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THE ACCELERATED BINOMIAL OPTION PRICING MODEL

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Abstract

This paper describes the application of a convergence acceleration technique to the binomial option pricing model, in the context of the valuation of the American put option on non-dividend paying stock. The resulting model, termed the accelerated binomial option pricing model, can also be viewed as an approximation to the Geske-Johnson model for the value of the American put. The new model is accurate and faster than the conventional binomial model. It is also likely to prove much more computationally convenient than the Geske-Johnson model. It is applicable to a wide range of option pricing problems.

The Accelerated Binomial Option Pricing Model

I Introduction

The binomial option pricing model, introduced by Cox, Ross and Rubinstein [3], is now widely used to value options, particularly where no analytic (closed form) solution exists, as in the benchmark case of the American put option. More recently, Geske and Johnson [5] introduced a method of valuing American put options based on the compound option model and utilising convergence acceleration techniques. As a result, their approach is a more efficient means of valuing such options than the binomial. In this paper we present a method, called the accelerated binomial option priding model, which is a hybrid of the binomial and Geske-Johnson models. It can be viewed as a binomial model incorporating the convergence acceleration techniques used by Geske and Johnson: equally it can be seen as a binomial approximation to the continuous time Geske-Johnson model. The purpose of this paper is to present the accelerated binomial option pricing method and to illustrate its accuracy, rather than to evaluate its computational efficiency vis-a-vis other methods. However, the results so far obtained with the accelerated binomial method show it to be more efficient than the unmodified binomial modèl and computationally simpler than the Geske-Johnson model. These issues are taken up again in the paper's conclusion. We begin by swiftly reviewing the binomial and Geske-Johnson models, then go on to present the accelerated binomial option pricing model.

We deal in this paper with American put options written on non-dividend paying stock. We make the usual assumptions - namely that the risk free interest rate, r, and the annualised standard deviation of the underlying stock price, σ, are both non-stochastic and constant over the life of the option. We denote time by the index t ($t = 0 \ldots T$, the maturity date of the option), the stock price at t by $S(t)$ and the exercise price by X.

II The Binomial Option Pricing Model

In the binomial option pricing model, the life of the option is divided into N discrete time periods, during each of which the price of the underlying asset is assumed to make a single move, either up or down. The magnitude of these movements is given by the multiplicative parameters u and d. The probability [1] of an upward movement is given by p, and the one period risk free rate we denote by q.

The binomial method approximates the continuous change in the option's value through time by valuing the option at a discrete set of nodes which together make a cone shaped grid. We identify each node in the cone by $\langle j,n \rangle$ where j indicates the number of upward stock moves required to generate the option's immediate non-negative exercise value at that node, given by

$$B_{j,n} = \max (0 \; ; \; X - u^j d^{n-j} S) \quad \text{(for a put)} \qquad (1)$$

and n is the period of the model ($n = 0 \ldots N$).

When valuing European options or American call options on non-dividend paying stock, it is only necessary to calculate the N+1 terminal exercise values of the option (i.e. the set B_{jN}, j=0 ... N in our notation). Since there is no probability of early exercise in these cases the intermediate values of the binomial process (for $0 < n < N$) need not be computed. Instead the binomial formula is used to 'jump backwards' from the terminal values to the initial option value (at node <0,0>). In Geske and Shastri's [6] analysis of approximation methods for option valuations, it was this feature of the binomial method that was chiefly responsible for its outperforming its competitors (finite difference methods) in terms of computing demands and expense by a considerable margin in the valuing of a call option on non-dividend paying stock (see, for example, Geske and Shastri [6, table 2, p.60, and figure 1, p.61]).

However, the application of the binomial method to the valuing of an American put option on non-dividend paying stock will be much less efficient. This is because the possibility of early exercise requires that both the holding value and the exercise value of the option be computed for each node in the process. We define the value of the option at the jn th node by

$$V_{jn} = \max \ (A_{jn}, B_{jn}) \tag{2}$$

where B_{jn} is as before and A_{jn} is the holding value of the option at that node:

$$A_{jn} = (p/q)V_{j+1,n+1} + ((1-p)/q)V_{j,n+1} \tag{3}$$

The binomial method entails the calculation of the values of all nodes in successively earlier periods, culminating in the value V_{oo} which is the option's value.

