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The A-D-E Classification of Minimal and $A_1^{(1)}$ Conformal Invariant Theories

A. Cappelli*, C. Itzykson, and J. B. Zuber

Service de Physique Théorique, CEN-Saclay, F-91191 Gif-sur-Yvette Cedex, France

Abstract. We present a detailed and complete proof of our earlier conjecture on the classification of minimal conformal invariant theories. This is based on an exhaustive construction of all modular invariant sesquilinear forms, with positive integral coefficients, in the characters of the Virasoro or of the $A_1^{(1)}$ Kac-Moody algebras, which describe the corresponding partition functions on a torus. A remarkable correspondence emerges with simply laced Lie algebras.

I. Introduction

1. The minimal conformal invariant models describe a class of massless two dimensional field theories, with known critical properties [1]. Their anomalous dimensions and operator content are encoded in the expression of the partition function on a torus. The sum over states decomposes into pairs of irreducible representations of the Virasoro algebra, with central charge c rational and smaller than 1, yielding a sesquilinear form in the characters χ_h,

$$Z(\tau)=\sum \mathcal{N}_{h,\bar{h}}\chi_h(\tau)\chi_{\bar{h}}^*(\tau) \ .$$

In this formula τ is the ratio of the two periods on the torus, and the summation extends over a finite table of known $(h,\bar{h})$ values. The non-negative integral coefficients $\mathcal{N}_{h,\bar{h}}$ yield the multiplicities of primary scaling operators $\varphi_{h,\bar{h}}$, which are in one to one correspondence with the products $\chi_h\chi_{\bar{h}}^*$ of characters. Cardy [2] noticed that modular invariance is a consistency condition on these partition functions.

Our aim here is to present a detailed proof of the classification of these positive modular invariants, announced in [3]. As these theories describe statistical models at criticality, this classifies the universality classes of two dimensional critical phenomena, pertaining to $c<1$, with finitely many primary observables. They include for instance the Ising and three-state Potts models.

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* Permanent address: Istituto Nazionale di Fisica Nucleare, Sezione di Firenze, Largo E. Fermi 2, I-50125 Firenze, Italy

Gepner [4] has observed that there exists a simpler and related problem of modular invariant sesquilinear forms in the characters of the $A_1^{(1)}$ affine Kac-Moody algebra – again with integral non-negative coefficients [5]. We address both problems.

Unexpectedly, a beautiful structure emerged [3], with a classification of these minimal invariants in terms of simply laced simple Lie algebras, or equivalently finite subgroups of SU_2 [6], the celebrated A-D-E classification. Our results were based on two conjectures. The first was a description of the commutant of the representation of the modular group afforded by the characters. In the meantime it was shown correct by Gepner and Qiu [7]. After recalling the expression and properties of the characters (Sect. II), we shall reproduce this proof for completeness, albeit in a slightly different formalism [8] (Sect. III). We were led to the second conjecture after tabulating those invariants involving non-negative integral coefficients that were constructed using the previous algorithms. When casting earlier findings by Cardy [2], ourselves [9, 10] and Gepner [4], in the appropriate notation, it was recognized that the diagonal entries of all these partition functions could be interpreted as the Coxeter exponents of simply laced simple Lie algebras, generalizing an observation of Kac [11].

We then checked that, within a natural ordering we could not find any further partition function at least up to a high order. Since the list exhausted the A-D-E classification, it was suggested that it was complete. This was comforted by V. Pasquier's construction of microscopic generalized solid-on-solid models, involving the Coxeter-Dynkin diagrams and exhibiting the predicted behavior [12]. We prove in Sect. IV that our lists are exhaustive. The method might be qualified as intrinsic, in the sense that it combines simple arithmetic remarks, but does not illuminate the nature of the correspondence with other A-D-E classifications. We suspect nevertheless that such a correspondence exists, and finding it remains a challenge. In the final Sect. V we study the representations of the modular group related to some of the positive invariants, and point out their connection with classical problems in algebra and number theory, according to the discussion given in F. Klein's treatise on the icosahedron [13].

The above method of classification can be extended to other families of conformal field theories, such as the minimal $N=1$ superconformal ones [14], or the Z_N-symmetric parafermionic models [15, 7]. Several authors have also related the exceptional affine invariants to simpler ones pertaining to higher rank Kac-Moody algebras [16].

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II. Preliminaries

2. As a matter of convenience we write $\mathbf{e}(x)$ for $\exp\{2i\pi x\}$, $\mathbb{C}$ is the set of complex numbers, $\mathbb{Z}$ the set of rational integers, $\mathbb{Z}/k\mathbb{Z}$ the set of integers modulo k, and $(\mathbb{Z}/k\mathbb{Z})^*$ its multiplicative subgroup of integers modulo k, prime to k. The notation

$p|k$ ($p\nmid k$) means that p divides k (does not divide k). For a set of integers $a, b, \ldots$ the symbol $(a, b, \ldots)$ stands for the largest (positive) common divisor of $a, b, \ldots$. In particular $(a, b) = 1$ means that a and b are coprime, in which case any rational integer may be represented as a linear combination $ra - sb$. We denote by $\sigma(n)$ the number of distinct positive divisors of n, including 1. We exclude 1 from the set of (positive) primes.

The modular group $\Gamma = PSL(2, \mathbb{Z})$ is the group of fractional linear transformations

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \tag{1}$$

with the integral coefficients, such that $ad - bc = 1$. A given element is therefore associated to a pair $\pm A$ of two by two matrices in $SL(2, \mathbb{Z})$. If in (1) the complex variable τ has a positive imaginary part, so has τ', and we shall henceforth assume that this is the case.

The modular group describes the effect of a change of basis on the ratio τ of two generators of a lattice Λ, with $\mathbb{C}/\Lambda$ identified with a torus. Γ is generated by two elements

$$T \quad \tau \to \tau + 1 \ , \qquad S \quad \tau \to -\tau^{-1} \ , \tag{2}$$

satisfying

$$S^2 = (ST)^3 = \text{identity} \ . \tag{3}$$

For any integer $k > 1$, Γ_k, the invariant subgroup of level k is such that $A \equiv \pm I \bmod k$. It is a non-trivial result that $\Gamma^k = PSL(2, \mathbb{Z})/\Gamma_k$ is isomorphic to the modular group on integers mod k, $PSL(2, \mathbb{Z}/k\mathbb{Z})$ [17]. Furthermore for $k > 2$

$$\text{order } \Gamma^k = \text{index } \Gamma_k = \frac{1}{2} k^3 \prod_{\substack{p \text{ prime} \\ p|k}} \left(1 - \frac{1}{p^2}\right) . \tag{4}$$

When $k = 2$, $A \equiv -A \bmod 2$, one has to omit the prefactor $1/2$ and the index of Γ_2 is six. If $(k_1, k_2) = 1$, the group $SL(2, \mathbb{Z}/k_1 k_2 \mathbb{Z})$ is isomorphic to the direct product $SL(2, \mathbb{Z}/k_1\mathbb{Z}) \times SL(2, \mathbb{Z}/k_2\mathbb{Z})$. Any element of the former group gives obviously rise to a pair of the latter, and the correspondence is clearly an injective homomorphism. Formula (4) with the prefactor $1/2$ omitted, shows that it is surjective. This entails that the representation theory for $SL(2, \mathbb{Z}/k\mathbb{Z})$ is in fact reduced to the case where k is the power of a prime. We shall not elaborate this point further, except to note that it obviously relates to the discussion of the following sections. In particular the center of $SL(2, \mathbb{Z}/k\mathbb{Z})$ is made of matrices of the form γI, with $\gamma^2 \equiv 1 \bmod k$. Let r denote the number of distinct odd prime divisors of k, and $a = 0$ if $k \not\equiv 0, 4$, $a = 1$ if $k \equiv 4$ and $a = 2$ if $k \equiv 0 \bmod 8$, the number of elements in the center is 2^{a+r}. In any representation these elements will belong to the commutant.

The simplest automorphic "form" under the modular group is Dedekind's function, defined for $\operatorname{Im} \tau > 0$ (the real axis is a natural boundary) by

$$\eta(\tau) = \mathbf{e}(\tau/24) \prod_{\ell=1}^{\infty} (1 - \mathbf{e}(\ell\tau)) \ . \tag{5}$$

It is convenient to use $q = \mathbf{e}(\tau)$, $|q| < 1$, giving a meaning to fractional powers of q. Omitting its prefactor, $\eta(\tau)^{-1}$ is the generating function of partitions. Euler's pentagonal identity gives the series expansion

$$\eta(\tau)=\sum_{\ell=-\infty}^{+\infty}(-1)^{\ell}\mathbf{e}\left(\tau\,\frac{(6\ell+1)^2}{24}\right). \tag{6}$$

From Poisson's formula, it follows that under a modular transformation

$$\begin{aligned} T\quad & \eta(\tau+1)=\mathbf{e}(1/24)\eta(\tau)\ , \\ S\quad & \eta(-\tau^{-1})=(\tau/i)^{1/2}\eta(\tau)\ , \end{aligned} \tag{7}$$

where the square root is 1 if $\tau=i$. For a general modular transformation

$$\eta(\tau')=\varepsilon_A(c\tau+d)^{1/2}\eta(\tau) \tag{8}$$

with ε_A a 24-th root of unity. The product representation shows that $\eta(\tau)$ never vanishes in the upper half plane $\operatorname{Im}\tau>0$.

3. The characters corresponding to the degenerate representations of the Virasoro algebra (abbreviated as conformal characters), follow from the work of Feigin and Fuchs [18], Rocha-Caridi [19], and Dobrev [20]. They are labelled by a pair c, h, with c the central charge, and h the highest weight. Let p and p' be a pair of coprime positive integers, both larger than 1. For $c<1$, the minimal degenerate series corresponds to central charges

$$c=1-\frac{6(p-p')^2}{pp'} \tag{9}$$

and highest weights given by

$$h(r,s)=\frac{(rp-sp')^2-(p-p')^2}{4pp'}\equiv h(p'-r,p-s)\ , \tag{10}$$

where the integers r and s are in the range $0<r<p'$, $0<s<p$ and may be further restricted by $sp'<rp$, if we assume for instance $p'<p$. When p, p' are successive integers $p=m+1$, $p'=m$, $m\geqq 2$, one has the discrete unitary series discovered by Friedan et al. [21]. In a given representation, let d_ℓ, $\ell\geqq 0$, be the dimension of the subspace with eigenvalue $h+\ell$ of the operator L_0 in the Virasoro algebra, and $q=\mathbf{e}(\tau)$, the character is defined as

$$\begin{aligned} \chi_{c,h}(\tau) &= \sum_{\ell=0}^{\infty} d_\ell q^{-c/24+h+\ell} \\ &= \eta(\tau)^{-1}\sum_{t=-\infty}^{+\infty}\left(q^{\frac{(2tpp'+rp-sp')^2}{4pp'}}-q^{\frac{(2tpp'+rp+sp')^2}{4pp'}}\right). \end{aligned} \tag{11}$$

We have included in $\chi_{c,h}$ a factor $q^{-c/24}$ to simplify formulas in the sequel, where c will be kept fixed, while h varies.

