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THE ABUNDANCE OF WILD HYPERBOLIC SETS AND NON-SMOOTH STABLE SETS FOR DIFFEOMORPHISMS

by SHELDON E. NEWHOUSE [1]

1. A fundamental problem in dynamical systems is to describe the orbit structures of a large set of diffeomorphisms of a compact manifold M. We write Diff^r M for the space of C^r diffeomorphisms of M with the uniform C^r topology and we assume dim $M>1$. The largest open set in Diff^r M whose orbit structures are well understood in the set of Ω-stable diffeomorphisms. For f in Diff^r M, a point x in M is *non-wandering* if for every neighborhood U of x in M, there is an integer $n>0$ so that $f^n(U)\cap U\neq\emptyset$. The set of non-wandering points of f is denoted $\Omega(f)$, and one says that f is Ω-stable if whenever g is close to f in Diff^r M, there is a homeomorphism $h:\Omega(f)\to\Omega(g)$ so that $gh=hf$. One would like to know how the set of Ω-stable diffeomorphisms sits in Diff^r M.

In an earlier paper [14], we described an open set of non Ω-stable diffeomorphisms of the two-dimensional sphere. Later [16], we showed how to use these mappings to give diffeomorphisms of any compact manifold with infinitely many periodic sinks. In fact, we gave residual subsets of open sets of diffeomorphisms each of whose elements has infinitely many sinks. For each of these diffeomorphisms the closure of the set of sinks is quite complicated, containing many different closed invariant infinite sets with dense orbits. The general structure of these sets remains to be described.

The most important property shared by these diffeomorphisms is the persistent presence of tangencies between the stable and unstable manifolds of an invariant hyperbolic set. We will call such a hyperbolic set *wild* (this will be made precise later). The main point of this paper is to show that wild hyperbolic sets occur quite frequently, and hence, that understanding them is of basic importance in the theory of dynamical systems.

Theorem 1. — Let M^2 *be a compact* C^∞ *two-dimensional manifold, and let* $r\geq 2$. *Assume* $f\in\text{Diff}^r M^2$ *has a hyperbolic basic set whose stable and unstable manifolds are tangent at some point* x. *Then* f *may be* C^r *perturbed into an open set* $U\subset\text{Diff}^r M^2$ *so that each* g *in* U *has a wild hyperbolic set near the orbit of* x.

Tangencies of the stable and unstable manifolds of hyperbolic basic sets as in theorem 1 occur quite naturally in isotopies passing from one diffeomorphism to another.

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For general examples, we refer to [19], [24], and [20]. More specifically, we note that recent numerical studies of Liénard's equations with periodic forcing terms show that such tangencies occur there [12]. Thus, the results of this paper describe some of the motions governed by those equations.

Theorem 1 has several interesting ramifications. While it was previously known that the set $\mathscr{A}$ of Ω-stable diffeomorphisms was not dense in Diff^r M, it seemed for a long time that rather strong conditions were needed to give open sets in $\mathrm{Diff}^r\,\mathrm{M}-\mathscr{A}$. In addition to the examples in [14], such open sets were given by Abraham and Smale [1], Shub [11], and Simon [21]. In each case one began with a special diffeomorphism in $\mathscr{A}$ and carefully modified it to move into the complement of the closure of $\mathscr{A}$. Theorem 1 shows that at least on surfaces very mild conditions yield such open sets after small perturbation.

In another direction, recall that in [18], [19], a general bifurcation theory is described for arcs of diffeomorphisms $\{f_t\}$, $0\leq t\leq 1$, with f_0 Morse-Smale. Under the assumptions that at the first bifurcation point b, the limit set of f_b consists of finitely many orbits and has an equidimensional cycle, it is shown [Theorem (4.1), 19] that, generically, given $\varepsilon>0$, there exist $\delta>0$ and a set $\mathrm{B}_\delta\subset[b, b+\delta)$ whose Lebesgue measure is less than $\varepsilon\delta$, such that f_t is structurally stable for t in $[b, b+\delta)-\mathrm{B}_\delta$. Also, it was conjectured that the set of t's in $[b, b+\delta)$ with f_t not structurally stable has measure zero. It follows from theorem 1 that this conjecture is false. In fact, a generic arc of C^2 diffeomorphisms $\{f_t\}$, $0\leq t\leq 1$, of M^2 containing an equidimensional cycle at $t=t_0$ in $(0, 1)$ will have the property that there are open intervals $(\alpha_i, \beta_i)\subset(0, 1)$ such that $t_0<\alpha_i<\beta_i$, $\beta_i\to t_0$ as $i\to\infty$, and f_t is not Ω-stable for $\alpha_i<t<\beta_i$. This, together with examples in [19], shows that open sets of non-Ω-stable diffeomorphisms occur even near the boundary of the Morse-Smale diffeomorphisms.

Since wild hyperbolic sets occur in many instances, it is important to develop a structure theory for diffeomorphisms containing them. The first remark in this direction, which follows from [16] (see [16 *a*] also), is that there is a residual set $\mathscr{B}\subset\mathrm{Diff}^r(\mathrm{M}^2)$ such that if $f\in\mathscr{B}$ and Λ is a wild hyperbolic set for f, then each point of Λ is a limit of infinitely many sinks or sources. In a subsequent paper we will develop a stable manifold theory and symbolic dynamics for certain non-hyperbolic invariant sets near wild hyperbolic sets. Here we will be content with showing that wild hyperbolic sets give rise to non-smooth stable sets.

Recall that if $x\in\mathrm{M}$ and d is a topological distance on M, one defines the stable set of x, $\mathrm{W}^s(x)=\mathrm{W}^s(x,f)$, to be the set of points y in M such that $d(f^n x, f^n y)\to 0$ as $n\to\infty$. The unstable set $\mathrm{W}^u(x,f)$ is defined to be $\mathrm{W}^s(x,f^{-1})$.

These sets play a fundamental role in the orbit structure of f. When f satisfies Axiom A it is known that each stable set is an injectively immersed submanifold of M which is diffeomorphic to a Euclidean space [9]. This led S. Smale to ask if the stable sets were smooth manifolds for a residual set of f's [25]. Our second theorem answers this question negatively.

Theorem 2. — *Suppose* U *is an open set in* $\mathrm{Diff}^r\, M^2$, $r \geq 2$, *so that each* f *in* U *has a wild hyperbolic set. Then there is a dense open set* $U_1 \subset U$ *so that each* f *in* U_1 *has a point whose stable set is locally the product of a Cantor set and an interval.*

One may ask if there are large sets of diffeomorphisms on any manifold whose stable sets are locally the product of a Cantor set and a disk whenever they fail to be smooth manifolds. In particular, is this true for a residual set or a set of full measure [17]? It is also interesting to ask whether the stable set of a point has positive measure if and only if it contains a periodic sink.

The main results of this paper deal with diffeomorphisms of 2-manifolds. It seems to us that theorems 1 and 2 are valid in higher dimensions, if one makes the obvious change in theorem 2 that the stable sets be locally the product of a Cantor set and a disk. However, the proofs involve several technical complications and will not be given here. On the other hand, the reader should notice that the standard method of embedding a two-disk normally hyperbolically in an n-disk (see [16]) enables one to get open sets of diffeomorphism with wild hyperbolic sets in any dimension greater than 1.

2. In this section and the next one we will prove theorem 1. First we give some definitions.

Recall that if $f: M \to M$ is a C^r diffeomorphism of a compact manifold M, $r \geq 1$, then a compact f-invariant set Λ is called *hyperbolic* if there are a continuous splitting $T_\Lambda M = E^s \oplus E^u$, a Riemann norm $|\,.\,|$ on TM, and a constant $0 < \lambda < 1$ so that for $x \in \Lambda$

(1) $$T_x f(E^s_x) = E^s_{fx}, \qquad T_x f(E^u_x) = E^u_{fx}$$

(2) $$|T_x f | E^s_x| < \lambda, \qquad |T_x f^{-1} | E^u_x| < \lambda.$$

Here $T_x f: T_x M \to T_{fx} M$ is the derivative of f at x. The hyperbolic set is called a *hyperbolic basic set* if there is a compact neighborhood U of Λ in M such that $\bigcap_{n \in \mathbf{Z}} f^n(U) = \Lambda$ and $f|\Lambda$ has a dense orbit. It can be proved ([2], [15], [4]) that for such a Λ, the periodic points of $f|\Lambda$ are dense in Λ. Also, for each $x \in \Lambda$, the stable set $W^s(x,f)$ is a copy of a Euclidean space C^r injectively immersed in M and tangent to E^s_x at x [9]. A similar statement holds for $W^u(x,f)$. For $\varepsilon > 0$, let

$$W^s_\varepsilon(x) = W^s_\varepsilon(x,f) = \{y \in M : d(f^n x, f^n y) \leq \varepsilon \text{ for } n \geq 0\},$$

and let $$W^u_\varepsilon(x) = W^u_\varepsilon(x,f) = \{y \in M : d(f^n x, f^n y) \leq \varepsilon \text{ for } n \leq 0\}.$$

If ε is small, then $W^u_\varepsilon(x)$ and $W^s_\varepsilon(x)$ are embedded disks in M as defined in the next paragraph with $W^u_\varepsilon(x) \subset W^u(x)$ and $W^s_\varepsilon(x) \subset W^s(x)$. We set

$$W^u(\Lambda) = W^u(\Lambda, f) = \bigcup_{x \in \Lambda} W^u(x,f),$$

$$W^s(\Lambda) = W^s(\Lambda, f) = \bigcup_{x \in \Lambda} W^s(x,f),$$

$$W^u_\varepsilon(\Lambda) = W^u_\varepsilon(\Lambda, f) = \bigcup_{x \in \Lambda} W^u_\varepsilon(x,f),$$

and $$W^s_\varepsilon(\Lambda) = W^s_\varepsilon(\Lambda, f) = \bigcup_{x \in \Lambda} W^s_\varepsilon(x,f).$$

The symbol $O(x) = O(x,f)$ will denote the orbit of x.

Given a positive integer σ, let B^σ be the closed unit ball in the Euclidean space $\mathbf{R}^\sigma$. A C^r embedding $\varphi: B^\sigma \to M$ will be called a C^r *σ-disk* or just a σ-disk. A disk in M is a σ-disk for some σ or a C^r embedding $\varphi: B^s \times B^u \to M$ where $s+u = \dim M$. Sometimes we will speak of a disk D or (D, φ) in M where D is the set $\varphi(B^\sigma)$ or $\varphi(B^s \times B^u)$; *i.e.* the image of φ. Given the disk $\varphi: B^s \times B^u \to M$, $D = \varphi(B^s \times B^u)$, define

$$\partial_s D = \varphi(B^s \times \partial B^u) \quad \text{and} \quad \partial_u D = \varphi(\partial B^s \times B^u)$$

where ∂B^σ is the boundary of B^σ. Also, for $\varphi: B^\sigma \to M$, $\varphi(B^\sigma) = D$, set $\partial D = \varphi(\partial B^\sigma)$.

Let (D^u, φ^u) and (D^s, φ^s) be a C^2 u-disk and a C^2 s-disk in M with $s+u = \dim M$. Suppose $z \in (D^u - \partial D^u) \cap (D^s - \partial D^s)$. We say that D^s and D^u have a *non-degenerate tangency* at z if there are C^2 coordinates $(x, y) = (x_1, \ldots, x_s, y_1, \ldots, y_u)$ near z with

$$D^s = \{(x, y) : y = 0\}$$

and a curve $t \mapsto \gamma(t)$ for t in an interval I about 0 such that:

(1) $\gamma(0) = z$;

(2) $\gamma(t) \in D^u$ for t in I;

(3) $0 \neq \gamma'(0)$ and $T_z D^s \cap T_z D^u$ is the one-dimensional subspace of $T_z M$ spanned by $\gamma'(0)$;

(4) $\gamma''(0) \neq 0$ and $\gamma''(0) \notin T_z D^s \cap T_z D^u$.

Here $\gamma'(0)$ and $\gamma''(0)$ are the first and second derivatives of γ at 0.

A non-degenerate tangency is a point of order one contact of D^s and D^u which is not of order two contact. It is the special case of a quasi-transversal intersection [19] of two submanifolds with complementary dimensions.

If z is a non-degenerate tangency of D^s and D^u, then the coordinates

$$(x, y) = (x_1, \ldots, x_s, y_1, \ldots, y_u)$$

above may actually be chosen so that $\gamma(t) = (x_1(t), \ldots, x_s(t), y_1(t), \ldots, y_u(t))$ where $x_1(t) = t$, $y_i(t) = 0 = x_j(t)$ for $i \geq 2$ and $j \geq 2$ and $y_1(t) = at^2$ with $a \neq 0$. From this one sees that if D_1^s and D_1^u are C^2 disks which are C^2 near D^s and D^u respectively, then near z, $D_1^s \cap D_1^u$ is either empty, a single non-degenerate tangency, or two transversal intersections.

If N_1 and N_2 are manifolds with $\dim N_1 + \dim N_2 = \dim M$, and $\varphi_1: N_1 \to M$, $\varphi_2: N_2 \to M$ are injective immersions, let us agree that $\varphi_1(N_1)$ and $\varphi_2(N_2)$ have a non-degenerate tangency at $z \in \varphi_1(N_1) \cap \varphi_2(N_2)$ if there are disks $D_1 \subset \varphi_1(N_1)$ and $D_2 \subset \varphi_2(N)$ with such a tangency at z.

Let Λ be a hyperbolic basis set for a C^r diffeomorphism $f: M \to M$ with $r \geq 2$ fixed and let U be a compact neighborhood of Λ with $\bigcap_n f^n(U) = \Lambda$. For g C^r near f, $\Lambda(g) = \bigcap_n g^n(U)$ is a hyperbolic basic set for g and there is a homeomorphism $h: \Lambda(f) \to \Lambda(g)$ with $gh = hf$. A non-degenerate tangency z of $W^u(x, f)$ and $W^s(y, f)$ for $x, y \in \Lambda$ will be called a *non-degenerate homoclinic tangency* for Λ or f. We will say that Λ is a *wild* hyperbolic set if each g C^r near f has the property that $\Lambda(g)$ has a non-degenerate homoclinic tangency.

In the remainder of this section and the next section, we assume $M = M^2$ is a compact two-dimensional manifold. Our goal is to prove theorem 1. We will actually prove a stronger result in the context of one-parameter families.

Let $\{f_t\}_{0 \leq t \leq 1}$ be a C^1 curve of C^r diffeomorphisms of M, $r \geq 3$. Suppose Λ_t is a hyperbolic basic set for f_t varying continuously with t. We say that $\{f_t\}$ creates a non-degenerate tangency of $W^u(\Lambda_t)$ and $W^s(\Lambda_t)$ at (t_0, x) if there are C^r curves $\gamma_t^u \subset W^u(\Lambda_t)$ and $\gamma_t^s \subset W^s(\Lambda_t)$ and a number $\varepsilon_0 > 0$ so that:

(1) $\gamma_{t_0}^s$ has a non-degenerate tangency with $\gamma_{t_0}^u$ at x;
(2) for $t_0 - \varepsilon_0 < t < t_0$, $\gamma_t^s \cap \gamma_t^u = \emptyset$;
(3) for $t_0 < t < t_0 + \varepsilon_0$, γ_t^s has two transverse intersections with γ_t^u;
(4) γ_t^s and γ_t^u vary differentiably with t.

Theorem 3. — With the above notation, suppose $\{f_t\}$ creates a non-degenerate tangency of $W^u(\Lambda_t)$ and $W^s(\Lambda_t)$ at (t_0, x_0) and f_{t_0} has a periodic point $p \in \Lambda_{t_0}$ of period n_0 such that $\det T_p f_{t_0}^{n_0} \neq 1$. *Then, given $\varepsilon > 0$, there is a t_ε with $|t_\varepsilon - t_0| < \varepsilon$ such that f_{t_ε} contains a wild hyperbolic set near the orbit of x_0.*

Theorem 3 implies theorem 1 since any f satisfying the hypothesis of theorem 1 can, after approximation, be embedded in a curve $\{f_t\}$ which satisfies the hypothesis of theorem 3 and creates a non-degenerate tangency near the original tangency for f.

Remarks. — 1. In a recent paper [8], M. Hénon studies numerically the polynomial mapping $f_t(x,y) = (y + t - ax^2, bx)$ of $\mathbf{R}^2$ with $a = 1.4$, $b = 0.3$, $t = 1$, and finds what appears to be a strange attractor. The actual existence of that attractor has not been proved. For certain values of the parameter t, one can verify that the curve $\{f_t\}$ has a hyperbolic saddle fixed point whose stable and unstable manifolds create a non-degenerate tangency. Thus, there are t's for which f_t has infinitely many periodic sinks, and it may be the case that Hénon has merely found a long periodic orbit.

2. In [7], Fatou proved that every rational function on $\mathbf{CP}^1 = S^2$ has only finitely many periodic sinks. Using the mapping in remark 1 or the related mapping

$$f_t(x,y) = (y, -ay^2 - bx + t) \quad \text{with} \quad a > 0, \quad 0 < b < 1,$$

one can give real polynomial mappings in two variables with infinitely many sinks. Having found a t_1 where $f_{t_1}(x,y)$ creates a non-degenerate tangency, it is clear that $\bar{f}_t = (y, -ay^2 - bx - \varepsilon x^2 + t)$, also creates one for some t near t_1 if $\varepsilon > 0$ is small. Thus, for some t_3, $\bar{f}_{t_3} = (y, -ay^2 - bx - \varepsilon x^2 + t_3)$ has infinitely many sinks as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$, and, hence, also as a map from $\mathbf{C}^2$ to $\mathbf{C}^2$. Setting $\frac{Y}{X} = x$, $\frac{Z}{X} = y$, we can projectivize $\bar{f}_{t_3}$ to the map

$$f(X, Y, Z) = (X^2, XZ, -aZ^2 - bXY - \varepsilon Y^2 + t_3 X^2)$$

which is a degree 2 polynomial mapping from $\mathbf{CP}^2$ to $\mathbf{CP}^2$ with infinitely many sinks.

3. It is frequently expressed that observable physical objects correspond to invariant sets of positive Lebesgue measure for their dynamical equations. Bowen and Ruelle have proved that zero-dimensional hyperbolic sets for C^2 diffeomorphisms have measure zero [4] (this is false for C^1 diffeomorphisms [5]). Using theorem 1 and [16], one could make even a hyperbolic set Λ of measure zero " observable " by creating a non-degenerate tangency of its stable and unstable manifolds. For then, one would get Λ as a limit of sinks (assuming Λ has a periodic point with determinant less than one) each of which would carry an open set of points permanently near Λ.

4. It is likely that theorem 3 holds with $r=2$. However, our proof makes use of a C^1 linearization of $f_{t_0}^{n_0}$ near p which is C^2 off $W_\varepsilon^u(p)$ (resp. $W_\varepsilon^s(p)$) if $\det T_p f_{t_0}^{n_0}<1$ (resp. $\det T_p f_{t_0}^{n_0}>1$) for some small $\varepsilon>0$. This uses the assumption that $r \geq 3$.

Before going to the proof of theorem 3, we need some preliminaries concerning Cantor sets.

By a *Cantor set* in $\mathbf{R}$ we mean a compact perfect totally disconnected subset F of $\mathbf{R}$. Given such an F, let F_0 be the smallest closed interval containing F. Then we may write $F_0-F=\bigcup_{i=0}^{\infty} U_i$ where $\mathrm{cl}\, U_i \cap \mathrm{cl}\, U_j = \emptyset$ if $i \neq j$ and each U_i is a bounded open interval. Let U_{-2} and U_{-1} be the unbounded components of $\mathbf{R}-F$. We call the U_i's, $i \geq -2$, the gaps of F, or, simply, F-*gaps*. For $i \geq 1$, set $F_i = F_0 - \bigcup_{0 \leq j \leq i-1} U_j$. Thus, $F_0 \supset F_1 \supset F_2 \ldots$, each F_i is a union of closed intervals, and $\bigcap_{i \geq 0} F_i = F$. Call the sequence $\{F_i\}_{i \geq 0}$ a *defining sequence* for F. It is determined merely by giving some enumeration $U_0, U_1, \ldots$, of the bounded gaps of F. For $i \geq 0$, if F_i^* is the component of F_i containing U_i, then $F_i^* - U_i$ is the union of two closed intervals c_i^ℓ and c_i^r which lie adjacent to U_i as in figure (2.1).

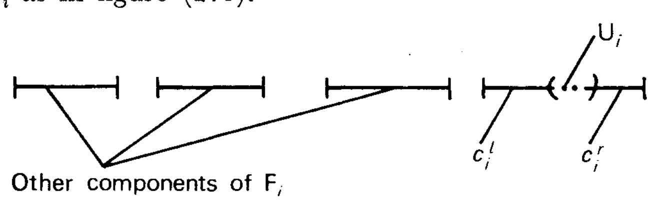

FIG. 2.1

In other words, c_i^ℓ and c_i^r are the components of F_{i+1} which meet the closure of U_i. We write c_i^ℓ for the component to the left of U_i and c_i^r for the component to the right of U_i. If I is an interval, let $\ell(I)$ be its length. Set $\tau(\{F_i\}) = \inf_{i \geq 0}\left\{\min\left(\frac{\ell c_i^\ell}{\ell U_i}, \frac{\ell c_i^r}{\ell U_i}\right)\right\}$ and set $\tau(F) = \sup\{\tau(\{F_i\}) : \{F_i\}$ is a defining sequence for $F\}$. We call $\tau(F)$ the *thickness* of F.

Observe that $0 \leq \tau(F) < \infty$, and it is an indicator of how *thick* F is. As an example, consider the middle α-set $F(\alpha)$. In this set one starts with a closed interval F_0 and deletes its middle open interval of length $\alpha.\ell(F_0)$ to obtain F_1. Having defined F_i, delete the middle open interval of length $\alpha.\ell(c)$ from each component c of F_i to define F_{i+1}. Let $F=\bigcap_{i\geq 0} F_i$. Of course, with this definition, $\{F_i\}$ is not a defining sequence for $F(\alpha)$ since $F_i - F_{i+1}$ consists of 2^i components for $i\geq 1$. That is unimportant. One easily shows that $\tau(F)=\frac{1-\alpha}{2\alpha}$. Let $H(F)$ be the *Hausdorff dimension* of a set F. Then, $H(F(\alpha))=\frac{-\log 2}{\log\left(\frac{1-\alpha}{2}\right)}$, and hence, $\tau(F(\alpha))=\tau(F(\beta))$ if and only if $H(F(\alpha))=H(F(\beta))$.

Also, for any Cantor set F with $\tau(F)>0$, if α is such that $\tau(F)=\tau(F(\alpha))$, then $H(F)\geq H(F(\alpha))$. From this we obtain the inequality $H(F)\geq\frac{-\log 2}{\log\left(\frac{\tau(F)}{1+2\tau(F)}\right)}$ for any such Cantor set F. Thus, $H(F)\to 1$ as $\tau(F)\to\infty$. On the other hand, it is easy to construct examples of Cantor sets F with $H(F)=1$ and $\tau(F)$ arbitrarily small.

We caution the reader that inclusion of Cantor sets $F\subset G$ does not necessarily imply that $\tau(F)\leq\tau(G)$. For example, let n be a positive integer and let

$$F(n, \alpha)=F(\alpha)\cup(n+F(\alpha))$$

where $F(\alpha)$ is the middle α-set constructed from the unit interval and

$$n+F(\alpha)=\{n+x: x\in F(\alpha)\}.$$

Then $\tau(F(n, \alpha))=\min\left\{\tau(F(\alpha)), \frac{1}{n-1}\right\}$. If $\frac{1}{n-1}<\tau(F(\alpha))$, we have $F(\alpha)\subset F(n, \alpha)$ and $\tau(F(\alpha))>\tau(F(n, \alpha))$.

The following simple lemma generalizes lemma (3.5) of [14].

Lemma 4. — Let F *and* G *be Cantor sets in* **R** *with* F *in no* G*-gap closure and* G *in no* F*-gap closure. If* $\tau(F).\tau(G)>1$, *then* $F\cap G\neq\emptyset$. *In fact, if* $\{F_i\}$ *and* $\{G_i\}$ *are defining sequences for* F *and* G, *respectively, such that* $\tau(\{F_i\}).\tau(\{G_i\})>1$, *then, for each* $i\geq 0$, $\operatorname{int}(F_i\cap G_i)\neq\emptyset$.

Remarks. — 1. The gap conditions in lemma 4 are fulfilled if F_0 and G_0 properly overlap that is, $\partial F_0\cap\operatorname{int} G_0\neq\emptyset$ and $\partial G_0\cap\operatorname{int} F_0\neq\emptyset$.

2. The word " closure " in the statement of lemma 4 could be deleted, and one would still get $F\cap G\neq\emptyset$, but the last statement would not necessarily hold. For the proof of theorem 2, it is this last statement which is important (see section 4).

The main part of the proof of lemma 4 is the

Gap Lemma. — Let F *and* G *be Cantor sets in* **R** *with defining sequences* $\{F_i\}$ *and* $\{G_i\}$. *Suppose that* G *is in no* F*-gap closure,* $\tau(\{F_i\}).\tau(\{G_i\})>1$, *and* c *is a component of* F_i *in no* G*-gap closure. Then one of the components of* $c\cap F_{i+1}$ *is in no* G*-gap closure.*

Proof of Gap lemma. — Let U_0, U_1, ... be the bounded F-gaps determining the sequence $\{F_i\}$ and let U_0^1, U_1^1, ... be the bounded G-gaps determining $\{G_i\}$.

If $c \subset F_{i+1}$, the result is trivial, so we can assume $c = F_i^*$ is the component of F_i containing the i-th F-gap U_i. Let c^ℓ be the left component of $c \cap F_{i+1}$ and c^r be the right component of $c \cap F_{i+1}$. We have

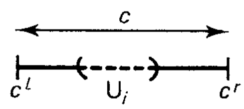

FIG. 2.1 *bis*

Assume, by way of contradiction, that c^ℓ and c^r are in G-gap closures. Let U_{-2}^1, U_{-1}^1 be the unbounded G-gaps. Thus, $c^\ell \subset \mathrm{cl}\, U_n^1$ and $c^r \subset \mathrm{cl}\, U_m^1$ for some $n, m \geq -2$. We may suppose $n \leq m$, as the other case is similar.

