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THE ACCURACY OF INFINITELY DIVISIBLE APPROXIMATIONS TO SUMS OF INDEPENDENT VARIABLES WITH APPLICATION TO STABLE LAWS

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1. Introduction and summary. Let $\{F_n\}$ be a sequence of distribution functions defined on the real line, and suppose $\{F_n(x)\}$ converges to some limiting distribution function $F(x)$. It is of interest to investigate the error involved in using $F(x)$ as an approximation to $F_n(x)$, that is to investigate the rate of convergence of $\{F_n\}$ to F. This leads to the problem of finding bounds on $M_n = \sup_{-\infty<x<\infty}|F_n(x)-F(x)|$. In particular, this problem has been studied by several authors for cases where $F_n(x)$ represents the distribution function of a certain sum of independent random variables.

For cases involving the classical forms of the central limit theorem Berry [1] and Esseen [3] have obtained certain bounds on M_n which have been reinvestigated and improved by many authors (c.f. [4] Chapter XVI).

Let (X_{nk}), $k = 1, 2, \cdots, k_n$; $n = 1, 2, \cdots$ be a system of random variables such that for each n, $X_{n1}, \cdots, X_{nk}$ are independent (we say the system is independent within each row). In [6], under suitable conditions, bounds have been obtained on M_n for the case where $F_n(x)$ is the distribution function of $S_n = X_{n1}, +\cdots+X_{nk}$ and $F(x)$ is an infinitely divisible distribution. A basic assumption made in [6] was that both X_{nk} and $F(x)$ have finite variances.

The purpose of this study is to extend the results of [6] to include the case where neither $F(x)$ nor X_{nk} need have finite variance. Our main theorem (Theorem 1) gives a bound on M_n under a mild assumption on X_{nk} and a certain assumption on the derivative of the infinity divisible distribution $F(x)$. It is shown in Section 4, that if $F(x)$ satisfies an additional condition which is considerably weaker than that having finite variance, then the bound obtained tends to zero as n becomes infinite under necessary and sufficient conditions that $\{F_n(x)\}$ converge to $F(x)$.

In Section 5 our general results are applied to the case of convergence of distribution functions of normed sums of independent identically distributed random variables to an arbitrary stable law with exponent α, $0 < \alpha < 2$.

2. Notation and preliminaries. Let $F(x)$ be an infinitely divisible distribution with characteristic function $\varphi(t)$. According to the Lévy-Khintchine formula we have

$$\log\varphi(t) = i\gamma t + \int_{-\infty}^{\infty}\left(e^{itu}-1-\frac{i+u}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG(u) \tag{2.1}$$

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where γ is a constant and $G(u)$ is a bounded nondecreasing function. The logarithm of $\varphi(t)$ can also be represented by Lévy's formula:

$$\log \varphi(t) = i\gamma t - \tfrac{1}{2}(t^2\sigma^2) + \int_{-\infty}^{0-} \left(e^{itx} - 1 - \frac{itx}{i+x^2}\right) dM(x) \tag{2.2}$$
$$+ \int_{0+}^{\infty} \left(e^{itx} - 1 - \frac{itx}{i+x^2}\right) dN(x),$$

where the connection between σ^2, M, N and G is given by

$$M(x) = \int_{-\infty}^{x} \frac{1+u^2}{u^2} dG(u) \qquad \text{for} \quad x < 0,$$
$$N(x) = -\int_{x}^{\infty} \frac{1+u^2}{u^2} dG(u) \qquad \text{for} \quad x > 0, \tag{2.3}$$
$$\sigma^2 = G(0+) - G(0-).$$

If an infinitely divisible distribution function F has finite variance, Kolmogorov's formula yields

$$\log \varphi(t) = i\mu t + \textstyle\int_{-\infty}^{\infty} (i^{itv} - 1 - itv)v^{-2}\, dK(v), \tag{2.4}$$

where μ is the mean of F, and K is a bounded nondecreasing function with $K(-\infty) = 0$ and $K(+\infty)$ equal to the variance of F. The relationship between (2.4) and (2.1) is given by

$$\mu = \gamma + \textstyle\int_{-\infty}^{\infty} u\, dG(u) \qquad \text{and} \tag{2.5}$$
$$K(v) = \textstyle\int_{-\infty}^{v} (1+u^2)\, dG(u).$$

A system of random variables (X_{nk}) as considered in Section 1 is said to be infinitesimal if for any $\varepsilon > 0$ $\lim_{n\to\infty} \max_{1 \leqq k \leqq k_n} P(|X_{nk}| > \varepsilon) = 0$. Given such a system, for any $a > 0$ let X_{nk}^a be defined by

$$X_{nk}^a = X_{nk} \qquad \text{if} \quad -a < x_{nk} \leqq a,$$
$$= 0 \qquad \text{otherwise};$$

and let $F_{nk}^a(x)$, $\varphi_{nk}^a(t)$, $\mu_{nk}(a)$, $\sigma_{nk}^2(a)$ denote respectively, the distribution function, characteristic function, mean and variance of X_{nk}^a. Let $S_n = X_{n1} + \cdots + X_{nk_n}$ and $S_n{}^a = X_{n1}^a + \cdots + X_{nk_n}^a$. Let $F_n(x)$ and $\varphi_n(t)$ denote the distribution function and characteristic function of S_n and let $F_n{}^a(x)$, $\varphi_n{}^a(t)$, $\mu_n(a)$ and $\sigma_n{}^2(a)$ denote respectively the distribution function, characteristic function, mean, and variance of $S_n{}^a$.