To make this more transparent, let us use the following notations. Define the even integer N through

$$N=2n=2pp' \qquad (p,p')=1 \qquad N\geqq 12\ , \tag{12}$$

and trade the weight h for an integer $\lambda \bmod N$

$$\lambda \equiv rp - sp' \bmod N \ . \tag{13}$$

If r and s are chosen as indicated before, λ lies in the range $0 < \lambda < n$, with multiples of p and p' excluded. The total number of possible values is therefore $\frac{1}{2}(p-1)(p'-1)$. The reason for these pecularities follows from the symmetries of characters as functions of $\lambda \bmod N$. To see this in detail, consider all possible pairs (r, s) leading to the same value of h, i.e. of $(rp - sp')^2$. We can think of these pairs as elements of a lattice $\mathscr{L}$, equipped with a Lorentzian metric and generated by two orthogonal vectors $\mathbf{a}_0$ and $\mathbf{a}_1$ such that $\mathbf{a}_0 \cdot \mathbf{a}_0 - 1 = \mathbf{a}_1 \cdot \mathbf{a}_1 + 1 = \mathbf{a}_0 \cdot \mathbf{a}_1 = 0$. We have the correspondence

$$\{r, s\} \to \boldsymbol{\lambda} = r\mathbf{a}_0 + s\mathbf{a}_1 \in \mathscr{L} \ , \tag{14}$$

Let W be the sublattice generated by

$$\mathbf{v}_+ = p'\mathbf{a}_0 + p\mathbf{a}_1 \ , \qquad \mathbf{v}_- = p'\mathbf{a}_0 - p\mathbf{a}_1 \ . \tag{15}$$

The index of W is $|\det(\mathbf{v}_+, \mathbf{v}_-)| = 2pp' = N$. By interchanging the roles of p and p', define also the dual sublattice $\tilde{W}$ generated by

$$\begin{aligned} &\mathbf{u}_+ = p\mathbf{a}_0 + p'\mathbf{a}_1 \ , &&\mathbf{u}_- = p\mathbf{a}_0 - p'\mathbf{a}_1 \ , \\ &\mathbf{u}_+ \cdot \mathbf{v}_+ = \mathbf{u}_- \cdot \mathbf{v}_- = 0 \ , &&\mathbf{u}_+ \cdot \mathbf{v}_- = \mathbf{u}_- \cdot \mathbf{v}_+ = N \ . \end{aligned} \tag{16}$$

Any vector $\boldsymbol{\lambda}$ can be represented by its scalar products on $\mathbf{u}_+, \mathbf{u}_-$ as

$$\lambda = \boldsymbol{\lambda} \cdot \mathbf{u}_+ = rp - sp' \qquad \lambda' = \boldsymbol{\lambda} \cdot \mathbf{u}_- = rp + sp' \tag{17}$$

and $\lambda = 0$ ($\lambda' = 0$) if and only if $\boldsymbol{\lambda} = \xi \mathbf{v}_+$ ($\boldsymbol{\lambda} = \xi \mathbf{v}_-$) for some integer ξ. Adding an element of W to $\boldsymbol{\lambda}$ leaves λ and λ' invariant mod N. In fact (i) as additive groups $\mathscr{L}/W$ and $\mathbb{Z}/N\mathbb{Z}$ are isomorphic, and (ii) there exists an integer $\omega_0 \bmod N$ such that

$$\omega_0^2 \equiv 1 \bmod 2N \qquad \lambda' \equiv \omega_0 \lambda \bmod N \ . \tag{18}$$

The mod $2N$ condition in the first equation is compatible with the fact that N being even a shift of ω_0 by a multiple of N changes ω_0^2 by a multiple of $2N$. Since p and p' are coprime, it is possible to find a pair (r_0, s_0) (an infinity of them) such that $r_0 p - s_0 p' = 1$.

Define $\boldsymbol{\lambda}_0 = r_0 \mathbf{a}_0 + s_0 \mathbf{a}_1$, then

$$\boldsymbol{\lambda}_0 \cdot \mathbf{u}_+ = 1 \ , \qquad \omega_0 = \boldsymbol{\lambda}_0 \cdot \mathbf{u}_- = r_0 p + s_0 p' \ , \tag{19}$$

and $\omega_0^2 - 1 = 4 r_0 s_0 pp' \equiv 0 \bmod 2N$. The map $\boldsymbol{\lambda} \bmod W \to \lambda = \boldsymbol{\lambda} \cdot \mathbf{u}_+ \bmod N$ is an homomorphism from $\mathscr{L}/W$ into $\mathbb{Z}/N\mathbb{Z}$. Since $(\boldsymbol{\lambda} - \lambda\boldsymbol{\lambda}_0) \cdot \mathbf{u}_+ = 0$, $\boldsymbol{\lambda}$ differs from $\lambda\boldsymbol{\lambda}_0$ by a multiple of $\mathbf{v}_+$, i.e. an element of W, thus proving (i) in the form

$$\boldsymbol{\lambda} \equiv \lambda \boldsymbol{\lambda}_0 \bmod W \ .$$

Multiplying both sides by $\mathbf{u}_-$, we get $\lambda' \equiv \omega_0 \lambda \bmod N$. The vector $\boldsymbol{\lambda}_0$ is defined up to a multiple of $\mathbf{v}_+ \in W$, hence ω_0 is defined mod N, which completes the proof of (18).

The factors p and p' are the smallest positive integers such that

$$\begin{aligned} \omega_0\lambda &\equiv \lambda \bmod N \quad \leftrightarrow \lambda \equiv 0 \bmod p \ , \\ \omega_0\lambda &\equiv -\lambda \bmod N \leftrightarrow \lambda \equiv 0 \bmod p' \end{aligned} \tag{20}$$

The requirement on the Virasoro representations is that $\omega_0 \not\equiv \pm 1 \bmod N$.

We can now rewrite the characters (11) in the form

$$\chi_\lambda(\tau) = \chi_{-\lambda}(\tau) = \chi_{\lambda+\xi N}(\tau) = -\chi_{\pm\omega_0\lambda}(\tau) = K_\lambda(\tau) - K_{\omega_0\lambda}(\tau) \tag{21}$$

with

$$K_\lambda(\tau) = K_{-\lambda}(\tau) = K_{\lambda+N}(\tau) = \eta(\tau)^{-1} \sum_{t=-\infty}^{+\infty} q^{\frac{(tN+\lambda)^2}{2N}} \ . \tag{22}$$

Under modular transformations the behavior of $K_\lambda(\tau)$ follows readily from Poissons's formula and from (7)

$$\begin{aligned} T \quad & K_\lambda(\tau+1) = \mathbf{e}\left(\frac{\lambda^2}{2N} - \frac{1}{24}\right) K_\lambda(\tau) \ , \\ S \quad & K_\lambda(-\tau^{-1}) = \frac{1}{\sqrt{N}} \sum_{\lambda' \in \mathbb{Z}/N\mathbb{Z}} \mathbf{e}\left(\frac{\lambda\lambda'}{N}\right) K_{\lambda'}(\tau) \ , \end{aligned} \tag{23a}$$

both relations being compatible with the symmetries indicated in (21). The equality $K_\lambda = K_{-\lambda}$ is crucial in insuring that (23a) defines a representation of the modular group. Similarly, $\omega_0^2 \equiv 1 \bmod 2N$ shows that χ_λ has identical transformation properties

$$\begin{aligned} T \quad & \chi_\lambda(\tau+1) = \mathbf{e}\left(\frac{\lambda^2}{2N} - \frac{1}{24}\right) \chi_\lambda(\tau) \ , \\ S \quad & \chi_\lambda(-\tau^{-1}) = \frac{1}{\sqrt{N}} \sum_{\lambda' \in \mathbb{Z}/N\mathbb{Z}} \mathbf{e}\left(\frac{\lambda\lambda'}{N}\right) \chi_{\lambda'}(\tau) \ . \end{aligned} \tag{23b}$$

4. In parallel with the treatment of the Virasoro characters, one can carry out a similar discussion involving integrable highest weight representations of the affine Lie algebra $A_1^{(1)}$ (the SU_2 current algebra) and their characters, henceforth referred to as affine ones. Those are labelled by a non-negative integer called the level k, and a lowest angular momentum (integer or half integer) ℓ, such that $0 \leqq 2\ell \leqq k$. To stress the analogy with the previous case, we define

$$N = 2\,(k+2) \geqq 4 \ , \qquad \lambda = 2\ell + 1 \ , \tag{24}$$

and write the affine character [22] as

$$\chi_\lambda^{\mathrm{aff}}(\tau) = \chi_{\lambda+N}^{\mathrm{aff}}(\tau) = -\chi_{-\lambda}^{\mathrm{aff}}(\tau) = \eta(\tau)^{-3} \sum_{t=-\infty}^{+\infty} (Nt+\lambda)\, q^{\frac{(Nt+\lambda)^2}{2N}} \ . \tag{25}$$

Here the role of $\lambda \to \omega_0\lambda$ is played by the involution $\lambda \to -\lambda$, under which the character is odd. The $k+1$ independent characters can be chosen with index $\lambda = 1, 2, \ldots, k+1$.

Under modular transformations

$$
\begin{aligned}
T \quad & \chi_\lambda^{\mathrm{aff}}(\tau+1) = \mathbf{e}\left(\frac{\lambda^2}{2N} - \frac{1}{8}\right) \chi_\lambda^{\mathrm{aff}}(\tau) \ , \\
S \quad & \chi_\lambda^{\mathrm{aff}}(-\tau^{-1}) = \frac{-i}{\sqrt{N}} \sum_{\lambda' \in \mathbb{Z}/N\mathbb{Z}} \mathbf{e}\left(\frac{\lambda\lambda'}{N}\right) \chi_{\lambda'}^{\mathrm{aff}}(\tau) \ .
\end{aligned}
\tag{26}
$$

Similar remarks apply here as it did for Eqs. (23). When $N=4$, $\lambda=1$, χ^{aff} reduces to 1, leading to a well known expression for η^3 due to Jacobi, the first in the series obtained by Macdonald and Dyson [22]. Similarly in the Virasoro case, when $N=12$, $p=3$, $p'=2$ and $r=2$, $s=1$, the representation is trivial, $\chi=1$, and Eq. (11) reduces to Euler's identity (6).

In [3] we have looked for the possible choices of phases ε_r and ε_s such that the transformations on integers mod N

$$
\begin{aligned}
T_{\lambda\lambda'} &= \varepsilon_r \, \mathbf{e}\left(\frac{\lambda^2}{2N}\right) \delta_{\lambda\lambda'} \\
S_{\lambda\lambda'} &= \varepsilon_s \frac{1}{\sqrt{N}} \, \mathbf{e}\left(\frac{\lambda\lambda'}{N}\right)
\end{aligned}
\tag{27}
$$

generate in the subspaces of even ($\varepsilon=+1$) or odd ($\varepsilon=-1$) vectors under $\lambda \to -\lambda$, a unitary finite dimensional representation of the modular group (keeping N even). Using Gauss'sum, it was found that there exist 12 possibilities with

$$
\varepsilon_s^2 = \varepsilon \ , \qquad (\varepsilon_s \varepsilon_r)^3 = \mathbf{e}(-1/8) \ , \tag{28}
$$

out of which two are realized in the previous cases. This enables one to get a better understanding of the phases which distinguish (23) from (26). It would be of interest to find representative problems for the ten remaining possibilities.

In both the conformal as well as the affine case, the group of level $2N$, Γ_{2N}, is represented by a multiple of the identity, a 24th root of unity in the Virasoro case, or an 8th root in the affine one[1]. As a consequence of this non-trivial property proved in [3], Eqs. (23) and (26) generate projective representations of Γ^{2N}, which are in general reducible. The phase is immaterial in the following discussion of invariant sesquilinear forms.

We now have all the elements to state the classification problem in both the conformal and affine case, referring to the literature for motivations and applications. To shorten notation we shall henceforth omit the suffix which distinguishes the affine from the conformal case, unless mandatory.

5. The partition functions (on tori) of critical models are sesquilinear forms in the characters

$$
Z(\tau) = \sum_{\lambda, \lambda' \in \mathscr{B}} \chi_\lambda^*(\tau) \mathscr{N}_{\lambda\lambda'} \chi_{\lambda'}(\tau) \ , \tag{29}
$$

[1] The role of the group Γ_{2N} in this problem had been foreseen by A. Schwimmer (private communication)

where the indices λ range over a fundamental domain $\mathscr{B}$ (within integers mod N) of the symmetry properties, i.e. $\lambda \to -\lambda$ in the affine case (where $\chi_0 = \chi_n = 0$) and $\lambda \to -\lambda$, $\lambda \to \pm\omega_0\lambda$ in the conformal one. The coefficients $\mathcal{N}_{\lambda\lambda'}$ should satisfy the following two conditions

(A) $Z(\tau)$ is modular invariant,
(B) $\mathcal{N}_{\lambda\lambda'}$ are non-negative integers.

An auxiliary normalization condition (unicity of the vacuum state) requires in the conformal case $\mathcal{N}_{p-p',p-p'} = 1$ (i.e. $\mathcal{N}_{1,1}$ in the unitary series). We take it to be $\mathcal{N}_{1,1} = 1$ in the affine case.

One could also generalize the problem to include frustrated partition functions, where (A) could be relaxed to a weaker condition of invariance under a subgroup of the modular group [23, 10]. This will not be considered here.

Corresponding to the two conditions (A) and (B), the problem subdivides itself into two parts to be treated successively: (A) To find the general form of a modular invariant, or equivalently to study the commutant of the representation; (B) To study the positivity and integrality restrictions, in the affine and conformal cases, respectively.