If $n = m$, then all of c lies in $\mathrm{cl}\, U_n^1$ contrary to the hypothesis that c is in no G-gap closure. Therefore, $n < m$.

Case 1. $n = -2$, $m = -1$. In this case c^ℓ and c^r are in different unbounded G-gap closures. This means that $G \subset \mathrm{cl}\, U_i$ contrary to the assumption that G is in no F-gap closure.

Case 2. $n < m$, $m \geq 0$. In this case c^r is in a bounded G-gap closure U_m^1. Since $n < m$, the interior of the component G_m^* of G_m containing U_m^1 misses c^ℓ. Also, int G_m^* is entirely to the right of c^ℓ. If G_m^ℓ is the component of $G_m^* - U_m^1$ to the left of U_m^1, then $G_m^\ell \subset \mathrm{cl}\, U_i$. Then we have something like figure 2.2

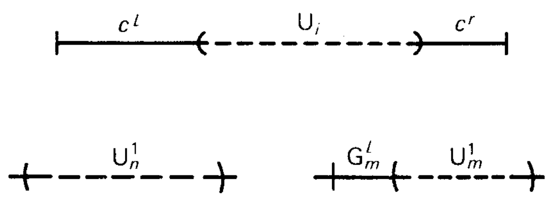

FIG. 2.2

But then $\tau(\{F_i\}) \cdot \tau(\{G_i\}) \leq \frac{\ell c^r}{\ell U_i} \cdot \frac{\ell G_m^\ell}{\ell U_m^1} \leq 1$ which is a contradiction and the Gap lemma is proved.

We can now prove lemma 4. Since neither F nor G is in a gap closure of the other, we have int $(F_0 \cap G_0) \neq \emptyset$. By the gap lemma with $c = F_0$, we can find a component c_1 of $c \cap F_1$ in no G-gap closure. Repeating the process inductively gives us a decreasing sequence c_1, c_2, c_3, ... of closed intervals so that each c_i is a component of $F_i \cap c_{i-1}$ and c_i is in no G-gap closure. But this means that int $c_i \cap$ int $G_m \neq \emptyset$ for all $m \geq 0$. In particular, $\emptyset \neq$ int $c_i \cap$ int $G_i \subset$ int $(F_i \cap G_i)$ as required.

We will use the notion of thickness of Cantor sets to define some invariants of a zero-dimensional hyperbolic basic set Λ for a C^2 diffeomorphism $f: M^2 \to M^2$. Let Λ be such a set. All estimates will be with respect to the distance and length functions induced by a fixed Riemann metric on M adapted to Λ.

There is a $\delta > 0$ so that each x in Λ has a neighborhood U in $M = M^2$ so that $U \cap \Lambda$ is homeomorphic to $(W^u_\delta(x,f) \cap \Lambda) \times (W^s_\delta(x,f) \cap \Lambda)$ ([4] or [23]). It follows that $W^u_\delta(\Lambda,f)$ is locally the product of a Cantor set and an interval. Let $\gamma: (-1, 1) \to M$ be a C^1 curve meeting $W^u(\Lambda,f)$ transversely at a point $y = \gamma(0)$. Choose an integer $n > 0$ so that $y \in f^n W^u_\delta(\Lambda,f)$. Then for $\varepsilon > 0$ small, y is contained in a Cantor set in $(\text{Image } \gamma \mid (-\varepsilon, \varepsilon)) \cap f^{n+1} W^u_\delta(\Lambda,f)$. See figure 2.3.

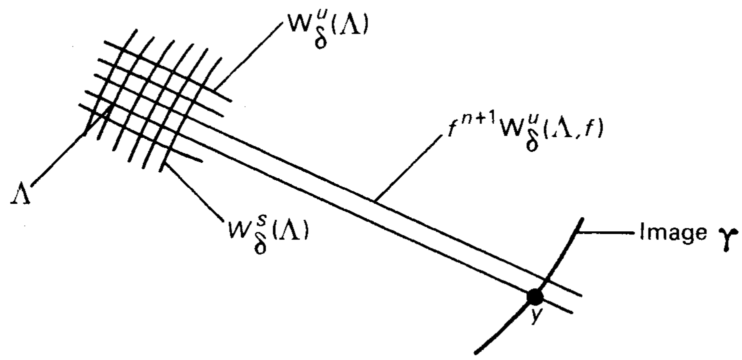

FIG. 2.3

Define $\tau^u(y, \gamma, \Lambda) = \inf_{\varepsilon > 0} \{\sup \tau(F) : F \text{ is a Cantor set in } (\text{Image } \gamma \mid (-\varepsilon, \varepsilon)) \cap f^{n+1} W^u_\delta(\Lambda,f)\}$. Clearly, $\tau^u(y, \gamma, \Lambda)$ does not depend on the n chosen so that $y \in f^n W^u_\delta(\Lambda, f)$.

Proposition 5. — *Suppose y and γ are as above, y_1 is another point in* $W^u(\Lambda,f)$ *and* $\gamma_1 : (-1, 1) \to M$ *is a* C^1 *curve meeting* $W^u(\Lambda,f)$ *transversely at* $y_1 = \gamma_1(0)$. *Then*

$$\tau^u(y, \gamma, \Lambda) = \tau^u(y_1, \gamma_1, \Lambda).$$

Furthermore, $\tau^u(y, \gamma, \Lambda)$ *is independent of the Riemann metric on* M.

Proof. — Let $E^s \oplus E^u = T_\Lambda M$ be the continuous splitting given by the definition of hyperbolicity. Since f is C^2 and $\dim E^s = \dim E^u = \dim M - 1 = 1$, E^u and E^s extend to C^1 Tf-invariant line fields $\bar{E}^u$ and $\bar{E}^s$ on a neighborhood U of Λ [Theorem (6.4) b, 9]. Integrating $\bar{E}^u$ and $\bar{E}^s$ locally, we obtain two C^1 foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ on a neighborhood U_1 of Λ such that, for $x \in U_1 \cap f^{-1} U_1 \cap f U_1$, $f\mathcal{F}^u_x \supset \mathcal{F}^u_{f(x)}$ and $f(\mathcal{F}^s_x) \subset \mathcal{F}^s_{f(x)}$. Also, the leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ are C^2 curves which vary continuously on compact sets in the C^2 topology.

Let y, γ, y_1, γ_1 be as in the statement of proposition 5. First suppose that $y_1 \in W^u(y,f)$. Choose an integer $N > 0$ so that $f^{-N+1}[y, y_1] \subset \text{int}\,(U_1 \cap W^u_\delta(\Lambda,f))$ where $[y, y_1]$ is the interval in $W^u(y,f)$ from y to y_1. Let V be a neighborhood of $[y, y_1]$ in $f^N U_1$ on which the foliation $f^N \mathcal{F}^u$ is trivial. Then there is a C^1 diffeomorphism $\varphi : \bar{\gamma} \to \bar{\gamma}_1$

defined by following the leaves of $f^N\mathscr{F}^u$, where $\bar{\gamma}$ is the component of γ in V containing y_1 and $\bar{\gamma}_1$ is the component of γ_1 in V containing y_1. Clearly, φ carries Cantor sets F in $\bar{\gamma}\cap f^N W^u_\delta(\Lambda,f)$ into Cantor sets φF in $\bar{\gamma}_1\cap f^N W^u_\delta(\Lambda,f)$, and $\tau(F)$ is near $\tau(\varphi F)$ for F small, by the mean-value theorem and the continuity of the derivative of φ. Thus

$$(1) \qquad \tau(y, \gamma, \Lambda)=\tau(y_1, \gamma_1, \Lambda) \quad \text{for} \quad y_1 \in W^u(y,f).$$

A second application of the mean-value theorem to $f|\gamma$ shows that

$$(2) \qquad \tau(y, \gamma, \Lambda)=\tau(fy, f\gamma, \Lambda).$$

Now, suppose y, γ, y_1, γ_1 are as above with $y_1 \notin W^u(y,f)$. It is known (see Smale [23] or Bowen [4]) that

$$W^u(y) \subset \text{Cl}\, W^u(O(y_1)) \quad \text{and} \quad W^u(y_1) \subset \text{Cl}\, W^u(O(y))$$

where $O(y)$ and $O(y_1)$ denote the orbits of y and y_1. Choose a sequence of integers $n_1<n_2<\dots$ and a sequence of points $\eta_i \in W^u(f^{n_i}y_1)\cap\gamma\cap f^N W^u_\delta(\Lambda,f)$ such that $\eta_i \to y$ as $i\to\infty$.

By (1) and (2), we have $\tau^u(\eta_i, \gamma, \Lambda)=\tau^u(y_1, \gamma_1, \Lambda)$ for all i. Thus,

$$\tau^u(y, \gamma, \Lambda) \geq \tau^u(y_1, \gamma_1, \Lambda).$$

Reversing the roles of y and y_1, we conclude that $\tau^u(y, \gamma, \Lambda)=\tau^u(y_1, \gamma_1, \Lambda)$.

To begin the proof of the second statement of proposition 5, let y and γ be as in the definition of $\tau^u(y, \gamma, \Lambda)$, and let g_1 and g_2 be two Riemann metrics on M.

Let

$$\alpha_1(t)=(g_{1\gamma(t)}(\gamma'(t), \gamma'(t)))^{\frac{1}{2}}$$

and

$$\alpha_2(t)=(g_{2\gamma(t)}(\gamma'(t), \gamma'(t)))^{\frac{1}{2}}$$

be the associated length functions of $\gamma'(t)=\frac{d\gamma}{dt}(t)$. Then $\alpha_1(t)$ and $\alpha_2(t)$ are nowhere zero differentiable functions of t (we assume t is close to 0). Given a C^1 curve ρ, let $\ell_1(\rho)$ and $\ell_2(\rho)$ denotes the lengths of ρ induced by g_1 and g_2, respectively. For $\varepsilon>0$ small, and $-\varepsilon<a<b_1<b<\varepsilon$, we have

$$\ell_1(\gamma|[a, b_1])=\int_a^{b_1}\alpha_1(t)\,dt=\alpha_1(t_1)(b_1-a),$$

$$\ell_1(\gamma|[b_1, b])=\alpha_1(t_2)(b-b_1), \qquad \ell_2(\gamma|[a, b_1])=\alpha_2(t_1')(b_1-a),$$

and

$$\ell_2(\gamma|[b_1, b])=\alpha_2(t_2')(b-b_1)$$

where t_1 and t_1' are in (a, b_1), and t_2 and t_2' are in (b_1, b).

Thus,

$$\frac{\ell_1(\gamma|[b_1, b])}{\ell_1(\gamma|[a, b_1])}=\frac{\alpha_1(t_2)(b-b_1)}{\alpha_1(t_1)(b_1-a)}$$

is near

$$\frac{\ell_2(\gamma\,|\,[b_1,b])}{\ell_2(\gamma\,|\,[a,b_1])}=\frac{\alpha_2(t_2')(b-b_1)}{\alpha_2(t_1')(b_1-a)}$$

for ε small, since $\frac{\alpha_1(t_2)}{\alpha_1(t_1)}$ and $\frac{\alpha_2(t_2')}{\alpha_2(t_1')}$ are near 1. Now the definition of $\tau^u(y,\gamma,\Lambda)$ implies that it is the same for g_1 and g_2.

In view of proposition 5, we may define $\tau^u(\Lambda)=\tau^u(\Lambda,f)=\tau^u(y,\gamma,\Lambda)$ for any y in $W^u(\Lambda,f)$ and any C^1 curve $\gamma:(-1,1)\to M$ meeting $W^u(\Lambda,f)$ transversely at $y=\gamma(0)$. Similarly, we set $\tau^s(\Lambda)=\tau^s(\Lambda,f)=\tau^u(\Lambda,f^{-1})$. We call

$\tau^u(\Lambda)$ the *unstable thickness* of Λ, and

$\tau^s(\Lambda)$ the *stable thickness* of Λ.

Observe that if Λ_1 and Λ_2 are two hyperbolic zero-dimensional basic sets for $f:M^2\to M^2$, and $\Lambda_1\subseteq\Lambda_2$, then $\tau^u(\Lambda_1)\leq\tau^u(\Lambda_2)$ and $\tau^s(\Lambda_1)\leq\tau^s(\Lambda_2)$.

Proposition 6. — 1) *If* Λ *is an infinite zero-dimensional hyperbolic basic set for a* C^2 *diffeomorphism* $f:M^2\to M^2$, *then* $0<\tau^u(\Lambda)<\infty$.

2) *If* g *is* C^2 *near* f, *then* $\tau^u(\Lambda(g))$ *is near* $\tau^u(\Lambda(f))$.

Proof. — We will prove that $\tau^u(\Lambda)>0$ by finding a special basic set $\Lambda_1\subset\Lambda$, and a number $S>0$, such that $\tau^u(\Lambda_1)>S$.

Let p be a periodic point of Λ of period $n_1\geq 1$. Since Λ is infinite, p has transversal homoclinic points. By a theorem due to Smale [22], there are a disk $D\subset M$ and integers $n_2, n_3>0$ such that if $\Lambda_2=\bigcap_{n\in\mathbf{Z}} f^{nn_1n_2}(D)$, then $p\in\Lambda_2\subset\Lambda$, and Λ_2 is a hyperbolic basic set for $f^{n_1n_2}$ on which $f^{n_1n_2}$ is topologically conjugate to a full shift automorphism on n_3 symbols. We will find a number $S>0$ so that $\tau^u(\Lambda_2,f^{n_1n_2})>S$. Then we set $\Lambda_1=\bigcup_{0\leq j\leq n_1n_2} f^j\Lambda_2$, and we will have $\tau^u(\Lambda_1,f)>S$ as required.

The *shift automorphism* just mentioned is defined as follows. Let $\Sigma_{n_3}=\{1,\ldots,n_3\}^{\mathbf{Z}}$ be the set of mappings from the integers $\mathbf{Z}$ into $\{1,\ldots,n_3\}$ with the compact open topology. Denote the elements of Σ_{n_3} by $\underline{a}=(\underline{a}_i)_{i\in\mathbf{Z}}$, and let $\sigma:\Sigma_{n_3}\to\Sigma_{n_3}$ be defined by $\sigma(\underline{a})_i=\underline{a}_{i+1}$. The mapping σ is called the full shift automorphism on n_3 symbols.

Now, D may be chosen so that if $(f^{n_1n_2}D)\cap D=A_1\cup\ldots\cup A_r$ is the decomposition of $(f^{n_1n_2}D)\cap D$ into connected components, then:

(1) $r=n_3$;

(2) each A_i is diffeomorphic to a disk and $\partial A_i\subset W^u(\Lambda_2,f^{n_1n_2})\cup W^s(\Lambda_2,f^{n_1n_2})$;

(3) the map $h:\Sigma_{n_3}\to\Lambda_2$ defined by $h(\underline{a})=\bigcap_{i\in\mathbf{Z}} f^{-in_1n_2}(A_{\underline{a}_i})$ is a homeomorphism such that $h\sigma=f^{n_1n_2}h$.

We note that although Smale's original paper [22] proved this under the assumption that f^{n_1} was linearizable near p, the result without this assumption may be obtained by combining the techniques of [22] with some well-known estimates as in [15], [13], or [16 a].

Let $g = f^{n_1 n_2}$.

We may label the A_i's such that

$$p = h(\underline{a}) \quad \text{where} \quad \underline{a}_i = 1 \quad \text{for all } i.$$

Let F_0 be the connected component of p in $W^s(p, g) \cap D$. Then $F = F_0 \cap \Lambda_2$ is a Cantor set, and we will get a lower bound on its thickness $\tau(F)$. First, we construct a defining sequence for F. Let $F^1, \ldots, F^{n_3}$ be the components of $F_0 \cap g(D)$. For each F^i, let $F^{i,1}, \ldots, F^{i,n_3}$ be the components of $F^i \cap g^2(D)$. Continuing, once $F^{i_1, i_2, \ldots, i_k}$ has been defined with $1 \leq i_j \leq n_3$, let $\{F^{i_1, \ldots, i_k, j}\}_{1 \leq j \leq n_3}$ be the components of $F^{i_1, \ldots, i_k} \cap g^{k+1} D$. Now let $U_1, \ldots, U_{n_3 - 1}$ be the components of $F_0 - \bigcup_{1 \leq j \leq n_3} F^j$, and once $U_{i_1, \ldots, i_k}$ has been defined with $1 \leq i_j \leq n_3 - 1$, let $U_{i_1, \ldots, i_k, \nu}$ be the components of

$$F^{i_1, \ldots, i_k} - \bigcup_{1 \leq j \leq n_3} F^{i_1, \ldots, i_k, j}$$

between $F^{i_1, \ldots, i_k, \nu}$ and $F^{i_1, \ldots, i_k, \nu+1}$. Then, $U_1, \ldots, U_{n_3-1}, U_{1,1}, U_{1,2}, \ldots, U_{1, n_3 - 1}, \ldots$ is a list of the gaps of F. We use it to get our defining sequence for F. To estimate the thickness of this defining sequence, we let

$$I = U_{i_1, \ldots, i_k} \quad \text{and} \quad J = F^{i_1, \ldots, i_{k-1}, j}$$

where $j = i_k$ or $i_k + 1$ and $k \geq 1$. Let $\rho(s)$, $a \leq s \leq b$, be a parametrization of $g^{-1}(I \cup J)$ by arc length, where we suppose that $\rho(s) \in g^{-1}(J)$ for $a \leq s \leq b_1 < b$ and $\rho(s) \in g^{-1}(I)$ for $b_1 \leq s \leq b$.

Then

$$\frac{\ell(J)}{\ell(I)} = \frac{\int_a^{b_1} |T_{\rho(s)} g(\rho'(s))| \, ds}{\int_{b_1}^{b} |T_{\rho(s)} g(\rho'(s))| \, ds} = \frac{|T_{\rho(s_1)} g(\rho'(s_1))| \, \ell(g^{-1}(J))}{|T_{\rho(s_2)} g(\rho'(s_2))| \, \ell(g^{-1}(I))}$$

where $a < s_1 < b_1$ and $b_1 < s_2 < b$.

Set $\alpha_{k-1} = |T_{\rho(s_1)} g(\rho'(s_1))|$ and $\beta_{k-1} = |T_{\rho(s_2)} g(\rho'(s_2))|$. Repeating this construction for negative powers $g^{-i} I$, $g^{-i} J$, we get

$$\frac{\ell(J)}{\ell(I)} = \frac{\alpha_{k-1} \cdot \alpha_{k-2} \cdots \alpha_1 \ell(g^{-k+1} J)}{\beta_{k-1} \cdot \beta_{k-2} \cdots \beta_1 \ell(g^{-k+1} I)}$$

where $\alpha_i = |T_{z_i} g(v_i)$, $\beta_i = |T_{z'_i} g(v'_i)|$, $z_i \in g^{i-k} J$, $z'_i \in g^{i-k} I$, v_i is a unit vector in $T_{z_i} g^{i-k} J$, and v'_i is a unit vector in $T_{z'_i} g^{i-k} I$.

In the following, we let $c_1, c_2, \ldots$ be constants independent of i and k which are defined by the first sentence or equation in which they appear. Occasionally we will explicitly define them.

Clearly, $|\beta_i| \geq c_1 > 0$ where $c_1 = \inf_{z \in D, |v|=1} |T_z g(v)|$. If c_2 is the infimum of the lengths of the components of $W^s(z, g)$ in $g(D) \cap D$ for $z \in \Lambda_2$, and c_3 is the supremum of the lengths of the components of $W^s(z, g)$ in $D - g(D)$ for $z \in \Lambda_2$, then

$$\frac{\ell(g^{-k+1}J)}{\ell(g^{-k+1}I)} \geq \frac{c_2}{c_3}.$$

Since Λ_2 is hyperbolic for g, there is a constant $0 < \lambda < 1$ so that

$$\ell(g^{i-k}(I \cup J)) \leq \lambda^{i-1} \ell(g^{-k+1}(I \cup J)) \leq \lambda^{i-1} \operatorname{diam} D \quad \text{for all} \quad 1 \leq i \leq k-1.$$

Now

$$\begin{aligned} |\beta_i - \alpha_i| &\leq \big||T_{z_i'} g(v_i')| - |T_{z_i'} g(v_i)|\big| + \big||T_{z_i'} g(v_i)| - |T_{z_i} g(v_i)|\big| \\ &\leq (c_4 c_5 + c_6) \ell(g^{i-k}(I \cup J)) \leq c_7 \lambda^i \quad \text{for} \quad 1 \leq i \leq k-1 \end{aligned}$$

where $c_4 = \sup_{z \in D} |T_z g|$, c_5 is an upper bound on the curvatures of $W^s(\Lambda_2) \cap D$, and c_6 depends on the C^2 size of g.

Thus, we get that $\left| \sum_{i=1}^{k-1} \frac{\beta_i - \alpha_i}{\beta_i} \right| \leq \frac{c_7 \lambda}{c_1(1-\lambda)}$. Since $\prod_{i=1}^{k-1} \frac{\alpha_i}{\beta_i} = \prod_{i=1}^{k-1} \left(1 - \frac{\beta_i - \alpha_i}{\beta_i}\right)$ we have that $\prod_{i=1}^{k-1} \frac{\alpha_i}{\beta_i} > c_8 > 0$ for all k. Thus

$$\frac{\ell(J)}{\ell(I)} > c_8 . \frac{c_2}{c_3}, \quad \text{so} \quad \tau(F) > \frac{c_8 c_2}{c_3} > 0.$$

This implies that if we put $S = \frac{c_8 . c_2}{c_3}$ then $\tau^u(\Lambda_1, f) > S$, and hence that $\tau^u(\Lambda) > S$.

To prove that $\tau^u(\Lambda) < \infty$, we need to go more deeply into the structure of zero-dimensional basic sets on 2-manifolds. From Hirsch and Pugh [9], we know that there is an $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ are diffeomorphic to closed intervals, and there is a $\delta(\varepsilon) > 0$ so that $W^u_\varepsilon(x) \cap W^s_\varepsilon(y)$ is a single point whenever $d(x, y) < \delta(\varepsilon)$.

Fix $\varepsilon = \varepsilon_0$. For $d(x,y) < \delta(\varepsilon)$, let $[x, y] = W^u_\varepsilon(x) \cap W^s_\varepsilon(y)$. By Smale [23], we know that if $x \in \Lambda$ and $\varepsilon_1, \varepsilon_2 > 0$ are positive numbers with $\varepsilon_i \leq \delta(\varepsilon)/2$, then the map $\varphi_{x, \varepsilon_1, \varepsilon_2} : (W^s_{\varepsilon_1}(x) \cap \Lambda) \times (W^u_{\varepsilon_2}(x) \cap \Lambda) \to \Lambda$ defined by $\varphi_{x, \varepsilon_1, \varepsilon_2}(y_1, y_2) = [y_1, y_2]$ is a homeomorphism onto a neighborhood of x in Λ. Since Λ is zero-dimensional, we may find a finite set of points $x_1, \ldots, x_N$ in Λ and small numbers $\{\varepsilon_{i1}, \varepsilon_{i2}\}$, $1 \leq i \leq N$, so that if we set $B_i = \text{Image } \varphi_{x_i, \varepsilon_{i1}, \varepsilon_{i2}}$, then:

(1) each B_i is an open and closed subset of Λ;

(2) $\Lambda = \bigcup_{i=1}^{N} B_i$;

(3) $B_i \cap B_j = \emptyset$ for $i \neq j$;

(4) for x and y in B_i, the map $\varphi_{x,y}: W^s_\varepsilon(x)\cap B_i \to W^s_\varepsilon(y)\cap B_i$ defined by $\varphi_{x,y}(y_1)=[y_1, y]$ for $y_1 \in W^s_\varepsilon(x)\cap B_i$ is a homeomorphism varying continuously with x and y.

Now for each i, let C_i be the smallest closed interval in $W^s_\varepsilon(x_i)$ containing $W^s_\varepsilon(x_i)\cap B_i$ and let D_i be the smallest closed interval in $W^u_\varepsilon(x_i)$ containing $W^u_\varepsilon(x_i)\cap B_i$.

For $x \in C_i$, let $C(x, \mathcal{F}^u_x)$ be the smallest closed interval in $\mathcal{F}^u_x$ containing x and $\mathcal{F}^u_x \cap W^s_\varepsilon(\partial D_i)$ where $W^s_\varepsilon(\partial D_i) = \bigcup_{y\in\partial D_i} W^s_\varepsilon(y)$ and let $\overline{B}_i = \bigcup_{x\in C_i} C(x, \mathcal{F}^u_x)$. Then each $\overline{B}_i$ is diffeomorphic to $C_i \times D_i$, and $\overline{B}_i \cap \overline{B}_j = \emptyset$ for $i \neq j$. Also, $\Lambda \subset \bigcup_{i=1}^{N} \overline{B}_i$. Here $\mathcal{F}^u$ is the foliation extending $W^u(\Lambda)$ defined in the proof of proposition 5.

Now consider the Cantor set $F = C_1 \cap \Lambda$. We know by Proposition 5 that $\tau^u(\Lambda) = \tau^u(x_1, C_1, \Lambda)$. Let $0<\lambda<1$ be as in the definition of hyperbolicity of Λ, and let $\varepsilon_1 > 0$ be such that $\lambda^{-1}\varepsilon_1 < \min_{i\neq j}\{\mathrm{dist}(\overline{B}_i, \overline{B}_j)\}$. Let $F^1, F^2, \ldots$ be a sequence of Cantor sets in F such that $F^i \to x_1$ and $\tau(F^i) \to \tau^u(x_1, C_1, \Lambda)$ as $i \to \infty$. Let $\{F^i_j\}_{j\geq 0}$ be a defining sequence for F^i such that $\tau(\{F^i_j\}) > \tau(F^i) - \frac{1}{i}$. We will find a number $T>0$ such that $\tau(\{F^i_j\}) < T$ for all i. This will prove that $\tau^u(\Lambda) = \tau^u(x_1, C_1, \Lambda) \leq T$.

Let us first prove the following auxiliary fact:

(5) Suppose G is a Cantor set whose convex hull is G_0 and let $\mathrm{diam}\, G_0 > \nu > 0$. There is a number $T(\nu, G) > 0$ so that any Cantor set $H \subset G$ with $\mathrm{diam}\, H \geq \nu$ is such that $\tau(H) < T(\nu, G)$.