Let $F(x)$ be an infinitely divisible distribution with corresponding $G(u)$ and γ given by (2.1). For each $a > 0$ such that $\pm a$ are continuity points of $G(u)$ we define

$$G^a(u) = 0, \qquad u \leqq -a \tag{2.6}$$
$$= G(u) - G(-a), \qquad -a < u \leqq a$$
$$= G(a) - G(-a), \qquad u > a$$

and

$$\gamma^a = \gamma - \int_{|u|>a} u^{-1}\, dG(u).$$

The nondecreasing function G^a and the constant γ^a determine a unique infinitely divisible distribution $F^a(x)$ through the formula (2.1). In [8] it is shown that if $F_n(x) \to F(x)$ then for any $a > 0$ (with $\pm a$ continuity points of G) $F_n{}^a(x) \to F^a(x)$ and that $\lim_{n\to\infty} \int_{-\infty}^{\infty} x^k\, dF_n{}^a(x) = \int_{-\infty}^{\infty} x^k\, dF^a(x)$ for any positive integer k (the result of [7] shows that limit is finite). We let $\varphi^a(t)$, $\mu(a)$, and $\sigma^2(a)$ denote the characteristic function mean, and variance of $F^a(x)$. From (2.4) we have

$$\log \varphi^a(t) = i\mu(a)t + \int_{-\infty}^{\infty} (e^{itv} - 1 - itv)v^{-2}\, dK^a(v) \tag{2.7}$$

where

$$\mu(a) = \gamma^a + \int_{-\infty}^{\infty} u\, dG^a(u) \qquad \text{and}$$

$$K^a(v) = \int_{-\infty}^{v} (1+u^2)\, dG^a(v).$$

Let

$$K_n{}^a(v) = \sum_{k=1}^{k_n} \int_{-\infty}^{v} x^2\, dF_{nk}^a(x + \mu_{nk}(a)).$$

For any $A > 0$ such that $-A$ and A are continuity points of $G(u)$, and hence continuity points of $K^a(v)$, let $0 < \delta \leqq 2A$ and define

$$m = m(A, \delta) = [2A/\delta] + 1.$$

Let $-A = x_0 < x_1 < \cdots < x_m = A$ be such that x_i is a continuity point of $K^a(v)$ and such that $\max_{1 \leqq i \leqq m}(x_i - x_{i-1}) < \delta$.
Define

$$\begin{aligned} E^a(n, t, m(A,\delta)) = {} & \tfrac{5}{4}\delta|t|^3(\sigma_n{}^2(a) + \sigma^2(a)) + \tfrac{1}{2}t^2 \textstyle\sum_{i=0}^{m} |K_n{}^a(x_i) - K^a(x_i)| \\ & + 2A^{-1}|t|(K_n{}^a(+\infty) - K_n{}^a(A) + K^a(+\infty) - K^a(A) \\ & + K_n{}^a(-A) + K^a(-A)). \end{aligned} \tag{2.8}$$

E^a is used in obtaining the desired bounds which will involve $g^a(n, m(A, \delta), r)$ defined by

$$\begin{aligned} g^a(n, m(A,\delta), r) = {} & [\tfrac{1}{3}\sigma_n{}^2(a) \max_{1 \leqq k \leqq k_n} \sigma_{nk}^2(a)]^{1/5} + [\tfrac{5}{6}\delta(\sigma_n{}^2(a) + \sigma^2(a)]^{1/4} \\ & + [\tfrac{1}{2} \textstyle\sum_{i=0}^{m} |K_n{}^a(x_i) - K^a(x_i)|]^{1/3} \\ & + \{4A^{-1}[K_n{}^a(+\infty) - K_n{}^a(A) + K^a(+\infty) - K^a(A) \\ & + K_n{}^a(-A) + K^a(-A)] + 2|\mu_n(a) - \mu(a)|\}^{1/2} \\ & + [8r^{-1} \textstyle\int_{|u|>a} |u|^r\, dG(u)]^{1/(1+r)} \end{aligned} \tag{2.9}$$

where r is a positive real number.

3. The general bound.

We require two lemmas.