III The Geske and Johnson Compound Option Approach

The Geske-Johnson analytic formula for the value of an American put option, which we denote by ϕ, can be written:

$$\sum_{n=1}^{\infty} \text{prob } (S_{ndt} < \bar{S}_{ndt}, S_{mdt} \geq \bar{S}_{mdt} \ \forall\, m < n)$$
$$.(X-E[S_{ndt} | S_{ndt} < \bar{S}_{ndt}, S_{mdt} > \bar{S}_{mdt} \ \forall\, m < n])/r^{ndt} \quad (4)$$

That is, the value of the option is given by the sum of the discounted conditional exercise values of the option at each instant during its life. The condition in question is that, at instant ndt, the stock price, S, should be below its critical value $\bar{S}$, not having fallen below its critical value at any previous instant, mdt. To use expression (4), then, entails the evaluation of an infinite sequence of successively higher order normal integrals, reflecting the fact that, at instant ndt, estimating the conditional expectation of the option's exercise value requires the evaluation of an n-variate normal integral.

Geske and Johnson [5] surmount this difficulty by defining a reduced number of early exercise instants during the life of the option and using Richardson's extrapolation to find an approximation to the true option value. Their three point extrapolation, for example, defines a set of option values, P(n), based on exercise opportunities restricted as follows: P(1) = option value based on exercise opportunities restricted to T; P(2) = option value based on exercise opportunities at T and T/2; P(3) = option value based on exercise opportunities at T, 2T/3 and T/3. The limit of this sequence, P(n) $n \to \infty$ is the option's value, ϕ. The approximation to ϕ is then given by

$$P = P(3) + 4.5*[P(3)-P(2)] - 0.5*[P(2)-P(1)] \qquad (5)$$

(see Geske and Johnson, [5, pp. 1518 and 1523]).

IV The Accelerated Binomial Option Pricing Model

The sequence of functions

$$P_a(S) = \sum_j^{N-a} \binom{N-a}{j} (p/q)^j \; ((1-p)/q)^{N-a-j} \; V_{j,N-a} \qquad (6)$$

where V is defined as earlier, converges to $P_N(S)$ $(=V_{oo})$ for any binomial model with N periods. Convergence is uniform from below (see appendix) and occurs at $P_m(S)$, where m is the earliest period in the model for which V_{om} takes its immediate exercise value rather than its holding value (Breen [2]). m will always be less than N unless the put should be exercised immediately. In other words, for a binomial option pricing model with fixed N, the value of the option is the limit of the sequence $P_a(S)$. $P_a(S)$ defines a sequence of option values with an increasing number of exercise opportunities. Thus, $P_o(S)$ is a European option, permitting exrcise only at period N. $P_1(S)$ is the value of an option permitting exercise at period N and period N-1; and so on. That the sequence converges to the option's value is true by definition $(P_N(S)=V_{oo})$. That convergence is from below is intuitively clear, insofar as, if this were not so, $P_j(S) > P_N(S)$ $(j < N)$ and $P_j(S)$ would be the option's value. But this would imply that exercise opportunities in periods earlier than period N-j would reduce the value of the option - an obvious contradiction.

Consider now the related sequence, $P'_n(S)$, or $P'(n)$ for short, defined as follows: $P'(1) = P_o(S)$; $P'(2)$ = binomial option value permitting exercise at N and N/2 only; $P'(3)$ = binomial option value permitting exercise at N, 2N/3 and N/3 only. Again, this sequence converges to the option's value, $P_N(S)$ from below - that is, $P_N(S)$ is the limit of the sequence $P'(n)$ as $n \to \infty$. It follows too that $P'(2) > P'(1)$ and that $P'(3) > P'(1)$, though not necessarily that $P'(3) > P'(2)$, although in practice this usually seems to be the case. [2]

To apply the Richardson extrapolation technique to the binomial we proceed by analogy with Geske and Johnson's exposition. The parallels between the sequences P and P' are clear: in both the number of exercise opportunities increases as we move down the sequence. Thus we apply formula (5) to the terms $P'(n)$, n= 1,2,3. The value of the option is then given as max $(P', X-S)$.