III. The Commutant

6. Let us first look at combinations such as (2.29), where the coefficients are arbitrary complex numbers, submitted only to condition (A), i.e. modular invariance. It is convenient to extend the range of summation for the indices λ and λ' to $\mathbb{Z}/N\mathbb{Z}$, provided the matrix $\mathcal{N}_{\lambda,\lambda'}$ satisfies the obvious symmetry relations. If ε, ε' take the values 0 or 1, those are

$$\mathcal{N}_{(-1)^\varepsilon\lambda,(-1)^{\varepsilon'}\lambda'} = (-1)^{\varepsilon+\varepsilon'}\mathcal{N}_{\lambda,\lambda'} \quad \text{affine case} \tag{1a}$$

$$\left.\begin{aligned} &\mathcal{N}_{(-1)^\varepsilon\lambda,(-1)^{\varepsilon'}\lambda'} = \mathcal{N}_{\lambda,\lambda'} \\ &\mathcal{N}_{\omega_0^\varepsilon\lambda,\omega_0^{\varepsilon'}\lambda'} = (-1)^{\varepsilon+\varepsilon'}\mathcal{N}_{\lambda,\lambda'} \end{aligned}\right\} \quad \text{conformal case} \tag{1b}$$

We now identify $\mathcal{N}$ with the matrix of an operator on N-dimensional vectors with components labelled by λ. Similarly we define two $N\times N$ unitary matrices (and operators)

$$\begin{aligned} T_{\lambda\lambda'} &= \delta_{\lambda,\lambda'}\,\mathbf{e}\left(\frac{\lambda^2}{2N}\right), \\ S_{\lambda\lambda'} &= \frac{1}{\sqrt{N}}\,\mathbf{e}\left(\frac{\lambda\lambda'}{N}\right). \end{aligned} \tag{2}$$

Here the δ symbol is understood mod N; S is the matrix of finite Fourier transform $S^2_{\lambda,\lambda'} = \delta_{\lambda,-\lambda'}$, $S^4_{\lambda,\lambda'} = \delta_{\lambda,\lambda'}$. Supplemented by the appropriate phases, and acting respectively in the even or odd subspace under $\lambda \to -\lambda$, we have seen that T and S generate the corresponding unitary representations of the modular group acting on conformal or affine characters. The phases drop out when we investigate the conditions,

$$T^\dagger \mathcal{N} T = S^\dagger \mathcal{N} S = \mathcal{N} \ , \tag{3}$$

which mean that $\mathcal{N}$ belongs to the commutant of S and T in view of unitarity. This is problem (A), where we may as well disregard the symmetry conditions (1) since they can easily be reinstated at the end, and are compatible with (3).

The clue to solve this question is provided by the requirement of commutation with T, and by the observation that elements of the commutant describe generalizations of the symmetries which were just said to be compatible with (3). Indeed $[\mathcal{N}, T]=0$ implies that off-diagonal elements of $\mathcal{N}_{\lambda,\lambda'}$ can be non-vanishing only if $\lambda^2 \equiv \lambda'^2 \bmod 2N$, which is consistent again with λ defined mod N because N is even. Thus, taking representative integers, $(\lambda'-\lambda)(\lambda'+\lambda)=2\xi N$. Apart from the obvious solutions $\lambda' \equiv \pm\lambda \bmod N$, this equation implies that $\lambda' \pm \lambda$ being of the same parity are both even, hence $\lambda'+\lambda = 2\bar{\xi}\bar{\delta}$, $\lambda'-\lambda=2\xi\delta$, with $\delta\bar{\delta}=n(=N/2)$, ξ, $\bar{\xi}$, δ, and $\bar{\delta}$ are integers, δ and $\bar{\delta}$ positive. Set $\alpha=(\delta,\bar{\delta})$, then α^2 divides n (hence N), $p=\bar{\delta}/\alpha$ and $p'=\delta/\alpha$ are coprime, and integers ϱ,σ exist such that $\varrho p - \sigma p' = 1$. (The reader will not confuse these integers with those entering the definition of c.) Defining $\omega \equiv \varrho p + \sigma p' \bmod N/\alpha^2$, we have $\omega^2 - 1 = 4\varrho\sigma p p' \equiv 0 \bmod 2N/\alpha^2$ and $\omega + 1 \equiv 2\varrho p \bmod N/\alpha^2$, $\omega - 1 \equiv 2\sigma p' \bmod N/\alpha^2$. Since $\lambda/\alpha = \bar{\xi}p - \xi p'$, $\lambda'/\alpha = \bar{\xi}p + \xi p'$, we find $\omega\lambda/\alpha \equiv \lambda'/\alpha \bmod N/\alpha^2$ or $\lambda' \equiv \omega\lambda \bmod N/\alpha$. This necessary condition is also sufficient for commutation with T. We conclude that for each divisor δ of n, $1 \leq \delta \leq n$, we can define a pair, $\alpha=(\delta, n/\delta)$ and ω such that $\omega^2 \equiv 1 \bmod 2N/\alpha^2$, and a symmetric matrix

$$\delta | n \ \rightarrow \ (\Omega_\delta)_{\lambda,\lambda'} = \begin{cases} 0 & \text{if} \quad \alpha \nmid \lambda \text{ or } \alpha \nmid \lambda' \\ \displaystyle\sum_{\xi \bmod \alpha} \delta_{\lambda', \omega\lambda + \xi N/\alpha} & \text{otherwise} \end{cases} \tag{4}$$

which commutes with T, and, as an immediate calculation shows, also with S. Among multiples of α, $\bar{\delta}$ is the smallest one left invariant mod N/α and δ the smallest one which changes sign mod N/α, and we can shift ω by a multiple of N/α^2 to make these properties hold mod N. Interchanging the roles of δ and $\bar{\delta}$ amounts simply to replacing ω by $-\omega$. For instance Ω_n corresponds to $\alpha=1$, $\omega=1$, i.e. $\Omega_n = I$, while Ω_1 corresponds to $\alpha=1$, $\omega=-1$, i.e. $(\Omega_1)_{\lambda,\lambda'} = \delta_{\lambda,-\lambda'}$. More generally Ω_δ and $\Omega_{n/\delta}$ will be linearly related when operating in the even or odd subspace with corresponding projector $(I \pm \Omega_1)/2$.

7. **Proposition 1** [7]. *The commutant of S and T is generated by the $\sigma(n)$ linearly independent operators Ω_δ.*

The thread of the argument is the following. We represent any operator as a polynomial in two basic ones obeying the simple commutation relations of finite quantum mechanics [8]. Elements of the commutant are obtained by averaging over the group $\bar{\Gamma}$ generated by S and T. This provides us with a basis $\{M_\delta\}$ which is equivalent to the set $\{\Omega_\delta\}$.

Introduce in the N-dimensional (Hilbert) space $\mathcal{H}$ of functions on $\mathbb{Z}/N\mathbb{Z}$ two unitary operators P and Q, which generate a representation of the finite Heisenberg group, through

$$(Q\psi)(\lambda)=\mathbf{e}\left(\frac{\lambda}{N}\right)\psi(\lambda) \qquad (P\psi)(\lambda)=\psi(\lambda-1) \ . \tag{5}$$

The analog of the canonical commutation relations reads

$$QP=\mathbf{e}(1/N)PQ \tag{6a}$$

and is supplemented by

$$Q^N=P^N=\mathbf{e}(1/N)^N I=I \ . \tag{6b}$$

Using Dirac's bra-ket notation $\langle\lambda|\psi\rangle\equiv\psi(\lambda)$, we have

$$Q|\lambda\rangle=\mathbf{e}(\lambda/N)|\lambda\rangle \qquad P|\lambda\rangle=|\lambda+1\rangle \ . \tag{7}$$

Polynomials in P and Q generate the full operator algebra in the form

$$M=\sum_{k,\ell \bmod N} P^k Q^\ell \frac{1}{N}\operatorname{Tr}\,(MQ^{-\ell}P^{-k}) \tag{8}$$

for any operator M, as can be shown in the case of a projector $|\lambda\rangle\langle\lambda'|$. This implies the irreducibility of the representation, and is in fact an adaptation of the Wigner representation in the continuum case. In view of (6a) we can assume a "normal ordering" with P's to the left of Q's.

The analogy with continuous quantum mechanics is pursued if we notice that S and T generate in their adjoint action the canonical group [i.e. transformations preserving (6)]

$$\begin{aligned} S^\dagger\begin{pmatrix}Q\\P\end{pmatrix}S&=\begin{pmatrix}P\\Q^{-1}\end{pmatrix},\\ T^\dagger\begin{pmatrix}Q\\P\end{pmatrix}T&=\begin{pmatrix}Q\\ \mathbf{e}(-1/2N)PQ^{-1}\end{pmatrix}. \end{aligned} \tag{9a}$$

More generally if we define the symbol

$$\{k,\ell\}\equiv\mathbf{e}(k\ell/2N)P^kQ^\ell \ , \tag{10}$$

$$S^\dagger\{k,\ell\}S=\{\ell,-k\} \ , \qquad T^\dagger\{k,\ell\}T=\{k,\ell-k\} \ . \tag{9b}$$

Clearly $\{k,\ell\}$ only depends on k,ℓ mod $2N$, but they are only independent mod N, since $\{k+aN,\ell+bN\}=(-1)^{kb-\ell n}\{k,\ell\}$.

Any automorphism $\mathscr{A}$ is a product of S's and T's acting as

$$\mathscr{A}^\dagger\{k,\ell\}\mathscr{A}=\{k',\ell'\} \qquad (k',\ell')=(k,\ell)\begin{pmatrix}a&b\\c&d\end{pmatrix}, \tag{11}$$

where the two-by-two matrix is an element of $\bar{\Gamma}^{2N}=SL(2,\mathbb{Z}/2N\mathbb{Z})$. In particular

$$S\to\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \qquad T\to\begin{pmatrix}1&-1\\0&1\end{pmatrix}. \tag{12}$$

Setting respectively $(k,\ell)=(1,0)$ or $(0,1)$, we get values of P', Q' which obviously verify (6), thus on the one hand the map $\{k,\ell\}\to\{k',\ell'\}$ is the resultant on operators of canonical transformations. On the other hand we know that the above two matrices in (12) generate $SL(2,\mathbb{Z})$. Thus we have a bijective map,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}/2N\mathbb{Z}) \leftrightarrow \mathscr{A} \ , \tag{13}$$

which allows us to identify both groups. The next step is to write Ω_δ in the Wigner form (8). Now recall that $\delta=\alpha p'$, $\bar\delta=n/\delta=\alpha p$ and $(p,p')=1$. The claim is that

$$\Omega_\delta=\frac{\delta}{n}\sum_{y,z \bmod n/\delta} P^{2\delta y}Q^{2\delta z}\,\mathbf{e}(\delta^2 yz/n) \ . \tag{14}$$

Indeed recall that $(\Omega_\delta)_{\lambda,\lambda'}$ is different from zero for

$$\lambda=x\bar\delta+y\delta \ , \quad \lambda'=x\bar\delta-y\delta \quad \bmod N$$

with $x\in\mathbb{Z}/2\delta\mathbb{Z}$, $y\in\mathbb{Z}/2\bar\delta\mathbb{Z}$, and

$$(\Omega_\delta)_{\lambda,\lambda'}=\frac{1}{2}\sum_{x \bmod 2\delta}\ \sum_{y \bmod 2\bar\delta}\delta_{\lambda,x\bar\delta+y\delta}\,\delta_{\lambda',x\bar\delta-y\delta} \ ,$$

where the factor 1/2 accounts for double counting. Inserting this in Eq. (8) yields

$$\begin{aligned}
\Omega_\delta&=\frac{1}{2N}\sum_{k,\ell \bmod N}P^kQ^\ell\sum_{\substack{x \bmod 2\delta\\ y \bmod 2\bar\delta}}\langle\bar\delta x+\delta y|Q^{-\ell}P^{-k}|\bar\delta x-\delta y\rangle\\
&=\frac{1}{2N}\sum_{k,\ell \bmod N}P^kQ^\ell\sum_{\substack{x \bmod 2\delta\\ y \bmod 2\bar\delta}}\mathbf{e}(\ell(\bar\delta x+\delta y)/N)\,\delta_{k,2\delta y \bmod N}\\
&=\frac{\delta}{N}\sum_{\substack{\ell \bmod N\\ y \bmod 2\bar\delta}}P^{2\delta y}Q^\ell\,\mathbf{e}(\ell y\delta/N)\,\delta_{\ell,0 \bmod 2\delta} \ .
\end{aligned}$$

Setting $\ell=2z\delta$, with $z \bmod \bar\delta$, we get (14). It is nice to verify, using (11) and (14), that S and T commute with Ω_δ.