LEMMA 1. *With $F_n(x)$ and $F_n{}^a(x)$ defined as in Section 2, it follows that*

$$|F_n(x)-F_n{}^a(x)| \leqq \sum_{k=1}^{k_n} \{F_{nk}(-a)+1-F_{nk}(a)\}.$$

PROOF. We have that

$$\begin{aligned} F_n(x) = P(S_n \leqq x) &= P(S_n \leqq x,\ X_{nk} \in (-a, a] \text{ for all } k) \\ &\quad + P(S_n \leqq x \text{ and } X_{nk} \notin (-a, a] \text{ for some } k) \\ &\leqq P(S_n{}^a \leqq x) + P(X_{nk} \notin (-a, a] \text{ for some } k) \\ &\leqq F_n{}^a(x) + \textstyle\sum_{k=1}^{k_n} P(X_{nk} \notin (-a, a]). \end{aligned}$$

Thus $F_n(x)-F_n{}^a(x) \leqq \sum_{k=1}^{k_n} [F_{nk}(-a)+1-F_{nk}(a)]$. A similar argument, starting with $F_n{}^a(x)$ proves the lemma.

LEMMA 2. *Let $F(x)$ be an infinitely divisible distribution with characteristic function $\varphi(t)$ and with corresponding $G(u)$ given by* (2.1). *Then for any real numbers a and r with $a > 1$, ($\pm a$ continuity points of G) and $0 < r \leqq 1$ we have $|\varphi^a(t)-\varphi(t)| \leqq 4|t|^r \int_{|u|>a} |u|^r \, dG(u)$ where $\varphi^a(t)$ is given by* (2.7).

PROOF. Since $\log \varphi^a(t)$ is given by formula (2.1) using $G^a(u)$ and γ^a given by (2.6), we have using Lemma 1 of [6] that

$$\begin{aligned} |\varphi^a(t)-\varphi(t)| &\leqq |\log \varphi^a(t) - \log \varphi(t)| \\ &= \left| -it \int_{|u|>a} u^{-1} \, dG(u) - \int_{|u|>a} \left(e^{itu} - 1 - \frac{itu}{1+u^2} \right) \frac{1+u^2}{u^2} \, dG(u) \right| \\ &= \left| \textstyle\int_{|u|>a} (e^{itu}-1) u^{-2}(1+u^2) \, dG(u) \right|. \end{aligned}$$

Now for $|u| > a > 1$ we have

$$\left| (e^{itu}-1) \frac{1+u^2}{u^2} \right| \leqq 2|e^{itu}-1| = 4|\sin \tfrac{1}{2}tu|.$$

Furthermore, for $|\frac{1}{2}tu| \geqq 1$ we have $4|\sin \frac{1}{2}tu| \leqq 4 \leqq 4|\frac{1}{2}tu|^r \leqq 4|tu|^r$, and for $|\frac{1}{2}tu| \leqq 1$ and $0 < r \leqq 1$ we have $4|\sin \frac{1}{2}tu| \leqq 4|\frac{1}{2}tu| \leqq 4|\frac{1}{2}tu|^r \leqq 4|tu|^r$.

It follows that $|\varphi^a(t)-\varphi(t)| \leqq 4|t|^r \int_{|u|>a} |u|^r dG(u)$, which proves the lemma.

We are now in a position to state and prove the theorem giving a general bound mentioned in Section 1. We use the notation developed in Section 2.

THEOREM 1. *Let $F(x)$ be an infinitely divisible distribution function with corresponding $G(u)$ given by the Levy-Khintchine formula* (2.1). *Let (X_{nk}) $k = 1, \cdots k_n$; $n = 1, 2, \cdots$ be a system of random variables independent within each row. Let $F_n(x)$ be the distribution function of the sum $S_n = X_{n1}, + \cdots + X_{nk_1}$ and suppose that*

$$dF(x)/dx = F'(x) \quad \textit{exists and} \quad |F'(x)| < B \quad \textit{for all} \quad x.$$

Assume that $\sigma_{nk}^2(a) \leqq 1$ *for all* n *and* k *(as will be seen in the proof of Theorem* 2, *this assumption is quite weak). Then it follows that for* $0 < r \leqq 1$ *and* $a \geqq 1$,

$$M_n = \sup_{-\infty < x < \infty} |F_n(x) - F(x)| \leqq k(B) g^a(n, m(A, \delta), r) + \textstyle\sum_{k=1}^{k_n} \{F_{nk}(-a) + 1 - F_{nk}(a)\}$$

where $k(B)$ *is a constant depending only on* B *and where* $g^a(n, m(A, \delta), r)$ *is given by* (2.9).