In practical terms the resulting accelerated binomial model is very easy to program. To give some idea of its accuracy we refer first to Table 1, where three sets of American option values for the data originally given by Cox and Rubinstein [4] and Parkinson [8] are shown. These three sets of values are based on, respectively, the unmodified binomial with 150 periods or Parkinson's numerical approach; the Geske-Johnson 'analytic' method using a four point extrapolation; and the accelerated binomial method presented here, using a three point extrapolation over a 150 period model. All three sets of values agree very closely. The largest error in the accelerated binomial method is of the order of one and a half cents compared with the binomial or numerical value. Clearly a four point extrapolation would be more accurate. What is most striking about the accelerated binomial method, however, is the reduction it brings about in the amount of computation required. The unmodified binomial method requires the calculation of $(N+1)^2$ node values, which, for N=150, is 22801. The accelerated binomial, on the other hand, calls for only 4N + 10 calculations - 610 for a 150 period model. Thus the accelerated binomial is very much faster than the binomial method, reducing the number of node value calculations by 97 per cent. A second source of comparison will be found in Table 2, which shows put option values calculated using the accelerated binomial together with values obtained using three other methods - the finite difference method, the Geske-Johnson method and Macmillan's [9] quadratic approximation. Comparing the three other methods against the finite difference values it can be seen that there is little to choose between them, although the accelerated binomial is, if anything marginally more accurate than either the Geske-Johnson model (which is here computed using a three point extrapolation) or Macmillan's. A similar conclusion is reached if we compare the accelerated binomial values in the present Table 1 with those given by Macmillan [9, pp. 131-132] for the same data using his own method.

[TABLES 1 AND 2 HERE]

V Conclusion

The only previous attempt to investigate the application of convergence acceleration techniques to the binomial option pricing model is contained in a paper by Omberg [7]. His approach differs from the present one insofar as he sought to find a means by which to accelerate the convergence of a sequence of binomial option models with increasing N (rather than, as in the approach used here, seeking to accelerate the convergence of a particular binomial model with fixed N). However, such a sequence converges in an oscillating, rather than uniform, manner, and Omberg showed that it was impossible to select the parameters of the binomial model in such a way as to ensure uniform convergence. Nevertheless, Omberg [7, p.464] notes that, if convergence acceleration could be applied to the binomial, then "binomial-pricing models might prove to be considerably more efficient than compound option models".

The present accelerated binomial model has advantages over both the binomial and Geske-Johnson models. On the one hand, it is much faster than the unmodified binomial, suggesting that it might prove to be more efficient than other numerical methods (such as finite difference methods) not only for valuing a small number of options, but also for valuing a large number. On the other hand, the accelerated binomial, viewed as an approximation to the Geske-Johnson model, removes the need to evaluate multivariate normal integrals of up to order three or four (in a four point extrapolation) - a computationally time consuming task. [3] That this is the main disadvantage of the Geske-Johnson method has been recognised by a number of authors, including Barone-Adesi and Whaley [1], Omberg [7] and also Selby and Hodges [10] who have demonstrated a means by which the integral evaluation problem can be reduced to more manageable proportions. The approach outlined here, however, is likely to prove far more convenient and accessible even than a Geske-Johnson model incorporating Selby and Hodges' modifications.

Since the binomial itself is an approximation to the true option value, our application of the Richardson extrapolation technique yields an approximation to an approximation. Nevertheless, this can be made as accurate as one desires, first by choosing a sufficiently large value for N, and, secondly, by extrapolating from a greater number of exercise points. Clearly, however, our choice of the conventional value of N (150) and of a simple three point extrapolation yields results which are sufficiently accurate for most practical purposes. Finally, because the model presented here is a modification to the binomial, it retains all the flexibility of the latter. Thus the accelerated binomial can be used to value all the variety of options (on foreign exchange, commodities, futures, and so on) for which the binomial itself is applicable.

