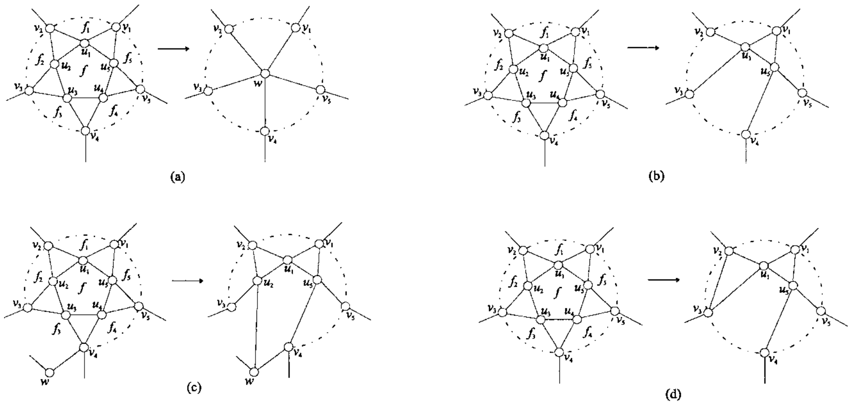

FIGURE 2

If f has no M-vertex, then put $H = G - \{u_1, ..., u_5\} + \{w, v_1w, v_2w, v_3w, v_4w, v_5w\}$—Fig. 2a. If f has only one M-vertex v_1, then put $H = G - \{u_2, u_3, u_4\} + \{u_1v_3, u_5v_4\}$—Fig. 2b. In both cases, $\delta(H) = 4$; $|V(H)| < |V(G)|$; $d_H(v_i) \geqslant 5$ for $i = 1, ..., 5$; and H contains no 4-face, no adjacent triangles, and no F_5^3-subgraph; contradicting the choice of G. Therefore f contains at least two M-vertices.

Suppose f has exactly two M-vertices v_1 and v_2 which are incident with the same M^*-face f_1 of f. If $d_G(f_3) \geqslant 6$, and w is the vertex other than u_3 such that wv_4 is an edge of f_3—Fig. 2c, then put $H = G - \{u_3, u_4\} + \{u_2w, u_5v_4\}$. If $d_G(f_2) \geqslant 6$, then put $H = G - \{u_2, u_3, u_4\} + \{u_5v_4, u_1v_3, v_2v_3\}$—Fig. 2d. The same contradiction as above establishes the latter part of Claim 1.

CLAIM 2. *Each $f \in \overline{F}_4$ has at least one M-vertex, and one of the two M^*-faces of f denoted by f_1 and f_3 in Fig.* 3b *is an M-face.*

The proof of Claim 2 is similar to that of Claim 1. If f has no M-vertex, then put $H = G - \{u_1, u_2, u_3\} + \{v_2u_5, v_3u_4\}$—Fig. 3a. If both $d_G(f_1) \geqslant 6$

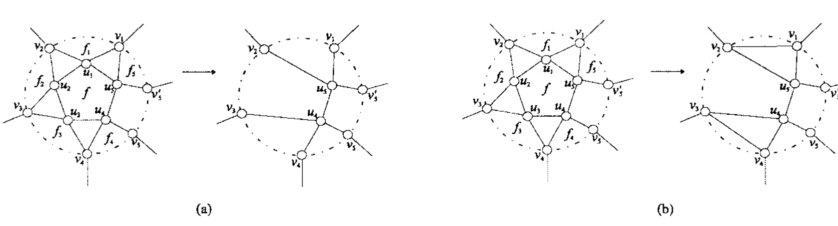

FIGURE 3

and $d_G(f_3) \geqslant 6$, then put $H = G - \{u_1, u_2, u_3\} + \{v_2v_1, v_2u_5, v_3u_4, v_3v_4\}$—Fig. 3b. This completes the proof of Claim 2.

Claim 3. *If u is an M-vertex of a face $f \in \bar{F}_4 \cup \bar{F}_5$ and is incident with a face $f^* \in F_5$, then f^* must be an M-face of f.*

Proof of Claim 3. Since u is an M-vertex of f, it is incident with one triangle and two non-triangular faces f' and f'', all of which share at least one vertex with f. The remaining two faces incident with u must be adjacent to either f' or f'' and therefore cannot be in F_5. Therefore if a face $f^* \in F_5$ is incident with u, then f^* has to be either f' or f'', and it is an M-face of f. This completes the proof of Claim 3.