PROOF. We have

$$|F_n(x) - F(x)| \leqq |F_n(x) - F_n^a(x)| + |F_n^a(x) - F(x)|. \tag{3.1}$$

Letting $\varphi(t)$ be the characteristic function of $F(x)$, from Lemma 2 it follows that

$$\begin{aligned} |\varphi_n^a(t) - \varphi(t)| &\leqq |\varphi_n^a(t) - \varphi^a(t)| + |\varphi^a(t) - \varphi(t)| \\ &\leqq |\varphi_n^a(t) - \varphi^a(t)| + 4|t|^r \textstyle\int_{|u|>a} |u|^r \, dG(u). \end{aligned}$$

If we let $T_n = [g^a(n, m(A, \delta), r)]^{-1}$ and restrict $|t| \leqq T_n$, by an argument analogous to the proof of Theorem 3 of [6], it follows that

$$\begin{aligned} |\varphi_n^a(t) - \varphi^a(t)| \leqq \tfrac{5}{8} t^4 \max_{1 \leqq k \leqq k_n} \sigma_{nk}^2(a) \sigma_n^{\,2}(a) \\ + |\mu_n(a) - \mu(a)|\,|t| + E^a(n, t, m(A, \delta)) \end{aligned}$$

where E^a is given by (2.8). Thus

$$\begin{aligned} |\varphi_n^a(t) - \varphi(t)| &\leqq \tfrac{5}{8} t^4 \max_{1 \leqq k \leqq k_n} \sigma_{nk}^2(a) \sigma^2(a) \\ &\quad + |\mu_n(a) - \mu(a)|\,|t| + E^a(n, t, m(A, \delta)) + 4|t|^r \textstyle\int_{|u|>a} |u|^r \, dG(u) \\ &\equiv h^a(t, n, m(A, \delta), r). \end{aligned}$$

From this it follows that

$$\textstyle\int_{-T_n}^{T_n} |t^{-1}(\varphi_n^a(t) - \varphi(t))| \, dt \leqq 2 \int_0^{T_n} t^{-1} [h^a(t, n, m(A, \delta), r)] \, dt \leqq g^a(n, m(A, \delta), r).$$

Now applying a Theorem of Esseen ([3] Theorem 2a, page 32), we have

$$M_n^{\,a} \equiv \sup_{-\infty < x < \infty} |F_n^a(x) - F(x)| \leqq (2\pi)^{-1} p g^a(n, m(A, \delta), r) + c(p) \cdot B / T_n$$

where $p > 1$ and $c(p)$ is a constant depending only on p ($p > 1$ is arbitrary). Thus

$$M_n^{\,a} \leqq (p/(2\pi) + c(p)B) g^a(n, m(A, \delta), r).$$

Applying Lemma 1 and (3.1) we have the proof of the Theorem.

4. Behavior of the estimate. In this section we examine, under suitable conditions, the behavior of the bound

$$D(n, A, \delta, a, r) = k(B) g^a(n, m(A, \delta), r) + \textstyle\sum_{k=1}^{k_n} \{F_{nk}(-a) + 1 - F_{nk}(a)\} \tag{4.1}$$

given in Theorem 1. Several lemmas will be needed.

LEMMA 3. *Let* $\{Q_n(a)\}$ $n = 0, 1, 2, \cdots$ *be defined for* $a > 0$ *and be such that*

(i) *for each* n, $Q_n(a) \geqq Q_n(b) \geqq 0$ *for* $a < b$.

(ii) $\lim_{a\to+\infty} Q_0(a) = 0$.

(iii) $\lim_{n\to\infty} Q_n(a) = Q_0(a)$ *at every continuity point of* $Q_0(a)$.

Then for any *nondecreasing sequence* $\{a_n\}$ *such that each* a_n *is a continuity point of* $Q_0(a)$, *and such that* $\lim_{n\to\infty} a_n = +\infty$ *we have* $\lim_{n\to\infty} Q_n(a_n) = 0$.

PROOF. Given $\varepsilon > 0$, let k be such that $Q_0(a_k) < \varepsilon/2$. Let n_k be such that $n > n_k$ implies $Q_n(a_k) < Q_0(a_k)+\varepsilon/2 < \varepsilon$. Now for $n > n_k$ we have $n > k$ so that by (i), $Q_n(a_n) \leqq Q_n(a_k) < \varepsilon$, which proves the lemma.

LEMMA 4. *Let* (X_{nk}), $F_n(x)$, $F(x)$ *and* $G(u)$ *be as in Theorem* 1. *If*

(i) *the system* (X_{nk}) *is infinitesimal,*

(ii) $\lim_{n\to\infty} F_n(x) = F(x)$ *at all continuity points of* $F(x)$,

then for any nondecreasing sequence $\{a_n\}$ *such that* $-a_n$ *and* a_n *are continuity points of* G *and* $\lim_{n\to\infty} a_n = \infty$ *we have*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \{F_{nk}(-a_n)+1-F_{nk}(a_n)\} = 0.$$

PROOF. Let $M(x)$ and $N(x)$ be given by (2.2) or (2.3) and let

$$Q_n(a) = \sum_{k=1}^{k_n} \{F_{nk}(-a)+1-F_{nk}(a)\} \quad \text{and} \quad Q_0(a) = M(-a)-N(a).$$

By Theorem 1, page 116 of [5] we have $\lim_{n\to\infty} Q_n(a) = Q_0(a)$ at continuity points of $Q_0(a)$ so that by Lemma 3, $\lim_{n\to\infty} Q_n(a_n) = 0$.