FOOTNOTES

1. By which we mean, of course, the probability within the binomial model implied by the risk neutrality assumption.

2. As with the sequence P(n), this means that P'(n) does not converge uniformly to its limit. In discussing the Geske-Johnson model Omberg [7, pp. 463-464] has suggested that uniform convergence of the sequence of P(n) would be desirable on the grounds that this would ensure that the convergence acceleration technique performs as intended. For both the sequences P(n) and P'(n) this could be accomplished by ensuring that each term in the sequence permits early exercise at every instant (or period, in the case of P'(n)) at which exercise was permitted in forming earlier terms. Thus, the term P(3) in the Geske-Johnson sequence would be amended to permit exercise at T, 3T/4, 2T/4 and T/4 - and analogously for P'(3). In what follows, however, we retain the original specifications of the terms of P and P'.

3. The accelerated binomial, unlike the Geske-Johnson approach, does not require the separate calculation of the critical stock price at each permitted exercise point.

APPENDIX

Uniform convergence is defined by Dini's theorem. In our case, to show that the sequence of functions $P_a(S)$ (a=0 ... N) converges uniformly we need only to demonstrate that, for all S, $P_a(S) \geqslant P_{a-1}(S)$.

Write $P_{a-1}(S)$ as

$$\sum_{j=0}^{n} \binom{n}{j} (p/q)^j \; ((1-p)/q)^{n-j} \; V_{jn}$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j} (p/q)^j \; ((1-p)/q)^{n-1-j} \; V_{jn} \; (1-p)/q$$

$$+ \sum_{j=1}^{n-1} \binom{n-1}{j-1} (p/q)^{j-1} \; ((1-p)/q)^{n-j} \; V_{jn} \; p/q \qquad \text{(a1)}$$

Write $P_a(S)$ as

$$\sum_{j=0}^{n-1} \binom{n-1}{j} (p/q)^j \; ((1-p)/q)^{n-1-j} \; V_{j\,n-1} \qquad \text{(a2)}$$

Since for all V_{jn}

$$V_{j,n-1} \geqslant V_{j+1,n}(p/q) + V_{jn} \; (1-p)/q$$

a2 is $\geqslant$

$$\sum_{j=0}^{n-1} \binom{n-1}{j} (p/q)^j \; ((1-p)/q)^{n-1-j} \; [V_{j+1,n}(p/q) + V_{jn} \; (1-p)/q]$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j} (p/q)^j \; ((1-p)/q)^{n-1-j} \; V_{jn} \; (1-p)/q$$

$$+ \sum_{j=0}^{n-1} \binom{n-1}{j} (p/q)^{j-1} \; ((1-p)/q)^{n-1-j} \; V_{j+1,n} \; p/q$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j} (p/q)^j \; ((1-p)/q)^{n-1-j} \; V_{jn} \; (1-p)/q$$

$$+ \sum_{j=1}^{n} \binom{n-1}{j-1} (p/q)^{j-1} \; ((1-p)/q)^{n-j} \; V_{jn} \; p/q$$

Thus $P_n(S) \geqslant P_{n-1}(S)$ and the sequence converges uniformly.

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TABLES

Table 1: Values of American Put Option using numerical, Geske-Johnson and Accelerated Binomial Methods

A. S=$40; r=1.05

X	σ	T	Binomial	Geske-Johnson	Accelerated Binomial
35.00	0.2000	0.08330	0.01000	0.006200	0.006000
35.00	0.2000	0.33330	0.20000	0.199900	0.198900
35.00	0.2000	0.58330	0.43000	0.432100	0.433800
40.00	0.2000	0.08330	0.85000	0.852800	0.851200
40.00	0.2000	0.33330	1.58000	1.580700	1.574000
40.00	0.2000	0.58330	1.99000	1.990500	1.984000
45.00	0.2000	0.08330	5.00000	4.998500	5.000000
45.00	0.2000	0.33330	5.09000	5.095100	5.102000
45.00	0.2000	0.58330	5.27000	5.271900	5.285000
35.00	0.3000	0.08330	0.08000	0.077400	0.077400
35.00	0.3000	0.33330	0.70000	0.696900	0.698500
35.00	0.3000	0.58330	1.22000	1.219400	1.224000
40.00	0.3000	0.08330	1.31000	1.310000	1.309000
40.00	0.3000	0.33330	2.48000	2.481700	2.476000
40.00	0.3000	0.58330	3.17000	3.173300	3.159000
45.00	0.3000	0.08330	5.06000	5.059900	5.063000
45.00	0.3000	0.33330	5.71000	5.701200	5.698000
45.00	0.3000	0.58330	6.24000	6.236500	6.239000
35.00	0.4000	0.08330	0.25000	0.246600	0.245000
35.00	0.4000	0.33330	1.35000	1.345000	1.350000
35.00	0.4000	0.58330	2.16000	2.156800	2.159000
40.00	0.4000	0.08330	1.77000	1.767900	1.766000
40.00	0.4000	0.33330	3.38000	3.363200	3.383000
40.00	0.4000	0.58330	4.35000	4.355600	4.339000
45.00	0.4000	0.08330	5.29000	5.285500	5.287000
45.00	0.4000	0.33330	6.51000	6.509300	6.505000
45.00	0.4000	0.58330	7.39000	7.383100	7.382000