Now by Euler's formula, $|V(G)| + |F(G)| - |E(G)| = 2$, we have

$$\sum_{v \in V(G)} \left(\frac{d_G(v)}{8} - \frac{1}{2}\right) + \sum_{f \in F(G)} \left(\frac{d_G(f)}{8} - \frac{1}{2}\right)$$
$$= \frac{1}{8}\left(\sum_{v \in V(G)} d_G(v) + \sum_{f \in F(G)} d_G(f)\right) - \frac{1}{2}(|V(G)| + |F(G)|)$$
$$= \frac{1}{8}(2\,|E(G)| + 2\,|E(G)|) - \frac{1}{2}(|E(G)| + 2) = -1.$$

For each $x \in V(G) \cup F(G)$, let $\sigma(x) = (d_G(x)/8) - (1/2)$ be a weight assigned to x. Then

$$\sum_{x \in V(G) \cup F(G)} \sigma(x) = -1. \tag{1}$$

We shall modify σ to a new weight σ^* according to the following rules:

(R-1) Every non-triangular face transfers 1/24 to each of its adjacent triangles.

(R-2) Let $k \geqslant 6$. Every k-vertex v transfers 1/12 to each of its incident faces f' adjacent to two triangles, each of which shares with f' a common edge incident with v; and transfers 1/24 to each of its incident faces f'' adjacent to exactly one triangle which shares with f'' a common edge incident with v.

(R-3) Let v be a 5-vertex of G.

(a) If v is incident with one 5-face $f_1 \in F_5$ and two 5-faces $f_2, f_3 \in F_4$, then transfer 1/12 to f_1 from v and transfer 1/48 to each of f_2 and f_3.

(b) If v is incident with one 5-face $f_1 \in F_5$ and up to one 5-face $f_2 \in F_4$, then transfer 1/12 to f_1 from v and transfer 1/24 to f_2, if necessary.

(c) If v is not incident with any 5-face in F_5, then transfer 1/24 from v to each face in F_4 which is incident with v.

Note that a 5-vertex v cannot be incident with two 5-faces in F_5 because otherwise there are adjacent triangles. Now, it follows that

$$\sum_{x \in V(G) \cup F(G)} \sigma^*(x) = \sum_{x \in V(G) \cup F(G)} \sigma(x) = -1. \tag{2}$$

If v is a vertex of degree at least 6, then v is incident with $l \leqslant \lfloor d_G(v)/2 \rfloor$ faces which receives 1/12 from v, and incident with at most $\lfloor \frac{2}{3}(d_G(v) - 2l) \rfloor$ faces which receive 1/24 from v, where $\lfloor x \rfloor$ is the largest integer not exceeding x. Therefore,

$$\begin{aligned} \sigma^*(v) &\geqslant \sigma(v) - \frac{l}{12} - \left\lfloor \frac{2}{3}(d_G(v) - 2l) \right\rfloor \cdot \frac{1}{24} \\ &\geqslant \sigma(v) - \frac{l}{12} - \frac{1}{36}(d_G(v) - 2l) \\ &= \frac{7d_G(v) - 36 - 2l}{72} \\ &= \frac{(6d_G(v) - 36) + (d_G(v) - 2l)}{72} \geqslant 0. \end{aligned}$$

Let v be a 5-vertex. If v is incident with one 5-face in F_5 and two 5-faces in F_4, then $\sigma^*(v) \geqslant \sigma(v) - (1/12) - 2 \cdot (1/48) = 0$. If v is incident with one 5-face in F_5 and at most one 5-face in F_4, then $\sigma^*(v) \geqslant \sigma(v) - (1/12) - (1/24) = 0$. If v is not incident with any 5-face of F_5, then v is incident with at most three 5-faces in F_4 and $\sigma^*(v) \geqslant \sigma(v) - 3 \cdot (1/24) = 0$.