LEMMA 5. *Let* $g(n, a, \delta)$ *be nonnegative and be such that to each* a *there corresponds a sequence* $\{\delta_n(a)\}$ *of positive real numbers such that* $\lim_{n\to\infty} g(n, a, \delta_n(a)) = 0$. *Then there exists a nondecreasing sequence* $\{a_n\}$ *such that* $\lim_{n\to\infty} a_n = \infty$ *and a sequence* $\{\delta_n\}$ *such that*

$$\lim_{n\to\infty} g(n, a_n, \delta_n) = 0.$$

PROOF. Let $\{\varepsilon_k\}$ be a sequence such that $\varepsilon_k \downarrow 0$ and let $\{\bar{a}_n\}$ be such that $\bar{a}_n \uparrow \infty$. By hypothesis we can choose $\{n_k\}$ such that $n_k > n_{k-1}$ and such that $g(n, \bar{a}_k, \delta_n(\bar{a}_k)) < \varepsilon_k$ for $n > n_k$.
If we define

$$\begin{aligned} \delta_n &= \delta_1(\bar{a}_1) \quad \text{for} \quad n \leqq n_1, \\ &= \delta_n(\bar{a}_k) \quad \text{for} \quad n_k < n \leqq n_{k+1}, \qquad k = 1, 2, \cdots \end{aligned}$$

and

$$\begin{aligned} a_n &= \bar{a}_1 \quad \text{for} \quad n \leqq n_1 \\ &= \bar{a}_k \quad \text{for} \quad n_k < n \leqq n_{k+1}, \qquad k = 1, 2, \cdots \end{aligned}$$

it is not difficult to see that $\{a_n\}$ and $\{\delta_n\}$ satisfy the conclusion of the lemma.

We now show that under suitable conditions we can choose sequences $\{A_n\}$, $\{\delta_n\}$ and $\{a_n\}$ such that the bound, $D(n, A_n, \delta_n, a_n, r)$ given by (4.1) approaches zero as n becomes infinite.

Using the same notation as in Theorem 1, let $0 < \delta < 1$ be such that $\pm\delta^{-\frac{1}{2}}$ are continuity points of $G(u)$. Using (2.9) define $g(n, a, \delta)$ by

$$g^a(n, m(\delta^{-\frac{1}{2}}, \delta), r) = g(n, a, \delta) + \{8r^{-1} \textstyle\int_{|u|>a} |u|^r \, dG(u)\}^{1/(1+r)}. \tag{4.2}$$

This leads to the main result of this section.

THEOREM 2. *Let (X_{nk}), $F_n(x)$ and $F(x)$ satisfy the conditions of Theorem 1. Assume further that the random variables (X_{nk}) are infinitesimal, $F_n(x)$ converges to $F(x)$ at continuity points of F, and that for some r, (without loss of generality assume $0 < r \leqq 1$)*

$$\textstyle\int_{-\infty}^{\infty} |u|^r \, dG(u) < \infty.$$

Then there exist sequences $\{a_n\}$ and $\{\delta_n\}$ such that, for large n

$$M_n = \sup_{-\infty<x<\infty} |F_n(x) - F(x)| \leqq D(n, \delta_n^{-\frac{1}{2}}, \delta_n, a_n, r), \quad \textit{and} \tag{4.3}$$

$$\lim_{n\to\infty} D(n, \delta_n^{-\frac{1}{2}}, \delta_n, a_n, r) = 0. \tag{4.4}$$

(The function D is given by (4.1) and (2.9)).

PROOF. Since (X_{nk}) are infinitesimal, (X^a_{nk}) as well as $(X^a_{nk} - \mu_{nk}(a))$ are also infinitesimal. By Theorems 3 and 6 of [8], if $\pm a$ are continuity points of G, we have $\lim_{n\to\infty} F_n{}^a(x) = F^a(x)$ and $\lim_{n\to\infty} \sigma_n{}^2(a) = \sigma^2(a)$. It follows from the proof of Theorem 5 of [6] that there exists a sequence $\{\delta_n(a)\}$ such that

$$\lim_{n\to\infty} g(n, a, \delta_n(a)) = 0. \tag{4.5}$$

By Lemma 5 we can find sequences $\{a_n\}$ and $\{\delta_n\}$ such that

$$\lim_{n\to\infty} g(n, a_n, \delta_n) = 0, \tag{4.6}$$

$a_n \leqq a_{n+1}$, and $\lim_{n\to\infty} a_n = \infty$. (Note that from (4.2) and (2.9) we see that (4.5) and (4.6) imply $\lim_{n\to\infty} \max_k \sigma^2_{nk}(a) = \lim_{n\to\infty} \max_k \sigma^2_{nk}(a_n) = 0$. This justifies the parenthetical remark in the statement of Theorem 1.) Clearly from the proof of Lemma 5, a_n can be chosen so that $\pm a_n$ are continuity points of $G(u)$ so that the conclusion of Lemma 4 holds. Now since $\int_{-\infty}^{\infty} |u|^r \, dG(u) < \infty$ it follows that $\lim_{n\to\infty} \int_{|u|>a_n} |u|^r \, dG(u) = 0$. Since

$$D(n, \delta_n^{-\frac{1}{2}}, \delta_n, a_n, r) = k(B)\{g(n, a_n, \delta_n) + [8r^{-1} \textstyle\int_{|u|>a_n} |u|^r \, dG(u)]^{1/1+r}\}$$
$$+ \textstyle\sum_{k=1}^{k_n} \{F_{nk}(-a) + 1 - F_{nk}(a)\}$$

we see that (4.4) holds. Finally from Theorem 1, as soon as n is so large that $a_n \geqq 1$ and $\max_k \sigma^2_{nk}(a_n) \leqq 1$, it follows that (4.3) holds.