B. S=1; X=1; T=1

r	σ	Numerical	Geske-Johnson	Accelerated Binomial
1.133	0.5000	0.14800	0.14760	0.14720
1.083	0.4000	0.12600	0.12580	0.12510
1.046	0.3000	0.10100	0.10050	0.09990
1.020	0.2000	0.07100	0.07120	0.07090
1.005	0.1000	0.03800	0.03770	0.03770
1.094	0.3000	0.08600	0.08590	0.08560
1.041	0.2000	0.06400	0.06400	0.06390
1.010	0.1000	0.03600	0.03570	0.03560
1.083	0.2000	0.05300	0.05250	0.05240
1.020	0.1000	0.03300	0.03220	0.03220
1.127	0.2000	0.04400	0.04390	0.04460
1.030	0.1000	0.03000	0.02920	0.02920

Notes to Table 1:
For the binomial and accelerated binomial, N=150. Geske-Johnson value is based on four point extrapolation; Accelerated Binomial value is based on three point extrapolation. Geske-Johnson values are from Geske and Johnson [5, p.1519]. Values in column 4, panel A, are from Cox and Rubinstein [4, p.248]; values in column 3, panel 3, are from Parkinson [8, pp. 30-34].

Table 2: Values of American Put Option using Finite Difference, Geske-Johnson, Macmillan's Quadratic Approximation and Accelerated Binomial Methods (X=100)

r	σ	t	S	FD	GJ	MQ	AB
1.08	0.2	0.25	80	20.00	20.00	20.00	20.00
1.08	0.2	0.25	90	10.04	10.07	10.01	10.06
1.08	0.2	0.25	100	3.22	3.21	3.22	3.22
1.08	0.2	0.25	110	0.66	0.66	0.68	0.66
1.08	0.2	0.25	120	0.09	0.09	0.10	0.09
1.13	0.2	0.25	80	20.00	20.01	20.00	20.01
1.13	0.2	0.25	90	10.00	0.96	10.00	10.00
1.13	0.2	0.25	100	2.92	2.91	2.93	2.92
1.13	0.2	0.25	110	0.55	0.55	0.58	0.56
1.13	0.2	0.25	120	0.07	0.07	0.08	0.07
1.08	0.4	0.25	80	20.32	20.37	20.25	20.36
1.08	0.4	0.25	90	12.56	12.55	12.51	12.56
1.08	0.4	0.25	100	7.11	7.10	7.10	7.09
1.08	0.4	0.25	110	3.70	3.70	3.71	3.70
1.08	0.4	0.25	120	1.79	1.79	1.81	1.80
1.08	0.2	0.50	80	20.00	19.94	20.00	20.00
1.08	0.2	0.50	90	10.29	10.37	10.23	10.37
1.08	0.2	0.50	100	4.19	4.17	4.19	4.17
1.08	0.2	0.50	110	1.41	1.41	1.45	1.40
1.08	0.2	0.50	120	0.40	0.40	0.42	0.40

Key: FD finite difference method;
GJ Geske-Johnson method, basded on three point extrapolation;
MQ Macmillan's quadratic approximation;
AB accelerated binomial with N=150 and three point extrapolation.

Columns 1 to 7 from Barone-Adesi and Whaley [1, Table 4, p.315].