Let $f \in F(G)$. If $d_G(f) = 3$, then $\sigma^*(f) = \sigma(f) + 3 \cdot (1/24) = 0$. If $d_G(f) \geqslant 6$, then $\sigma^*(f) \geqslant \sigma(f) - d_G(f) \cdot (1/24) = (2d_G(f) - 12)/24 \geqslant 0$. If $d_G(f) = 5$ and $f \notin F_5 \cup F_4$, then $\sigma^*(f) \geqslant \sigma(f) - 3 \cdot (1/24) = 0$. If $f \in F_5 \backslash \bar{F}_5$, then there is at least one vertex of degree 5 or more incident with f and $\sigma^*(f) \geqslant \sigma(f) + (1/12) - 5 \cdot (1/24) = 0$. If $f \in F_4 \backslash \bar{F}_4$, then there is also at least one vertex of degree 5 or higher incident with f. Suppose u is such a vertex.

If 1/24 or more is transferred from u to f, which will be the case if $d_G(u) > 5$ or if transfer is done according to (R-3)(b) or (R-3)(c), then $\sigma^*(f) \geqslant \sigma(f) + (1/24) - 4 \cdot (1/24) = 0$. Otherwise $d_G(u) = 5$ and u is incident with a face f_1 in F_5 and another face f_2 in F_4 in addition to f. Thus we have the situation of Fig. 4 and $d_G(w) \geqslant 5$ because otherwise the triangles T and T' are adjacent. It follows that $\sigma^*(f) \geqslant \sigma(f) + (1/48) + (1/48) - 4 \cdot (1/24) = 0$.

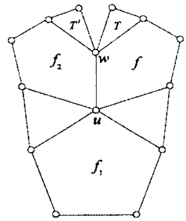

FIGURE 4

Up to now, we have $\sigma^*(x) \geqslant 0$ for each $x \in V(G) \cup F(G) - \bar{F}_5 \cup \bar{F}_4$, $\sigma^*(x) = -1/12$ for each $x \in \bar{F}_5$ and $\sigma^*(x) = -1/24$ for each $x \in \bar{F}_4$.

For each $f \in \bar{F}_5 \cup \bar{F}_4$, let $M^*(f) = \{$vertex(face) which is an M-vertex(face) of $f\}$. If we can prove that for each $f \in \bar{F}_5 \cup \bar{F}_4$, the inequality

$$\sigma^*(f) + \sum_{x \in M^*(f)} \frac{\sigma^*(x)}{m^*(x)} \geqslant 0, \tag{3}$$

holds, then we have $\sum_{x \in V(G) \cup F(G)} \sigma^*(x) \geqslant 0$. This will contradict (2) and will complete the proof.

Suppose $f \in \bar{F}_5$ as in Fig. 1a. Then $\sigma^*(f) = -1/12$. Let $r_{i,j}$ be the value transferred from v_i to f_j according to (R_2) and (R_3), where $i, j = 1, \ldots, 5$.

By Claim 1, f has at least two M-vertices. Suppose v_1 is an M-vertex of f. In the sequel, we want to show that

$$\frac{\sigma^*(v_1)}{m^*(v_1)} + \frac{\sigma^*(f_5)}{m^*(f_5)} + \frac{\sigma^*(f_1)}{m^*(f_1)} \geqslant \frac{1}{24}. \tag{4}$$

(i) If $f_1 \in F_5$, then $r_{1,1} = r_{2,1} = 1/12$ and $\sigma^*(f_1) \geqslant 1/12$. Hence $(\sigma^*(f_1)/m^*(f_1)) \geqslant (\sigma^*(f_1)/2) \geqslant 1/24$. Similarly, if $f_5 \in F_5$, then $(\sigma^*(f_5)/m^*(f_5)) \geqslant 1/24$. In the following, we shall assume that f_1 and f_5 are both not in F_5, and hence by Claim 3, v_1 is not incident with any face in F_5.