Choose a finite set $\{U_1, \ldots, U_m\}$ of bounded gaps of G such that if $x, y \in G$, $x<y$, and $y-x>\nu$, then some U_i is in (x, y). Let

$$\gamma_1 = \max_{1\leq i\leq m}\{\mathrm{dist}(U_i, \partial G_0)\}, \quad \text{and let} \quad \gamma_2 = \min_{1\leq i\leq m} \ell U_i.$$

If $H \subset G$ is such that $\mathrm{diam}\, H > \nu$, we may choose points $x, y \in H$ with $x<y$ and $y-x>\nu$. Let H_0 be the smallest closed interval containing H. If U_i is a gap of G in (x, y), then there is some gap V of H such that $V \supset U_i$. Then, clearly

$$\tau(H) \leq \frac{\mathrm{dist}(V, \partial H_0)}{\ell V} \leq \frac{\mathrm{dist}(U_i, \partial G_0)}{\ell U_i} \leq \frac{\gamma_1}{\gamma_2},$$

and this proves (5).

Now, for each i, let $k_i = \inf\{j : \mathrm{diam} f^{-j} F^i \geq \varepsilon_1\}$. Dropping the first few F^i's if necessary and relabeling, we may assume $k_i \geq 1$ for all i. Since $\mathrm{diam} f^{-k_i+1} F^i < \varepsilon_1$, we have $\mathrm{diam} f^{-k_i} F^i < \lambda^{-1}\varepsilon_1$, so there is a $j_i \in \{1, \ldots, N\}$ such that $f^{-k_i} F^i \subset B_{j_i}$. By (4) and (5), there is a real number $T_1 > 0$ so that if G is a Cantor set in $W^s_\varepsilon(x) \cap B_i$ for some $x \in B_i$ and some $1 \leq i \leq N$, and $\mathrm{diam}\, G \geq \varepsilon_1$, then $\tau(G) < T_1$. Thus, for all i, $\tau(f^{-k_i} F^i) < T_1$, and hence

$$\tau(f^{-k_i}\{F^i_j\}) < T_1.$$

We wish to estimate $\tau(\{F_j^i\})$. For any $j \geq i$ and any i, let $\eta = \eta_{i,j} = F_j^i - F_{j+1}^i$ and let $\xi = \xi_{i,j}$ be a component of F_{j+1}^i adjacent to η. Then let $\xi_0 = f^{-k_i}\xi$ and $\eta_0 = f^{-k_i}\eta$. We may choose ξ and η so that $\frac{\ell(\eta_0)}{\ell(\xi_0)} < T_1$ since $\tau(f^{-k_i}\{F_j^i\}) < T_1$. Now

$$\tau(\{F_j^i\}) \leq \frac{\ell(\eta)}{\ell(\xi)},$$

and as in the proof of the fact that $\tau^u(\Lambda) > 0$, we have

$$\frac{\ell(\eta)}{\ell(\xi)} = \left(\prod_{\nu=1}^{k_i} \frac{\alpha_\nu}{\beta_\nu}\right) \frac{\ell(\eta_0)}{\ell(\xi_0)} \quad \text{where} \quad |\beta_\nu| \geq \inf_{z \in M} |T_z f|$$

and $|\beta_\nu - \alpha_\nu| \leq c_9 \lambda^\nu$ for all $\nu \geq 1$ where c_9 is a constant depending on the diameters of the $\overline{B}_i$, the C^1 and C^2 sizes of f, and on the curvatures of the curves $W_\varepsilon^s(x)$ for $x \in \Lambda$. Moreover

(6) there are constants c_{10}, c_{11}, $c_{12} > 0$, so that

$$0 < c_{10} < \left| \prod_{\nu=1}^{\infty} \frac{\alpha_\nu}{\beta_\nu} \right| \leq c_{11} e^{c_{12}(1/(1-\lambda))}$$

for any sequences of numbers α_ν, β_ν with $|\beta_\nu| \geq \frac{1}{2} \inf_{z \in M, |v|=1} |T_z f(v)|$ and $|\beta_\nu - \alpha_\nu| \leq 2c_9 \lambda^\nu$. Thus, $\tau(\{F_j^i\} \leq \frac{\ell(\eta)}{\ell(\xi)} \leq c_{11} e^{c_{12}(1/1-\lambda)} T_1$, and we may take $T = c_{11} e^{c_{12}(1/(1-\lambda))} T_1$.

We now prove Prop. (6.2).

It is clear from the preceding proof that there is a neighborhood U_1 of f in $\mathrm{Diff}^2(M^2)$ so that if $g \in U_1$, then $S < \tau^u(\Lambda(g)) < T$. We will assume $S < 1 < T$.

For g near f, all of the structures defined for f may be defined for g so that they vary continuously with g in the natural topologies. We denote the g-structures by $\Lambda(g)$, $\mathscr{F}^u(g)$, $\overline{B}_i(g)$, etc.

Let $U_2 \subset U_1$ be a neighborhood of f in $\mathrm{Diff}^2(M^2)$ so that for g in U_2, $x \in \bigcup_{1 \leq i \leq N} \overline{B}_i(g)$,

(7) $$|T_x g | T_x \mathscr{F}_x^s(g)| < \lambda$$

and

(8) $$\lambda^{-1} \varepsilon_1 < \min_{i \neq j} \{\mathrm{dist}(\overline{B}_i(g), \overline{B}_j(g))\}.$$

Let g be near f and let $h : \Lambda(f) \to \Lambda(g)$ be the homeomorphism so that $gh = hf$. If $x \in \Lambda(f)$ and $I \subset W_\varepsilon^s(x, f)$ is a closed interval with $\partial I \subset \Lambda(f)$, then there is an interval $I(g) \subset W_\varepsilon^s(hx, g)$ such that $\partial I(g) = h(\partial I)$.

Let $0 < \varepsilon_4 < 1$ be arbitrary. Choose $0 < \varepsilon_2 < \frac{1}{2}$ small enough so that

(9) $$\frac{1 - \varepsilon_2}{1 + \varepsilon_2} - (1 - \varepsilon_4) > \frac{\varepsilon_4}{2}.$$

Then choose an integer $\nu_0>0$ so that for any sequences α_ν, β_ν as in (6), we have

$$(10\,a)\qquad 1-\varepsilon_2<\prod_{\nu\geq\nu_0}\frac{\alpha_\nu}{\beta_\nu}<1+\varepsilon_2;$$

$$(10\,b)\qquad (\sup_z |T_z f^{-1}|)\,.\,\lambda^{\nu_0}\varepsilon_1<\lambda^{-1}\varepsilon_1.$$

Now, let $\varepsilon_3>0$ be small enough so that

$$(11)\qquad \varepsilon_3(1-\varepsilon_2)<\frac{\varepsilon_4}{2}S.$$

Finally, we pick a neighborhood $U_3(\varepsilon_3)$ of f in U_2 so that if $g\in U_3(\varepsilon_3)$, $x\in\Lambda(f)$, and I and J are intervals in $W^s_\varepsilon(x,f)$ with $\min(\ell I, \ell J)\geq\dfrac{\lambda^{\nu_0}\varepsilon_1}{1+2T}\cdot\dfrac{S}{2}$, then

$$(12)\qquad \left|\frac{\ell(I)}{\ell(J)}-\frac{\ell(J(g))}{\ell(I(g))}\right|<\varepsilon_3,$$

and

(13) for any $k\geq 0$ and $y\in\Lambda(g)$, if z_1 and z_2 are in $g^k W^s_\varepsilon(y,g)$ and v_1 and v_2 are unit tangent vectors with $v_1\in T_{z_1}g^kW^s_\varepsilon(y,g)$ and $v_2\in T_{z_2}g^kW^s_\varepsilon(y,g)$, then

$$|T_{z_i}g(v_i)|>\frac{1}{2}\inf_{z\in M,|v|=1}|T_z f(v)|,\qquad\text{and}\qquad ||T_{z_1}gv_1|-|T_{z_2}gv_2||<2c_9\lambda^k.$$

Now suppose $x\in\Lambda(f)$ and $F^1(f), F^2(f), \ldots$ is a sequence of Cantor sets in $W^s_\varepsilon(x,f)$ converging to x so that $\tau(F^i(f))\to\tau^u(x, W^s_\varepsilon(x,f), \Lambda(f))=\tau^u(\Lambda(f))$ as $i\to\infty$. Let $I_1(f), I_2(f), \ldots$ and $J_1(f), J_2(f), \ldots$ be sequences of intervals in $W^s_\varepsilon(x,f)$ so that:

(14) $J_i(f)$ is a gap in $F^i(f)$ and $I_i(f)$ is an adjacent component of $J_i(f)$ in some defining sequence of $F^i(f)$;

$$(15)\qquad \frac{\ell(I_i(f))}{\ell(J_i(f))}\to\tau^u(\Lambda(f))\qquad\text{as } i\to\infty;$$

$$(16)\qquad S<\frac{\ell(I_i(f))}{\ell(J_i(f))}<T\qquad\text{for all } i.$$

Suppose $g\in U_3(\varepsilon_3)$. We will prove

$$(17)\qquad \liminf_{i\to\infty}\left|\frac{\ell(I_i(g))}{\ell(J_i(g))}\right|\geq(1-\varepsilon_4)\tau^u(\Lambda(f)).$$

Once (17) is proved, it will follow that $\tau^u(\Lambda(g))\geq(1-\varepsilon_4)\tau^u(\Lambda(f))$ since

$$\tau^u(\Lambda(g))\geq\limsup_{i\to\infty}\tau(hF^i(f))\geq\limsup_{i\to\infty}\frac{\ell(I_i(g))}{\ell(J_i(g))}.$$

Then, reversing the roles of f and g and repeating the argument will give that

$$\tau^u(\Lambda(f)) \geq (1-\varepsilon_4)\tau^u(\Lambda(g)) \quad \text{for} \quad g \in U_3(\varepsilon_3).$$

But then $(1-\varepsilon_4)\tau^u(\Lambda(f)) \leq \tau^u(\Lambda(g)) \leq (1-\varepsilon_4)^{-1}\tau^u(\Lambda(f))$, and as ε_4 was arbitrary, we will have proved prop. (6.2).

We now prove (17).

For each i, let $k_i = \inf\left\{j \geq 0 : \ell(f^{-j}J_i)(f) \geq \dfrac{\lambda^{\nu_0}\varepsilon_1}{1+2T}\right\}$. We assume each $k_i \geq 1$.

As before, we have

$$\text{(18 a)} \qquad \frac{\ell(I_i(f))}{\ell(J_i(f))} = \prod_{\nu=\nu_0}^{k_i+\nu_0-1} \frac{\alpha_\nu(f)}{\beta_\nu(f)} \cdot \frac{\ell(f^{-k_i}I_i f)}{\ell(f^{-k_i}J_i f)}$$

and

$$\text{(18 b)} \qquad \frac{\ell(I_i g)}{\ell(J_i g)} = \prod_{\nu=\nu_0}^{k_i+\nu_0-1} \frac{\alpha_\nu(g)}{\beta_\nu(g)} \cdot \frac{\ell(g^{-k_i}I_i g)}{\ell(g^{-k_i}J_i g)}.$$

Because of (13), the numbers $\alpha_\nu(f)$, $\beta_\nu(f)$, $\alpha_\nu(g)$, and $\beta_\nu(g)$ all satisfy the conditions in (6).

By (10 a), (16), (18 a) and the fact that $\varepsilon_2 < \frac{1}{2}$, we have

$$\frac{S}{2} < \frac{\ell(f^{-k_i}I_i f)}{\ell(f^{-k_i}J_i f)} < 2T.$$

Thus,

$$\begin{aligned}\ell(f^{-k_i}(I_i f \cup J_i f)) &= \ell(f^{-k_i}I_i f) + \ell(f^{-k_i}J_i f)\\ &\leq (2T+1)\ell(f^{-k_i}J_i f) \leq \lambda^{-1}\varepsilon_1\end{aligned}$$

by the definition of k_i, and (10 b).

This means that there is some j_i such that $f^{-k_i}(I_i f \cup J_i f) \subset \overline{B}_{j_i}$. Also

$$\ell(f^{-k_i}J_i f) \geq \frac{\lambda^{\nu_0}\varepsilon_1}{1+2T} > \frac{\lambda^{\nu_0}\varepsilon_1}{1+2T} \cdot \frac{S}{2}$$

and

$$\ell(f^{-k_i}I_i f) \geq \frac{S}{2}\ell(f^{-k_i}J_i f) \geq \frac{\lambda^{\nu_0}\varepsilon_1}{1+2T} \cdot \frac{S}{2}.$$

Since $(f^{-k_i}I_i f)(g) = g^{-k_i}I_i(g)$, and $(f^{-k_i}J_i f)(g) = g^{-k_i}J_i(g)$, (12) gives us

$$\left|\frac{\ell(f^{-k_i}I_i f)}{\ell(f^{-k_i}J_i f)} - \frac{\ell(g^{-k_i}I_i g)}{\ell(g^{-k_i}J_i g)}\right| < \varepsilon_3 \quad \text{for each } i.$$

Hence,

$$\begin{aligned}\frac{\ell I_i(g)}{\ell J_i(g)} &= \frac{\ell(g^{-k_i}I_i g)}{\ell(g^{-k_i}J_i g)} \prod_{\nu=\nu_0}^{k_i} \frac{\alpha_\nu(g)}{\beta_\nu(g)}\\ &\geq (1-\varepsilon_2)\left(\frac{\ell(f^{-k_i}I_i f)}{\ell(f^{-k_i}J_i f)} - \varepsilon_3\right)\\ &\geq (1-\varepsilon_2)\left((1+\varepsilon_2)^{-1}\frac{\ell I_i f}{\ell J_i f} - \varepsilon_3\right).\end{aligned}$$

This implies that

$$\liminf_{i\to\infty}\frac{\ell I_i g}{\ell J_i g}\geq(1-\varepsilon_2)((1+\varepsilon_2)^{-1}\tau^u(\Lambda(f))-\varepsilon_3).$$

By the choices of ε_3 and ε_2, we have

$$\begin{aligned}\varepsilon_3(1-\varepsilon_2)&<\frac{\varepsilon_4}{2}S<\frac{\varepsilon_4}{2}\tau^u(\Lambda(f))\\&<\left(\frac{1-\varepsilon_2}{1+\varepsilon_2}-(1-\varepsilon_4)\right)\tau^u(\Lambda(f))\\&=\left(\frac{1-\varepsilon_2}{1+\varepsilon_2}\right)\tau^u(\Lambda(f))-(1-\varepsilon_4)\tau^u(\Lambda(f)),\end{aligned}$$

so

$$\begin{aligned}(1-\varepsilon_4)\tau^u(\Lambda(f))&<\left(\frac{1-\varepsilon_2}{1+\varepsilon_2}\right)\tau^u(\Lambda(f))-\varepsilon_3(1-\varepsilon_2)\\&\leq\liminf_{i\to\infty}\frac{I_i g}{J_i g}\end{aligned}$$

which is (17).

This completes the proof of Proposition 6.

We now proceed to prove theorem 3. We begin by stating several lemmas whose proofs will be deferred to section 5.

The first lemma is the main step in the proof of theorem 3.

Lemma 7. — *Suppose $\{f_t\}$ and $p=p_{t_0}$ are as in the statement of theorem 3. Given $\varepsilon>0$, there is a t_1 with $|t_1-t_0|<\varepsilon$ such that f_{t_1} has two infinite hyperbolic basic sets $\Lambda_1(t_1)$ and $\Lambda_2(t_1)$ near $O(x_0)$ satisfying:*

(1) $p_{t_1}\in\Lambda_1(t_1)$;

(2) $W^u(\Lambda_1(t_1))$ *has transverse intersections with* $W^s(\Lambda_2(t_1))$, *and* $W^u(\Lambda_2(t_1))$ *has transverse intersections with* $W^s(\Lambda_1(t_1))$;

(3) *there is a point x_1 near x_0 such that $\{f_t\}$ creates a non-degenerate tangency of* $W^u(\Lambda_2(t_1))$ *and* $W^s(\Lambda_2(t_1))$ *at* (t_1, x_1);

(4) $\tau^u(\Lambda_1(t_1))\cdot\tau^s(\Lambda_2(t_1))>1$.

In lemma 7, p_{t_1} refers to the unique periodic point of f_{t_1} near $p=p_{t_0}$ for t_1 near t_0.

Lemma 8. — *Suppose Λ_1 and Λ_2 are two hyperbolic basic sets for $f: M^2\to M^2$ such that $W^u(\Lambda_1)$ meets $W^s(\Lambda_2)$ transversely at z_1, $W^u(\Lambda_2)$ meets $W^s(\Lambda_1)$ transversely at z_2, and $\{z_1, z_2\}\cap(\Lambda_1\cup\Lambda_2)=\emptyset$. Then there is a hyperbolic basic set Λ_3 for f such that $\Lambda_1\cup\Lambda_2\subset\Lambda_3$.*

Remark. — Lemma 8 should be thought of as a two-dimensional generalization of Smale's homoclinic point theorem [22]. In that theorem, $\Lambda_1 = \Lambda_2$ is a single hyperbolic periodic orbit. The lemma remains true for a C^1 diffeomorphism $f: M \to M$, dim M arbitrary, provided one assumes $\dim \Lambda_1 = \dim \Lambda_2 = 0$. Instead of the Markov coverings we use in our proof of lemma 8 (which do not always exist in higher dimensions), one uses the semi-invariant disk families of [10] in a careful way.

Lemma 9. — *Suppose γ_1 and γ_2 are two C^2 curves having a non-degenerate tangency at a point x_0 in M. Let $\mathscr{F}_1$, $\mathscr{F}_2$ be two families of C^2 curves such that:*

(1) *$\mathscr{F}_1$ and $\mathscr{F}_2$ are foliations of a neighborhood of x_0;*

(2) $\gamma_1 \in \mathscr{F}_1$, $\gamma_2 \in \mathscr{F}_2$;

(3) *for $i = 1, 2$, the map $x \mapsto T_x\mathscr{F}_{ix}$ is a C^1 map for x near x_0 where $T_x\mathscr{F}_{ix}$ is the unit tangent vector to the leaf $\mathscr{F}_{ix}$ at x.*

Then, $\mathscr{F}_1$ and $\mathscr{F}_2$ are tangent near x_0 along a C^1 curve $\gamma(\mathscr{F}_1, \mathscr{F}_2)$ which is transverse to γ_1 and γ_2 at x_0. Moreover, if $\mathscr{F}_1^1$ and $\mathscr{F}_2^1$ are two foliations whose unit tangent fields are C^1 close to those of $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively, then $\gamma(\mathscr{F}_1^1, \mathscr{F}_2^1)$ is C^1 close to $\gamma(\mathscr{F}_1, \mathscr{F}_2)$.

Proof of theorem 3. — Let t_1, $\Lambda_1(t_1)$ and $\Lambda_2(t_1)$ be as in lemma 7. By lemma 8, there is a hyperbolic basic set $\Lambda_3(t_1)$ so that $\Lambda_3(t_1) \supset \Lambda_1(t_1) \cup \Lambda_2(t_1)$. Then, $\{f_t\}$ creates a non-degenerate tangency of $W^u(\Lambda_3(t))$ and $W^s(\Lambda_3(t))$ at (t_1, x_1), and

$$\tau^u(\Lambda_3(t_1)) \,.\, \tau^s(\Lambda_3(t_1)) > 1.$$

Let $E^s \oplus E^u = T_{\Lambda_3(t_1)}M$ be the continuous splitting of $T_{\Lambda_3(t_1)}M$ given in the definition of hyperbolicity. As in the proof of proposition 5, we may use theorem (6.4 *b*) of [9] to give two f_t-invariant foliations $\mathscr{F}^u(t)$, $\mathscr{F}^s(t)$ on U which extend $W^u(\Lambda_3(t))$, $W^s(\Lambda_3(t))$ for t near t_1. Choose integers $n_1 < 0 < n_2$ such that $f_{t_1}^n(x_1) \in U$ for $n \leq n_1$, and $f_{t_1}^n(x_1) \in U$ for $n \geq n_2$. Let $\mathscr{F}_1(t) = f_t^{-n_1}\mathscr{F}^u(t)$ and $\mathscr{F}_2(t) = f_t^{-n_2}\mathscr{F}^s(t)$. Then, near x_1, $\mathscr{F}_1(t_1)$ and $\mathscr{F}_2(t_1)$ satisfy the hypotheses of lemma 9. So we get a C^1 curve $\gamma(\mathscr{F}_1(t_1), \mathscr{F}_2(t_1)) = \gamma(t_1)$ through x_1 transverse to $\mathscr{F}_{1x_1}(t_1)$ and $\mathscr{F}_{2x_1}(t_1)$ along which $\mathscr{F}_1(t_1)$ and $\mathscr{F}_2(t_1)$ are tangent. Since $\tau^u(\Lambda_3(t_1)) \,.\, \tau^s(\Lambda_3(t_1)) > 1$, there are Cantor sets $F(t_1) \subset W^u(\Lambda_3(t_1)) \cap \gamma(t_1)$ and $G(t_1) \subset W^s(\Lambda_3(t_1)) \cap \gamma(t_1)$ very near x_1 with

$$\tau(F(t_1)) \,.\, \tau(G(t_1)) > 1.$$

Also, all of these objects are defined for t near t_1 and vary nicely with t. In particular, $\tau(F(t))$ and $\tau(G(t))$ vary continuously with t. This can be seen as follows. First project $F(t)$ along the leaves of $\mathscr{F}_1(t)$ into some $W_\varepsilon^s(z_t, f_t) \cap \Lambda_3(t)$ with $z_t \in \Lambda_3(t)$, and project $G(t)$ along the leaves of $\mathscr{F}_2(t)$ into some $W_\varepsilon^u(z_t', f_t) \cap \Lambda_3(t)$ with $z_t' \in \Lambda_3(t)$. Call the projected Cantor sets $\overline{F}(t)$ and $\overline{G}(t)$. Using C^2 continuous dependence of $W_\varepsilon^s(z_t, f_t)$ and $W_\varepsilon^u(z_t', f_t)$ on t, and the arguments in the proof of proposition 6, one sees that $\tau(\overline{F}(t))$ and $\tau(\overline{G}(t))$ are continuous in t. Projecting back from $\overline{F}(t)$, $\overline{G}(t)$ to $F(t)$,

$G(t)$ gives the continuity of $\tau(F(t))$ and $\tau(G(t))$. Clearly, there are t's near t_1 for which $F_0(t)$ and $G_0(t)$ properly overlap. Again the arguments of the proof of proposition 6 will show that such t's may be found so that $\tau(F(t)).\tau(G(t))>1$. Applying lemma 4 to such a t will give that $\Lambda_3(t)$ is a wild hyperbolic set, and complete the proof of theorem 3.

There is a slight subtlety here. The sets $F(t_1)$ and $G(t_1)$ are very small, and a perturbation to $F(t)$ and $G(t)$ sufficient to make $F_0(t)$ and $G_0(t)$ properly overlap might destroy the thickness conditions. To insure this does not happen, one proceeds as follows. Let $B_\varepsilon(x_1)$ denote the ball of radius ε about x_1. Pick $\delta>0$ so that

$$\tau^u(\Lambda_3(t_1)).\tau^s(\Lambda_3(t_1))>1+2\delta.$$

Then choose $\varepsilon_1>0$ small enough so that

$$(1-\varepsilon_1)^4((1-\varepsilon_1)^4(1+\delta)-\varepsilon_1)>1+\frac{\delta}{2}.$$

Now, fix integers n_1, $n_2>0$ and a small $\varepsilon_2>0$ so that the next property holds. If $|t-t_1|<\varepsilon_2$ and F and G are Cantor sets such that:

a) $F\subset W^u(\Lambda_3(t))\cap\gamma(t)\cap B_{\varepsilon_2}(x_1)$,

b) $\tau(F)>\dfrac{\tau^u(\Lambda_3(t))}{2}$,

c) $G\subset W^s(\Lambda_3(t))\cap\gamma(t)\cap B_{\varepsilon_2}(x_1)$ and

d) $\tau(G)>\dfrac{\tau^s(\Lambda_3(t))}{2}$,

then $\dfrac{\tau(\overline{F})}{\tau(F)}$, $\dfrac{\tau(\overline{G})}{\tau(G)}$, $\dfrac{\tau(f_t^{-n_1}\overline{F})}{\tau(\overline{F})}$, and $\dfrac{\tau(f_t^{n_2}\overline{G})}{\tau(\overline{G})}$ are closer to 1 than ε_1. Upper bars here mean projection along leaves of $\mathscr{F}_1(t)$ and $\mathscr{F}_2(t)$. Once n_1, n_2, and ε_2 are fixed, take $F(t_1)\subset W^u(\Lambda_3(t_1))\cap\gamma(t_1)\cap B_{\varepsilon_2}(x_1)$ and $G(t_1)\subset W^s(\Lambda_3(t_1))\cap\gamma(t_1)\cap B_{\varepsilon_2}(x_1)$ so that:

$$\tau(F(t_1))>\frac{\tau^u(\Lambda_3(t_1))}{2}, \qquad \tau(G(t_1))>\frac{\tau^s(\Lambda_3(t_1))}{2},$$

and

$$\tau(F(t_1)).\tau(G(t_1))>1+\delta.$$

Finally, pick $\varepsilon_3<\varepsilon_2$ so that $|t-t_1|<\varepsilon_3$ implies

$$\tau(f_t^{-n_1}\overline{F}(t)).\tau(f_t^{n_2}\overline{G}(t))>\tau(f_{t_1}^{-n_1}\overline{F}(t_1)).\tau(f_{t_1}^{n_2}\overline{G}(t_1))-\varepsilon_1.$$

Then, for $|t-t_1|<\varepsilon_3$ with $F_0(t)$ and $G_0(t)$ properly overlapping, we also have

$$\begin{aligned}\tau(F(t)).\tau(G(t)) &\geq (1-\varepsilon_1)^4\tau(f_t^{-n_1}\overline{F}(t)).\tau(f_t^{n_2}\overline{G}(t))\\ &\geq (1-\varepsilon_1)^4(\tau(f_{t_1}^{-n_1}\overline{F}(t_1)).\tau(f_{t_1}^{n_2}\overline{G}(t_1))-\varepsilon_1)\\ &\geq (1-\varepsilon_1)^4((1-\varepsilon_1)^4\tau(F(t_1)).\tau(G(t_1))-\varepsilon_1)\\ &> (1-\varepsilon_1)^4((1-\varepsilon_1)^4(1+\delta)-\varepsilon_1)>1+\frac{\delta}{2}.\end{aligned}$$

5. This section contains the proofs of lemmas 7, 8 and 9.

Proof of lemma 7. — Fix $\varepsilon > 0$ small. Let us first show that:

(1) there are a ν with $|\nu - t_0| < \varepsilon$ and an integer $0 \leq k < n_0$ such that $\{f_t\}$ creates a non-degenerate tangency of $W^s(p_\nu)$ and $W^u(f_\nu^k p_\nu)$ at a point (ν, x_ν) with the orbit of x_ν passing near x_0.