5. Application to stable laws. The basic properties of stable distributions can be found in [4] or [5]. We recall that a distribution function $F(x)$ is stable if and only if there exists a sequence of independent identically distributed random variables $\{X_n\}$ and constants A_n and $B_n > 0$ such that

$$\lim_{n\to\infty} P\left\{\frac{X_1+\cdots+X_n}{B_n} - A_n \leqq x\right\} = F(x). \tag{5.1}$$

Since stable distributions are infinitely divisible, by letting $X_{nk} = X_k/B_n$ it can be seen that our previous results can be applied to limit theorems of the type (5.1).

As is well known to every stable distribution, there corresponds an exponent $\alpha(0 < \alpha \leqq 2)$. The case $\alpha = 2$ corresponds to the normal distribution and will not be discussed here. From the theorem on page 164 of [5] we know that for stable distributions with exponent α, $(0 < \alpha < 2)$, the functions M and N and the constant σ^2 in (2.3) are given

$$\begin{aligned} M(x) &= c_1 |x|^{-\alpha}, \qquad x < 0 \\ N(x) &= -c_2 x^{-\alpha}, \qquad x > 0 \\ \sigma^2 &= 0 = G(0+)-G(0-) \end{aligned} \tag{5.2}$$

where c_1 and c_2 are nonnegative and $c_1+c_2 > 0$. From Section 36 of [5] it follows that all proper stable distributions (and hence all stable distributions with $0 < \alpha < 2$) have bounded derivatives of all orders of every point. Thus for (proper) stable distributions the assumption in Theorem 1 on the derivative of $F(x)$ is always satisfied.

The next lemma removes one of the hypotheses of Theorem 2.

LEMMA 6. *If $F(x)$ is a stable distribution function with corresponding function $G(u)$ given by the formula* (2.1), *then there exists a real number $r > 0$ such that*

$$\int_{-\infty}^{+\infty} |u|^r \, dG(u) < +\infty.$$

PROOF. From (5.2) and (2.3) we note that

$$\begin{aligned} dG(u) &= c_1 \frac{|u|^{1-\alpha}}{1+u^2} du, \qquad \text{for} \quad u < 0 \\ &= c_2 \frac{u^{1-\alpha}}{1+u^2} du, \qquad \text{for} \quad u > 0. \end{aligned} \tag{5.3}$$

Thus
$$\int_{-\infty}^{+\infty} |u|^r \, dG(u) = (c_1+c_2) \int_{0+}^{+\infty} \frac{x^{1-(\alpha-r)}}{1+x^2} dx$$

which is finite if $r < \alpha$.

The next two lemmas will be used to simplify the general estimate given in Theorem 1 to the stable case.

LEMMA 7. *Let $F(x)$ be a stable distribution function with representation given by* (2.2) *and* (5.2). *Then using the notation of Section* 2,

$$\mu(a) = \gamma+(c_1-c_2)\left\{\int_a^{+\infty}\frac{du}{(1+u^2)u^\alpha}-\int_0^a\frac{u^{2-\alpha}du}{1+u^2}\right\}, \tag{5.4}$$

$$\sigma^2(a) = (c_1+c_2)\frac{a^{2-\alpha}}{2-\alpha}, \tag{5.5}$$

and

$$\begin{aligned} K^a(v) &= 0, & v &< -a,\\ &= c_1\left(\frac{a^{2-\alpha}}{2-\alpha}-\frac{|v|^{2-\alpha}}{2-\alpha}\right), & -a &\leqq v < 0,\\ &= c_1\left(\frac{a^{2-\alpha}}{2-\alpha}\right)+c_2\left(\frac{v^{2-\alpha}}{2-\alpha}\right), & 0 &\leqq v < a\\ &= (c_1+c_2)\left(\frac{a^{2-\alpha}}{2-\alpha}\right), & a &\leqq v. \end{aligned} \tag{5.6}$$

PROOF. From (5.3) and (2.6) we have

$$\gamma^a = \gamma+(c_1-c_2)\int_a^{+\infty}\frac{du}{(1+u^2)u^\alpha}.$$

Hence, from (2.7)

$$\begin{aligned}\mu(a) &= \gamma^a+\textstyle\int_{-a}^{a} u\,dG(u)\\ &= \gamma^a+\int_{-a}^{0} uc_1\frac{|u|^{1-\alpha}}{1+u^2}du+\int_0^a uc_2\frac{u^{1-\alpha}}{1+u^2}du\\ &= \gamma^a-(c_1-c_2)\int_0^a\frac{u^{2-\alpha}}{1+u^2}du\\ &= \gamma+(c_1-c_2)\left\{\int_a^{+\infty}\frac{du}{(1+u^2)u^\alpha}-\int_0^a\frac{u^{2-\alpha}}{1+u^2}du\right\}.\end{aligned}$$