(ii) If $f_1 \in F_4$ and $f_5 \notin F_4$, then $r_{1,1} = 1/24$, $r_{1,5} = 0$, and at most 1/24 is transferred to one non-triangular face (other than f_1 and f_5) incident with v_1. Hence $\sigma^*(v_1) \geqslant 1/24$ and $(\sigma^*(v_1)/m^*(v_1)) \geqslant (\sigma^*(v_1)/2) \geqslant 1/48$. Because $f_1 \in F_4$, we have $r_{2,1} \geqslant 1/48$. If $m^*(f_1) = 1$, then $(\sigma^*(f_1)/m^*(f_1)) = \sigma^*(f_1) \geqslant 1/48$. If $m^*(f_1) = 2$, then f_1 is incident with three vertices with degree 5 or higher. Hence there is vertex $w \notin \{v_1, v_2\}$ incident with f_1 and $d_G(w) \geqslant 5$, so each of w and v_2 will transfer at least 1/48 to f_1 and $(\sigma^*(f_1)/m^*(f_1)) = (\sigma^*(f_1)/2) \geqslant 1/48$. It follows that $(\sigma^*(v_1)/m^*(v_1)) + (\sigma^*(f_1)/m^*(f_1)) \geqslant 1/24$. Similarly, if $f_1 \notin F_4$ and $f_5 \in F_4$, then $(\sigma^*(v_1)/m^*(v_1)) + (\sigma^*(f_5)/m^*(f_5)) \geqslant 1/24$.

(iii) If $f_1 \notin F_4$ and $f_5 \notin F_4$, then $r_{1,1} = r_{1,5} = 0$ and v_1 will transfer at most 1/24 to one non-triangular face incident with it, and therefore $\sigma^*(v_1) \geqslant 1/12$. It follows that $(\sigma^*(v_1)/m^*(v_1)) \geqslant (\sigma^*(v_1)/2) \geqslant 1/24$.

Thus the proof of (4) is complete.

Similarly, if v_3 is an M-vertex of f, then $(\sigma^*(v_3)/m^*(v_3)) + (\sigma^*(f_2)/m^*(f_2)) + (\sigma^*(f_3)/m^*(f_3)) \geqslant 1/24$. Combining this with (4), we get (3). An analogous conclusion can be made if v_4 is an M-vertex of f.

Suppose that f has exactly two M-vertices v_1 and v_2. By Claim 1, f_2, f_3, f_4 and f_5 are all 5-faces. We also have $d_G(v_j) \geqslant 6$ for $j = 3$, 4 and 5. If $f_3 \in F_5$, then as in (i) above, we have $\sigma^*(f_3) \geqslant 1/12$. If $f_3 \notin F_5$, then $r_{3,3} \geqslant 1/24$ and $r_{4,3} \geqslant 1/24$, and hence $\sigma^*(f_3) \geqslant 1/24$. Similarly, we have $\sigma^*(f_4) \geqslant 1/24$. It follows that $(\sigma^*(f_3)/m^*(f_3)) + (\sigma^*(f_4)/m^*(f_4)) \geqslant (\sigma^*(f_3)/2) + (\sigma^*(f_4)/2) \geqslant 1/24$. Combining this with the argument made on v_1, we also have (3).

Now suppose $f \in \bar{F}_4$ as in Fig. 1b. Then $\sigma^*(f) = -1/24$. Let $r_{i,j}$ be the weight transferred from v_i to f_j according to (R_2) and (R_3), where $i = 1, ..., 4$ and $j = 1, ..., 5$.

By Claim 2, f has at least one M-vertex. If v_2 is an M-vertex of f, then we can show as before that $(\sigma^*(v_2)/m^*(v_2)) + (\sigma^*(f_1)/m^*(f_1)) + (\sigma^*(f_2)/m^*(f_2)) \geqslant 1/24$, and (3) follows. Similarly if v_3 is an M-vertex of f, then $(\sigma^*(v_3)/m^*(v_3)) + (\sigma^*(f_2)/m^*(f_2)) + (\sigma^*(f_3)/m^*(f_3)) \geqslant 1/24$ and (3) follows also. If both v_2 and v_3 are not M-vertices of f, then without loss of generality, we may assume that v_1 is an M-vertex of f. Also by Claim 2, either f_1 or f_3 is an M-face of f.