There is nothing to prove if Λ_{t_0} reduces to the single orbit $O(p_{t_0})$, so we may assume that Λ_t is infinite for t near t_0.

From the local product structure of $f_t|\Lambda_t$ and the fact that $f_t|\Lambda_t$ has a dense orbit, it follows that $W^u(O(p_t))$ is dense in $W^u(\Lambda_t)$ and $W^s(O(p_t))$ is dense in $W^s(\Lambda_t)$ where $O(p_t)$ is the orbit of p_t. Let $x, y \in \Lambda_{t_0}$ be such that $x_0 \in W^u(x, f_{t_0}) \cap W^s(y, f_{t_0})$. For t near t_0, $z \in \Lambda_{t_0}$, let $z_t = h_t z$ where $h_t : \Lambda_{t_0} \to \Lambda_t$ is the unique homeomorphism near the inclusion such that $f_t h_t = h_t f_{t_0}$. By continuous dependence on compact sets of $W^u(z, f_t)$ and $W^s(z, f_t)$ for $z \in \Lambda_t$ and t near t_0, we may choose $\delta > 0$ such that if $z_1 \in W^s_\delta(x, f_{t_0}) \cap \Lambda_{t_0}$ and $z_2 \in W^u_\delta(y, f_{t_0}) \cap \Lambda_{t_0}$, then there are a $t(z_1, z_2)$ and a point $x(z_1, z_2)$ with $|t(z_1, z_2) - t_0| < \varepsilon$ such that $\{f_t\}$ creates a non-degenerate tangency of $W^u(z_{1t}, f_t)$ and $W^s(z_{2t}, f_t)$ at $(t(z_1, z_2), x(z_1, z_2))$. Pick integers $0 \leq n_1 \leq n_2 < n_0$ such that $W^u(f_{t_0}^{n_1} p_{t_0}, f_{t_0})$ meets $W^s_\delta(x, f_{t_0})$ and $W^s(f_{t_0}^{n_2} p_{t_0}, f_{t_0})$ meets $W^u_\delta(y, f_{t_0})$. Let $z_1 \in W^u(f_{t_0}^{n_1} p_{t_0}, f_{t_0})) \cap W^s_\delta(x, f_{t_0})$ and let $z_2 \in W^s(f_{t_0}^{n_2} p_{t_0}, f_{t_0}) \cap W^u_\delta(y, f_{t_0})$. Since $\{f_t\}$ creates a non-degenerate tangency of $W^u(z_{1t}, f_t)$ and $W^s(z_{2t}, f_t)$ at $(t(z_1, z_2), x(z_1, z_2))$, it also creates one of $W^s(f_t^{-n_2} z_{2t}, f_t)$ and $W^u(f_t^{-n_2} z_{1t}, f_t)$ at (ν, x_ν) with $\nu = t(z_1, z_2)$ and $x_\nu = f_\nu^{-n_2}(x(z_1, z_2))$. This proves (1).

To prove lemma 7, we wish to produce hyperbolic sets $\Lambda_1(t_1)$, $\Lambda_2(t_1)$ near $O(x_\nu)$ for some t_1 with $|t_1 - \nu| < \varepsilon$ satisfying (7.1)-(7.4).

We will assume $\det T_{p_{t_0}} f_{t_0}^{n_0} < 1$ as the other case follows replacing $\{f_t\}$ by $\{f_t^{-1}\}$. Since there is a t near ν for which $W^s(p_t, f_t)$ has transverse intersections with $W^u(f_t^k p_t, f_t)$ near x_ν, it follows from Smale's homoclinic point theorem (or the generalization in lemma 8 below) that for such a t there is an infinite hyperbolic basic set $\Lambda_1(t)$ near $O(x_\nu)$ with $O(p_t) \subset \Lambda_1(t)$. Also, we may assume $\Lambda_1(t)$ small enough so that there are points z_1, z_2 not in $\Lambda_1(t)$ such that $W^u(p_t)$ meets $W^s(\Lambda_1(t))$ transversely at z_1 and $W^s(p_t)$ meets $W^u(\Lambda_1(t))$ transversely at z_2. Further, an analysis of how hyperbolic sets near $O(x)$ are created as t moves near ν shows that we may choose a t near ν so that $W^s(p_t, f_t)$ and $W^u(f_t^k p_t, f_t)$ are tangent near x_ν and f_t has an infinite hyperbolic set $\Lambda_1(t)$ as just described. (For more information on the creation of hyperbolic sets near a tangency, see [19].) Relabeling, we assume that $\{f_t\}$ creates a non-degenerate tangency at (ν, x_ν) of $W^s(p_\nu)$ and $W^u(f_\nu^k p_\nu)$, and f_ν has an infinite hyperbolic basic set $\Lambda_1(\nu)$ so that $W^s(\Lambda_1(\nu))$ has a transverse intersection z_1 with $W^u(p_\nu)$, $W^u(\Lambda_1(\nu))$ has a transverse intersection z_2 with $W^s(p_\nu)$ and $\{z_1, z_2\} \cap \Lambda_1(\nu) = \emptyset$. For convenience of notation in case $n_0 = 1$ below, we assume $1 \leq k \leq n_0$. By proposition (6.1), $\tau^u(\Lambda_1(\nu)) > 0$. Let $T > 2\tau^u(\Lambda_1(\nu))^{-1}$. We will show there are t's near ν so that f_t has a basic set $\Lambda_2(t)$ near $O(x_\nu)$ satisfying (7.2) and (7.3) and having $\tau^s(\Lambda_2(t))$ bigger than T. Since $\tau^u(\Lambda_1(t))$ is near $\tau^u(\Lambda_1(\nu))$ for t near ν (Proposition (6.2)), we will thus also obtain (7.4).

Let (u, v) be coordinates on $\mathbf{R}^2$. Choose coordinates (U, φ) about p_ν so that $\varphi : U \to \mathbf{R}^2$ is a C^r diffeomorphism such that

$$\varphi^{-1}((v=0)) \subset W^s(p_\nu, f_\nu), \qquad \varphi^{-1}((u=0)) \subset W^u(p_\nu, f_\nu), \qquad \varphi(p_\nu) = (0, 0),$$

and
$$\{f_\nu^{n_0 j}(x_\nu), f_\nu^{-n_0 j - k}(x_\nu)\} \subset U \quad \text{for} \quad j \geq 0$$

where n_0 is the period of p_ν. Suppose $x_\nu = (u_0, 0)$ and $f_\nu^{-k}(x_\nu) = (0, v_0)$. Since the tangency at x_ν is non-degenerate, we have, for (u, v) in some small neighborhood V of $(0, v_0)$, $\varphi f_\nu^k \varphi^{-1}(u, v) = (g_1(u, v), g_2(u, v))$ where g_1 and g_2 are C^r mappings from V to $\mathbf{R}$ such that $g_1(0, v_0) = u_0$, $g_2(0, v_0) = 0$, $g_{1v}(0, v_0) \neq 0$, $g_{2v}(0, v_0) = 0$ and $g_{2vv}(0, v_0) \neq 0$. We assume $u_0 < 0$, $v_0 > 0$, $g_{1v}(0, v_0) > 0$, and $g_{2vv}(0, v_0) < 0$, the other cases being similar. Here and below $g_{i\sigma}$ is the partial derivative of g_i with respect to the variable σ, and $g_{i\sigma\tau}$ is the second order partial derivative of g_i with respect to σ and τ.

We have the following figure

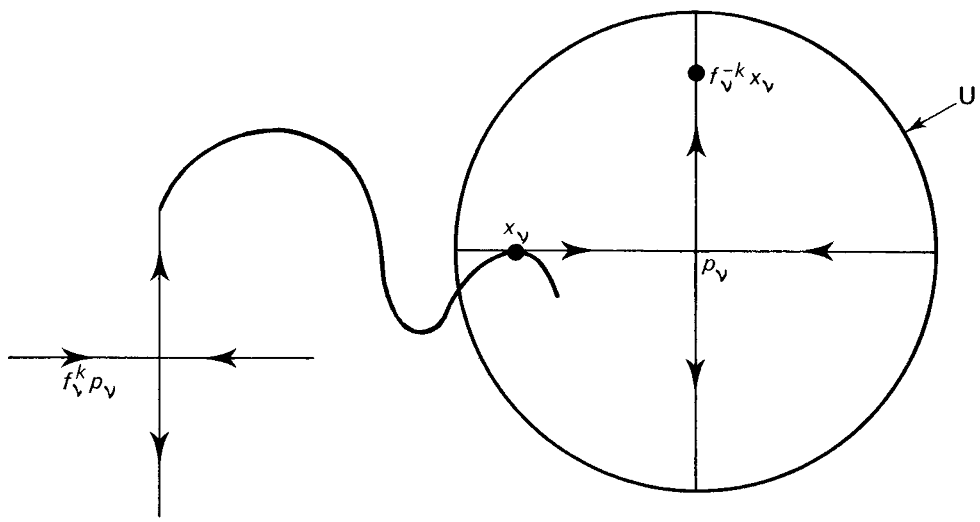

FIG. 5.1

Since $\{f_t\}$ creates a non-degenerate tangency of $W^u(f_t^k p_t)$ and $W^s(p_t)$ at (ν, x_ν) we have that for some t's near ν, $W^u(f_t^k p_t)$ has two transverse intersections with $W^s(p_t)$ near x_ν. We may as well suppose that such t's are larger than ν.

Then we have the next figure where t is near ν and x_t is near x_ν

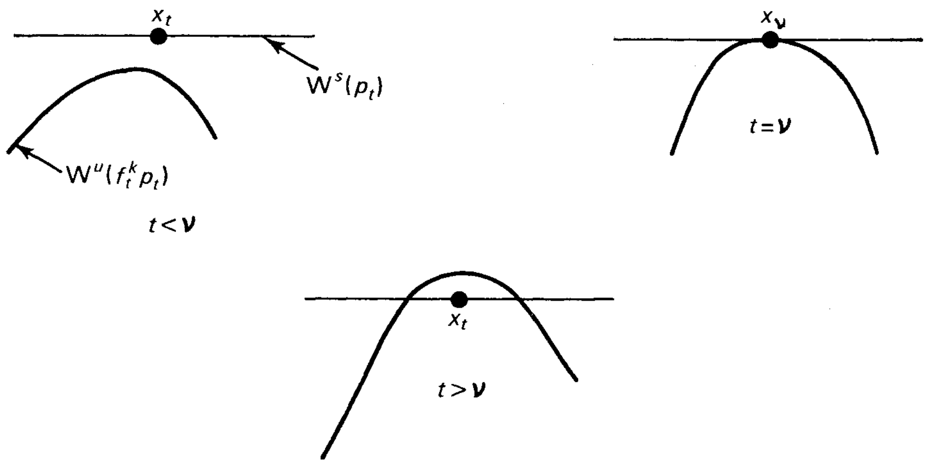

FIG. 5.2

Geometrically it is easy to motivate the construction of the required set $\Lambda_2(t)$. Looking at figures 5.1 and 5.2, one sees that we may choose a disk D near x_ν, a large integer n, and a t near ν so that f_t^n maps D to V as a Smale horseshoe diffeomorphism. That is, D and $f_t^n D$ are as in figure 5.3 with $A'=f_t^n A$, $B'=f_t^n B$, etc.

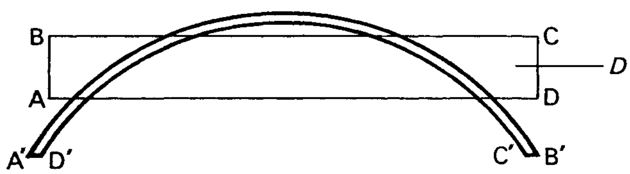

FIG. 5.3

Thus, $\Lambda_2(t)=\bigcap_j f_t^{nj} D$ will be a hyperbolic basic set. Since $\det T_{p_\nu} f_\nu^{n_0} < 1$, for large n, f_t^n contracts D horizontally much more than it expands D vertically.

So, if we arrange for f_t^n to map the right side of D nearly tangent to the top of D and $f_t^n D - D$ to be small relative to the size of $f_t^n D$, we have something like figure 5.4 *a*.

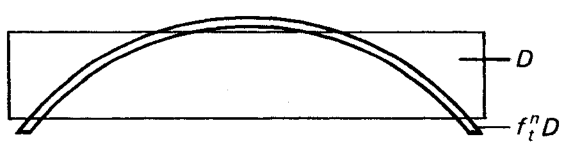

FIG. 5.4 *a*

Hence, $f_t^{-n}(D \cap f_t^n D)$ looks like figure 5.4 *b*

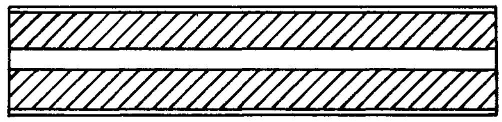

FIG. 5.4 *b*

where the non-shaded areas have small heights.

For such t, n, and D let $\Lambda_{21}(t) = \bigcap_{-\infty<j<\infty} f_t^{nj} D$ be the largest f_t^n invariant subset of D. A little thought makes it reasonable that $\Lambda_{21}(t)$ is hyperbolic for f_t^n, and $\tau^s(\Lambda_{21}(t)) \to \infty$ as $n \to \infty$. We pick n large enough to give $\tau^s(\Lambda_{21}(t)) > 2T$. We then show that (7.2) holds for $\Lambda_{21}(t)$, *i.e.* $W^u(\Lambda_{21}(t), f_t^n)$ meets $W^s(O(p_t), f_t)$ transversely and $W^s(\Lambda_{21}(t), f_t^n)$ meets $W^u(O(p_t), f_t)$ transversely. Next, we decrease t slightly to t_1 so that $W^u(\Lambda_{21}(t_1), f_{t_1}^n)$ and $W^s(\Lambda_{21}(t_1), f_{t_1}^n)$ become tangent at a single point y_{t_1} near the top of D. Then $\Lambda_{21}(t_1)$ is no longer hyperbolic for $f_{t_1}^n$, but if we let U_{t_1} be a small neighborhood of y_{t_1}, then $\Lambda_{22}(t_1) = \bigcap_{-\infty<j<\infty} f_{t_1}^{nj}(D - U_{t_1})$ is hyperbolic for $f_{t_1}^n$, satisfies (7.2) and (7.3) for $f_{t_1}^n$, and has $\tau^s(\Lambda_{22}(t_1)) > T$ [1]. Finally, we will set

$$\Lambda_2(t_1) = \bigcup_{0 \le j \le n} f_{t_1}^j(\Lambda_{22}(t_1))$$

and (7.1)-(7.4) will be satisfied.

We proceed to the analytic details. To simplify the notation, we assume $n_0 = k = 1$, so that $f_\nu(p_\nu) = p_\nu$ and $x_\nu \in W^s(p_\nu) \cap W^u(p_\nu)$.

Let $T > 0$. We will produce a basic set $\Lambda(t)$ near x_ν for some power f_t^{n+1}, t near ν, such that $\tau^s(\Lambda(t)) > T$, $W^u(\Lambda(t), f_t^{n+1})$ has a transverse intersection with $W^s(p_t)$, and $W^u(p_t)$ has a transverse intersection with $W^s(\Lambda(t), f_t^{n+1})$.

To prove hyperbolicity and to estimate the thickness $\tau^s(\Lambda(t))$ of $\Lambda(t)$ we will need to estimate the derivatives $T_z f_t^n$ of iterates of f_t for t near ν as a function of certain z's near p_t. For this, our methods require that f_t be linear near p_t and that the linearizations vary continuously with t.

Since we need something like $z \mapsto T_z f_t^n$ to be Lipschitz in z, C^1 linearizations (which always exist if f is C^2) are not enough. Using the assumptions that f_ν is C^3 and $\det T_{p_\nu} f_\nu < 1$, we can produce C^1 linearizations φ_t for f_t near p_t which are C^2 off $W^u(p_t)$, and this enables us to carry out the necessary estimates.

We begin with the construction of the linearization φ_ν for f_ν near p_ν. We will let U be a neighborhood of p_ν in M which is small enough for each statement involving U to hold. First, since $W^u(p_\nu, f_\nu)$ and $W^s(p_\nu, f_\nu)$ are one-dimensional, we may use Sternberg's theorem on linearizations for contractions [26] and simple extension procedures to produce a diffeomorphism $\varphi : U \to \mathbf{R}^2$ so that $\varphi(p_\nu) = (0, 0)$ and $\varphi f_\nu \varphi^{-1}$ is

[1] The neighborhood U_{t_1} has to be chosen, of course, so that $\Lambda_{22}(t_1)$ is a basic set for $f_{t_1}^n$.

linear on $\varphi(W^u(p_\nu, f_\nu) \cup W^s(p_\nu, f_\nu))$ near $(0, 0)$. This can be done with φ of class C^r as long as f is C^r $(r \geq 2)$.

Identifying f with $\varphi f_\nu \varphi^{-1}$ and p_ν with $(0, 0)$, we assume f_ν is linear on $W^u(p_\nu, f_\nu) \cup W^s(p_\nu, f_\nu)$ near p_ν.

Since we will take large iterates of f to construct $\Lambda(t)$, we may as well assume the eigenvalues of $T_{p_\nu} f_\nu$ are both positive. We will construct the linearization using foliations defined by Tf-invariant vector fields. Without the positivity assumption on the eigenvalues one would have to use line fields.

Let D^s (D^u) be a small neighborhood of p_ν in $W^s(p_\nu)$ $(W^u(p_\nu))$. Let F^u be a Tf_ν-invariant C^2 vector field on a neighborhood of $D^s - f_\nu D^s$ which is transverse to D^s. This is constructed by taking a C^2 vector field X on a neighborhood U_1 of ∂D^s, a C^2 vector field Y on a neighborhood U_2 of $D^s - f_\nu D^s$, and a C^∞ bump function ψ which is one on a neighborhood of $\partial D^s \cup f_\nu \partial D^s$ and whose support is in $U_1 \cup f_\nu U_1$. Let $\overline{X}$ be the vector field on $U_1 \cup f_\nu U_1$ which is equal to X on U_1 and $Tf_\nu \circ X \circ f_\nu^{-1}$ on $f_\nu U_1$. Then let F^u be the vector field $\psi \overline{X} + (1 - \psi) Y$. If X and Y are transverse to D^s and point to the same side of D^s, then F^u is transverse to D^s and Tf_ν-invariant. Iterating F^u by Tf_ν^n, $n > 0$, gives a C^2 Tf_ν-invariant vector field, also denoted F^u, on $U - D^u$ for some small neighborhood U of p_ν.

Let $\overline{F}^u_z = \dfrac{F^u_z}{|F^u_z|}$ be the unit field determined by F^u_z for $z \in U - D^u$. By the λ-lemma, $\{\overline{F}^u_z\}_{z \in U - D^u}$ extends to a continuous unit vector field $\overline{F}^u$ on U where $\overline{F}^u_z$, $z \in D^u$, is a unit vector tangent to D^u at z. Using the methods in the proof of theorem (6.3 *b*) of [9], one sees that $\overline{F}^u$ is a C^1 vector field on U. This is done as follows. Write $f = f_\nu$ and let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be the derivative of $\varphi f \varphi^{-1}$ at $(0, 0)$. Using a suitable C^∞ function $\tilde{\psi} : \varphi U \to \mathbf{R}$ with compact support and value one at $(0, 0)$, and replacing f by $\tilde{\psi} . \varphi f \varphi^{-1} + (1 - \tilde{\psi}) L$, we may assume that f is a C^3 diffeomorphism of $\mathbf{R}^2$ which is uniformly C^1 near L. Let E be the vector bundle over $\mathbf{R}^2$ whose fiber at $z \in \mathbf{R}^2$ is the space $L(\mathbf{R}^1, \mathbf{R}^1)$ of linear maps from $\mathbf{R}^1$ to $\mathbf{R}^1$. Use the norm $|(u, v)| = \max\{|u|, |v|\}$ on $\mathbf{R}^2$. The line field of $\overline{F}^u$ on $\mathbf{R}^2$ may be thought of as a line field whose value at $z \in \mathbf{R}^2$ is the graph of an element $\sigma_z \in L(\mathbf{R}^1, \mathbf{R}^1)$. Thus, we are looking for a section σ of E over $\mathbf{R}^2$ whose graph is Tf-invariant. This σ must satisfy the equation $T_z f(\text{graph } \sigma_z) = \text{graph } \sigma_{f(z)}$, $z \in \mathbf{R}^2$. This equation defines a C^1 map $G : E \to E$ which covers f. That is, the diagram below commutes (with $\pi : E \to \mathbf{R}^2$ the natural projection)

$$\begin{array}{ccc} E & \xrightarrow{G} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbf{R}^2 & \xrightarrow{f} & \mathbf{R}^2 \end{array}$$

If μ and λ are the eigenvalues of L with $0 < \mu < 1 < \lambda$, then the fiber Lipschitz constant of G is $(\mu + \varepsilon)(\lambda - \varepsilon)^{-1}$ and the Lipschitz constant of f^{-1} is $\mu^{-1} + \varepsilon$ where ε is small. Thus, letting Lip S denote the Lipschitz constant of a map S, we have $(\sup_z \text{Lip}\, G_z) . \text{Lip} f^{-1} < 1$ for ε small. This implies that G has a unique invariant section which is C^1.

Replacing f_ν by f_ν^{-1}, and repeating the above procedure, we may construct a unit vector field $\bar{F}^s$ on U which contains the tangent field to D^s and whose line field is Tf^{-1}-invariant. Since f is C^3 and $\det T_{p_\nu} f_\nu < 1$, the previous method and theorem (3.2) in [11] show that $\bar{F}^s$ is C^2. Indeed, in this case, $\sup_z \text{Lip}\, G_z < (\mu^{-1} - \varepsilon)^{-1}(\lambda^{-1} + \varepsilon)$ and $\text{Lip}(f^{-1})^{-1} \leq \lambda + \varepsilon$. Since $\mu\lambda < 1$, one has $(\mu^{-1} - \varepsilon)^{-1}(\lambda^{-1} + \varepsilon)(\lambda + \varepsilon)^2 < 1$ for ε small. Also, the induced map G is C^2 since f is C^3. Thus, the invariant section is C^2, and this implies that $\bar{F}^s$ is C^2.

Let $\bar{\mathscr{F}}^s$ and $\bar{\mathscr{F}}^u$ be the foliations of U obtained by integrating $\bar{F}^s$ and $\bar{F}^u$, respectively. The mapping $\varphi_\nu : U \to D^s \times D^u$ defined by $\varphi_\nu(z) = (\bar{\mathscr{F}}_z^u \cap D^s) \times (\bar{\mathscr{F}}_z^s \cap D^u)$ is our linearization which is C^1 on U and C^2 off D^u. It is clear from the construction that φ_t is defined for t near ν and varies continuously in the appropriate topologies.

Now, in $\varphi_\nu U$, $\varphi_\nu f_\nu \varphi_\nu^{-1}$ is given by $(u_1, v_1) = (\mu u, \lambda v)$ with $0 < \mu < 1 < \lambda$ and $\mu\lambda < 1$. Suppose that $(u_0, 0) = \varphi_\nu(x_\nu)$, $(0, v_0) = \varphi_\nu f_\nu^{-1}(x_\nu)$ and for (u, v) near $(0, v_0)$,

$$\varphi_\nu f_\nu \varphi_\nu^{-1}(u, v) = (g_1(u, v), g_2(u, v))$$

as before. We identify f_ν with $\varphi_\nu f_\nu \varphi_\nu^{-1}$, U with $\varphi_\nu U = D^s \times D^u$, etc. We assume D^s and D^u each have length two.

In what follows, α will be a small positive number chosen so that each equation in which it appears is true for all $n \geq 0$, and $c_1, c_2, \ldots$ will be constants independent of n, each defined by the first equation in which it appears.

Let γ be a smooth curve through x_ν transverse to $W^s(p_\nu)$ and $W^u(p_\nu)$. Choose $n_1 > 0$ so that $f_\nu^{-n}\gamma \cap \gamma \cap U \neq \emptyset$ for $n \geq n_1$. Let $z_n \in f_\nu^{-n-1}\gamma \cap \gamma$ and $n+1 \geq n_1$ be such that $f_\nu^j(z_n) \in U$ for $0 \leq j \leq n+1$. Assume the coordinates φ_ν are chosen so that $v = -a(u-u_0)^2 + r_1(u)$ represents $\varphi_\nu W^u(p_\nu, f_\nu)$ near $(u_0, 0)$ where $\lim_{u \to u_0} \frac{|r_1(u)|}{|u-u_0|^2} = 0$ and $a > 0$.

Let $d_1 = g_{1v}(0, v_0)$ and set

$$\varepsilon_{1n} = \frac{(2+\alpha)\lambda^{-n}}{ad_1^2}. \tag{1}$$

Then,
$$a(d_1 \varepsilon_{1n})^2 - 2\lambda^{-n}\varepsilon_{1n} = \alpha\lambda^{-n}\varepsilon_{1n} \tag{2}$$

and
$$\frac{d_1 \varepsilon_{1n}}{\left(\frac{\alpha\lambda^{-n}}{2}\varepsilon_{1n}\right)^{1/2}} \geq 2T. \tag{3}$$

Writing $z_n = (u_n, v_n)$ in the φ_ν-coordinates, let $D_1^s \subset D^s$ be the interval centered at u_n with length $2d_1\varepsilon_{1n}$ and let $D_1^u \subset D^u$ be the interval centered at v_n with length $2\lambda^{-n}\varepsilon_{1n}$. Then set $D_n = D_1^s \times D_1^u$.