To find a general element of the commutant Gepner and Qiu's idea is to average the adjoint action of the group $\bar\Gamma^{2N}=SL(2,\mathbb{Z}/2N\mathbb{Z})$ on an arbitrary element. From the representation (8) it suffices to average this adjoint action on each P^kQ^ℓ. Let $|\bar\Gamma^r|$ denote the order of $\bar\Gamma^r$. Set

$$\begin{aligned}
M'_{k,\ell}&=\frac{1}{|\bar\Gamma^{2N}|}\sum_{\bar\Gamma^{2N}}\mathscr{A}^\dagger P^kQ^\ell\mathscr{A}\\
&=\frac{1}{|\bar\Gamma^{2N}|}\sum_{\bar\Gamma^{2N}}\mathbf{e}\left(\frac{abk^2+cd\ell^2+2bck\ell}{2N}\right)P^{ak+c\ell}Q^{bk+d\ell} \ ,
\end{aligned}\tag{15}$$

where $\mathscr{A}\leftrightarrow\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\bar\Gamma^{2N}$. The second expression follows from the definitions (10) and (11). Clearly any element of the commutant is a linear combination of the $M'_{k,\ell}$

with k and ℓ ranging mod N (i.e. $M'_{k+\xi N,\ell+\xi' N}=M'_{k,\ell}$). Furthermore $M'_{k\ell}$ vanishes if k and ℓ are not both even. Indeed the kernel of the map $\bar{\Gamma}^{2N}\to\bar{\Gamma}^N$ is given by the eight matrices

$$\begin{pmatrix} 1+\alpha N & \beta N \\ \gamma N & 1+\alpha N \end{pmatrix} \quad \alpha,\beta,\gamma=0,1 \ .$$

Averaging (15) over this invariant subgroup leaves $M'_{k\ell}$ invariant and multiplies the coefficient of $P^{ak+c\ell}Q^{bk+d\ell}$ by

$$\frac{1}{8}\sum_{\alpha,\beta,\gamma\in\mathbb{Z}/2\mathbb{Z}} \mathbf{e}\left[\frac{\beta}{2}(ak+c\ell)^2+\frac{\gamma}{2}(bk+d\ell)^2\right] .$$

This is non-vanishing only if both $k'=ak+c\ell$ and $\ell'=bk+d\ell$ are even. The linear transformation is invertible mod 2, hence k and ℓ are also even. Thus in (15) we may as well assume the indices even which allows us to write $M_{k,\ell}=M'_{2k,2\ell}$ as

$$M_{k,\ell}=\frac{1}{|\bar{\Gamma}^n|}\sum_{\bar{\Gamma}^n}\mathbf{e}\left(\frac{abk^2+cd\ell^2+2bck\ell}{n}\right)P^{2(ak+c\ell)}Q^{2(bk+d\ell)} \ , \tag{16}$$

where the structure of the formula has in fact reduced the average to $\bar{\Gamma}^n$, and k and ℓ are defined mod n. It also follows from the preceding that for any element in $\bar{\Gamma}^n$,

$$M_{k,\ell}=\mathbf{e}\left(\frac{abk^2+cd\ell^2+2bck\ell}{n}\right)M_{ak+c\ell,\,bk+d\ell} \ . \tag{17}$$

Pick representatives k and ℓ in the range 1 to n, let $d=(k,\ell)$; then m_1 and m_2 exist such that $m_1k+m_2\ell=d$. The matrix

$$\begin{pmatrix} \ell/d & m_1 \\ -k/d & m_2 \end{pmatrix}$$

is unimodular,

$$(k,\ell)\begin{pmatrix} \ell/d & m_1 \\ -k/d & m_2 \end{pmatrix}=(0,d) \ , \qquad M_{k,\ell}=\mathbf{e}\left(\frac{-k\ell}{n}\right)M_{0,d} \ .$$

The process can be repeated by introducing $\delta=(n,d)=(k,\ell,n)$,

$$M_{k,\ell}=\mathbf{e}\left(\frac{-k\ell}{n}\right)M_{0,\delta} \ .$$

We conclude that $\sigma(n)$ linearly independent operators $M_\delta\equiv M_{0,\delta}$ can be defined, labelled by divisors of n, $1\leqq\delta\leqq n$,

$$M_\delta=\frac{1}{|\bar{\Gamma}^n|}\sum_{\bar{\Gamma}^n}P^{2\delta c}Q^{2\delta d}\,\mathbf{e}\left(\frac{\delta^2 cd}{n}\right) \ , \tag{18}$$

generating the commutant.

Linear independence follows from the fact that two distinct divisors δ and δ' have disjoint orbits under $\bar{\Gamma}^n$ mod n. Indeed if δ and δ' divisors of n in the range 1 to n were on the same orbit, we would have

$$c\delta=\xi n \ , \qquad d\delta=\delta'+\xi' n \ .$$

Thus, since $\delta|n$, δ' would be a multiple of δ and interchanging their roles, δ a multiple of δ', an impossibility if δ is distinct from δ'.

It is now easy to complete the proof. We can split Ω_δ as a sum,

$$\Omega_\delta = \frac{\delta}{n} \sum_{y,z=1}^{n/\delta} M_{\delta(y,z,n/\delta)} , \tag{19}$$

using the fact that $\mathscr{A}^\dagger \Omega_\delta \mathscr{A} = \Omega_\delta$ for any element of $\bar{\Gamma}^{2N}$. [Recall that (x, y, z) is the greatest common divisor of x, y, z.] This relation is of a triangular form, $\Omega_\delta = \sum_{\delta|\delta'} A_{\delta\delta'} M_{\delta'}$ with $A_{\delta\delta'} = \frac{\delta\delta'}{n^2} |\bar{\Gamma}^{n/\delta'}|$. It can therefore be inverted to yield M_δ in terms of $\Omega_{\delta'}$, so that the Ω's as well as the M's can be used to generate the commutant, thus completing the proof of Proposition 1.

When n is prime, $\sigma(n) = 2$, and the representation splits into two irreducible ones acting on the even or odd subspaces.

IV. A-D-E Classification

8. In this section we derive the classification of partition functions in both the affine and conformal cases. The difficulty stems in each case from the oddness property under multiplication by Ω_1, or Ω_{ω_0} respectively, if we keep the convention to write the invariants in terms of χ_λ, $\lambda \bmod N$.

We first look at the *affine case*, writing Z as

$$Z(\tau) = \frac{1}{2} \sum_{\lambda, \lambda' \bmod N} \chi_\lambda^*(\tau) \left(\sum_{\delta|n} c_\delta \Omega_\delta \right)_{\lambda\lambda'} \chi_{\lambda'}(\tau) . \tag{1a}$$

Recall that $\chi_{\xi n} = 0$. Divide the integers mod N different from zero mod n into two disjoint sets: U and L with representatives lying respectively in the intervals $1 \leqq \lambda \leqq n-1$ and $n+1 \leqq \lambda \leqq 2n-1$. Therefore $L \equiv -U \bmod N$, and a fundamental domain $\mathscr{B}$ is U. We have $(\Omega_\delta \chi)_\lambda = (\Omega_{n/\delta}\chi)_{-\lambda} = -(\Omega_{n/\delta}\chi)_\lambda$ and $(\Omega_\delta)_{\lambda\lambda'} = (\Omega_\delta)_{-\lambda, -\lambda'}$. This allows us for each factorization $n = \delta\bar{\delta}$ to replace in (1a) $c_\delta \Omega + c_{\bar{\delta}} \Omega_{\bar{\delta}}$ by $(c_\delta - c_{\bar{\delta}})\Omega_\delta$ or $(c_{\bar{\delta}} - c_\delta)\Omega_{\bar{\delta}}$. We use whichever of the two combinations has a non-negative coefficient, and rewrite Z as

$$\begin{aligned} Z(\tau) &= \sum_{\substack{\lambda \in U \\ \lambda' \bmod N}} \chi_\lambda^*(\tau) \left(\sum_{\delta|n} c_\delta \Omega_\delta \right)_{\lambda, \lambda'} \chi_{\lambda'}(\tau) \\ &= \sum_{\lambda, \lambda' \in U} \chi_\lambda^*(\tau) \left\{ \sum_{\delta|n} c_\delta [(\Omega_\delta)_{\lambda,\lambda'} - (\Omega_\delta)_{\lambda, -\lambda'}] \right\} \chi_{\lambda'}(\tau) \end{aligned} \tag{1b}$$

with $c_\delta \geqq 0$, and $c_\delta > 0$ implying $c_{n/\delta} = 0$. The coefficient of $\chi_1^* \chi_1$ should be one. But only Ω_n and Ω_1 contribute to it, and with the above conventions $c_1 = 0$, $c_n = 1$. The matrices Ω have non-negative integral coefficients. We want to ensure that this is also true for the matrix within curly brackets in the second expression (1b). Using the previous conventions, we have for the required solutions the following result, announced in [3] as a conjecture.

Proposition 2. *For the affine partition functions the following set of possibilities is exhaustive:*

$$
\begin{array}{lll}
n\geqq 2 & \Omega_n & (A_{n-1}) \\
n \text{ even} \geqq 6 & \Omega_n+\Omega_2 & (D_{n/2+1}) \\
n=12 & \Omega_{12}+\Omega_3+\Omega_2 & (E_6) \\
n=18 & \Omega_{18}+\Omega_3+\Omega_2 & (E_7) \\
n=30 & \Omega_{30}+\Omega_5+\Omega_3+\Omega_2 & (E_8)
\end{array}
\tag{2}
$$

In (2) we give the combination $\sum c_\delta \Omega_\delta$ occurring in (1). We have two infinite series, labelled A and D, and three exceptional cases, labelled E. The index on A, D or E is the rank of the corresponding simple Lie algebra. The correspondence is clarified in Table 1, which gives the expanded form of the partition function. The coefficient of the terms $\chi_\lambda^* \chi_\lambda$, $1\leqq\lambda\leqq n-1$, is the multiplicity of λ in the list of the Coxeter exponents for the corresponding algebra, and n is its Coxeter number.

The A series starts at $n=2$, for which the only character is the trivial one $\chi_1=1$. Similarly the D series starts with $n=6$. When $n=4$, the corresponding formula yields the same result as the A_3 invariant.

It is explicit in Table 1 that the above set of partition functions fulfills all requirements. What we shall now show is that the list is exhaustive.

The Ω_δ's occurring in (2), apart from $\Omega_n\equiv I$, have indices with very low prime values 2, 3 or 5. This is reminiscent of the orders of the prime cyclic subgroups of the rotation symmetry groups of regular solids. There exists a close connection between simply laced simple Lie algebras and finite subgroups of SU_2 up to conjugation. We return to this point in the next section.

9. The proof of Proposition 2 is constructive and involves three steps, Lemmas 1–3.

We set aside the case of the principal invariant with $c_\delta=0$ except $c_n=1$. For each divisor $\delta|n$, $1<\delta<n$, such that $c_\delta>0$, we define $\alpha(\delta)=(\delta, n/\delta)$ and $\omega(\delta)$ as before,

Table 1. List of affine partition functions in terms of $A_1^{(1)}$ characters

$n\geqq 2$	$\sum_{\lambda=1}^{n-1} \lvert\chi_\lambda\rvert^2$	A_{n-1}	Ω_n
$n=4\varrho+2$, $\varrho\geqq 1$	$\sum_{\substack{\lambda \text{ odd}=1\\ \lambda\neq 2\varrho+1}}^{4\varrho+1} \lvert\chi_\lambda\rvert^2+2\lvert\chi_{2\varrho+1}\rvert^2+\sum_{\lambda\text{ odd}=1}^{2\varrho-1}(\chi_\lambda\chi_{4\varrho+2-\lambda}^*+\text{c.c.})$ $=\sum_{\lambda \text{ odd}=1}^{2\varrho-1}\lvert\chi_\lambda+\chi_{4\varrho+2-\lambda}\rvert^2+2\lvert\chi_{2\varrho+1}\rvert^2$	$D_{2\varrho+2}$	$\Omega_n+\Omega_2$
$n=4\varrho$ $\varrho\geqq 2$	$\sum_{\lambda\text{ odd}=1}^{4\varrho-1}\lvert\chi_\lambda\rvert^2+\lvert\chi_{2\varrho}\rvert^2+\sum_{\lambda\text{ even}=2}^{2\varrho-2}(\chi_\lambda\chi_{4\varrho-\lambda}^*+\text{c.c.})$	$D_{2\varrho+1}$	$\Omega_n+\Omega_2$
$n=12$	$\lvert\chi_1+\chi_7\rvert^2+\lvert\chi_4+\chi_8\rvert^2+\lvert\chi_5+\chi_{11}\rvert^2$	E_6	$\Omega_{12}+\Omega_3+\Omega_2$
$n=18$	$\lvert\chi_1+\chi_{17}\rvert^2+\lvert\chi_5+\chi_{13}\rvert^2+\lvert\chi_7+\chi_{11}\rvert^2+\lvert\chi_9\rvert^2$ $+[(\chi_3+\chi_{15})\chi_9^*+\text{c.c.}]$	E_7	$\Omega_{18}+\Omega_3+\Omega_2$
$n=30$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2+\lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	E_8	$\Omega_{30}+\Omega_5+\Omega_3+\Omega_2$

such that $\omega^2(\delta) \equiv 1 \bmod 2N/\alpha^2$. Consider in (1b) the factor of χ_1^*

$$\chi_1 + \sum_{\delta, \alpha(\delta)=1} c_\delta \chi_{\omega(\delta)} ,$$

arising from those δ's such that $\alpha(\delta)=1$ and $\omega(\delta) \in (\mathbb{Z}/N\mathbb{Z})^*$, which for $1<\delta<n$ are all distinct from ± 1. It follows from our previous requirements that $\omega(\delta) \in U$. Indeed if $\omega(\delta) \in L$, $c_\delta \chi_{\omega(\delta)} = -c_\delta \chi_{-\omega(\delta)}$ would be a negative contribution to Z, and there would be needed a δ', with $\alpha(\delta')=1$ and $\omega(\delta') = -\omega(\delta) = \omega$, such that $(c_{\delta'} - c_\delta)\chi_\omega$ is a positive contribution to Z, i.e. $c_{\delta'} - c_\delta > 0$. But $\alpha = 1$, $\omega' = -\omega$ means $\delta' = n/\delta$, and this was excluded by convention. We conclude that all the $\omega(\delta)$'s such that $\alpha(\delta)=1$ have to belong to U, the corresponding coefficients being positive integers. Similar reasoning will recur frequently.