This proves (5.4). For $-a < v < 0$, we have

$$\begin{aligned}K^a(v) &= \textstyle\int_{-\infty}^{v}(1+u^2)\,dG^a(u) = \int_{-a}^{v}(1+u^2)\,dG(u)\\ &= \int_{-a}^{v} c_1|u|^{1-\alpha}du = c_1\left(\frac{a^{2-\alpha}}{2-\alpha}-\frac{v^{2-\alpha}}{2-\alpha}\right).\end{aligned}$$

For $0 < v < a$, we have

$$\begin{aligned}K^a(v) &= \textstyle\int_{-\infty}^{v}(1+u^2)\,dG^a(u)\\ &= \textstyle\int_{-a}^{0}(1+u^2)\,dG(u)+[G(0+)-G(0-)]+\int_0^v(1+u^2)\,dG(u)\\ &= c_1\left(\frac{a^{2-\alpha}}{2-\alpha}\right)+c_2\left(\frac{v^{2-\alpha}}{2-\alpha}\right),\end{aligned}$$

using (5.3) and the fact that $G(0+)-G(0-) = \sigma^2 = 0$ from (5.2). From this (5.6) follows. Formula (5.5) follows from (5.6) and the fact that $\sigma^2(a) = K^a(+\infty)$.

LEMMA 8. *Let $F(x)$ be a stable distribution function as given in Lemma 7. If $0 < r < \alpha$, then*

$$\int_{|u|>a} |u|^r \, dG(u) \leqq \frac{c_1+c_2}{(\alpha-r)a^{\alpha-r}}.$$

PROOF. Using (5.3) we have

$$\begin{aligned}\int_{|u|>a} |u|^r \, dG(u) &= (c_1+c_2)\int_a^{+\infty} \frac{u^{1+r-\alpha}}{1+u^2}\, du \\ &\leqq (c_1+c_2)\int_a^{+\infty} \frac{u^{1+r-\alpha}}{u^2}\, du \\ &= \frac{c_1+c_2}{(\alpha-r)a^{\alpha-r}},\end{aligned}$$

which proves the lemma.

As was done at the beginning of this section, let $\{X_n\}$, $n = 1, 2, \cdots$ be a sequence of independent random variables with a common distribution function $\bar{F}(x)$. For each n, let

$$S_n = \frac{X_1+\cdots+X_n}{B_n},$$

where B_n are suitably chosen positive constants. Let $F_n(x)$ denote the distribution function of S_n. To apply our general result, we let

$$X_{nk} = X_k/B_n.$$

The results expressed in (5.7)–(5.13) follow easily.

$$\begin{aligned} X^a_{nk} &= X_k/B_n \qquad -aB_n < X_k \leqq aB_n, \\ &= 0 \qquad \text{otherwise.}\end{aligned} \tag{5.7}$$

$$F_{nk}(x) = \bar{F}(xB_n). \tag{5.8}$$

$$\begin{aligned} F^a_{nk}(x) &= 0 && \text{for } x \leqq -a, \\ &= \bar{F}(xB_n)-\bar{F}(-aB_n) && \text{for } -a < x < 0, \\ &= \bar{F}(xB_n)+1-\bar{F}(aB_n) && \text{for } 0 \leqq x \leqq a, \\ &= 1 && \text{for } a \leqq x.\end{aligned} \tag{5.9}$$

$$\mu_{nk}(a) = B_n^{-1}\textstyle\int_{-aB_n}^{aB_n} x\, d\bar{F}(x) \quad \text{and} \quad \mu_n(a) = Bn_n^{-1}\int_{-aB_n}^{aB_n} x\, d\bar{F}(x). \tag{5.10}$$

$$\sigma^2_{nk}(a) = B_n^{-2}\{\textstyle\int_{-aB_n}^{aB_n} x^2\, d\bar{F}(x) - (\int_{-aB_n}^{aB_n} x\, d\bar{F}(x))^2\} \quad \text{and} \tag{5.11}$$

$$\sigma_n{}^n(a) = Bn_n^{-2}\{\textstyle\int_{-aB_n}^{aB_n} x^2\, d\bar{F}(x) - (\int_{-aB_n}^{aB_n} x\, d\bar{F}(x))^2\}.$$