For t near ν, let D^u_{1t}, D^s_{1t}, etc., be the structures for f_t defined as for f_ν. Since $\varphi_t^{-1}(\{u_{nt}\}\times D^u_{1t})$ is a C^2 curve which is nearly orthogonal to $\varphi_t^{-1}D^s_t$, we have that $f^n_t\varphi_t^{-1}(\{u_{nt}\}\times D^u_{1t})$ is C^2 near its projection on $\varphi_t^{-1}D^u_t$. Therefore, the curve

$$\varphi_t f_t^{n+1}\varphi_t^{-1}(\{u_{nt}\}\times D^u_{1t})$$

is C^2 close to $W^u(p_t, f_t)$ near x_ν. Let $v(t)$ be the maximum value of the v-coordinates of this curve, and suppose this maximum is assumed at the point $\bar{x}(t)$ in U.

Let t_n be so that

$$(4) \qquad v(t_n)=\lambda^{-n}(v_0+\varepsilon_{1n})+\frac{\alpha\lambda^{-n}\varepsilon_{1n}}{2}.$$

We claim:

(5) for large n,

a) $\Lambda_n=\bigcap_j f_{t_n}^{(n+1)j}D_{n,t_n}$ is a hyperbolic basic set for $f_{t_n}^{n+1}$.

b) $W^u(\Lambda_n, f_{t_n}^{n+1})$ has a transversal intersection with $W^s(p_{t_n})$ and $W^s(\Lambda_n, f_{t_n}^{n+1})$ has a transversal intersection with $W^u(p_{t_n})$.

c) $\tau^s(\Lambda_n)>\frac{3}{2}T$.

In the following, we agree to take n large enough for each statement involving n to be true. Further, we will write $a\approx b$ to mean a is approximately the same as b where we leave to the reader the task of making the precise statement. Also, in most of our expressions, we will leave out the dependence on t_n of our structures and write D_n for D_{n,t_n}, f^{n+1} for $f_{t_n}^{n+1}$, etc. For $z=(u(z), v(z))\in D_n$, let $D^s_{nz}=D^s_1\times\{v(z)\}$ and let $D^u_{nz}=\{u(z)\}\times D^u_1$. Thus $\ell(D^s_{nz})=2d_1\varepsilon_{1n}$ and $\ell(D^u_{nz})=2\lambda^{-n}\varepsilon_{1n}$ for each $z\in D_n$. Also, for $z\in D_n$, $f^{n+1}D^u_{nz}$ is a C^2 curve whose distance from $\bar{x}(t_n)$ is near $g_{1u}(0, v_0)\mu^n d_1\varepsilon_{1n}$, and, if $w\in D_n$, the endpoints of $f^{n+1}D^u_{nz}$ are closer to the endpoints of $f^{n+1}D^u_{nw}$ than $3g_{1u}(0, v_0)\mu^n d_1\varepsilon_{1n}$.

Let $\pi^u: U\to D^u$ and $\pi^s: U\to D^s$ be the natural projections. For any curve γ C^2 near some part of $f^{n+1}D^u_1$ we have

$$c_1\ell(\pi^u\gamma)^{1/2}\leq\ell(\gamma)\leq c_2\ell(\pi^u\gamma)^{1/2} \quad\text{and}\quad c_3\ell(\pi^s\gamma)\leq\ell(\gamma)\leq c_4\ell(\pi^s\gamma)$$

with $c_1>0$, $c_2>0$ and c_3 and c_4 near 1.

By (2) and (4), for any $z\in D_n$, we have:

$$(6)\ a) \qquad \pi^u f^{n+1}D^u_{nz}\supset D^u_1=\pi^u D_n$$

and

$$b) \qquad \frac{\frac{1}{2}\ell(f^{n+1}D^u_{nz})-\ell(f^{n+1}D^u_{nz}-D_n)}{2\ell(f^{n+1}D^u_{nz}-D_n)}>2T.$$

For (6 *b*) note that the first expression of the inequality is bigger than

$$\frac{c_5\varepsilon_{1n} - c_6\left(\frac{\alpha\lambda^{-n}\varepsilon_{1n}}{2}\right)^{1/2}}{2c_6\left(\frac{\alpha\lambda^{-n}\varepsilon_{1n}}{2}\right)^{1/2}}.$$

Also, we have $\ell f^{n+1}\mathrm{D}^s_{nz} < c_7\mu^n d_1\varepsilon_{1n} < \frac{\alpha\lambda^{-n}\varepsilon_{1n}}{4}$ for n large since $\mu\lambda < 1$. So $f^{n+1}\mathrm{D}_n$ and D_n look as in figure 5.5

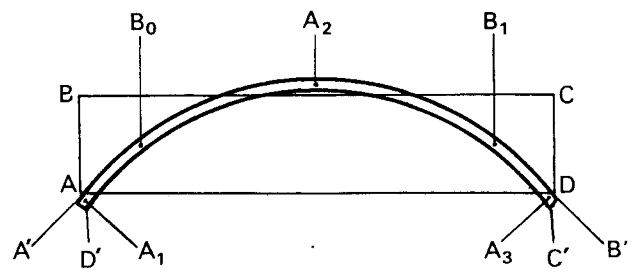

FIG. 5.5

where $\mathrm{A}' = f^{n+1}\mathrm{A}$, $\mathrm{B}' = f^{n+1}\mathrm{B}$, etc.

For later use let us label the two components of $f^{n+1}\mathrm{D}_n \cap \mathrm{D}_n$ by B_0 and B_1 and the three components of $f^{n+1}\mathrm{D}_n - \mathrm{D}_n$ by A_1, A_2, and A_3 as in the figure.

If m and j are integers with $m \geq 1$ and $0 \leq j \leq m-1$ and $z \in \Lambda_n$, let $\mathrm{D}^u_{nz}(m)$ denote the largest interval about z in $\mathrm{D}^u_{nz} \cap \bigcap_{0 \leq j \leq m-1} f^{-(n+1)j}\mathrm{D}_n$. We will need to prove:

(7) for large n and any $m > 0$, if $z \in \Lambda_n$ and $1 \leq j \leq m$, then $f^{(n+1)j}\mathrm{D}^u_{nz}(m)$ is C^2 near a part of the curve $v = v(t_n) - a(u-u_0)^2$.

We will defer the proof of (7).

To prove that Λ_n is hyperbolic we wish to apply theorem (3.1) of [18]. We need to find a sector S_{nz} in each $\mathrm{T}_z\mathrm{M}$, $z \in \Lambda_n$, so that for some $m(z) > 0$, $\mathrm{T}_z f^{(n+1)m(z)}$ maps S_{nz} into $\mathrm{S}_{n, f^{(n+1)m(z)}(z)}$, and for some $\beta > 1$, $\mathrm{T}_z f^{(n+1)m(z)} | \mathrm{S}_{nz}$ and $\mathrm{T}_y f^{-(n+1)m(z)} | (\mathrm{T}_y\mathrm{M} - \mathrm{S}_{ny})$ are β-expansions, where $y = f^{(n+1)m(z)}(z)$. Once this is done, a compactness argument allows one to choose $m(z)$ independent of z, and then (*e.g.* as in lemma (4.7) of [19]), one can show that the splitting $\mathrm{T}_z\mathrm{M} = \mathrm{T}_z\mathrm{D}^s \oplus \mathrm{T}_z\mathrm{D}^u$ is an almost hyperbolic splitting for Λ_n. By theorem (3.1) of [18], it follows that Λ_n is hyperbolic.

Unfortunately, our proof that Λ_n is hyperbolic is rather cumbersome. It would be good to find a simpler proof.

We proceed to find the sectors S_{nz}. For $z = (u(z), v(z)) \in \mathrm{D}_n \cap f^{n+1}(\mathrm{D}_n)$, and $w = f^{-n-1}z$, the curve $f^{n+1}\mathrm{D}^u_{nw}$ is C^2 near the curve

$$v = \lambda^{-n}(v_0 + \varepsilon_{1n}) + \frac{\alpha\lambda^{-n}\varepsilon_{1n}}{2} - a(u-u_0)^2 = v(t_n) - a(u-u_0)^2.$$

Thus, a vector $v=(v_1, v_2)\in T_w f^{n+1} D^u_{nw}$ satisfies

$$|v_2| \geq (2-\delta) a^{1/2} |v(t_n)-v(z)|^{1/2} |v_1| = (2-\delta) a |u(z)-u_0| \, |v_1| \tag{8}$$

where δ is a small number which we specify later. We set

$$\varepsilon(z)=(2-2\delta)a^{1/2}|v(t_n)-v(z)|^{1/2}$$

and take

$$S_{nz}=\{v=(v_1, v_2)\in T_z M : |v_2| \geq \varepsilon(z)|v_1|\}.$$

Note that C^2 estimates near D_n are legitimate since our linearization φ_{t_n} is C^2 on a neighborhood of D_n.

Now, if $v=(v_1, v_2)\in S_{nz}$, $z\in D_n \cap f^{n+1} D_n \cap f^{-n-1} D_n$, $v_1 \neq 0$, and $T_z f^n(v)=(v_1', v_2')$, then

$$\left|\frac{v_2'}{v_1'}\right| \geq \frac{\lambda^n}{\mu^n}\left|\frac{v_2}{v_1}\right| \geq \frac{\lambda^n}{\mu^n}(2-2\delta)a^{1/2}\left(\frac{\alpha}{2}\lambda^{-n}\varepsilon_{1n}\right)^{1/2} \geq c_8 \mu^{-n}.$$

Since $\mu\lambda < 1$, this implies that (for n large) $T_z f^{n+1}(v)\in S_{n, f^{n+1}(z)}$. In other words, the sectors $\{S_{nz}\}$ are $T_z f^{n+1}$-invariant. We now prove there is an integer $m(z)$ so that $T_z f^{(n+1)m(z)}$ is a $\left(\frac{9}{8}\right)^{1/2}$-expansion on S_{nz}. Once this is done, a similar argument replacing f by f^{-1} shows that $T_y f^{-(n+1)m(z)}$ expands $T_y M - S_{ny}$ where $y=f^{(n+1)m(z)}(z)$. In view of our previous discussion, this will prove that Λ_n is hyperbolic.

We use the norm $|(v_1, v_2)| = \max(|v_1|, |v_2|)$.

First observe that if $z\in\Lambda_n$, $0\neq v=(v_1, v_2)\in S_{nz}$, and we set

$$v(t_n)-v(z)=\left(\frac{\alpha}{2}+\tilde{\alpha}\right)\lambda^{-n}\varepsilon_{1n},$$

then

$$\frac{|T_z f^{n+1} v|}{|v|} \geq (2-2\delta)\left(\frac{\alpha}{2}+\tilde{\alpha}\right)^{1/2}(2+\alpha)^{1/2}. \tag{9}$$

This follows from the definitions of S_{nz}, ε_{1n}, and the facts that $|v|=|v_1|$ and $|T_z f^{n+1} v| \approx d_1 \lambda^n |v_2|$.

If $\alpha \geq \frac{1}{4}$, then (9) implies that $|T_z f^{n+1} v| \geq \left(\frac{9}{8}\right)^{1/2} |v|$ for δ small, and we may take $m(z)=1$. Henceforth, assume $\alpha < \frac{1}{4}$. Set $H_0=\left\{z\in D^n : |v(t_n)-v(z)| \leq \frac{\lambda^{-n}\varepsilon_{1n}}{4}\right\}$, and $H_1 = D_n - H_0$. Assume first that $z\in H_1 \cap \Lambda_n$ and $v\in S_{nz}$. Then $\frac{\alpha}{2}+\tilde{\alpha} \geq \frac{1}{4}$ and again by (9), we may take $m(z)=1$. Now we assume $z\in H_0\cap\Lambda_n$, so $\tilde{\alpha} < \frac{1}{4}$.

Since $z\in H_0\cap\Lambda_n$, and n is large, it is clear from figure 5.5 that $f^{n+1} z\in H_1$. If $f^{(n+1)j} z\in H_1$ for $j\geq 1$, then $|T_z f^{(n+1)j} v| \geq c_9 (2+\alpha)^{j/2} |v|$ for $v\in S_{nz}$, so an $m(z)$ can easily be found. The remaining case is when $z \in H_0$ and there is a $j > 1$ such that $f^{(n+1)j}z \in H_0$. Let $m(z)$ be the least such j. This means that $v(t_n) - v(f^{(n+1)j}z) > \frac{\lambda^{-n}\varepsilon_{1n}}{4}$ for $j = 1, 2, \ldots, m(z)-1$.

Let $z_j = f^{(n+1)j}z$ for $1 \leq j \leq m(z)-1$ and write $z_j = (u_j, v_j)$. Let $\eta(z_1)$ be the expansion of $T_{z_1}f^{(n+1)(m(z)-1)}$ along the curve $f^{(n+1)}D^u_{nz}$. If $v \in S_{nz}$, then $T_z f^{n+1}v$ is nearly tangent to $f^{n+1}D^u_{nz}$ at z_1. Thus, by (9),

$$|T_z f^{(n+1)m(z)}v| \geq \eta(z_1)(2-2\delta)\left(\frac{\alpha}{2}+\tilde{\alpha}\right)^{1/2}(2+\alpha)^{1/2}|v|.$$

We will prove:

$$(10) \qquad \eta(z_1)(2-2\delta)\left(\frac{\alpha}{2}+\tilde{\alpha}\right)^{1/2}(2+\alpha)^{1/2} > 1.11 > \left(\frac{9}{8}\right)^{1/2}$$

for δ small and $\alpha < \frac{1}{4}$.

The distance from z to the top of D_n is $\tilde{\alpha}\lambda^{-n}\varepsilon_{1n}$, so the distance from $z_1 = f^{n+1}z$ to the bottom of D_n measured along $f^{n+1}D^u_{nz} \cap B_1$ is nearly $\tilde{\alpha}d_1\varepsilon_{1n}$. This means that $|u_1 - u_0| \approx (1-\tilde{\alpha})d_1\varepsilon_{1n}$. Since $\tilde{\alpha} < \frac{1}{4}$, we have $|u_1 - u_0| \geq \frac{3}{4}d_1\varepsilon_{1n}$.

To estimate the expansion $\eta(z_1)$, we introduce functions $\psi_0, \psi_1, \ldots, \psi_{m(z)-1}$ from $[0, 1]$ to $\mathbf{R}$ as follows. Let $v(0) = v(t_n)$, and let ξ_0 be the curve $v = v(t_n) - a(u-u_0)^2$, $|u - u_0| \leq d_1\varepsilon_{1n}$. Let ξ_j be the curve which contains z_j and is parallel to ξ_0. Thus, ξ_j has the equation $v = v(j) - a(u-u_0)^2$, $|u-u_0| \leq d_1\varepsilon_{1n}$, where $v(j)$ is near $v(t_n)$. Let π^s be the projection $\pi^s(u, v) = u$, and let $u \in \pi^s D_n$. If $\zeta = \frac{|u-u_0|}{d_1\varepsilon_{1n}}$, set

$$\psi_j(\zeta) = \frac{|\pi^s f^{n+1}(u, v(j) - a(u-u_0))^2 - u_0|}{d_1\varepsilon_{1n}}$$

for $0 \leq j \leq m(z)-1$. We claim $\psi_j(\zeta) \approx (2+\alpha)\zeta^2 - \left(1+\frac{\alpha}{2}\right)$ for each j and $\frac{3}{4} \leq \zeta \leq 1$.

Let us prove the claim. Since each ξ_j is nearly horizontal and we are using the maximum norm, the expansion $T_{\bar{z}}f^{n+1}$ along ξ_j at a point $\bar{z} = (u, v)$ on ξ_j is nearly $\frac{d}{du}\pi^s f^{n+1}(u, v(j) - a(u-u_0)^2)$ at u. By (7), this is nearly

$$2ad_1\lambda^n|u-u_0| = 2ad_1\lambda^n\zeta d_1\varepsilon_{1n} = 2(2+\alpha)\zeta.$$

This means that $\psi_j'(\zeta) \approx 2(2+\alpha)\zeta$. For $\frac{3}{4} \leq \zeta \leq 1$, we may think of $\psi_j(\zeta)$ as follows. Take a point (u, v) on ξ_j with $|u-u_0| = \zeta d_1\varepsilon_{1n}$ and let $\psi_j(\zeta)d_1\varepsilon_{1n}$ be the distance from $f^{n+1}(u, v)$ to the line $u = u_0$. If $\bar{u}$ is the u-coordinate of the point where ξ_0 crosses the line $v = \lambda^{-n}v_0$, then

$$\frac{|\bar{u}-u_0|}{d_1\varepsilon_{1n}} = \left(\frac{1+\frac{\alpha}{2}}{2+\alpha}\right)^{1/2}$$

and $\psi_0\left(\frac{|\bar{u}-u_0|}{d_1\varepsilon_{1n}}\right)=0$. This means that $\psi_0(\zeta)\approx(2+\alpha)\zeta^2-\left(1+\frac{\alpha}{2}\right)$. Similarly, each $\psi_j(\zeta)\approx(2+\alpha)\zeta^2-\left(1+\frac{\alpha}{2}\right)$ also.

Let w be a point where ξ_0 meets the bottom of H_0. Write $w=(u(w),v(w))$ and let $\theta d_1\varepsilon_{1n}=|u(w)-u_0|$. Then,

$$\theta d_1\varepsilon_{1n}=\left(\frac{\lambda^{-n}\varepsilon_{1n}}{4a}\right)^{1/2},$$

so $\theta=\frac{1}{2(2+\alpha)^{1/2}}$.

Set $\theta_1=\psi_0^{-1}(\theta)$ and $\theta_2=\psi_0^{-1}(\theta_1)=\psi_0^{-2}(\theta)$. Then,

$$(2+\alpha)\theta_1^2-\left(1+\frac{\alpha}{2}\right)\approx\frac{1}{2(2+\alpha)^{1/2}}$$

or

$$\theta_1^2\approx\frac{\frac{1}{2(2+\alpha)^{1/2}}+1+\frac{\alpha}{2}}{2+\alpha}.$$

For $0\leq\alpha\leq\frac{1}{4}$, this gives $.81\leq\theta_1\leq.82$. Also,

$$\frac{1+\frac{1}{8}+.81}{2+\frac{1}{4}}\leq\frac{1+\frac{\alpha}{2}+.81}{2+\alpha}\leq\theta_2^2\leq\frac{1+\frac{\alpha}{2}+.82}{2+\alpha}<\frac{1.82}{2}.$$

so, $.86\leq\theta_2^2\leq.91$ or $.93\leq\theta_2\leq.95$.

Let η_0 be the expansion of T_zf^{n+1} on S_{nz}. We have seen in (9) that

$$\eta_0\geq(2-2\delta)\left(\frac{\alpha}{2}+\tilde{\alpha}\right)^{1/2}(2+\alpha)^{1/2}\geq2\sqrt{2}\,\tilde{\alpha}^{1/2}$$

for δ small.

Let $\zeta_1=\frac{|u_1-u_0|}{d_1\varepsilon_{1n}}$ and, for $2\leq j\leq m(z)-1$, let $\zeta_j=\psi_{j-1}\circ\ldots\circ\psi_1(\zeta_1)$. Note that $|u_j-u_0|=\zeta_jd_1\varepsilon_{1n}$. For $1\leq j\leq m(z)-1$, let η_j be the expansion of $T_{z_j}f^{n+1}$ along the curve $f^{(n+1)j}D^u_{nz}$ at z_j. Then,

$$\eta_j\approx2ad_1\lambda^n|u_j-u_0|=2ad_1^2\lambda^n\zeta_j\varepsilon_{1n}=2(2+\alpha)\zeta_j,$$

so $\eta_j\approx\psi_j'(\zeta_j)$ and $\eta(z_1)\approx\eta_{m(z)-1}\cdot\eta_{m(z)-2}\cdots\eta_1$. Also, since each $z_j\in H_1$ for

$$1\leq j\leq m(z)-1,$$

(9) gives us that $\eta_j\geq\sqrt{2}$ for δ small.

To prove (10), we consider three cases.

Case 1: $\theta \leq \zeta_1 \leq \theta_1$.

Recall that $\zeta_1 \approx 1-\tilde{\alpha}$, so $\frac{3}{4} \leq \zeta_1 \leq \theta_1$. In this case $m(z)=2$ and $\eta(z_1) \approx \psi_1'(\zeta_1)$.

Thus,
$$\eta(z_1) \cdot \eta_0 \geq 2(2+\alpha)\zeta_1 \cdot 2\sqrt{2}\,\tilde{\alpha}^{1/2}$$
$$\geq 2(2+\alpha)2\sqrt{2}\,\zeta_1(1-\zeta_1)^{1/2}.$$

Since $\zeta \mapsto \zeta(1-\zeta)^{1/2}$ is decreasing on $\left[\frac{3}{4}, 1\right]$, we have
$$\eta(z_1) \cdot \eta_0 \geq 2(2+\alpha)2\sqrt{2}\,\theta_1(1-\theta_1)^{1/2}$$
$$\geq 2(2+\alpha)2\sqrt{2}(.82)(.18)^{1/2}$$
$$\geq 3.94.$$

Case 2: $\theta_1 \leq \zeta_1 \leq \theta_2$.

In this case, $m(z) \geq 3$. Now,
$$\eta(z_1) \cdot \eta_0 \approx \eta_{m(z)-1} \cdots \eta_1 \cdot \eta_0$$
$$\geq \sqrt{2}\,\psi_1'(\zeta_1) \cdot \eta_0$$
$$\geq 4\psi_1'(\zeta_1)(\tilde{\alpha})^{1/2}$$
$$\approx 4(2)(2+\alpha)\zeta_1(1-\zeta_1)^{1/2}$$
$$\geq 16\theta_2(1-\theta_2)^{1/2}$$
$$\geq 16(.95)(.05)^{1/2} \approx 3.4.$$

Case 3: $\theta_2 \leq \zeta_1$.

Let $n(z)$ be the least integer j such that $\zeta_j < \theta_2$. Thus, $\zeta_{n(z)} < \theta_2$, but $\theta_2 \leq \zeta_{n(z)-1}$. Since $\zeta_1 \geq \theta_2$, one has $n(z) \geq 2$ and $m(z) \geq n(z)+2$. Now $\psi_{n(z)-1} \circ \psi_{n(z)-2} \circ \ldots \circ \psi_1$ maps the interval $[1-\tilde{\alpha}, 1]$ onto an interval containing $[\theta_2, 1]$. Thus, for some $1-\tilde{\alpha} < \zeta' < 1$, $(\psi_{n(z)-1} \circ \ldots \circ \psi_1)'(\zeta')\tilde{\alpha} > 1-\theta_2$. Set $\zeta_1' = \zeta'$, $\zeta_j' = \psi_{j-1} \circ \ldots \circ \psi_1(\zeta')$ for $j \geqslant 2$, and $\eta_j' = \psi_j'(\zeta_j')$. Then,
$$\prod_{j=1}^{n(z)-1} \psi_j'(\zeta_j') = \prod_{j=1}^{n(z)-1} \eta_j' > \frac{1-\theta_2}{\tilde{\alpha}}.$$

and
$$\prod_{j=1}^{n(z)-1} \eta_j = \prod_{j=1}^{n(z)-1} \frac{\eta_j}{\eta_j'} \cdot \eta_j' \geq \frac{1-\theta_2}{\tilde{\alpha}} \prod_{j=1}^{n(z)-1} \frac{\eta_j}{\eta_j'}.$$

We will show that $\prod_{j=1}^{n(z)-1} \frac{\eta_j}{\eta_j'} \geq .89$. From this we get
$$\prod_{j=0}^{m(z)-1} \eta_j \geq \eta_{m(z)-1} \cdot \eta_{m(z)-2} \prod_{j=0}^{n(z)-1} \eta_j$$
$$\geq 2(.89)\left(\frac{1-\theta_2}{\tilde{\alpha}}\right) \cdot \eta_0$$
$$\geq 2(.89)2\sqrt{2}\left(\frac{1-\theta_2}{\tilde{\alpha}^{1/2}}\right).$$

But, $1-\zeta_1 \approx \tilde{\alpha} \leq 1-\theta_2$, so $\tilde{\alpha}^{-1/2} \geq (1-\theta_2)^{-1/2}$ and $\frac{1-\theta_2}{\tilde{\alpha}^{1/2}} \geq (1-\theta_2)^{1/2} \geq (.05)^{1/2} \approx .22$. Thus, $\prod_{j=0}^{m(z)-1} \eta_j \geq 2(.89)2\sqrt{2}(.22) \approx 1.11$ which proves (10).

We now show $\prod_{j=1}^{n(z)-1} \frac{\eta_j}{\eta_j'} \geq .89$. We have $\prod_{j=1}^{n(z)-1} \frac{\eta_j}{\eta_j'} = \exp \sum_{j=1}^{n(z)-1} \ln\left(1 - \frac{\eta_j'-\eta_j}{\eta_j'}\right)$. Now, $\zeta_{n(z)-1}$ and $\zeta'_{n(z)-1}$ lie in $[\theta_2, 1]$. In this interval $\psi_j'(\zeta) \approx 2(2+\alpha)\zeta \geq 2(2+\alpha)\theta_2$, so

$$\psi_j'(\zeta)^{-1} \leq (2(2+\alpha)\theta_2)^{-1} \leq (2(2+\alpha)(.93))^{-1}$$
$$\leq .27 \quad \text{for all } j.$$

Thus
$$|\zeta_j - \zeta_j'| \leq (.27)|\zeta_{j+1} - \zeta'_{j+1}|$$

and
$$\sum_{i=1}^{n(z)-1} |\zeta_j - \zeta_j'| \leq \frac{1}{1-.27}(1-\theta_2)$$
$$\leq \frac{.07}{.73} = .10.$$

Also, $\eta_j' \geq \psi_j'(\theta_2) \geqslant 2(2+\alpha)(.93)$. Thus,

$$\frac{|\eta_j - \eta_j'|}{\eta_j'} \leq \frac{2(2+\alpha)|\zeta_j - \zeta_j'|}{\eta_j'} \leq \frac{.07}{.93} \approx .08.$$

Now, $\ln(1-x) > \frac{-10}{9}x$ for $0 < x < .1$, so

$$\ln\left(1 - \frac{\eta_j'-\eta_j}{\eta_j'}\right) \geq \frac{-10}{9}\left|\frac{\eta_j'-\eta_j}{\eta_j'}\right|$$

and
$$\sum_{j=1}^{n(z)-1} \ln\left(1 - \frac{\eta_j'-\eta_j}{\eta_j'}\right) \geq \frac{-10}{9}\left(\frac{100}{93}\right)\sum_j |\zeta_j - \zeta_j'|$$
$$\geq -\left(\frac{10}{9}\right)\left(\frac{100}{93}\right)(.1) \approx -.12.$$

So $\exp \sum_j \ln\left(1 - \frac{\eta_j'-\eta_j}{\eta_j'}\right) \geq e^{-.12} \approx .89$. This completes the proof that Λ_n is hyperbolic (except for (7)).