The identity relating affine characters for levels $k=n-2$ and $k' = \frac{n}{\alpha^2} - 2$, $\alpha^2 | n$, [3]

$$\sum_{\xi \bmod \alpha} \chi_{\alpha\lambda + \xi N/\alpha}(\tau; N) = \alpha \chi_\lambda(\tau; N/\alpha^2) , \qquad \lambda \bmod N/\alpha^2 \tag{3}$$

shows that both sides vanish if $n=\alpha^2$, since $\chi_\lambda(\tau; N=2)=0$. Hence in this case $(\Omega_\alpha \chi)_\lambda = 0$, and the corresponding term may be disregarded in (1).

Define $\alpha_{\min}$ as Inf $\{\alpha(\delta); \delta \neq n, c_\delta > 0\}$. We have

Lemma 1 (i) $\alpha_{\min} = 1$ *or* 2, (ii) *if* $\alpha_{\min} = 2$, *the unique possible partition function corresponds to* $\Omega_n + \Omega_2$ $(n \equiv 0 \bmod 4)$.

It is useful to represent geometrically the integers mod N on a circle of radius $N/2\pi$ as regularly spaced at distance 1. The upper semi-circle represents U, the lower one L. For $\alpha(\delta) = \alpha > 1$, the points $\lambda' = \omega\lambda + \xi N/\alpha$ are the vertices of a regular polygon with α edges and vertices, in short an α-gon. It is clear that if $\alpha \geq 4$ at least two vertices belong to $L \cup \{0, n\}$, one of them being certainly in L. These negative contributions to Z have to be compensated by positive ones. Consider the factor of $\chi^*_{\alpha_{\min}}(\tau)$ in Z. By definition of $\alpha_{\min}$, only Ω_δ's such that $\alpha(\delta) = \alpha_{\min}$ contribute and the factor is

$$\chi_{\alpha_{\min}}(\tau) + \sum_{\delta, \alpha(\delta) = \alpha_{\min}} c_\delta \sum_{\xi \bmod \alpha_{\min}} \chi_{\omega\alpha_{\min} + \xi N/\alpha_{\min}} .$$

Suppose first $\alpha_{\min} \geq 4$. Then each α-gon involves at least two terms in L which have to be compensated. The case where only one term is in L would require $\alpha_{\min} = 4$, and a set of indices $\omega\alpha_{\min} + \xi N/\alpha_{\min}$ ranging over 0, $n/2$, n, $3n/2 \bmod N$ $(n \equiv 0 \bmod 4^2)$. Since ω is invertible mod $n/4$ $(=2N/\alpha^2_{\min})$, $\alpha^2_{\min} = 4^2$ is a multiple of n, the only possibility being $n = \alpha^2_{\min}$, the case discarded by Eq. (3).

Since each α-gon in the above sum has at least two terms in L, and the coefficient of $\chi_{\alpha_{\min}}(\tau)$ is 1, two possibilities of cancellations are open: (i) the negative terms are compensated by positive ones of the same α-gon; (ii) a negative term pertaining to δ is compensated by a positive one from a δ' contribution. In case (ii) the corresponding multipliers ω and ω' have to satisfy

$$\omega' \alpha_{\min} \equiv -\omega \alpha_{\min} \quad \bmod N/\alpha_{\min} ,$$

or equivalently $\omega' \equiv -\omega \bmod N/\alpha^2_{\min}$. But this is excluded since it would imply $\delta\delta' = n$. In case (i) we have $2\omega \equiv 0 \bmod N/\alpha^2_{\min}$, hence $2 \equiv 0 \bmod N/\alpha^2_{\min}$, and this is possible only for $n=\alpha_{\min}^2$, $\alpha(\delta)=\alpha_{\min}$, also excluded. Thus if $\alpha_{\min}$ were to be larger or equal to 4, one would find negative coefficients in the factor of $\chi_{\alpha_{\min}}^*$.

If $\alpha_{\min}=3$, beyond the two previous cases, which as above are excluded, there exists a third one with a 3-gon (an equilateral triangle) with a unique point in L compensated by $\chi_{\alpha_{\min}}\equiv\chi_3$ with the c_δ coefficient being 1. But then $\omega\equiv-1$ mod $N/9$, so that the corresponding contribution is $\chi_{-3}+\chi_{-3+N/3}+\chi_{-3-N/3}$ and $\pm N/3-3\in U\cup\{0,n\}$. Thus $3+2n/3\geqq n$, meaning $9\geqq n$ and since $9|n$, $n=9$. Again we find an excluded case: $n=\alpha_{\min}^2$, $\alpha(\delta)=\alpha_{\min}$. This proves part (i) of the lemma.

Assume now $\alpha_{\min}=2$, and consider the coefficient of χ_2^*. This is

$$\chi_2+\sum_{\delta,\alpha(\delta)=2} c_\delta(\chi_{2\omega}+\chi_{2\omega+n}) \; .$$

The two points 2ω and $2\omega+n$ can be in three sets of positions. In the first one, these are the points $\pm n/2$ mod N, i.e. $4\omega=n$ mod N. This is the excluded possibility $n=4$. The second possibility is that $2\omega=0$ mod n, excluded by $\omega^2\equiv1$ mod n. The configuration where terms from δ and δ' compensate each other is excluded as before. The only remaining case is that there exists a unique δ, such that $\alpha(\delta)=2$, $c_\delta=1$, and by choosing the representative ω mod $N/4$, we have $2\omega=-2$ mod N, $2\omega+n\in U$, meaning that the negative term $\chi_{2\omega}$ is compensated by χ_2. Hence $\delta=2$, $n=4k$, $\omega\equiv-1$ mod $2k$.

The partition function contains $\Omega_n+\Omega_2$ plus a sum over δ's such that $\alpha(\delta)\geqq3$. For $\lambda\in U$, $(\Omega_n+\Omega_2)\chi_\lambda$ is equal to χ_λ if λ is odd and to $\chi_{n-\lambda}$ for λ even. To discuss the occurrence of other Ω's we can retrace the steps of the proof of part (i), replacing the contribution of Ω_n by the one of $\Omega_n+\Omega_2$, which amounts to replacing χ_λ by $\chi_{n-\lambda}$ if λ is even in the factor of χ_λ^*. The same arguments exclude any $\alpha(\delta)\geqq3$, and proves part (ii) of the lemma.

Lemma 2. *If n is odd the unique possibility is Ω_n.*

Assume on the contrary that there exist additional possibilities. Since n is odd, we know from Lemma 1 that $\alpha_{\min}=1$. Let us show that this leads to a contradiction. Consider the coefficient of $\chi_{2^\gamma}^*$ with $2^\gamma<n$. Since n is odd the only contributions are from δ's such that $\alpha(\delta)=1$, which by hypothesis must be present. The coefficient reads

$$\chi_{2^\gamma}+\sum_{\alpha(\delta)=1} c_\delta\chi_{\omega 2^\gamma} \; .$$

We know already that all these ω's $\in U$, and choose representatives $0<\omega<n$. Let us show that $2^\gamma\omega$ has also to belong to U. Suppose on the contrary that some δ is such that $2^\gamma\omega\in L$. Then the corresponding negative contribution has to be compensated by some ω' (possibly 1), requiring $2^\gamma(\omega+\omega')\equiv0$ mod N. Let first $\gamma=1$, thus $\omega+\omega'$ is smaller than N and $2(\omega+\omega')$ smaller than $2N$. This leads to $2(\omega+\omega')=N$, i.e. $\omega+\omega'=n$. But ω and ω' are odd, by $\omega^2\equiv1$ mod $2N$, thus $\omega+\omega'$ is even while n is odd. We conclude that $\omega<n/2$. The argument can be iterated. For instance if $4<n$, we cannot have $4\omega\in L$. Again an ω' would be needed for compensation, and $4(\omega+\omega')\equiv0$ mod N. By the previous bound, $0<4(\omega+\omega')<2N$, so that the only possibility is $2(\omega+\omega')=n$, where the right-hand side is odd, leading to a contradiction. Hence ω is smaller than $n/4$. Thus for any γ such that $2^\gamma\in U$, $2^\gamma\omega\in U$, and a positive representative is smaller than $n/2^\gamma$. Let now γ be such that $2^\gamma < n < 2^{\gamma+1}$, pick $0<\omega<n/2^\gamma$, and recall that if $\bar{\delta}=n/\delta$, $\omega\bar{\delta}\equiv\bar{\delta} \bmod N$. But δ is a divisor of n larger than 2, since n is odd. Hence $\bar{\delta}<2^\gamma\left(\frac{1}{\delta/2}\right)<2^\gamma$ and $\omega\bar{\delta}<\frac{n}{2^\gamma}2^\gamma=n$. Thus $\omega\bar{\delta}\equiv\bar{\delta} \bmod N$ means $\omega\bar{\delta}=\bar{\delta}$, $\omega=1$ contrary to the assumption that it corresponds to a divisor $\delta>2$. The lemma is proved.

The two previous lemmas restrict the search of further non-trivial solutions to the cases n even and $\alpha_{\min}=1$. We assume in the sequel n even, and will study the effect of multiplication by ω's, such that $\omega^2\equiv 1 \bmod 2N$ $(\alpha=1)$, $\omega\in U$, on integers λ belonging to $U^*=(\mathbb{Z}/N\mathbb{Z})^*\cap U$. Let also $L^*=(\mathbb{Z}/N\mathbb{Z})^*\cap L$. The following is in fact the crucial arithmetical observation.

Lemma 3. *Let n be even, $n\neq 12$ and 30, $N=2n$, $\omega\in U^*$, $\omega^2\equiv 1 \bmod 2N$, $\omega\neq 1$, and $\omega\neq n-1$ if $n=2 \bmod 4$, then there exists $\lambda\in U^*$ such that $\omega\lambda\in L^*$.*

For each such ω there is an associated factorization of $n=\delta\bar{\delta}$, $(\delta,\bar{\delta})=1$. The following pairs are excluded by hypothesis: $\{n,1\}$ and $\{1,n\}$ ($\omega=1$ or $\omega=-1\notin U^*$) for any n even; $\{2,n/2\}$ for $n=2 \bmod 4$ ($\alpha=1$, $\omega=n-1$); $\{3,4\}$ for $n=12$; $\{2,15\}$, $\{3,10\}$, $\{5,6\}$ for $n=30$. The cases to consider are $2<\delta<\bar{\delta}<n$ or $2<\bar{\delta}<\delta<n$.

We search a representative of $\lambda\in U^*$ in the range $0<\lambda<n$, such that a representative λ' of $\omega\lambda$ is in the range $-n<\omega\lambda<0$. Since δ and $\bar{\delta}$ are coprimes, we look for λ and λ' in the form

$$\begin{cases}\lambda\ =\mu\bar{\delta}+\varrho\delta\\ \lambda'=\mu\bar{\delta}-\varrho\delta\end{cases}\tag{4}$$

with $0<\mu<\delta$, $0<\varrho<\bar{\delta}$, μ and ϱ prime respectively to δ and $\bar{\delta}$. Since n is even, one in the pair δ, $\bar{\delta}$ is even, the other odd. The above conditions imply λ prime to δ and $\bar{\delta}$, hence to $n=\delta\bar{\delta}$, hence to $N=2n$ since n is even. Requiring $0<\lambda<n$ and $-n<\lambda'<0$ yields

$$\begin{cases}0<\dfrac{\mu}{\delta}+\dfrac{\varrho}{\bar{\delta}}<1\\[2ex] -1<\dfrac{\mu}{\delta}-\dfrac{\varrho}{\bar{\delta}}<0\end{cases}.\tag{5a}$$

As $\frac{\mu}{\delta}$ and $\frac{\varrho}{\bar{\delta}}$ should be positive irreducible fractions, smaller than one, the two lower bounds are irrelevant. If a solution μ, ϱ exists, then a fortiori so does the solution $1,\varrho$. Thus it is sufficient to look for ϱ in the range $0<\varrho<\bar{\delta}$, prime to $\bar{\delta}$, such that

$$\frac{1}{\delta}<\frac{\varrho}{\bar{\delta}}<1-\frac{1}{\delta}\,.\tag{5b}$$

It is easy to convince oneself that no such ϱ exists in the excluded cases of the lemma. In all other cases we exhibit a solution ϱ. We distinguish several possibilities:

(i) $2<\bar{\delta}<\delta<n=\delta\bar{\delta}$, set $\varrho=1$ $\left(\text{Eq. (5b) holds since } \frac{1}{\bar{\delta}}<\frac{1}{2}<1-\frac{1}{\delta}\right)$.