Suppose f_1 is an M-face of f. If $f_1 \in F_5$ then $r_{1,1} = r_{2,1} = 1/12$. So $\sigma^*(f_1) \geqslant 1/12$ and $(\sigma^*(f_1)/m^*(f_1)) \geqslant (\sigma^*(f_1)/2) \geqslant 1/24$. Suppose $f_1 \in F_4$, then by Claim 3, v_1 is not incident with any face in F_5 and therefore $r_{1,1} = 1/24$. If v_1 is not incident with two triangles adjacent to f_1, then v_2 is incident with two triangles adjacent to f_1. Since v_2 is not an M-vertex of f, $d_G(v_2) \geqslant 6$ and so $r_{2,1} = 1/12$. Hence $\sigma^*(f_1) \geqslant 1/12$. It follows that $(\sigma^*(f_1)/m^*(f_1)) \geqslant (\sigma^*(f_1)/2) \geqslant 1/24$. If v_1 is incident with two triangles adjacent to f_1, then f_5 is adjacent to at least two non-triangular face and hence $r_{1,5} = 0$. Since we transfer at most 1/24 from v_1 to one non-triangular face (other than f_1 and f_5) incident with v_1, we have $\sigma^*(v_1) \geqslant 1/24$. Since $r_{2,1} \geqslant 1/24$, we have $\sigma^*(f_1) \geqslant 1/24$. It follows that $(\sigma^*(f_1)/m^*(f_1)) + (\sigma^*(v_1)/m^*(v_1)) \geqslant (\sigma^*(f_1)/2) + (\sigma^*(v_1)/2) \geqslant 1/24$. If $f_1 \notin F_5 \cup F_4$, then $r_{1,1} = 0$ and because $d_G(v_2) \geqslant 6$, $r_{2,1} \geqslant 1/24$. It follows that $\sigma^*(v_1) \geqslant 1/24$ and $\sigma^*(f_1) \geqslant 1/24$, and that $(\sigma^*(v_1)/m^*(v_1)) + (\sigma^*(f_1)/m^*(f_1)) \geqslant 1/24$. As before, (3) holds.

Suppose f_1 is not an M-face of f, then $r_{1,1} = 0$ and $\sigma^*(v_1) \geqslant 1/24$. By Claim 2, f_3 must be an M-face of f. If $f_3 \in F_5$, or if v_4 is also an M-vertex of f, the same argument as above leads to (3). Suppose $f_3 \notin F_5$ and v_4 is not an M-vertex of f, then $d_G(v_4) \geqslant 6$ and therefore $r_{4,3} \geqslant 1/24$. Because v_3 is not an M-vertex of f, we also have $r_{3,3} \geqslant 1/24$ and hence $\sigma^*(f_3) \geqslant 1/24$. It follows that $(\sigma^*(v_1)/m^*(v_1)) + (\sigma^*(f_3)/m^*(f_3)) \geqslant (\sigma^*(v_1)/2) + (\sigma^*(f_3)/2) \geqslant 1/24$ and that (3) again holds.

3. PROOF OF THEOREM 1

Suppose that G is a counterexample of minimum order, then $\delta(G) = 4$. Because G is C_4-free, G has no adjacent triangles and has no 4-face. By Lemma 1, G has a F_5^3-subgraph H with

$$V(H) = \{u_1, ..., u_6\} \text{ and } E(H) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_6u_1, u_2u_6\}.$$

Let $L = (L(v) \mid v \in V(G))$ be a color-list of G in which each list contains 4 colors. Then $G' = G - V(H)$ admits a list coloring ϕ' with color-list L restricted to G'.

For all $v \in V(H)$, let $L^0(v) = L(v) \backslash \{\phi'(u) \mid u \in V(G') \text{ and } vu \in E(G)\}$. Then, $|L^0(u_i)| \geqslant 2$, $i = 1, 3, 4, 5$, $|L^0(u_2)| \geqslant 3$ and $|L^0(u_6)| \geqslant 3$. Let L^* be a subset of $L^0(u_3)$ with $|L^*| = 2$. We shall choose at u_2 a color $c_2 \in L^0(u_2) \backslash L^*$, at u_1 a color $c_1 \in L^0(u_1) \backslash \{c_2\}$, at u_6 a color $c_6 \in L^0(u_6) \backslash \{c_1, c_2\}$, at u_5 a color $c_5 \in L^0(u_5) \backslash \{c_6\}$, at u_4 a color $c_4 \in L^0(u_4) \backslash \{c_5\}$ and at u_3 a color $c_3 \in L^* \backslash \{c_4\}$.

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