To see that Λ_n is actually a basic set is easy. From the definition of D_n and figure 5.5, one has that f^{n+1} maps the top and bottom of D_n below D_n and f^{-n-1} maps the sides of D_n away from D_n. Hence $\Lambda_n \subset \operatorname{int} D_n$. Evidently f^{n+1} maps D_n to its image as the familiar Smale horseshoe diffeomorphism in [22]. So, if $C = \{0, 1\}^{\mathbf{Z}}$

and $\sigma : C \to C$ is the usual shift map defined by $\sigma(\underline{a})(i) = \underline{a}(i+1)$, $\underline{a} \in C$, then the map $h : C \to \Lambda_n$ defined by

$$h(\underline{a}) = \bigcap_{j \in \mathbf{Z}} f^{-(n+1)j} B_{\underline{a}(j)}$$

is a homeomorphism and $h\sigma = f^{n+1} h$. Since σ has a dense orbit, so does $f^{n+1} | \Lambda_n$. This proves (5a).

To prove (5b), first notice, from figure 5.5, that f^{n+1} will have a fixed point q in B_0 with $T_q f^{n+1}$ having positive eigenvalues. Let ξ be the curve in $W^u(q)$ joining q to the bottom of D_n. Then $f^{n+1}\xi \supset \xi$ and extends at least a fixed distance $(\beta - 1)\ell\xi$ below ξ with $\beta > 1$. The same statement holds if we replace ξ by $f^{n+1}\xi$ and successive iterates unless some part of the iterate crosses $W^s(p_{t_n})$.

But this means that, indeed, iteration of ξ by powers of f^{n+1} eventually makes it cross $W^s(p_{t_n})$ transversely. Similarly, $W^s(q)$ meets $W^u(p_{t_n})$ transversely.

We now estimate $\tau^s(\Lambda_n)$ to prove (5c). For convenience of notation, let us write f for f_{t_n} and g for $f_{t_n}^{n+1}$.

Fix $z \in D_n = D_{n,t_n}$ and let $\gamma = D^u_{nz}$. Consider the Cantor set $F = W^s(\Lambda_n) \cap \gamma$. Let F_0 be the smallest closed interval in γ which contains F. Let F_1 be F_0 minus the component of $\gamma - F$ which meets $g^{-1}A_2$. Thus, F_1 is a union of two closed intervals F_{11}, F_{12} so that $F_{11} \subset g^{-1}B_0$ and $F_{12} \subset g^{-1}B_1$. Let F_2 be F_1 minus the union of the components of $\gamma - F$ which meet $\gamma - g^{-2}A_2$. Then F_2 is the union of four closed intervals F_{21}, F_{22}, F_{23}, F_{24} such that $F_{2j} \subset g^{-1}B_{a_1} \cap g^{-2}B_{a_2}$ with $a_1 = 0$ or 1 and $a_2 = 0$ or 1. Continuing, by induction, if $F_i = F_{i1} \cup \ldots \cup F_{i2^i}$ has been defined, let F_{i+1} be F_i minus the union of the components of $\gamma - F$ which meet $g^{-(i+1)}A_2$. Then, for each i, F_i is a union of 2^i components each of which lies in exactly one $g^{-1}B_{a_1} \cap g^{-2}B_{a_2} \cap \ldots \cap g^{-i}B_{a_i}$ with $a_j = 0$ or 1 for $1 \leq j \leq i$.

We will show that n large implies that the thickness of the defining sequence F_0, F_{11}, F_{12}, F_{21}, F_{22}, F_{23}, ... for F is bigger than T. This will complete the proof of (5c) and hence of lemma 7.

Let I be some F_{ij}, and let J be the interval in $F_{i-1} - F_i$ adjacent to I. Then $g^i(J)$ is a curve meeting A_2 (in fact in the component of $g(D_n) - W^s(\Lambda_n)$ which contains A_2) whose length is less than twice the diameter of A_2 (α small and n large), and $g^i(I) \subset B_0 \cup B_1$.

Also, $\ell g^i I$ is near $\operatorname{diam} B_0$. Let $I_0 = g^i I = f^{(n+1)i} I$ and $J_0 = g^i J = f^{(n+1)i} J$.

As in the proof of proposition 6, we write

$$\frac{\ell I}{\ell J} = \frac{\alpha_i \bar{\alpha}_i \alpha_{i-1} \bar{\alpha}_{i-1} \ldots \alpha_1 \bar{\alpha}_1 \ell I_0}{\beta_i \bar{\beta}_i \beta_{i-1} \bar{\beta}_{i-1} \ldots \beta_1 \bar{\beta}_1 \ell J_0}$$

where $\alpha_j = |T_{z_j} f^{-n} v_j|$, $\beta_j = |T_{z'_j} f^{-n} v'_j|$, $\bar{\alpha}_j = |T_{\bar{z}_j} f^{-1} \bar{v}_j|$, $\bar{\beta}_j = |T_{\bar{z}'_j} f^{-1}(\bar{v}'_j)|$, with $\bar{z}_j \in g^{-j+1} I_0$, $z_j \in f^{-1} g^{-j+1} I_0$, $\bar{z}'_j \in g^{-j+1} J_0$, $z'_j \in f^{-1} g^{-j+1} J_0$, and $\bar{v}_j$, v_j, $\bar{v}'_j$, v'_j are unit vectors tangent to $g^{-j+1} I_0$, $f^{-1} g^{-j+1} I_0$, $g^{-j+1} J_0$, and $f^{-1} g^{-j+1} J_0$, respectively.

We assert that:

(11) there is a constant $c_{10}>0$ such that $\prod_i \frac{\alpha_i}{\beta_i} \cdot \prod_i \frac{\bar{\alpha}_i}{\bar{\beta}_i} > c_{10}$.

Let us show that (5 c) follows from (11). In the notation of the proof of (5 a), if $z \in \Lambda_n \cap H_0$ and $m(z)$ is the least integer j such that $f^{(n+1)j} z \in H_0$, then $m(z)$ is independent of n (for large n). This implies that:

(12) there are constants $c_{11}>0$ and $\lambda_1>1$ independent of n such that $T_z f^{(n+1)j} | S_{nz}$ and $T_z f^{(-n-1)j} | T_z M - S_{nz}$ are $c_{11}\lambda_1^j$-expansions for $z \in \bigcap_{-j \leq i \leq j} f^{(n+1)i} D_n$ and $0 \leq j < \infty$.

From (12), we have that

$$\begin{aligned} \ell(J_0) &\leq \operatorname{diam} A_2 + c_{11}^{-1} \sum_{j=0}^{i} \lambda_1^{-j} \max(\operatorname{diam} A_1, \operatorname{diam} A_2) \\ &\leq c_{12}(\alpha\lambda^{-n}\varepsilon_{1n})^{1/2}\left(1+\frac{1}{1-\lambda_1^{-1}}\right) \\ &= c_{13}(\alpha\lambda^{-n}\varepsilon_{1n})^{1/2} \end{aligned}$$

and

$$\begin{aligned} \ell(I_0) &\geq c_{14} d_1 \varepsilon_{1n} - c_{11}^{-1} \sum_{j=0}^{\infty} \lambda_1^{-j} \max(\operatorname{diam} A_1, \operatorname{diam} A_2, \operatorname{diam} A_3) \\ &\geq c_{14} d_1 \varepsilon_{1n} - c_{15}(\alpha\lambda^{-n}\varepsilon_{1n})^{1/2}. \end{aligned}$$

If α is small enough, then $\frac{\ell(I_0)}{\ell(J_0)} \geq \frac{3}{2c_{10}} T$, so (11) implies $\frac{\ell(I)}{\ell(J)} > \frac{3}{2} T$ which, in turn, implies (5 c).

We proceed to prove (11). We have

$$\begin{aligned} |\bar{\alpha}_j - \bar{\beta}_j| &\leq \big| |T_{\bar{z}_j} f^{-1} \bar{v}_j| - |T_{\bar{z}_j} f^{-1} \bar{v}_j'| \big| + \big| |T_{\bar{z}_j} f^{-1} \bar{v}_j'| - |T_{\bar{z}_j'} f^{-1} \bar{v}_j'| \big| \\ &\leq (K_{2j} K_{3j} + K_{4j}) \ell(g^{-j+1}(I_0 \cup J_0)) \end{aligned}$$

where K_{2j} is bounded by $\sup_z |T_z f^{-1}|$, K_{3j} is bounded by the curvature of $g^{-j+1}(I_0 \cup J_0)$, and K_{4j} is bounded by the C^2 size of f^{-1}. By (7) the K_{3j}'s are uniformly bounded, and, by (12), $\ell(g^{-j+1}(I_0 \cup J_0)) \leq c_{11}^{-1} \lambda_1^{-j+1} \ell(I_0 \cup J_0)$. Thus, $\sum_j |\bar{\alpha}_j - \bar{\beta}_j| \leq c_{16} \lambda^{-n}$, and, we have

$$\begin{aligned} \prod_j \frac{\bar{\alpha}_j}{\bar{\beta}_j} = \prod_j \left(1 - \frac{\bar{\beta}_j - \bar{\alpha}_j}{\bar{\beta}_j}\right) &= \exp\left(\sum_j \ln\left(1 - \frac{\bar{\beta}_j - \bar{\alpha}_j}{\bar{\beta}_j}\right)\right) \\ &\geq \exp(-c_{17}\lambda^{-n}) > \frac{1}{2} \quad \text{for } n \text{ large.} \end{aligned}$$

The estimate for $\prod_j \frac{\alpha_j}{\beta_j}$ is similar once one proves:

(13) there are constants $0 < c_{18} < c_{19}$ such that $c_{18} \leq |T_z f^n(v)| \leq c_{19}$ if v is a unit vector tangent to the curve $g^{-j}(I_0 \cup J_0)$ with $1 \leq j \leq i$, and

(14) $\sum_{j \geq 1} |\alpha_j - \beta_j| \leq c_{20}$ for some constant $c_{20} > 0$.

Indeed, (13) implies that the β_j are uniformly bounded below, and with (14), this gives a lower bound for $\prod_j \frac{\alpha_j}{\beta_j}$. Let us prove (13) and (14).

By (7), if $z=(u,v)\in g^{-j}(I_0\cup J_0)$, $1\le j\le i$, and v is a unit vector tangent to that curve, we have $|T_z f^n(v)|\approx 2a\lambda^n|u-u_0|$. But $\frac{|u-u_0|}{d_1\varepsilon_{1n}}$ is bounded above and below as is $\frac{\lambda^{-n}}{\varepsilon_{1n}}$. This gives (13). For (14) we write

$$|\alpha_j-\beta_j| = ||T_{z_j}f^{-n}v_j|-|T_{z'_j}f^{-n}v'_j||.$$

Let $y_j=f^{-n}z_j$ and $y'_j=f^{-n}z'_j$ and let u_j be the u-coordinate of y_j and u'_j be the u-coordinate of y'_j. Since y_j and y'_j are on the curve $g^{-j}(I_0\cup J_0)$, (7) gives us $\alpha_j^{-1}\approx\lambda^n 2a|u_j-u_0|$ and $\beta_j^{-1}\approx\lambda^n 2a|u'_j-u_0|$. Thus,

$$\begin{aligned}|\alpha_j-\beta_j| &\approx \lambda^{-n}(2a)^{-1}||u_j-u_0|^{-1}-|u'_j-u_0|^{-1}|\\ &= \lambda^{-n}(2a)^{-1}(\bar u_j-u_0)^{-2}|u_j-u'_j|\end{aligned}$$

where $\bar u_j$ is some number between u_j and u'_j. Since $|\bar u_j-u_0|\lambda^n$ is bounded above and below, we get

$$|\alpha_j-\beta_j|\le c_{21}\lambda^n \ell(g^{-j}(I_0\cup J_0)).$$

By (12), we have $\ell(g^{-j}(I_0\cup J_0))\le c_{11}^{-1}\lambda_1^{-j}\ell(I_0\cup J_0)\le c_{22}\lambda_1^{-j}\lambda^{-n}$. Thus, $|\alpha_j-\beta_j|\le c_{21}c_{22}\lambda_1^{-j}$ and (14) follows.

We now prove (7).

It suffices to show that if ξ is a piece of the curve $v=-\tilde a(u-u_0)^2+\tilde v$ with $|\tilde v - v(t_n)|\le\frac{\alpha}{4}\lambda^{-n}\varepsilon_{1n}$, $\tilde a$ near a, and $\xi\subset D_n$, then $f^n\xi$ approaches its projection on $(u=0)$ in the C^2 topology. We will, in fact, show that this C^2 convergence is faster than $(\mu+\varepsilon)^n(\lambda+\varepsilon)^n\varepsilon^{-1/2}$ where ε is small.

Pick an honest C^2 coordinate system $\tilde\varphi$ on the neighborhood U in which we have $\tilde\varphi f\tilde\varphi^{-1}(u,v)=(f_1(u,v),f_2(u,v))$ with $f_{1u}(0,0)=\mu$, $f_{1v}(0,0)=0=f_{2u}(0,0)$, and $f_{2v}(0,0)=\lambda$. As before, we identify D^s, D^u, etc. with $\tilde\varphi^{-1}D^s$, $\tilde\varphi^{-1}D^u$, etc.

Write ξ as the graph of a function $s:\tilde D^u\to D^s$ with $\tilde D^u$ an interval in D^u. Then, $f(\xi)$ is the graph of the function $\Gamma_f(s)$ defined by $\Gamma_f(s)=f_1\circ(s,1)\circ(f_2\circ(s,1))^{-1}$ where $(s,1)(v)=(s(v),v)$. Here we use the graph transform notation of Hirsch and Pugh [9], although our coordinates are interchanged from theirs. Since $f^n\xi$ is the graph of $\Gamma_f^n(s)$, it suffices to show that $\Gamma_f^n(s)$ is C^2 closer to its projection on $(u=0)$ than $(\mu+\varepsilon)^n(\lambda+\varepsilon)^n\varepsilon^{-1/2}$.

We will use $s'(v)$ and $s''(v)$ to denote the first and second derivative of s at v and we will use similar notations for the corresponding derivatives of other functions. Also, we write $s''(v)(w)^2=s''(w)^2$ for the second derivative of a function s at v on the pair (w,w).

Since $\xi \subset D_n$, we have that if (u, v) is a point in ξ, then $|v-\widetilde{v}| \geq \frac{\alpha}{4}\lambda^{-n}\varepsilon_{1n}$. This implies that

$$s'(v) = \frac{1}{2}\widetilde{a}^{-1/2}|v-\widetilde{v}|^{-1/2} \leq \frac{1}{2}\widetilde{a}^{-1/2}\left(\frac{\alpha}{4}\lambda^{-n}\varepsilon_{1n}\right)^{-1/2} \leq \left(\frac{d_1^2 a}{\alpha(2+\alpha)\widetilde{a}}\right)^{1/2}\lambda^n.$$

To estimate the first and second derivatives of $\Gamma_f^n(s)$ we first need to make the second derivatives of f small on U. This is done with a scale change $u = \varepsilon u_1$, $v = \varepsilon v_1$ where ε is a small number. Note that in the (u_1, v_1)-coordinates d_1, λ, and μ remain the same. Also, ξ has the equation $v_1 = \widetilde{v}_1 - \widetilde{a}\varepsilon(u_1-u_{10})^2$, so $|s'(v_1)| = \frac{1}{2}\left(\frac{|v_1-\widetilde{v}_1|}{\widetilde{a}\varepsilon}\right)^{-1/2}\frac{1}{\widetilde{a}\varepsilon}$. But $|\varepsilon v_1 - \varepsilon\widetilde{v}_1| \geq \frac{\alpha}{4}\lambda^{-n}\varepsilon_{1n}$, so $\frac{|v_1-\widetilde{v}_1|}{\widetilde{a}\varepsilon} \geq \frac{\alpha}{4}\lambda^{-n}\varepsilon_{1n}\frac{1}{\widetilde{a}\varepsilon^2}$. This gives

$$|s'(v_1)| \leq \frac{1}{2}\widetilde{a}^{-1/2}\left(\frac{\alpha}{4}\lambda^{-n}\varepsilon_{1n}\right)^{-1/2}$$

which is the same upper bound we had for s' in the (u, v)-coordinates.

Let us use ε to denote various small, possibly different, numbers which come up in the equations below.

We dispense with the subscripts and assume ξ has the form $v = \widetilde{v} - \widetilde{a}\varepsilon(u-u_0)^2$ or $s(v) = \left(\frac{v-\widetilde{v}}{-\widetilde{a}\varepsilon}\right)^{1/2} + u_0$ with $\widetilde{a}$ near a. Then,

$$|s'(v)| \leq c_{23}\lambda^n \quad \text{and} \quad |s''(v)| \leq \frac{c_{23}}{\varepsilon^{1/2}}\lambda^{3n}$$

where c_{23} is a constant independent of n and ε.

Let $\zeta(v) = \zeta_s(v) = (f_2 \circ (s, 1))^{-1}(v)$. Then, $\zeta_s'(v) = (f_{2u}(s(v), v)s'(v) + f_{2v}(s(v), v)^{-1}$. Since f leaves $(v = 0)$ invariant, $f_{2u}(u, 0) \equiv 0$, so

$$|f_{2u}(s(v), v)| = |f_{2u}(s(v), v) - f_{2u}(s(v), 0)| \leq \varepsilon|v| \leq \varepsilon\lambda^{-n}$$

where we use the fact that the second derivatives of f are small on U. This gives $|f_{2u}(s(v), v)s'(v)| \leq \varepsilon$ and $|\zeta'(v)| \leq (\lambda-\varepsilon)^{-1}$. Strictly speaking this estimate only has been proved for $(u, v) \in D_n$ since we used $|v| \leq c\lambda^{-n}$ and $|s'(v)| \leq c_{23}\lambda^n$. The estimates for $(u, v) \in f^j D_n$ are similar, $1 \leq j \leq n$.

Now, if $\psi(v) = (f_2 \circ (s, 1))(v)$, we have

$$\psi(\zeta(v)) = v, \qquad \psi'(\zeta(v)) \circ \zeta'(v) = \text{Id},$$

and

$$\psi''(\zeta(v))(\zeta'(v))^2 + \psi'(\zeta(v)) \circ \zeta''(v) = 0.$$

Thus,

$$\begin{aligned}\zeta''(v) &= -\psi'(\zeta(v))^{-1}\psi''(\zeta(v))(\zeta'(v))^2 \\ &= -\zeta'(v)\psi''(\zeta(v))(\zeta'(v))^2.\end{aligned}$$

Also

$$\Gamma_f(s)' = (f_1 \circ (s, 1))' \circ \zeta'(v) = (f_{1u} \circ s' + f_{1v}) \circ \zeta'(v)$$

and

$$\Gamma_f(s)'' = f_1''((s, 1)' \circ \zeta'(v))^2 + f_1' \circ (s, 1)''(\zeta'(v))^2 + f_1' \circ (s, 1)' \circ \zeta''(v).$$

Using these formulas, and setting $|\tilde{s}|_0$ to be the C^0 size of a function $\tilde{s}$, one can calculate that

$$|\Gamma_f^n(s)| \leq c_{24}(\mu+\varepsilon)^n |s|_0$$
$$|\Gamma_f^n(s)'| \leq c_{25}(\mu+\varepsilon)^n(\lambda-\varepsilon)^{-n}|s'| + c_{26}|\Gamma_f^n(s)|_0,$$
and
$$|\Gamma_f^n(s)''| \leq c_{27}(\mu+\varepsilon)^n(\lambda-\varepsilon)^{-2n}|s''| + c_{28}|\Gamma_f^n(s)'| + c_{29}|\Gamma_f^n(s)|_0.$$

The easy way to do this is to let J_0, J_1, J_2 be the spaces of zero, one, and two-jets of functions s from intervals in D^u to D^s. Then, Γ_f induces mappings $H_i : J_i \to J_i$ and H_i is a fiber contraction over H_{i-1} in the sense of Hirsch and Pugh [9]. The Lipschitz constant of $H_0 = \Gamma_f$ is $\mu+\varepsilon$ and the fiber Lipschitz constants of H_1 and H_2 are $(\mu+\varepsilon)(\lambda-\varepsilon)^{-1}$ and $(\mu+\varepsilon)(\lambda-\varepsilon)^2$, respectively. Then the above estimates follow since the attractive fixed point of the map H_2 is the 2-jet of the zero function from D^u to D^s.

In any event, using the estimates $|s'| \leq c_{23}\lambda^n$ and $|s''| \leq c_{23}\varepsilon^{-1/2}\lambda^{3n}$, one sees that $|\Gamma_f^n(s)|$, $|\Gamma_f^n(s)'|$, and $|\Gamma_f^n(s)''|$ are all bounded by $c_{30}(\mu+\varepsilon)^n(\lambda+\varepsilon)^n\varepsilon^{-1/2}$ with ε small.

We have thus proved (5).

The set Λ_n plays the role of $\Lambda_{21}(t)$ in the motivating discussion near figure 5.4. Recall that $f_{t_n}^{n+1}$ has a fixed point $q(t_n)$ in $B_0(t_n)$ so that $T_{q(t_n)}f_{t_n}^{n+1}$ has positive eigenvalues. Let t'_n be the largest number less than t_n for which $W^u(q(t'_n), f_{t'_n}^{n+1})$ is tangent to $W^s(q(t'_n), f_{t'_n}^{n+1})$ near the top of $D_n(t'_n)$. Let the point of tangency be $y(t'_n)$. If U_n is a small neighborhood of $y(t'_n)$, then estimates similar to those in the proof of (5) show that $\Lambda_{22}(t'_n) = \bigcap_j f_{t'_n}^{j(n+1)}(D_n(t'_n) - U_n)$ satisfies (5 *a*) and (5 *b*) for $f_{t'_n}^{n+1}$ and $\tau^s(\Lambda_{22}(t'_n)) > T$. Also, $\{f_t^{n+1}\}$ creates a non-degenerate tangency of $W^u(\Lambda_{22}(t), f_t^{n+1})$ and $W^s(\Lambda_{22}(t), f_t^{n+1})$ at $(y(t'_n), t'_n)$. Then, taking $t_1 = t'_n$ and

$$\Lambda_2(t_1) = \bigcup_{0 \leq j \leq n+1} f_{t_1}^j(\Lambda_{22}(t_1)),$$

we have (7.1)-(7.4). This completes the proof of lemma 7.

Before beginning the proof of lemma 8, we will construct some special coverings of a zero-dimensional hyperbolic basic set Λ for a C^2 diffeomorphism $f: M^2 \to M^2$.

Let $\mathscr{F}^u$ and $\mathscr{F}^s$ be the foliations on a neighborhood U of Λ constructed in the proof of Proposition 5. A *Markov cover* of Λ is a finite set $\mathscr{A} = \{A_1, \ldots, A_s\}$ of disks in M such that:

(1) there is a C^1 diffeomorphism $\varphi_i : B^1 \times B^1 \to M$ with image $\varphi_i = A_i$ and

$$\varphi_i(\{x\} \times B^1) \subset \mathscr{F}^u(\varphi_i(\{x\} \times \{0\})), \quad x \in B^1,$$
$$\varphi_i(B^1 \times \{y\}) \subset \mathscr{F}^s(\varphi_i(\{0\} \times \{y\})), \quad y \in B^1;$$

(2) $$\Lambda \subset \bigcup_{i=1}^{s} \operatorname{int} A_i;$$

(3) $$f(\bigcup_{i=1}^{s} \partial_s A_i) \cap \bigcup_{i=1}^{s} A_i = \emptyset;$$

(4) $$f^{-1}(\bigcup_{i=1}^{s} \partial_u A_i) \cap \bigcup_{i=1}^{s} A_i = \emptyset.$$

Recall that if $\varphi_i : B^1 \times B^1 \to M$ is a disk, then $\partial_s(\text{image } \varphi_i) = \text{image}(\varphi_i | (B^1 \times \partial B^1))$ and $\partial_u(\text{image } \varphi_i) = \text{image}(\varphi_i | (\partial B^1 \times B^1))$.

Given such a Markov cover, we define $\operatorname{diam} \mathscr{A} = \sup_{1 \leq i \leq s} \{\operatorname{diam} A_i\}$. We shall prove that if f and Λ are as above and $\varepsilon > 0$, then there exist Markov coverings $\mathscr{A}$ of Λ with $\operatorname{diam} \mathscr{A} < \varepsilon$.

First we recall the definition of a subshift of finite type (see [4]). Let N be an integer greater than one, and let A be an $N \times N$ matrix whose entries are zeroes and ones. Write the (i,j)-th entry of A as A_{ij}. Let $\Sigma_A = \{\underline{a} \in \Sigma_N : A_{\underline{a}_i \underline{a}_{i+1}} = 1 \text{ for all } i \in \mathbf{Z}\}$ and let $\sigma : \Sigma_A \to \Sigma_A$ be the restriction of the shift map on Σ_N. Here $\Sigma_N = \{1, \dots, N\}^{\mathbf{Z}}$ is as in the proof of Proposition 6. The pair (σ, Σ_A) is called *a subshift of finite type.*

According to Bowen and Lanford ([3], [6]), there are an integer N, a matrix A, and a homeomorphism $h : \Sigma_A \to \Lambda$ such that $fh = h\sigma$. Fix $\varepsilon > 0$. Given $\delta > 0$ small, choose an integer $n_\delta > 0$ so that for any sequence $(b_{-n_\delta}, b_{-n_\delta+1}, \dots, b_0, \dots, b_{n_\delta})$ such that $b_i \in \{1, \dots, N\}$ for $|i| \leq n_\delta$, $\operatorname{diam}(h\{\underline{a} \in \Sigma_A : \underline{a}_i = b_i \text{ for } |i| \leq n_\delta\}) < \frac{\delta}{2}$. Let $\{B_1, \dots, B_{2r_\delta+1}\}$ be the collection of sets $h\{\underline{a} \in \Sigma_A : \underline{a}_i = b_i \text{ for } |i| \leq n_\delta\}$ where the b_i run independently from 1 to N. Then $\{B_1, \dots, B_{2r_\delta+1}\}$ forms a Markov partition of Λ in the sense of Bowen [4], and $\operatorname{diam} B_i < \frac{\delta}{2}$ for all i. Note that $r_\delta = N^{2n_\delta+1}$.