(ii) $2<\delta<\bar{\delta}<n$, δ is even. Then $\bar{\delta}$ is odd, of the form $\bar{\delta}=1+2k$, $k>1$. Set $\varrho=k=\frac{\bar{\delta}-1}{2}$. Clearly $(\varrho,\bar{\delta})=1$. Since $\bar{\delta}>\delta>2$, Eq. (5b) is satisfied.

(iii) $2<\delta<\bar{\delta}<n$, δ odd, $\bar{\delta}=0 \bmod 4$. Choose $\varrho=\frac{\bar{\delta}}{2}-1$ for $\bar{\delta}>4$.

The case $\bar{\delta}=4$, hence $\delta=3$ is one of the excluded possibilities, then $\bar{\delta}=4k$, $k>1$, $\varrho=2k-1>1$. Any common factor of ϱ and $\bar{\delta}$ would have to divide $\bar{\delta}-2\varrho=2$, hence, ϱ being odd, we conclude $(\varrho,\bar{\delta})=1$ and $\varrho/\bar{\delta}<1$. Since δ is odd larger than 2, $\delta\geqq 3$, so $\frac{1}{\delta}\leqq\frac{1}{3}$ and $1-1/\delta\geqq 2/3$, while $\frac{\varrho}{\bar{\delta}}=\frac{1}{2}-\frac{1}{4k}$ is for $k>1$ bounded by $\frac{1}{2}-\frac{1}{8}\leqq\frac{\varrho}{\bar{\delta}}<\frac{1}{2}$, i.e. $\frac{1}{\delta}\leqq\frac{1}{3}<\frac{3}{8}\leqq\frac{\varrho}{\bar{\delta}}<\frac{1}{2}<\frac{2}{3}\leqq 1-\frac{1}{\delta}$.

(iv) $2<\delta<\bar{\delta}<n$, δ odd, $\bar{\delta}=2 \bmod 4$. The case where $\bar{\delta}=6$ and δ odd, prime to $\bar{\delta}$ in the interval $2<\delta<\bar{\delta}$, requires $\delta=5$ which is excluded by hypothesis. Thus $\bar{\delta}>6$. If $\bar{\delta}=10$ the possible δ's are 3, 7, 9. The pair $\delta=3$, $\bar{\delta}=10$ is excluded by hypothesis. For $\delta=7$ or 9, $\varrho=3$ is a solution. We can now assume $\bar{\delta}=2k$, k odd$\geqq 7$. Take $\varrho=k-2$ odd. Again any common divisor of ϱ and $\bar{\delta}$ divides 4 and since ϱ is odd this common divisor has to be 1, thus $(\varrho,\bar{\delta})=1$ and $0<\varrho/\bar{\delta}<1$. Now $1/\delta\leqq 1/3$, $1-1/\delta\geqq 2/3$. On the other hand $\frac{\varrho}{\bar{\delta}}<\frac{1}{2}<1-\frac{1}{\delta}$, while the condition $\varrho/\bar{\delta}>\frac{1}{3}\geqq\frac{1}{\delta}$ means $3k-6>2k$, i.e. $k>6$ which is the case. The lemma is proved.

10. We complete the proof of Proposition 2. For any n larger than 2 (recall that $n=2$ is the trivial case with a unique $\chi_1=1$), the choice Ω_n leads to the principal invariant (type A) and is the unique possibility for n odd according to Lemma 2. We then look for additional terms Ω_δ with $\omega\neq\pm 1$, when n is even $\geqq 4$. By Lemma 1, these additional terms are such that $\alpha_{\min}=\operatorname{Inf}\alpha(\delta)=1$ or 2. If $\alpha_{\min}=2$ the only possibility is $\Omega_n+\Omega_2$ (and $n\equiv 0 \bmod 4$). Thus we are left with the case where n is even and $\alpha_{\min}=1$.

Consider the coefficient of $\chi_\lambda^*(\tau)$ in (1b) for any $\lambda\in U^*$. Only those Ω_δ's such that $\alpha(\delta)=1$ contribute, and by hypothesis some of them occur with a positive coefficient (in which case $\bar{\delta}=n/\delta$ does not occur). To those δ's correspond ω's which all must have the property that for every $\lambda\in U^*\to\omega\lambda\in U^*$ (we include $\omega=1$, corresponding to Ω_n). Indeed if $\omega\lambda\in L^*$, then for positivity, another ω' must occur, such that $\omega'\lambda\in U^*$ and $\omega\lambda+\omega'\lambda\equiv 0 \bmod N$. Since λ is invertible mod N by hypothesis, this requires $\omega+\omega'=0 \bmod N$, or $\delta\delta'=n$, a case excluded by construction. Lemma 3 controls this property. We study in turn $n\equiv 0$ or 2 mod 4.

(i) $n\equiv 0 \bmod 4$. If $n\neq 12$, all terms Ω_δ, $\delta\neq n$ with $\alpha_{\min}=1$ are excluded. Then $\alpha_{\min}=2$, $\Omega_n+\Omega_2$ is the only non-trivial possibility (D type). For $n=12$ an additional solution $\Omega_{12}+\Omega_3+\Omega_2$ can be found by inspection (E_6 type).

(ii) $n\equiv 2 \bmod 4$, $\alpha_{\min}$ cannot be 2 ($2^2\nmid n$), hence for a non-trivial solution $\alpha_{\min}=1$. According to Lemma 3, if $n\neq 30$ the only possible additional term with $\alpha(\delta)=1$ is Ω_2 ($\omega=n-1$). Its coefficient has to be unity [if we look say at the coefficient of $\chi_2^*(\tau)$, i.e. $(\Omega_n+c_2\Omega_2)\chi_2(\tau)=(1-c_2)\chi_2(\tau)$]. Thus beside Ω_n (A type) and $\Omega_n+\Omega_2$ (D type), we have in this case, if $n\neq 30$, as only further possibilities $\Omega_n+\Omega_2+\sum_{\alpha(\delta)\geqq 3} c_\delta\Omega_\delta$, where all α's are odd.

We are going to show that for $n \neq 18$, all c_δ's must vanish. The proof parallels the one of Lemma 1. Let again $\tilde{\alpha}_{\min} \geqq 3$ be the lowest possible value among those occurring in the additional terms. Look at the coefficient of $\chi^*_{\tilde{\alpha}_{\min}}(\tau)$, where the first term contributes $\chi_{\tilde{\alpha}_{\min}}(\tau)+\chi_{n-\tilde{\alpha}_{\min}}(\tau)$, and only those Ω_δ's with $\alpha(\delta)=\tilde{\alpha}_{\min}$ will yield further contributions. Recall that the corresponding index of χ will range over the vertices of a regular polygon with $\tilde{\alpha}_{\min}$ vertices. As in Lemma 1, indices λ belonging to L from these polygons cannot be compensated by the positive ones from the same polygon or from those of another polygon (with the same number $\tilde{\alpha}_{\min}$ of vertices).

This leaves as the only non-trivial possibility a unique polygon (a unique Ω_δ), the negative terms being compensated by $\tilde{\alpha}_{\min}$ and $n-\tilde{\alpha}_{\min}$ with $\tilde{\alpha}_{\min}=3$ or 5 (with at most two vertices in L). The coefficient of Ω_δ has to be one. Let ω correspond to δ. If the polygon has two vertices in L, we must have

$$\begin{aligned} \omega\tilde{\alpha}_{\min}+\xi N/\tilde{\alpha}_{\min} &\equiv -\tilde{\alpha}_{\min} \bmod N\ , \\ \omega\tilde{\alpha}_{\min}+(\xi-1)N/\tilde{\alpha}_{\min} &\equiv -(n-\tilde{\alpha}_{\min}) \bmod N \end{aligned}$$

for some $\xi \bmod \tilde{\alpha}_{\min}$. Set $n=2\tilde{\alpha}^2_{\min}q \geqq 18$, and subtract both terms, obtaining $N/\tilde{\alpha}_{\min}=n-2\tilde{\alpha}_{\min}+\varrho N$ for some integer ϱ, i.e. $q[(2\varrho+1)\tilde{\alpha}_{\min}-2]=1$. Thus $q=1$ and $\tilde{\alpha}_{\min}=3$, excluding $\tilde{\alpha}_{\min}=5$. If $\tilde{\alpha}_{\min}=3$, it is not possible for the equilateral triangle to have a single term in L compensated either by $\tilde{\alpha}_{\min}=3$ or $n-3$ as in the proof of Lemma 1. The only possibility left is therefore $n=18$.

The only exceptional cases are $n=18$ and $n=30$, which are readily studied separately, with the result quoted in Proposition 2. This concludes the main proof of this paper.

11. Similar results hold in the conformal case. For the proof we refer to [3, 4]. It is simpler to return to the original notation, where the conformal characters are labelled by two integers r, s mod $2p'$ and $2p$ respectively [and $(p,p')=1$], with the appropriate symmetries. Then the role of $N=2n$ in the affine case is now played by a pair $\{2p', 2p\}$.

Proposition 3. (i) *Acting on the conformal characters, the general elements of the commutant are equivalent to tensor products $\Omega_{\delta'} \otimes \Omega_\delta$ in an obvious notation where $\delta'|p', \delta|p$.*

(ii) *As a result the conformal partition functions can be specified by a pair of elements in Table 1, where the role of n is played by p' and p. Since p and p' are coprime, one of them is odd, the corresponding invariant being of the A-type. This yields two infinite series and three exceptional pairs of models.*

These partition functions are reproduced in Table 2. As special case we have the unitary series, with p and p' replaced by two consecutive integers [21].

The method extends to minimal supersymmetric conformal theories, presented in detail in reference [14]. For completeness Table 3 gives the corresponding exhaustive list of positive integral invariants for the unitary theories.

Similarly Gepner and Qiu have applied the affine invariants to parafermionic theories [15, 7].

Table 2. List of partition functions in terms of conformal characters. The unitary series corresponds to $p'=m-1$, $p=m$ or $p=m-1$, $p'=m$, $m=3,4,\ldots$

	$\frac{1}{2}\sum_{r=1}^{p'-1}\sum_{s=1}^{p-1}\lvert\chi_{rs}\rvert^2$	$(A_{p'-1},A_{p-1})$
$p'=4\varrho+2$, $\varrho\geqq 1$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{\substack{r\ \mathrm{odd}=1\\ r\neq 2\varrho+1}}^{4\varrho+1}\lvert\chi_{rs}\rvert^2+2\lvert\chi_{2\varrho+1\ s}\rvert^2+\sum_{r\ \mathrm{odd}=1}^{2\varrho-1}(\chi_{rs}\chi^*_{p'-r\ s}+\mathrm{c.c.})\right\}$	$(D_{2\varrho+2},A_{p-1})$
$p'=4\varrho$, $\varrho\geqq 2$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{r\ \mathrm{odd}=1}^{4\varrho-1}\lvert\chi_{rs}\rvert^2+\lvert\chi_{2\varrho\ s}\rvert^2+\sum_{r\ \mathrm{even}=2}^{2\varrho-2}(\chi_{rs}\chi^*_{p'-r\ s}+\mathrm{c.c.})\right\}$	$(D_{2\varrho+1},A_{p-1})$
$p'=12$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1s}+\chi_{7s}\rvert^2+\lvert\chi_{4s}+\chi_{8s}\rvert^2+\lvert\chi_{5s}+\chi_{11s}\rvert^2\}$	(E_6,A_{p-1})
$p'=18$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1s}+\chi_{17s}\rvert^2+\lvert\chi_{5s}+\chi_{13s}\rvert^2+\lvert\chi_{7s}+\chi_{11s}\rvert^2+\lvert\chi_{9s}\rvert^2$ $+[(\chi_{3s}+\chi_{15s})\chi^*_{9s}+\mathrm{c.c.}]\}$	(E_7,A_{p-1})
$p'=30$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1s}+\chi_{11s}+\chi_{19s}+\chi_{29s}\rvert^2+\lvert\chi_{7s}+\chi_{13s}+\chi_{17s}+\chi_{23s}\rvert^2\}$	(E_8,A_{p-1})