Pick $x_i \in B_i$ for each i. As in the proof of Proposition 6, let C_i be the smallest closed interval in $W^s_\delta(x_i)$ containing $W^s_\delta(x_i) \cap B_i$ and D_i be the smallest closed interval in $W^u_\delta(x_i)$ containing $W^u_\delta(x_i) \cap B_i$, and let $\overline{B}_i = \bigcup_{x \in C_i} C(x, \mathscr{F}^u_x)$. Then $\overline{B}_i$ is diffeomorphic to $C_i \times D_i$, $\overline{B}_i \cap \overline{B}_j = \emptyset$ for $i \neq j$, and $\Lambda \subset \bigcup_{i=1}^{2r_\delta+1} \overline{B}_i$ as before. Clearly, for δ small, $\operatorname{diam} \overline{B}_i < \frac{\varepsilon}{2}$. For each i, let $\partial_s \overline{B}_i = \bigcup_{x \in C_i} \partial C(x, \mathscr{F}^u_x)$ and $\partial_u \overline{B}_i = \bigcup_{x \in \partial C_i} C(x, \mathscr{F}^u_x)$. From the construction of the $\overline{B}_i$'s we have

$$f\left(\bigcup_i \partial_s \overline{B}_i\right) \subset \bigcup_i \partial_s \overline{B}_i$$

and

$$f^{-1}\left(\bigcup_i \partial_u \overline{B}_i\right) \subset \bigcup_i \partial_u \overline{B}_i.$$

If we extend each $\overline{B}_j$ to a disk A_j by moving $\partial_u \overline{B}_j$ and $\partial_s \overline{B}_j$ out slightly, then using the facts that f expands each $\mathscr{F}^u_x$, contracts each $\mathscr{F}^s_x$, and the invariance of $\mathscr{F}^s$ and $\mathscr{F}^u$, one sees that $\{A_1, \dots, A_{2r_\delta+1}\}$ is a Markov cover of Λ.

Proof of lemma 8. — Suppose f, Λ_1, Λ_2, z_1 and z_2 are as in the statement of that lemma.

For $i = 1, 2$, let $\mathscr{F}^u_i$, $\mathscr{F}^s_i$ be the f-invariant foliations on a neighborhood U_i of Λ_i extending $W^u(\Lambda_i)$, $W^s(\Lambda_i)$. Choose U_i small enough so that $\mathscr{F}^s_i$ and $\mathscr{F}^u_i$ are trivial on each component of U_i and $\{z_1, z_2\} \cap (U_1 \cup U_2) = \emptyset$. Let $\mathscr{B}_i = \{B^i_1, \dots, B^i_{s_i}\}$ be a Markov covering of Λ_i so that $\bigcup_{j=1}^{s_i} B^i_j \subset \operatorname{int} U_i$, $O(z_k) \cap \bigcup_{j=1}^{s_i} B^i_j = \emptyset$ for $i, k = 1, 2$. Suppose $U_1 \cap U_2 = \emptyset$.

Choose an integer $n_0>0$ so that, for $n\geq n_0$,

$$\{f^{-n}z_2, f^n z_1\}\subset \operatorname{int} \bigcup_{j=1}^{s_2} B_j^2,$$

and

$$\{f^n z_2, f^{-n} z_1\}\subset \operatorname{int} \bigcup_{j=1}^{s_1} B_j^1.$$

For a set $H\subset M$ and $x\in H$, let $C(x, H)$ denote the connected component of x in H. Let $B_\varepsilon(x)$ denote the closed ball of radius ε about x. Pick $\varepsilon>0$ small enough so that

$$f^{n_0}B_\varepsilon(z_2)\cup f^{-n_0}B_\varepsilon(z_1)\subset \operatorname{int}\bigcup_{j=1}^{s_1} B_j^1,$$

$$f^{-n_0}B_\varepsilon(z_2)\cup f^{n_0}B_\varepsilon(z_1)\subset \operatorname{int}\bigcup_{j=1}^{s_2} B_j^2, \quad fB_\varepsilon(z_1)\cap B_\varepsilon(z_1)=\emptyset,$$

and

$$fB_\varepsilon(z_2)\cap B_\varepsilon(z_2)=\emptyset.$$

For $x\in B_\varepsilon(z_2)$, let

$$\mathscr{F}_x^u=C(x, f^{n_0}\mathscr{F}^u_{2f^{-n_0}x}\cap B_\varepsilon(z_2)) \quad \text{and} \quad \mathscr{F}_x^s=C(x, f^{-n_0}\mathscr{F}^s_{1f^{n_0}x}\cap B_\varepsilon(z_2)).$$

Similarly, for $x\in B_\varepsilon(z_1)$, let

$$\mathscr{F}_x^u=C(x, f^{n_0}\mathscr{F}^u_{1f^{-n_0}x}\cap B_\varepsilon(z_1)) \quad \text{and} \quad \mathscr{F}_x^s=C(x, f^{-n_0}\mathscr{F}^s_{2f^{n_0}x}\cap B_\varepsilon(z_1)).$$

For ε small, $\mathscr{F}^s$ and $\mathscr{F}^u$ will be transverse foliations of $B_\varepsilon(z_1)\cup B_\varepsilon(z_2)$. Extend $\mathscr{F}^u$, $\mathscr{F}^s$ to $\bigcup_{-n_0\leq j\leq n_0} f^jB_\varepsilon(z_1)\cup\bigcup_{-n_0\leq j\leq n_0} f^jB_\varepsilon(z_1)$ via iteration by powers of f. Define

$$\mathscr{F}_x^s=\mathscr{F}_{1x}^s,\ \mathscr{F}_x^u=\mathscr{F}_{1x}^u \quad \text{for} \quad x\in U_1-\bigcup_{-n_0\leq j\leq n_0} f^jB_\varepsilon(z_1),$$

and

$$\mathscr{F}_x^s=\mathscr{F}_{2x}^s,\ \mathscr{F}_x^u=\mathscr{F}_{2x}^u \quad \text{for} \quad x\in U_2-\bigcup_{-n_0\leq j\leq n_0} f^jB_\varepsilon(z_2).$$

Let $V=U_1\cup U_2\cup\bigcup_{-n_0\leq j\leq n_0} f^jB_\varepsilon(z_1)\cup\bigcup_{-n_0\leq j\leq n_0} f^jB_\varepsilon(z_2)$. Then the tangents to $\mathscr{F}^s$ and $\mathscr{F}^u$ give a (discontinuous) splitting of T_VM, say $T_VM=E_1^s\oplus E_1^u$ with $E_{1x}^s=T_x\mathscr{F}_x^s$, $E_{1x}^u=T_x\mathscr{F}_x^u$.

For $x\in R\subset V$, let $W^u(x, R)=C(x, \mathscr{F}_x^u\cap R)$ and $W^s(x, R)=C(x, \mathscr{F}_x^s\cap R)$. If R is small, and $x, y\in R$, $[x, y]\equiv W^u(x, R)\cap W^s(y, R)$ is at most one point.

Motivated by Bowen [4], we will call a set $R\subset V$ a *rectangle* if:

(1) $R=\mathrm{Cl}(\operatorname{int} R)$;

(2) for each $x\in R$, the map $[\ ,\ ]: W^u(x, R)\times W^s(x, R)\to R$ is a homeomorphism, and $W^u(x, R)$ and $W^s(x, R)$ are homeomorphic to closed real intervals.

It follows that each rectangle $R\subset V$ is a C^1 disk in M. Note that each B_j^i in the Markov cover $\mathscr{B}^i$ is a rectangle. Also, each component of $\bigcap_{n_1\leq j\leq n_2} f^jB_j^i$ is a rectangle, where $-\infty<n_1<n_2<\infty$, $i=1, 2$. If R is a rectangle we set

$$\partial^u R=\{z\in R: z\notin \operatorname{int} R\cap W^s(z, R)\} \quad \text{and} \quad \partial^s R=\{z\in R: z\notin \operatorname{int} R\cap W^u(z, R)\}.$$

If $R=\{R_1, \dots, R_k\}$ is a collection of rectangles, define $\partial^u R = \bigcup_i \partial^u R_i$ and $\partial^s R = \bigcup_i \partial^s R_i$. From the definition of Markov covering, we have that

(3) $f\partial^s \mathscr{B}^i \cap \bigcup_{B^i_j \in \mathscr{B}^i} B^i_j = \emptyset$ and $f^{-1}\partial^u \mathscr{B} \cap \bigcup_{B^i_j \in \mathscr{B}^i} B^i_j = \emptyset$.

We define the *s-diameter* of R to be $\omega_s R = \sup_{x\in R} \ell W^s(x, R)$, and the *u-diameter* of R to be $\omega_u R = \sup_{x\in R} \ell W^u(x, R)$. For n large, set

$$B^{1u}_n = \bigcap_{0\leq k\leq n} f^k(\bigcup_{j=1}^{s_1} B^1_j), \quad B^{1s}_n = \bigcap_{-n\leq k\leq 0} f^k(\bigcup_{j=1}^{s_1} B^1_j),$$

$$B^{2u}_n = \bigcap_{0\leq k\leq n} f^k(\bigcup_{j=1}^{s_2} B^2_j), \quad \text{and} \quad B^{2s}_n = \bigcap_{-n\leq k\leq 0} f^k(\bigcup_{j=1}^{s_2} B^2_j).$$

Then B^{1u}_n and B^{2u}_n are finite unions of rectangles whose s-diameters go to zero as $n\to\infty$ while B^{1s}_n and B^{2s}_n are finite unions of rectangles whose u-diameters go to zero as $n\to\infty$.

Set $C(z_2, f^{n_0}(B^{2u}_n) \cap f^{-n_0-n} B^{1u}_n) = R_{z_2,n}$, and $C(z_1, f^{n_0} B^{1u}_n \cap f^{-n_0-n} B^{2u}_n = R_{z_1,n}$. Then $R_{z_1,n}$ is a rectangle in $B_\varepsilon(z_1)$ and $R_{z_2,n}$ is a rectangle in $B_\varepsilon(z_2)$ for n large. Also, $z_1 \in \operatorname{int} R_{z_1,n}$ and $z_2 \in \operatorname{int} R_{z_2,n}$.

Modifying $\mathscr{B}^1$ and $\mathscr{B}^2$ slightly if necessary, we may assume for n large that

(4)
$$f^{n_0}\partial^s B^{1u}_n \cap R_{z_1,n} = \emptyset, \quad f^{n_0}\partial^s B^{2u}_n \cap R_{z_2,n} = \emptyset,$$
$$f^{-n-n_0}\partial^u B^{1u}_n \cap R_{z_2,n} = \emptyset \quad \text{and} \quad f^{-n-n_0}\partial^u B^{2u}_n \cap R_{z_1,n} = \emptyset.$$

Finally, let

$$V_n = B^{2u}_n \cup \Big(\bigcup_{-n_0\leq j\leq n_0+n} f^j R_{z_2,n}\Big) \cup B^{1u}_n \cup \Big(\bigcup_{-n_0\leq j\leq n_0+n} f^j R_{z_1,n}\Big)$$

and

$$\Lambda_{3,n} = \bigcap_{-\infty<k<\infty} f^k V_n.$$

We claim that for n large, $\Lambda_{3,n}$ is our required basic set. We agree to take n large in the following without further mention.

Clearly, $\Lambda_{3,n}$ is a closed f-invariant set containing $\Lambda_1 \cup \Lambda_2 \cup \{z_1, z_2\}$. The splitting $E^s_1 \oplus E^u_1$ of $T_{V_n}M$ is an almost hyperbolic splitting for f on V_n in the sense of [18]. Thus, $\Lambda_{3,n}$ is hyperbolic by theorem (3.1) in [18]. Moreover, V_n is a union of rectangles R such that $\partial^u R$ is in the forward orbit of $\partial^u \mathscr{B}^1 \cup \partial^u \mathscr{B}^2$ and $\partial^s R$ is in the backward orbit of $\partial^s \mathscr{B}^1 \cup \partial^s \mathscr{B}^2$. From these facts and (3) and (4), it follows that $\Lambda_{3,n} \cap \partial V_n = \emptyset$, so $\Lambda_{3,n} \subset \operatorname{int} V_n$. To show that Λ_{3n} is a hyperbolic basic set it remains to show that $f|\Lambda_{3n}$ has a dense orbit.

For this it suffices to prove that:

(5) if U is any relatively open subset of $\Lambda_{3,n}$, then $\bigcup_{j\geq 0} f^j U$ is dense in $\Lambda_{3,n}$.

Indeed, if (5) holds, and $G_1, G_2, \dots$ is a countable basis for the topology of $\Lambda_{3,n}$, then any point in $\bigcap_{i=1}^{\infty} \bigcup_{j\geq 0} f^j G_i$ will have a dense orbit.

Let $T_{\Lambda_{3,n}}M = E^s \oplus E^u$ be the hyperbolic splitting of TM over $\Lambda_{3,n}$, and let $0 < \lambda < 1$ be as in the definition of hyperbolicity. There is an $\varepsilon_0 > 0$ so that for $0 < \varepsilon \leq \varepsilon_0$ and $y \in \Lambda_{3,n}$, $W^u_\varepsilon(y)$ and $W^s_\varepsilon(y)$ are closed 1-disks in M, tangent at y to E^u_y and E^s_y resp., $\ell f W^u_\varepsilon(y) > \lambda^{-1} \ell W^u_\varepsilon(y)$, and $\ell f^{-1} W^s_\varepsilon(y) > \lambda^{-1} \ell W^s_\varepsilon(y)$.

Fix $0 < \varepsilon \leq \varepsilon_0$ and $x \in \Lambda_{3,n}$. We will prove that $\bigcup_{j \geq 0} f^j(\Lambda_{3,n} \cap B_\varepsilon(x))$ is dense in $\Lambda_{3,n}$. In fact, we will prove that if $0 < \delta < \varepsilon$, and $x_1 \in \Lambda_{3,n}$, then the forward orbit of $W^u_\delta(x)$ in V_n meets $W^s_\delta(x_1)$. This will prove (5) and complete the proof of lemma 8.

Since $\ell W^u_\delta(f^{j+1}x) > \lambda^{-1} \ell W^u_\delta(f^j x)$ for $j \geq 0$, there are an $n_1 > 0$ and a $B^{i_1}_{k_1}$ so that

(6) $$f^{n_1} W^u_\delta(x) \supset C(f^{n_1}x, B^{i_1}_{k_1} \cap W^u(f^{n_1}(x))).$$

From the definition of V_n, we have $f^{-j} C(f^{n_1}x, B^{i_1}_{k_1} \cap W^u(f^{n_1}x)) \subset V_n$ for $0 \leq j \leq n_1$. If n is large and y is in $B^{i_1}_{k_1} \cap \Lambda_{3,n}$, then

(7) $C(y, B^{i_1}_{k_1} \cap W^u(y))$ is near $W^u(y, B^{i_1}_{k_1})$.

Hence, we may assume that $C(f^{n_1}x, B^{i_1}_{k_1} \cap W^u(f^{n_1}x)) \cap W^s(y, B^{i_1}_{k_1})$ is a unique point for each $y \in B^{i_1}_{k_1}$.

Since $f|\Lambda_1$ and $f|\Lambda_2$ have dense orbits, the forward orbit of $C(f^{n_1}x, B^{i_1}_{k_1} \cap W^u(f^{n_1}x))$ in V_n meets each $W^s(y, B^{i_1}_k)$ for $y \in B^{i_1}_k$, $k = 1, \ldots, s_i$. So the forward orbit of $C(f^{n_1}x, B^{i_1}_{k_1} \cap W^u(f^{n_1}x))$ in V_n meets each $W^s(y, R_{z_1,n})$ and $W^s(y, R_{z_2,n})$ as well, and hence meets each $W^s(y, B^i_k)$ for $i = 1, 2$ and $k = 1, \ldots, s_i$. To sum up, if

$$\mathscr{B} = \mathscr{B}^1 \cup \mathscr{B}^2 \cup \{R_{z_1,n}\} \cup \{R_{z_2,n}\},$$

then the forward orbit of $W^u_\delta(x)$ in V_n meets $W^s(y, B)$ for each y in B and B in $\mathscr{B}$. Similarly, the backward orbit of $W^s_\delta(x_1)$ in V_n meets each $W^u(y, B)$ and by (7) it meets $C(f^{n_1}x, B^{i_1}_{j_1} \cap W^u(f^{n_1}x))$. But then (6) shows that the forward orbit of $W^u_\delta(x)$ in V_n meets $W^s_\delta(x_1)$ as required.

Proof of lemma 9. — In the notation of the statement of that lemma, pick C^2 coordinates (x_1, x_2) near x_0, so that $x_1(x_0) = x_2(x_0) = 0$ and $(x_2 = 0) \subset \gamma_1$. Let $v_{1x} = T_x\mathscr{F}_{1x}$ and $v_{2x} = T_x\mathscr{F}_{2x}$ be the C^1 unit tangent fields with $x = (x_1, x_2)$. In the (x_1, x_2) coordinates we have

$$v_{1x} = v_1(x_1, x_2) = a_1(x_1, x_2)\frac{\partial}{\partial x_1} + a_2(x_1, x_2)\frac{\partial}{\partial x_2}$$

and $$v_{2x} = v_2(x_1, x_2) = b_1(x_1, x_2)\frac{\partial}{\partial x_1} + b_2(x_1, x_2)\frac{\partial}{\partial x_2}$$

where $$a_1(0, 0) = 1 = b_1(0, 0) \quad \text{and} \quad a_2(0, 0) = 0 = b_2(0, 0).$$

Consider the C^1 function $\varphi : x \mapsto \det(v_{1x}, v_{2x}) = a_1 . b_2 - a_2 . b_1$. If we show that $\varphi_{x_1}(0, 0) \neq 0$, then the implicit function theorem will produce the curve γ as the set $\{(g(x_2), x_2)\}$ where g is a C^1 real-valued function defined for x_2 near 0.

Now $\varphi_{x_1}=a_{1x_1}.b_2+a_1.b_{2x_1}-a_{2x_1}.b_1-a_2.b_{1x_1}$ which at (o, o) is $a_1.b_{2x_1}-a_{2x_1}.b_1$. Since $\gamma_1\subset(x_2=0)$, $a_2(x_1,0)=0$ for all x_1, so $a_{2x_1}(0,0)=0$. Thus,

$$\varphi_{x_1}(0,0)=a_1(0,0).b_{2x_1}(0,0)=b_{2x_1}(0,0).$$

For (x_1, x_2) near o, let $f(x_1, x_2)$ be the real number so that $(x_1, f(x_1,x_2))\in\mathscr{F}_{2(0,x_2)}$. Then the mapping $x_1\mapsto(x_1,f(x_1,x_2))$ is a C^2 parametrization of $\mathscr{F}_{2(0,x_2)}$ and the tangent map $(x_1,x_2)\mapsto(1,f_{x_1}(x_1,x_2))$ is C^1. Also,

$$b_2(x_1,x_2)=\frac{f_{x_1}(x_1,x_2)}{\sqrt{1+(f_{x_1}(x_1,x_2))^2}}.$$

Since γ_1 and γ_2 have a non-degenerate tangency at x_0, $f_{x_1x_1}(0,0)\neq 0$. But one easily computes that

$$\varphi_{x_1}(0,0)=b_{2x_1}(0,0)=f_{x_1x_1}(0,0)\neq 0.$$

The C^1 continuous dependence of γ on $\mathscr{F}_1$ and $\mathscr{F}_2$ also follows from the implicit fonction theorem.

6. In this section we will prove theorem 2. Before doing this it seems worthwhile to give some conditions on $f: M\to M$ insuring that some stable set $W^s(x,f)$ is not a manifold where M is any manifold with $\dim M>1$.

If E and F are normed linear spaces, $S\subset E$ and $\lambda>1$, a linear map $A: E\to F$ is called a *λ-expansion* on S is $|Av|\geq\lambda|v|$ for all v in S. If $E=E_1\oplus E_2$ and $\varepsilon>0$, the *ε-sector* about E_1 relative to E_2 is the set $S_\varepsilon(E_1)=\{(v_1,v_2)\in E_1\oplus E_2: |v_2|\leq\varepsilon|v_1|\}$. The set $S_\varepsilon(E_2)$ is defined similarly. In the next definition $\varphi_i: B^s\times B^u\to D_i$ will be a disk in M with $s+u=\dim M$. For a point $\xi=\varphi_i(x,y)\in D_i$ with $x\in B^s$, $y\in B^u$, denote the ε_i-sector about $T_\xi\varphi_i(\{x\}\times B^u)$ relative to $T_\xi\varphi_i(B^s\times\{y\})$ by $S_{\varepsilon_i}D_{i,\xi}$.

Consider a sequence $D_1, D_2, \ldots$ of disjoint disks in M (the images of embeddings $\varphi_i: B^s\times B^u\to M$). The sequence (D_i, φ_i) is called *f-controlled* if there are a constant $\lambda>1$, integers $0=n_0<n_1<n_2<\ldots$, and real numbers $\varepsilon_i>0$ $(i=1,2,\ldots)$ such that for each x in B^s, y in B^u, and $i\geq 1$, if we put $z=\varphi_i(x,y)$ and $w=f^{n_i-n_{i-1}}z$, then the following conditions are satisfied:

(1) $f^{n_i-n_{i-1}}(\varphi_i(\{x\}\times B^u))\cap D_{i+1}$ consists of two connected components each of which projects diffeomorphically onto $\varphi_{i+1}(\{0\}\times B^u)$ via $\varphi_{i+1}(u,v)\mapsto\varphi_{i+1}(0,v)$;

(2) $T_zf^{n_i-n_{i-1}}S_{\varepsilon_i}D_{i,z}\subset S_{\varepsilon_{i+1}}D_{i+1,w}$ and the map $T_zf^{n_i-n_{i-1}}$ is a λ-expansion on $S_{\varepsilon_i}D_{i,z}$;

(3) $f^{n_{i-1}-n_i}(\varphi_{i+1}(B^s\times\{y\}))\cap D_i$ consists of two components each projecting diffeomorphically onto $\varphi_i(B^s\times\{0\})$ via $\varphi_i(u,v)\mapsto\varphi_i(u,0)$;

(4) $T_wf^{n_{i-1}-n_i}(T_wD_{i+1}-S_{\varepsilon_{i+1}}D_{i+1,w})\subset T_zD_i-S_{\varepsilon_i}D_{i,z}$ and $T_wf^{n_{i-1}-n_i}$ is a λ-expansion on $T_wD_{i+1}-S_{\varepsilon_{i+1}}D_{i+1,w}$;

(5) if we put $r_i=\max\{\operatorname{diam} f^k(f^{n_i-n_{i-1}}D_i\cap D_{i+1}\cap f^{n_i-n_{i+1}}D_{i+2}): 0\leq k\leq n_{i+1}-n_i\}$, then *a)* $\sup_i r_i\leq\varepsilon_1$

b) $r_i\to 0$ as $i\to\infty$;

(6) if we define $\bar{D}_k = \bigcap_{i \geq k} f^{-n_i} D_{i+1}$ for $k \geq 0$ and if $z_1 \in \bar{D}_k$ and $z_2 \in M - \bar{D}_k$, then there is an integer $n > 0$ such that $\text{dist}(f^n z_1, f^n z_2) > \varepsilon_1$.

To illustrate, let us consider the geometric meaning of conditions (1)-(5) in two dimensions. We have a sequence of disks D_i and integers n_i as in figure 6.1.

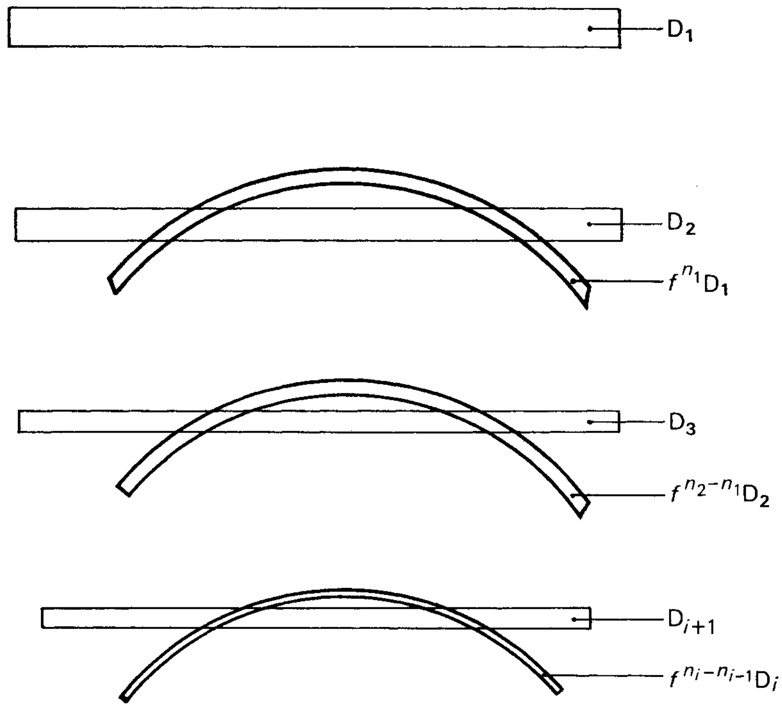

FIG. 6.1

Each D_i is mapped near D_{i+1} by $f^{n_i - n_{i-1}}$ as in a horseshoe diffeomorphism. Because of (1)-(5), as i increases, the vertical heights of the D_i's will shrink and the two components of the intersection $f^{n_i - n_{i-1}} D_i \cap D_{i+1}$ will approach each other.

Lemma 10. — $\bar{D}_0$ *is locally the product of a Cantor set and an s-disk. All points of* $\bar{D}_0$ *are in the same stable set, and if* z *is in* $\bar{D}_k$, $k \geq 0$, *then* $\bar{D}_k = W^s_{\varepsilon_1}(z, f) \cap D_k$.

We remind the reader that

$$W^s_{\varepsilon_1}(z, f) = \{y \in M : \text{dist}(f^n y, f^n z) \leq \varepsilon_1 \text{ for } n \geq 0\}.$$

Proof. — For each $k \geq 0$, let $\pi^s_k : D_k \to \varphi_k(B^s \times \{0\})$ be the natural projection.

By (1)-(4), $f^{n_{k-1} - n_k} D_{k+1} \cap D_k$ consists of two connected components, say C_{k0} and C_{k1}, each of which is a union of s-disks which project diffeomorphically onto $\varphi_k(B^s \times \{0\})$ by π^s_k. Also, $\partial(C_{ki}) - \partial_u D_k$ is a union of s-disks in $f^{n_{k-1} - n_k} \partial_s D_{k+1}$, for $i = 0, 1$.

Applying (1)-(4) again, one sees that

$$f^{n_{k-2} - n_{k-1}}(f^{n_{k-1} - n_k} D_{k+1} \cap D_k) \cap D_{k-1} = f^{n_{k-2} - n_k} D_{k+1} \cap f^{n_{k-2} - n_{k-1}} D_k \cap D_{k-1}$$

consists of 2^2 components each of which is a union of s-disks which π^s_{k-1} projects diffeomorphically and each of whose boundary off $\partial_u D_{k-1}$ is in $f^{n_{k-2}-n_k}\partial_s D_{k+1}$. Continuing by induction, we get that $\bar{D}_k = f^{-n_k}D_{k+1} \cap f^{-n_{k-1}}D_k \cap \ldots \cap D_1$ consists of 2^k components, say $C^k_1, \ldots, C^k_{2^k}$, with the following properties:

a) Each C^k_i a union of s-disks which π^s_1 maps diffeomorphically onto $\varphi_1(B^s \times \{0\})$.

b) For each i, $\partial C^k_i - \partial_u D_1 \subset f^{-n_k}\partial_s D_{k+1}$.

For $z = \varphi_1(x,y) \in C^k_i$, let $C^k_{iz} = C^k_i \cap \varphi_1(\{x\} \times B^u)$. We call $\sup_{z \in C^k_i} \operatorname{diam} C^k_{iz}$ the u-diameter of C^k_i and denote it by $d_u(C^k_i)$. We will show that $d_u(C^k_i) \leq C\lambda^{-k+1}$ for some constant C.

Thus the u-diameters of the components of $\bar{D}_k$ will tend to zero as k approaches infinity. This in conjunction with *a)* and *b)* will prove that $\bar{D}_0$ is locally the product of a Cantor set and an s-disk because $f^{-n_k}\partial_s D_{k+1} \cap \bar{D}_0 = \emptyset$ for all k.

By (1) and (2), if $z \in C^k_i$, then $\operatorname{diam} C^k_{iz} \leq \lambda^{-k+1} \operatorname{diam} f^{-n_{k-1}} C^k_{iz}$. However, since $f^{-n_{k-1}}C^k_{iz}$ is a C^r disk in $f^{n_{k-1}-n_{k-2}}D_{k-1} \cap D_k \cap f^{n_{k-1}-n_k}D_{k+1}$ whose tangent vectors at $\varphi_k(x,y)$ lie in $S_{\varepsilon_k} T\varphi_k(\{x\} \times B^u)$ we have that

$$\operatorname{diam} f^{-n_{k-1}} C^k_{iz} \leq C \operatorname{diam}(f^{n_{k-1}-n_{k-2}}D_{k-1} \cap D_k \cap f^{n_{k-1}-n_k}D_{k+1})$$

for some constant C depending on the Riemann metric on M. From (5 *a*) we have then that $\operatorname{diam} C^k_{iz} \leq \lambda^{-k+1} C\varepsilon_1$ as needed.

The fact that $W^s(x,f) = W^s(y,f)$ for x and y in $\bar{D}_0$ follows immediately from (5 *b*).

To prove the final statement of the lemma, let $z \in \bar{D}_k$ with $k \geq 0$. By (5 *a*) we have $\bar{D}_k \subset D_k \cap W^s_{\varepsilon_1}(z,f)$. If $z_1 \in D_k - \bar{D}_k$, then for some $i \geq k$, $f^{n_i}z_1 \notin D_{i+1}$. Since $f^{n_i}z \in D_{i+1}$, we may use (6) to find an integer $n > 0$ so that $\operatorname{dist}(f^{n+n_i}z, f^{n+n_i}z_1) > \varepsilon_1$. Thus $z \notin W^s_{\varepsilon_1}(z,f)$ and lemma 10 is proved.

Notice that if $z \in \bar{D}_0$, then $W^s(z,f) = \bigcup_{n \geq 0} f^{-n}W^s_{\varepsilon_1}(z,f)$. Thus, $W^s(z,f)$ is not a manifold. It naturally inherits a topology from $\bigcup_{n \geq 0} f^{-n}W^s_{\varepsilon_1}(z,f)$ making it locally the product of a Cantor set and an s-disk. Our next lemma shows that f-controlled disks occur under rather mild assumptions.

Lemma 11. — Suppose $f: M \to M$ is a C^2 diffeomorphism, Λ is a hyperbolic basic set for f, and there are points x, y in Λ so that $W^u(x)$ and $W^s(y)$ have a non-degenerate tangency at z. Let D^u and D^s be C^2 disks in $W^u(x)$ and $W^s(y)$, respectively, containing z in their interiors. Suppose there are disks $D^u_i \subset W^u(\Lambda)$ converging to D^u in the C^2 topology so that z is a limit of transversal intersections of D^u_i and D^s. Then z is a limit of f-controlled disks. Hence, there are points near z whose stable sets are locally the product of a Cantor set and an s-disk.

Proof. — Recall there are an $\varepsilon > 0$ and semi-invariant disc families $\{\widetilde{W}^u_\varepsilon(x)\}$, $\{\widetilde{W}^u_\varepsilon(x)\}$ for x in some neighborhood U_1 of Λ, as in [10]. This means that:

a) for each x, $\widetilde{W}^u_\varepsilon(x)$ is a C^r u-disk containing x and $\widetilde{W}^s_\varepsilon(x)$ is a C^r s-disk containing x;

b) $\widetilde{W}^u_\varepsilon(x)$ and $\widetilde{W}^s_\varepsilon(x)$ vary C^r continuously with x in U_1;

c) $f\widetilde{W}^u_\varepsilon(x) \supset \widetilde{W}^u_\varepsilon(fx)$ and $f\widetilde{W}^s_\varepsilon(x) \subset \widetilde{W}^s_\varepsilon(fx)$ for x in $U_1 \cap f^{-1}U_1$;

d) for $x \in \Lambda$, $\widetilde{W}^u_\varepsilon(x) = W^u_\varepsilon(x)$ and $\widetilde{W}^s_\varepsilon(x) = W^s_\varepsilon(x)$;

e) for $x \in \operatorname{int} U_1$, $\bigcup_{y \in \widetilde{W}^s_\varepsilon(x)} \widetilde{W}^u_\varepsilon(y)$ and $\bigcup_{y \in \widetilde{W}^u_\varepsilon(x)} \widetilde{W}^s_\varepsilon(y)$ are neighborhoods of x.

It is not known if $\{\widetilde{W}^u_\varepsilon(x)\}$ and $\{\widetilde{W}^s_\varepsilon(x)\}$ can always be chosen to be foliations on a neighborhood of Λ, but fortunately, we do not need this.

Pick a compact neighborhood U_2 of Λ such that $U_2 \subset \operatorname{int} U_1$, $\bigcap_n f^n(U_2) = \Lambda$ and $\{f^n(z) : n \in \mathbf{Z}\} \cap \partial U_2 = \emptyset$. Choose integers $k_1, k_2 > 0$ so that $f^{-n}(z) \in \operatorname{int} U_2$ for $n > k_1$, $f^{-k_1}(z) \notin U_2$, $f^n(z) \in \operatorname{int} U_2$ for $n > k_2$, and $f^{k_2}z \notin U_2$.

Set $U = \bigcup_{-k_1 \leq j \leq k_2} f^j(U_2)$ and define the families $\{\widetilde{W}^u_\varepsilon(x)\}$, $\{\widetilde{W}^s_\varepsilon(x)\}$ for $x \in U$ by iteration. We may choose U so that there is a finite set $\{x_1, \ldots, x_\nu\} \subset \Lambda$ such that $U \subset \bigcup_{i=1}^{\nu} \operatorname{int} H_i$ where

$$H_i = \bigcup_x \{\widetilde{W}^u_\varepsilon(x) : x \in W^s_\varepsilon(x_i)\} \cap \bigcup_x \{\widetilde{W}^s_\varepsilon(x) : x \in W^u_\varepsilon(x_i)\}.$$

If the families $\{\widetilde{W}^u_\varepsilon(x)\}$ and $\{\widetilde{W}^s_\varepsilon(x)\}$ were foliations, H_i would be like a local product neighborhood in which the foliations were trivial.

Observe that

$$z \in \operatorname{int}(f(U) - U) \cap \operatorname{int}(f^{-1}(U) - U).$$

Let V be a small compact neighborhood of z so that $\operatorname{dist}(V, U) > 0$. Shrinking D^s and D^u if necessary we may assume that $D^s \cup D^u \subset \operatorname{int} V$ and $D^s \cap D^u = \{z\}$. Let $\pi_i : V_1 \to D^s$ and $\pi_2 : V_2 \to D^u$ be tubular neighborhood retractions with V_1, V_2 compact and $V_1 \cup V_2 \subset \operatorname{int} V$. Thus $D^s \subset V_1$ and there is a diffeomorphism $\psi_i : V_1 \to B^s \times B^u$ so that $\psi_1(D^s) = B^s \times \{0\}$ and $\psi_1(\pi_1^{-1}x) = \{\psi_1(x)\} \times B^u$ for $x \in D^s$. Also $D^u \subset V_2$ and there is a diffeomorphism $\psi_2 : V_2 \to B^u \times B^s$ with $\psi_2(D^u) = B^u \times \{0\}$ and $\psi_2(\pi_2^{-1}x) = \{\psi_2(x)\} \times B^s$ for $x \in D^u$.

We wish to define a family of embeddings $\varphi_i : B^s \times B^u \to M$ so that $\{(\varphi_i(B^s \times B^u), \varphi_i)\}$ is f-controlled.

For each large i, let $\mu_i > 0$ be small enough so that if F^u is a u-disk μ_i—C^2-close to D^u_i, then $F^u \cap D^s$ consists of two transverse intersections. Since $f|\Lambda$ has a dense orbit, $W^u(\Lambda) \subset \operatorname{Cl} \bigcup_{n \geq 0} f^n D^u_i$ for each i. Hence there are sequences $y_i \in D^u_i$ and $n_i > 0$ so that $\operatorname{dist}(f^{n_i - n_{i-1}} y_i, y_{i+1}) < \frac{\mu_i}{2}$, $f^j(y_i) \in U$ for $0 < j < n_i - n_{i-1}$, and $y_i \to z$ as $i \to \infty$.

The disk D_i will be chosen to be the component of $f^{n_{i-1} - n_i}(V_2) \cap V_1$ containing y_i. The embedding $\varphi_i : B^s \times B^u \to D_i$ is defined so that for $x \in B^s$, $\varphi_i(\{x\} \times B^u)$ is contained in some fiber of π_1 and, for $y \in B^u$, $f^{n_i - n_{i-1}} \varphi_i(B^s \times \{y\})$ is contained in some fiber of π_2.

A generalization of the λ-lemma to basic sets (as stated in Proposition (2.3) of [19]) gives that the forward iterates of the fibers $\{\pi_1^{-1}x\}_{x \in D^s}$ will be mapped C^r near disks in $W^u_\varepsilon(\Lambda) = \bigcup_{x \in \Lambda} W^u_\varepsilon(x)$ and backward iterates of $\{\pi_2^{-1}x\}_{x \in D^u}$ will be mapped C^r near disks in $W^s_\varepsilon(\Lambda)=\bigcup_{x\in\Lambda}W^s_\varepsilon(x)$. This guarantees that for large n_i conditions (1)-(5) in the definition of an f-controlled sequence of disks may be obtained with λ large and ε_1 small.

To obtain (6), we must be careful to say how small ε_1 must be taken. If necessary, we will agree to increase n_i and therefore make the D_i's approach D^s and their u-diameters get smaller without further mention.

The Riemann metric on M induces one on each $\widetilde{W}^u_\varepsilon$ and $\widetilde{W}^s_\varepsilon$ and on D^u and D^s. If F is a submanifold and $z_1, z_2\in F$, let us write $d(F; z_1, z_2)$ for the distance between z_1 and z_2 in the induced metric on F. Let us write $\mathrm{dist}(F; A, z_1)$ for the induced distance between a point $z_1\in F$ and a set $A\subset F$.

Now, first pick $\varepsilon_1'>0$ so that:

f) if $d(M; w, fz)<2\varepsilon_1'$, then $\mathrm{dist}(D^s; f^{-1}\widetilde{W}^u_\varepsilon(w)\cap D^s, z)<\frac{1}{2}d(D^s; \partial D^s, z)$, and

g) if $d(M; w, f^{-1}z)<2\varepsilon_1'$ and $w_1\in\widetilde{W}^u_\varepsilon(w)$ with $d(\widetilde{W}^u_\varepsilon(w); w, w_1)<2\varepsilon_1'$, then fw and fw_1 lie in the same component of $f\widetilde{W}^u_\varepsilon(w)\cap V_2$.

We may choose U so that:

h) there is a constant $c_1>0$ such that for $z_1, z_2\in U$, $y\in\widetilde{W}^s_\varepsilon(z_1)$ and $z_2\in\widetilde{W}^u_\varepsilon(y)$, we have $d(M; z_1, z_2)\geq c_1 d(\widetilde{W}^u_\varepsilon(y); y, z_2)$.

Now we choose $\varepsilon_1>0$ such that:

i) $\varepsilon_1<\mathrm{dist}(M; O_+(z)\cup\Lambda, M-U)$ where $O_+(z)=\{f^n z : n>0\}$,

j) $\varepsilon_1<\min(\varepsilon_1', c_1\varepsilon_1')$, and

k) if $d(M; z_1, z_2)\leq\varepsilon_1$, and $z_2\in f^{-1}\widetilde{W}^u_\varepsilon(y)$ with $y\in U$, then $d(M; y, fz_1)<\varepsilon_1'$.

We now prove that *b)* can be obtained with this ε_1. Suppose $z_1\in\overline{D}_k$ and $z_2\notin\overline{D}_k$ for some $k>0$, and suppose, by way of contradiction, that $z_2\in W^s_{\varepsilon_1}(z_1)$. Since $z_2\notin\overline{D}_k$, there is an integer $i\geq k$ so that $f^{n_i}(z_2)\notin D_{i+1}$.

But $f^{n_i}(z_1)\in D_{i+1}$ and $d(M; f^{n_i}z_1, f^{n_i}z_2)\leq\varepsilon_1$. If there is a j with $n_i<j<n_{i+1}$ such that $f^j z_2\notin U$, then $d(M; f^j z_2, f^j z_1)>\mathrm{dist}(M; f^j z_1, M-U)$.

For the D_j's close to z, $f^j z_1$ will be near $\Lambda\cup O_+(z)$, so $\mathrm{dist}(M; f^j z_1, M-U)>\varepsilon_1$ by *i)*. Then $d(M; f^j z_1, f^j z_2)>\varepsilon_1$, which contradicts $z_2\in W^s_{\varepsilon_1}(z_1)$.

Now suppose $f^j z_2\in U$ for $n_i<j<n_{i+1}$. Since $d(M; f^{n_i+1}z_2, f^{n_i+1}z_1)\leq\varepsilon_1$, there is a y in $\widetilde{W}^s_\varepsilon(f^{n_i+1}z_1)$ such that $f^{n_i}(z_2)\in f^{-1}\widetilde{W}^u_\varepsilon(y)$ and (by *k)*) $d(M; y, f^{n_i+1}z_1)<\varepsilon_1'$. If we assume $d(f^{n_i+1}z_1, fz)<\varepsilon_1'$, then $d(M; y, fz)<2\varepsilon_1'$. By *f)*,

$$\mathrm{dist}(D^s; f^{-1}\widetilde{W}^u_\varepsilon(y)\cap D^s, z)<\frac{1}{2}d(D^s; \partial D^s, z).$$

By *f)* and *g)*, and the fact that D_{i+1} is the component of $f^{n_i-n_{i+1}}(V_2)\cap V_1$ containing $f^{n_i}(z_1)$ (and hence $f^{-1}y$), any point $w\in f^{-1}\widetilde{W}^u_\varepsilon(y)$ such that $d(f^{j-1}\widetilde{W}^u_\varepsilon(y); f^j w, f^j y)\leq\varepsilon_1'$ for $0\leq j\leq n_{i+1}-n_i$ must be in the component of y in $f^{-1}\widetilde{W}^u_\varepsilon(y)\cap D_{i+1}$. In particular such a point w is in D_{i+1}. Thus, since $f^{n_i}(z_2)\notin D_{i+1}$, there is a j with $n_i<j<n_{i+1}$ and

$$d(\widetilde{W}^u_\varepsilon(f^{j-n_i-1}y); f^j z_2, f^{j-n_i-1}y)>\varepsilon_1'.$$

But, by *(h)*, since $f^j z_1 \in \widetilde{W}^s_\varepsilon(f^{j-n_i-1}y)$, we have

$$d(M; f^j z_2, f^j z_1) \geq c_1 d(\widetilde{W}^u_\varepsilon f^{j-n_i-1}y; f^j z_2, f^{j-n_i-1}y) \geq c_1 \varepsilon_1' > \varepsilon_1$$

which is a contradiction.

It is easy to give examples of open sets of diffeomorphisms satisfying the hypotheses of lemma 11 on manifolds of dimension bigger than two. To give a specific example let us consider a variation of a class of diffeomorphisms which has already been studied by Simon [21]. Begin with a diffeomorphism $f_1 : B^2 \to B^2$ having a one-dimensional basic set Λ as in Plykin [27]. Let $f_2 : \mathbf{R} \to \mathbf{R}$ be the linear mapping $y \mapsto ay$ where $a > \max(\sup_{x \in B^2} |T_x f_1|^2, 1) \geq 1$. Set $f = f_1 \times f_2 : B^2 \times \mathbf{R} \to B^2 \times \mathbf{R}$. Then $\Lambda_1 = \Lambda \times \{0\}$ is a one-dimensional basic set for f having a fixed point p such that $\dim W^u(p) = 2$ and $\dim W^s(p) = 1$. Also, $W^s(\Lambda_1, f)$ fills up a two-dimensional neighborhood of Λ_1 in $B^2 \times \{0\}$. Let $x \in W^s(\Lambda_1, f) - \Lambda_1$. The picture is as in figure 6.2.

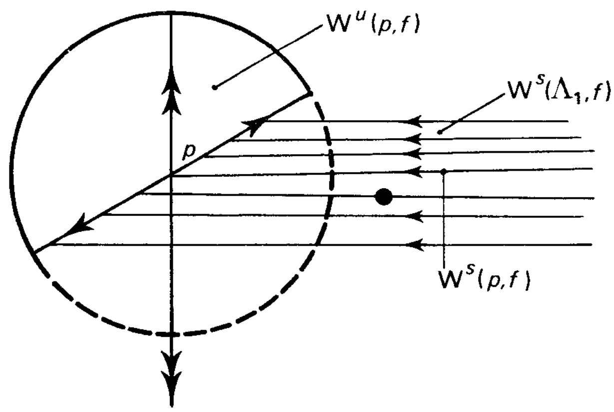

Fig. 6.2

With an isotopy supported off Λ_1, move a piece of $W^u(p,f)$ around to create a non-degenerate tangency with $W^s(p,f)$. Somewhat further in the isotopy gives the example as is clear from figure 6.3.

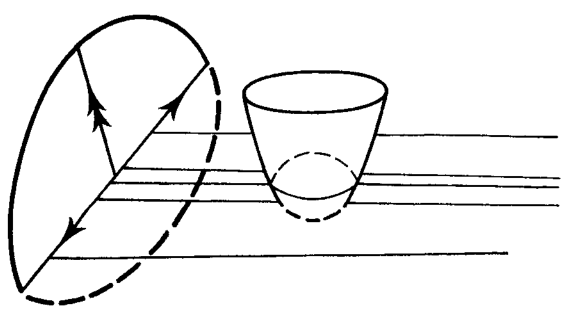

Fig. 6.3

The definition of the number a insures that for g C^2 near f, $W^s(\Lambda_1(g))$ is a C^2 manifold fibered by the C^2 curves $W^s(y, g)$, $y \in \Lambda_1(g)$. One may globalize in the usual way by putting f in a coordinate chart in any M with $\dim M > 2$. This method produces the conditions of lemma 11 if Λ_1 is any basic set such that $W^u(p) \cap \Lambda_1$ contains a C^2 curve. It probably also works whenever $\dim \Lambda_1 > 0$.

We now restrict to a two-dimensional manifold M^2 and proceed to prove theorem 2. Fix $r \geq 2$ and let $f \in \mathrm{Diff}^r M^2$ be a diffeomorphism having an infinite zero-dimensional hyperbolic basic set Λ. Suppose $W^u(\Lambda, f)$ and $W^s(\Lambda, f)$ have a non-degenerate tangency at a point z. Let $\gamma: [-1, 1] \to M$ be a C^1 curve transverse to $W^u(\Lambda, f)$ and $W^s(\Lambda, f)$ at z such that $\gamma(-1)$ and $\gamma(1)$ miss $W^u(\Lambda) \cup W^s(\Lambda)$. For $\varepsilon > 0$ small, choose $N > 0$ so that $f^{-N+1} z \in W^u_\varepsilon(\Lambda, f)$ and $f^{N-1} z \in W^s_\varepsilon(\Lambda, f)$. Write $W^u_N(\Lambda, f) = f^N W^u_\varepsilon(\Lambda, f)$ and $W^s_N(\Lambda, f) = f^{-N} W^s_\varepsilon(\Lambda, f)$. Then $W^u_N(\Lambda) \cap \gamma$ and $W^s_N(\Lambda) \cap \gamma$ are Cantor sets in γ. Let $I^u(\gamma)$ be the smallest closed interval in γ containing $W^u_N(\Lambda) \cap \gamma$, and let $I^s(\gamma)$ be the smallest closed interval in γ containing $W^s_N(\Lambda) \cap \gamma$.

Suppose:

(1) $I^u(\gamma)$ and $I^s(\gamma)$ overlap in the sense that

$$\partial I^u(\gamma) \cap \mathrm{int}\, I^s(\gamma) \neq \emptyset, \quad \text{and} \quad \partial I^s(\gamma) \cap \mathrm{int}\, I^u(\gamma) \neq \emptyset, \quad \text{and}$$

(2) $$\tau(W^u_N(\Lambda) \cap \gamma) . \tau(W^s_N(\Lambda) \cap \gamma) > 1.$$

By lemma 4, we know that $W^u_N(\Lambda) \cap \gamma \cap W^s_N(\Lambda) \neq \emptyset$. Assume that:

(3) each point of $W^u_N(\Lambda) \cap \gamma \cap W^s_N(\Lambda)$ is a non-degenerate tangency of $W^u_N(\Lambda)$ and $W^s_N(\Lambda)$.

By proposition 6 and the C^2 continuous dependence of $W^u_N(\Lambda)$ and $W^s_N(\Lambda)$ on f, we have that the set V of diffeomorphisms f for which there are a hyperbolic Λ for f and a curve γ satisfying (1), (2) and (3) is open in $\mathrm{Diff}^r M^2$.

Let $U \subset \mathrm{Diff}^r M^2$ be an open set of diffeomorphisms each of whose elements has a wild hyperbolic set. By the proof of theorem 1, the set $U_1 = U \cap V$ is dense and open in U. To prove theorem 2, it suffices to show that each f in V satisfies the hypotheses of lemma 11.

Given f in V, let Λ be a hyperbolic set and γ be a curve satisfying (1), (2) and (3). Let $\{F^u_i\}_{i \geq 0}$ and $\{F^s_i\}_{i \geq 0}$ be defining sequences for $W^u_N(\Lambda) \cap \gamma$ and $W^s_N(\Lambda) \cap \gamma$ respectively. By the last statement in lemma 4, we have $\mathrm{int}(F^u_i \cap F^s_i) \neq \emptyset$ for all $i \geq 0$. Let $z \in W^u_N(\Lambda) \cap \gamma \cap W^s_N(\Lambda)$ and let C^u_i and C^s_i be components of F^u_i and F^s_i such that $\mathrm{int}\, C^u_i \cap C^s_i \neq \emptyset$ and $C^u_i \cup C^s_i \to z$ as $i \to \infty$.

By (3) there are closed intervals $D^u \subset W^u(z)$ and $D^s \subset W^s(z)$ containing z in their interiors and having z as a non-degenerate tangency. To satisfy the conditions of lemma 11, we only need to find a sequence D^u_i of closed intervals in $W^u_N(\Lambda)$ converging to D^u such that each D^u_i has a transverse intersection z_i with D^s and $z_i \to z$ as $i \to \infty$. Since Λ is an infinite basic set, D^u is a C^2 limit of infinitely many disks in $W^u_N(\Lambda)$, and

D^s is a C^2 limit of infinitely many disks in $W^s_N(\Lambda)$. The only way such a sequence D^u_i would not exist is for $W^u_N(\Lambda)$ to accumulate on $W^u(z)$ from only one side and $W^s_N(\Lambda)$ to accumulate on $W^s(z)$ from only the other side as in figure 6.4.

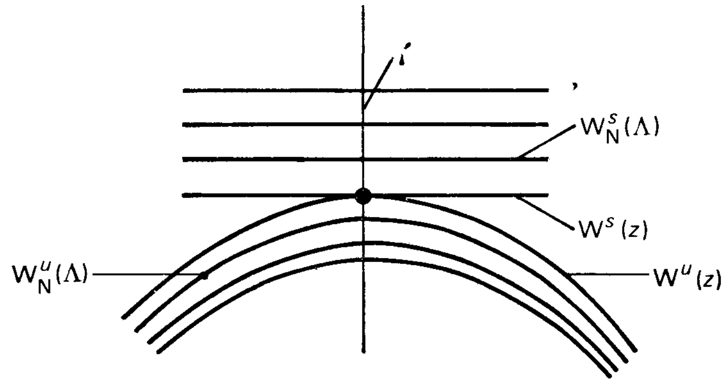

Fig. 6.4

But this cannot happen since $\partial C^u_i \subset W^u_N(\Lambda)$, $\partial C^s_i \subset W^s_N(\Lambda)$, $C^u_i \cup C^s_i \to z$, and $\text{int}\,(C^u_i \cap C^s_i) \neq \emptyset$. Thus, the hypotheses of lemma 11 are satisfied by f.

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