Table 3. List of unitary superconformal partition functions: χ, $\tilde{\chi}$ are characters of highest weight representations of the superconformal algebra, in the Neveu-Schwarz sector; $\tilde{\chi}$ includes minus signs for fermionic descendants. $\hat{\chi}$ are characters in the Ramond sector. The indices (p',p) are $(m,m+2)$, or $(m+2,m)$ by exchanging $r\leftrightarrow s$. A detailed description is given in [14]

$m\geqq 3$	$\frac{1}{4}\sum_{\substack{r=1\\ r-s\ \mathrm{even}}}^{m-1}\sum_{s=1}^{m+1}\{\lvert\chi_{rs}\rvert^2+\lvert\tilde{\chi}_{rs}\rvert^2\}+\frac{1}{4}\sum_{\substack{r=1\\ r-s\ \mathrm{odd}}}^{m-1}\sum_{s=1}^{m+1}\lvert\hat{\chi}_{rs}\rvert^2$	(A_{m-1},A_{m+1})
$p=4\varrho+2$, $\varrho\geqq 1$	$\frac{1}{4}\sum_{\substack{r=1\\ \mathrm{odd}}}^{p'-1}\left\{\sum_{\substack{s=1\\ \mathrm{odd}}}^{2\varrho-1}\lvert\chi_{r,s}+\chi_{r,p-s}\rvert^2+2\lvert\chi_{r,2\varrho+1}\rvert^2+(\chi\to\tilde{\chi})\right\}$ $+\frac{1}{4}\sum_{\substack{r=2\\ \mathrm{even}}}^{p'-2}\left\{\sum_{\substack{s=1\\ \mathrm{odd}}}^{2\varrho-1}n\,\lvert\hat{\chi}_{r,s}+\hat{\chi}_{r,p-s}\rvert^2+2\lvert\hat{\chi}_{r,2\varrho+1}\rvert^2\right\}$	$(A_{p'-1},D_{2\varrho+2})$
$p=4\varrho$, $\varrho\geqq 2$	$\frac{1}{4}\sum_{\substack{s=1\\ \mathrm{odd}}}^{p-1}\sum_{\substack{r=1\\ \mathrm{odd}}}^{p'-1}(\lvert\chi_{r,s}\rvert^2+\lvert\tilde{\chi}_{rs}\rvert^2)+\frac{1}{4}\sum_{\substack{r=2\\ \mathrm{even}}}^{p'-2}\left\{\lvert\chi_{r,2\varrho}\rvert^2+\sum_{\substack{r=2\\ \mathrm{even}}}^{2\varrho-2}(\chi_{rs}\chi^*_{p'-r\ s}+\mathrm{c.c.})+(\chi\to\tilde{\chi})\right\}$ $+\frac{1}{4}\sum_{\substack{r=2\\ \mathrm{even}}}^{p'-2}\sum_{\substack{s=1\\ \mathrm{odd}}}^{p-1}\lvert\hat{\chi}_{rs}\rvert^2+\frac{1}{4}\sum_{\substack{r=1\\ \mathrm{odd}}}^{p'-1}\left\{\lvert\hat{\chi}_{r,2\varrho}\rvert^2+\sum_{\substack{s=2\\ \mathrm{even}}}^{2\varrho-2}(\hat{\chi}_{rs}\chi^*_{p'-r,s}+\mathrm{c.c.})\right\}$	$(A_{p'-1},D_{2\varrho+1})$
$p=12$	$\frac{1}{4}\sum_{\substack{r=1\\ \mathrm{odd}}}^{p'-1}\{\lvert\chi_{r1}+\chi_{r7}\rvert^2+\lvert\chi_{r5}+\chi_{r11}\rvert^2+(\chi\to\tilde{\chi})\}$	

Table 3 (continued)

	$+\frac{1}{4}\sum_{\substack{r=2\\ \text{even}}}^{p'-2}\{|\chi_{r4}+\chi_{r8}|^2+(\chi\to\tilde{\chi})\}$	
	$+\frac{1}{4}\sum_{\substack{r=2\\ \text{even}}}^{p'-2}\{|\hat{\chi}_{r1}+\hat{\chi}_{r7}|^2+|\hat{\chi}_{r5}+\hat{\chi}_{r11}|^2\}+\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}|\hat{\chi}_{r4}+\hat{\chi}_{r8}|^2$	$(A_{p'-1}, E_6)$
$p=12$	$\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\{|\chi_{r1}+\chi_{r5}+\chi_{r7}+\chi_{r11}|^2+|\tilde{\chi}_{r1}+\tilde{\chi}_{r5}+\tilde{\chi}_{r7}+\tilde{\chi}_{r11}|^2$	
	$+2|\hat{\chi}_{r4}+\hat{\chi}_{r8}|^2\}$	$(D_{p'/2+1}, E_6)$
$p=18$	$\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\{|\chi_{r1}+\chi_{r17}|^2+|\chi_{r5}+\chi_{r13}|^2+|\chi_{r7}+\chi_{r11}|^2+|\chi_{r9}|^2$	
	$+[(\chi_{r3}+\chi_{r15})\chi_{r9}^*+\text{c.c.}]\}$	
	$+\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\{\chi\to\tilde{\chi}\}+\frac{1}{4}\sum_{\substack{r=2\\ \text{even}}}^{p'-2}\{\chi\to\hat{\chi}\}$	$(A_{p'-1}, E_7)$
$p=30$	$\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\{|\chi_{r1}+\chi_{r11}+\chi_{r19}+\chi_{r29}|^2+|\chi_{r7}+\chi_{r13}+\chi_{r17}+\chi_{r23}|^2\}$	
	$+\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\{\chi\to\tilde{\chi}\}+\frac{1}{4}\sum_{\substack{r=2\\ \text{even}}}^{p'-2}\{\chi\to\hat{\chi}\}$	$(A_{p'-1}, E_8)$

V. Miscellanea

12. Some of the invariants listed in Tables 1 and 2 are related, as a consequence of the triviality of lowest characters (Jacobi's and Euler's identities)

$$\chi_1^{\text{aff}}(\tau; N=4)=\eta^{-3}(\tau)\sum_{t=-\infty}^{+\infty}(4t+1)\,\mathbf{e}\left(\tau\,\frac{(4t+1)^2}{8}\right)=1\ , \tag{1a}$$

$$\chi_1^{\text{conf}}(\tau; N=12)=\eta^{-1}(\tau)\sum_{t=-\infty}^{+\infty}(-1)^t\,\mathbf{e}\left(\tau\,\frac{(6t+1)^2}{24}\right)=1\ , \tag{1b}$$

in conjunction with the formulae

$$\sum_{\xi \bmod \alpha}\chi_{\alpha(\lambda+\xi N)}^{\text{aff}}(\tau; N\alpha^2)=\alpha\chi_\lambda^{\text{aff}}(\tau; N)\ , \tag{2a}$$

$$\sum_{\xi \bmod \alpha}\chi_{\alpha(\lambda+\xi N)}^{\text{conf}}(\tau; N\alpha^2)=\chi_\lambda^{\text{conf}}(\tau; N)\ . \tag{2b}$$

Taking $\lambda=1$, $N=4$, 12 in (2a) and (2b) respectively, yields

$$\sum_{\ell=0}^{\alpha-1}\chi_{(4\ell+1)\alpha}^{\text{aff}}(\tau; 4\alpha^2)=\alpha\ , \tag{3a}$$

$$\sum_{\ell=0}^{\alpha-1}\chi_{(12\ell+1)\alpha}^{\text{conf}}(\tau; 12\alpha^2)=1\ . \tag{3b}$$

The only cases relevant here are obtained for low values of α,

$$\begin{array}{c} k=6 \\ (N=16) \end{array} \quad \chi_2^{\mathrm{aff}}-\chi_6^{\mathrm{aff}}=2\;, \quad Z_{A_7}^{\mathrm{aff}}-Z_{D_5}^{\mathrm{aff}}=4\;, \tag{4a}$$

$$\begin{array}{c} k=16 \\ (N=36) \end{array} \quad \chi_3^{\mathrm{aff}}+\chi_{15}^{\mathrm{aff}}-\chi_9^{\mathrm{aff}}=3\;, \quad Z_{D_{10}}^{\mathrm{aff}}-Z_{E_7}^{\mathrm{aff}}=9\;, \tag{4b}$$

$$\begin{array}{c} p=8,\ p'=3 \\ (N=48) \end{array} \quad \chi_2^{\mathrm{conf}}-\chi_{10}^{\mathrm{conf}}=1\;, \quad Z_{A_7,A_2}^{\mathrm{conf}}-Z_{D_5,A_2}^{\mathrm{conf}}=1\;. \tag{4c}$$

For any higher value of α, it may be seen that the identities (3) relate invariants with indefinite signs.

In view of such linear relations one may at first think that our classification is redundant. Let us stress on the contrary, that we have classified all modular invariant sesquilinear forms in the characters, and that it is non-trivial that a constant can be expressed in such a form. The physical interpretation of Eq. (4) is more obscure. They imply that the spectra of eigenvalues of $L_0, \bar{L}_0$ in the models of type D_{10} and E_7, for instance, differ by nine copies of the state with $h=\bar{h}=c/24 =k/8(k+2)=1/9$. These remain to be understood in terms of concrete realizations.

13. Should we have expected to find an A-D-E classification? What is the precise relation with Lie algebras? Unfortunately we have no clear answer to this question. Amazingly there exist other A-D-E classifications, such as the one of discrete subgroups of SU_2 [6], isomorphic to factor groups of the modular group. Hence it is tempting to study in some detail the representations of the latter afforded by the combinations occurring in the partition functions.

Consider in particular the three exceptional affine partition functions corresponding to $n=12, 18, 30$. The corresponding SU_2 subgroups are (the covering groups of) the tetrahedral group ($\mathscr{A}_4$ the alternate group of permutations on four objects), the octahedral group ($\mathscr{S}_4$), the icosahedral group ($\mathscr{A}_5$).

We start with the E_6 case.

$$Z_{E_6}=|\chi_1+\chi_7|^2+|\chi_5+\chi_{11}|^2+|\chi_4+\chi_8|^2\;. \tag{5}$$

Call the successive combinations y_1, y_2, y_3. They can be parametrized, using the Dedekind function as

$$y_1+y_2=\chi_1+\chi_7+\chi_5+\chi_{11}=\left\{q^{-1/48}\prod_1^\infty(1+q^{n-1/2})\right\}^5=\left\{\frac{\eta(\tau/2)}{\eta(\tau)}\right\}^5\;,$$

$$y_1-y_2=\chi_1+\chi_7-\chi_5-\chi_{11}=\left\{q^{-1/48}\prod_1^\infty(1-q^{n-1/2})\right\}^5=\left\{\mathbf{e}\left(\frac{-1}{48}\right)\frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}\right\}^5\;,$$

$$y_3=\chi_4+\chi_8=4\left\{q^{1/24}\prod_1^\infty(1+q^n)\right\}^5=4\left\{\frac{\eta(2\tau)}{\eta(\tau)}\right\}^5\;, \tag{6}$$

so that we can also write

$$Z_{E_6}=\frac{1}{2|\eta(\tau)|^{10}}\left\{\left|\eta\left(\frac{\tau}{2}\right)\right|^{10}+\left|\eta\left(\frac{\tau+1}{2}\right)\right|^{10}+|\sqrt{2}\,\eta(2\tau)|^{10}\right\}, \tag{7}$$

exhibiting in a straighforward fashion modular invariance, since the three terms are permuted under a modular transformation. To prove (6) use the fact that both sides transform identically, and that their ratio is bounded in the upper imaginary τ plane. It is easier to express the action of the modular group on the combinations

$$\begin{aligned} u &= \mathbf{e}(5/48)\,\frac{y_1+y_2}{\sqrt{2}}=\frac{1}{\sqrt{2}}\,\mathbf{e}(5/48)(\chi_1+\chi_7+\chi_5+\chi_{11}) , \\ v &= \frac{y_1-y_2}{\sqrt{2}}=\frac{1}{\sqrt{2}}(\chi_1+\chi_7-\chi_5-\chi_{11}) , \\ w &= y_3=\chi_4+\chi_8 , \end{aligned} \tag{8}$$

in which case under S and $R=TS$ we have

$$S \quad \begin{aligned} u&\to u \\ v&\to w , \\ w&\to v \end{aligned} \qquad R=TS \quad \begin{aligned} u&\to w \\ v&\to \mathbf{e}(-5/24)u \\ w&\to \mathbf{e}(5/24)v . \end{aligned} \tag{9a}$$

It is clear that these generate permutations up to 24th roots of unity on u, v, w and that $Z_{E_6}=|u|^2+|v|^2+|w|^2$, as well as $uvw=2\,\mathbf{e}(5/48)$ are invariant. Moreover $S^2=R^3=1$ is obvious. Remarkably, if we look at the action of the modular group, not on u, v, w but on their 12th powers, we find that

$$S \quad \begin{aligned} u^{12}&\to u^{12} \\ v^{12}&\to w^{12} , \\ w^{12}&\to v^{12} \end{aligned} \qquad R \quad \begin{aligned} u^{12}&\to w^{12} \\ v^{12}&\to -u^{12} . \\ w^{12}&\to -v^{12} \end{aligned} \tag{9b}$$

These transformations can be interpreted on the vector with coordinates u^{12}, v^{12}, and w^{12}, as a reflection (determinant $=-1$) in a plane through the first axis and at 45 degeees to the 2 and 3 axis for S, and as a rotation of $2\pi/3$ (determinant $=+1$) around an axis with coordinates $(1,-1,1)$. Take a cube with center at the origin, with vertices $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$; $\varepsilon_1=\pm 1$. Inscribe a regular tetrahedron with a subset of these vertices such that $\varepsilon_1\varepsilon_2\varepsilon_3=+1$. Then (9b) generates the group $\mathscr{S}_4$ of permutations of these four points, the full symmetry group of the tetrahedron including reflections. The invariant subgroup of proper rotations, the tetrahedral group $\mathscr{A}_4$, is generated by transformations containing an even number of S's. Symmetric functions of 24th powers of u, v, w are rational function of $j(\tau)$, the invariant modular function. For instance,

$$u^{24}+v^{24}+w^{24}=-2^{-8}\{15j^2+50\cdot 16^2 j+3\cdot 16^4) , \tag{10}$$

which follows from the fact that the equation

$$X^3-j^{1/3}X+16=0 \tag{11a}$$

has for its three solutions

$$\begin{aligned} X_1 &= -q^{-1/6} \prod_1^\infty (1+q^{n-1/2})^8 , \\ X_2 &= q^{-1/6} \prod_1^\infty (1-q^{n-1/2})^8 , \\ X_3 &= 16 q^{1/3} \prod_1^\infty (1+q^n)^8 , \\ 2^{-12}(X_1^{15} &+ X_2^{15} + X_3^{15}) = u^{24}+v^{24}+w^{24} . \end{aligned} \tag{11b}$$

In any case we see some loose connection between Z_{E_6} and the tetrahedral group $\mathscr{A}_4$ isomorphic to $PSL\ (2, \mathbb{Z}/3\mathbb{Z})$.

In the E_7 case, $n=18$, the partition function is in fact related to the D_{10} one as in (4b)

$$Z_{E_7} = |\chi_1+\chi_{17}|^2 + |\chi_5+\chi_{13}|^2 + |\chi_7+\chi_{11}|^2 + |\chi_9|^2 + \{\chi_9^*(\chi_3+\chi_{15})+cc\} = Z_{D_{10}} - 9 . \tag{12}$$

If we set

$$Z_{E_7} = 3Z-3 , \quad Z_{D_{10}} = 3Z+6 , \tag{13a}$$

then

$$Z = \left|1+\frac{1}{3}\frac{\eta^3\left(\frac{\tau}{9}\right)}{\eta^3(\tau)}\right|^2 + \left|1+\frac{1}{3}\frac{\eta^3\left(\frac{\tau+1}{9}\right)}{\eta^3(\tau+1)}\right|^2 + \left|1+\frac{1}{3}\frac{\eta^3\left(\frac{\tau+2}{9}\right)}{\eta^3(\tau+2)}\right|^2 + \left|1+9\frac{\eta^3(9\tau)}{\eta^3(\tau)}\right|^2 . \tag{13b}$$

Define

$$\begin{aligned} y_\ell &= \frac{1}{3}\mathbf{e}\left(\frac{1-\ell}{9}\right)\left\{(\chi_1+\chi_{17}) + \mathbf{e}\left(\frac{\ell-1}{3}\right)(\chi_5+\chi_{13}) - \mathbf{e}\left(\frac{2(\ell-1)}{3}\right)(\chi_7+\chi_{11})\right\} \\ &= 1+\frac{1}{3}\frac{\eta^3\left(\frac{\tau+\ell-1}{9}\right)}{\eta^3(\tau+\ell-1)} , \qquad \ell = 1,2,3 , \\ y_4 &= 1+\chi_9 = 1+9\frac{\eta^3(9\tau)}{\eta^3(\tau)} . \end{aligned} \tag{14}$$

The action of the modular group becomes

$$S \quad \begin{matrix} y_1 \to y_4 \\ y_2 \to \mathbf{e}(1/3)y_3 \\ y_3 \to \mathbf{e}(-1/3)y_2 \\ y_4 \to y_1 \end{matrix} , \qquad T \quad \begin{matrix} y_1 \to y_2 \\ y_2 \to y_3 \\ y_3 \to \mathbf{e}(-1/3)y_1 \\ y_4 \to y_4 \end{matrix} . \tag{15}$$

We see that on the cubes y_ℓ^3, S and T generate the tetrahedral group $\mathscr{A}_4$ again, if we let y_ℓ^3 correspond to the four vertices of a regular tetrahedron. In this case we fail to find a correspondence with the octahedral group. Symmetric functions in the y_ℓ^3 are modular invariant, and expressible in terms of j.

Finally we look at the E_8 case, $n=30$.

$$Z_{E_8}=|\chi_1+\chi_{11}+\chi_{19}+\chi_{29}|^2+|\chi_7+\chi_{13}+\chi_{17}+\chi_{23}|^2 \ . \tag{16}$$

Set

$$y_1=\chi_1+\chi_{11}+\chi_{19}+\chi_{29} \ , \qquad y_2=\chi_7+\chi_{13}+\chi_{17}+\chi_{23} \ , \tag{17}$$

where the indices run over $(\mathbb{Z}/30\mathbb{Z})^*$ in two groups.

Since under T

$$y_1\to\mathbf{e}\left(\frac{1}{120}-\frac{1}{8}\right)y_1 \ , \qquad y_2\to\mathbf{e}\left(\frac{49}{120}-\frac{1}{8}\right)y_2 \tag{18}$$

T^5 acts as multiplication by $\mathbf{e}(5/12)$. It is a theorem [17] for such a low power that the conjugates of T^5 generate all of Γ_5, and therefore $\Gamma^5=PSL(2,\mathbb{Z}/5\mathbb{Z})$ acts on the ratio of the two y's. This group is in fact isomorphic to the icosahedral group, itself isomorphic to $\mathscr{A}_5$.

Indeed with

$$z(\tau)=\frac{y_2(\tau)}{y_1(\tau)} \qquad \omega=\mathbf{e}(1/5) \ , \qquad u=\omega+\omega^{-1}=\frac{\sqrt{5}-1}{2} \ , \tag{19}$$

$$\begin{aligned} T \quad & z(\tau+1)=\omega^2 z(\tau) \ , \\ S \quad & z(-\tau^{-1})=\frac{-uz(\tau)+1}{z(\tau)+u} \ . \end{aligned} \tag{20}$$

If as in Klein [13] we introduce the "isobaric" polynomials in z (i.e. polynomials which under a modular transformation are multiplied by a power of the denominator)

$$\begin{aligned} V&=z(z^{10}-11z^5-1) \ , \\ E&=z^{30}+1-522(z^{25}-z^5)-10005(z^{20}+z^{10}) \ , \\ F&=z^{20}+1+228(z^{15}-z^5)+494z^{10} \end{aligned} \tag{21}$$

of respective degrees 12 (adding ∞ as a root of V), 30 and 20, their zeroes on the Riemann sphere are the vertices, the mid-edge points, and the mid-face points of a regular icosahedron. The following identity holds

$$F^3=E^2+1728\,V^5 \ . \tag{22}$$

Thus in this case, there is a definite indication of a relation to the icosahedral group. The ratios E^2/F^3 and V^5/F^3 are modular invariants related to j.

We would describe further examples, without giving a neat solution to the problem raised at the beginning of this section. It is therefore left as an open question to unravel the connection between integrable and/or critical two dimensional field theories, simple Lie algebras and finite rotation groups.

References

1. Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B.: Nucl. Phys. B**241**, 333 (1984)
2. Cardy, J.L.: Nucl. Phys. B**270** [FS16], 186 (1986)
3. Cappelli, A., Itzykson, C., Zuber, J.B.: Nucl. Phys. B**280** [FS18], 445 (1987)

4. Gepner, D.: Nucl. Phys. B**287**, 111 (1987)
5. Gepner, D., Witten, E.: Nucl. Phys. B**278**, 493 (1986)
6. Slodowy, P.: In: Algebraic geometry. Dolgachev, J. (ed.), Lecture Notes in Mathematics, Vol. 1008. Berlin, Heidelberg, New York: Springer 1983, and further references therein
7. Gepner, D., Qiu, Z.: Nucl. Phys. B**285** [FS19], 423 (1987)
8. Balian, R., Itzykson, C.: C. R. Acad. Sci. Paris **303**, 773 (1986)
9. Itzykson, C., Zuber, J.B.: Nucl. Phys. B**275** [FS17], 580 (1986)
10. Zuber, J.B.: Proceedings of/"Symposium on topological and geometrical methods in field theory," Espoo, Finland, June 1986. Hietarinta, J., Westerholm, J. (eds.). Singapore: World Scientific 1987
11. Kac, V.G.: Private communication
12. Pasquier, V.: Nucl. Phys. B**285** [FS19], 162 (1987); J. Phys. A**20**, L217, L221 (1987) Saclay preprints S. Pht 87/014, 062
13. Klein, F.: The icosahedron and the solution of equations of the fifth degree, New York: Dover 1956
14. Cappelli, A.: Modular invariant partition functions of superconformal theories. Phys. Lett. **185**, 82 (1987)
    Kastor, D.: Nucl. Phys. B**280** [FS18], 304 (1987)
    Matsuo, Y., Yahikozawa, S.: Superconformal field theory with modular invariance on a torus. Phys. Lett. B**178**, 211 (1987)
15. Fateev, V.A., Zamolodchikov, A.B.: Zh. E. T. F. **89**, 380 (1985) (Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in Z_N-symmetric statistical systems. Soviet Phys. JETP **62**, 215 (1985))
16. Bouwknegt, P., Nahm, W.: Realizations of the exceptional modular invariant $A_1^{(1)}$ partition functions. Phys. Lett. **184**B, 359 (1987)
    Bernard, D.: Private communication
    Bouwknegt, P.: Amsterdam preprint ITFA-87-05
    Bowcock, P., Goddard, P.: Nucl. Phys. B**285** [FS19], 651 (1987)
17. Gunnings, R.C.: Lectures on Modular forms. Princeton, NJ: Princeton University Press 1962
    Magnus, W.: Non-Euclidean tesselations and their groups. New York: Academic Press 1974
    Knopp, M.I.: Modular functions in analytic number theory. Chicago: Markham 1970
    Coxeter, H.S.M., Moser, W.O.J.: Generators and relations for discrete groups. Berlin, Heidelberg, New York: Springer 1965
18. Feigin, B.L., Fuchs, D.B.: Funkts. Anal. Prilozh. **17**, 91 (1983) (Funct. Anal. Appl. **17**, 241 (1983)), and Moscow preprint; In: Topology. Proceedings of Leningrad Conference, 1982. Faddeev, L.D., Mal'cev, A.A. (eds.) Lecture Notes in Mathematics, Vol. 1060. Berlin, Heidelberg, New York: Springer 1984
19. Rocha-Caridi, A.: In: Vertex operators in mathematics and physics. Leposwky, J., Mandelstam, S., and Singer, I. (eds.). Berlin, Heidelberg, New York: Springer 1985
20. Dobrev, V.K.: Trieste preprint IC/86/123
21. Friedan, D., Qiu, Z., and Shenker, S.: Phys. Rev. Lett. **52**, 1575 (1984) and In: Vertex operators in mathematics and physics. Op. cit., Commun. Math. Phys. **107**, 535 (1986)
22. Kac, V.G., Peterson, D.: Adv. Math. **53**, 125 (1984)
    Kac, V.G.: Infinite dimensional Lie algebras. London: Cambridge University Press 1985
23. Zuber, J.B.: Discrete symmetries of conformal theories. Phys. Lett. **176**B, 127 (1986)
    Cardy, J.: Nucl. Phys. B**275**, [FS17], 200 (1986)

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