$$
\begin{aligned}
K_n{}^a(v) &= n\textstyle\int_{-\infty}^{v+\mu_{nk}(a)}(u-\mu_{nk}(a))^2\,dF_{nk}^a(u) \\
&= 0, \qquad v+\mu_{nk}(a) \leqq -a, \\
&= n\textstyle\int_{-a}^{v+\mu_{nk}(a)}(u-\mu_{nk}(a))^2\,d\bar{F}(uB_n), \qquad -a < v+\mu_{nk}(a) < 0, \\
&= n\textstyle\int_{-a}^{v+\mu_{nk}(a)}(u-\mu_{nk}(a))^2\,d\bar{F}(uB_n) \\
&\quad +[\mu_{nk}(a)]^2[1-\bar{F}(aB_n)+\bar{F}(-aB_n)], \qquad 0 \leqq v+\mu_{nk}(a) < a, \\
&= n\textstyle\int_{-a}^{a}(u-\mu_{nk}(a))^2\,d\bar{F}(uB_n) \\
&\quad +[\mu_{nk}(a)]^2[1-\bar{F}(aB_n)+\bar{F}(-aB_n)], \qquad a < v+\mu_{nk}(a).
\end{aligned} \tag{5.12}
$$

Finally we have

$$\textstyle\sum_{k=1}^{k_n}\{F_{nk}(-a)+1-F_{nk}(a)\} = n\{\bar{F}(-aB_n)+1-\bar{F}(aB_n)\}. \tag{5.13}$$

The following theorem is an immediate consequence of Theorem 1, Lemmas 7 and 8, and (5.7)–(5.13).

THEOREM 3. *Let* $\{X_n\}$, $n = 1, 2, \cdots$ *be a sequence of independent random variables with a common distribution function* $\bar{F}(x)$. *For each n, let*

$$S_n = \frac{X_1+\cdots+X_n}{B_n},$$

where B_n *are positive constants. Let* $F_n(x)$ *denote the distribution function of* S_n. *Let* $F(x)$ *be a stable distribution function with exponent* α *given by* (2.2) *and* (5.2). *Assume that*

$$B_n{}^{-2}\{\textstyle\int_{-aB_n}^{aB_n}x^2\,dF(x)-(\int_{-aB_n}^{aB_n}x\,dF(x))^2\} \leqq 1.$$

(We note that this is $\sigma_{nk}^2(a) \leqq 1$ so that by the proof of Theorem 2 this is a weak assumption.)

Then for, $0 < r \leqq 1, a > 1$, *we have*

$$
\begin{aligned}
M_n &= \sup_{-\infty<x<+\infty}|F_n(x)-F(x)| \\
&\qquad \leqq kg^a(n, m(A,\delta), r)+n\{\bar{F}(-aB_n)+1-\bar{F}(aB_n)\},
\end{aligned}
$$

where k is a constant depending only on the bound of the derivative of F, and

$$
\begin{aligned}
&g^a(n, m(A,\delta), r) \\
&\quad = \left\{\frac{n}{3B_n{}^4}\left[\int_{-aB_n}^{aB_n}x^2\,dF(x)-\left(\int_{-aB_n}^{aB_n}x\,dF(x)\right)^2\right]^2\right\}^{1/5} \\
&\quad +\left\{\frac{5}{6}\delta\left[\frac{n}{B_n{}^2}\left\{\int_{-aB_n}^{aB_n}x^2\,dF(x)-\left(\int_{-aB_n}^{aB_n}x\,dF(x)\right)^2\right\}+\frac{(c_1+c_2)a^{2-\alpha}}{2-\alpha}\right]\right\}^{1/4} \\
&\quad +\{\tfrac{1}{2}\textstyle\sum_{i=0}^{m}|K_n{}^a(x_i)-K^a(x_i)|\}^{1/3} \\
&\quad +\left\{\frac{4}{A}\left[\frac{n}{B_n{}^2}\left(\int_{-aB_n}^{aB_n}x^2\,dF(x)-\left(\int_{-aB_n}^{aB_n}x\,dF(x)\right)^2\right)-K_n{}^a(A)+\frac{(c_1+c_2)a^{2-\alpha}}{2-\alpha}\right.\right. \\
&\qquad \left.-K^a(A)+K_n{}^a(-A)+K^a(-A)\right]+2\left|\frac{n}{B_n}\int_{-aB_n}^{aB_n}x\,dF(x)-\gamma-(c_1-c_2)\right. \\
&\qquad \left.\left.\times\left(\int_a^{+\infty}\frac{du}{(1+u^2)u^\alpha}-\int_0^a\frac{u^{2-\alpha}\,du}{1+u^2}\right)\right|\right\}^{1/2}+\left\{\frac{8(c_1+c_2)}{r(\alpha-r)a^{\alpha-r}}\right\}^{1+r^{-1}}.
\end{aligned}
$$

The functions $K_n^a(v)$ and $K^a(v)$ are given by (5.12) and (5.6), and A, δ, $m(A, \delta)$ are given in Section 2.

6. An example. As an example we consider a sequence of independent random variables $\{X_n\}$, $n = 1, 2, \cdots$ with a common distribution function $\bar{F}(x)$. Let $\bar{F}(x)$ have density function

$$\bar{f}(x) = \pi^{-1}(1-\cos x),$$

and consider the normed sums

$$S_n = n^{-1}(X_1 + \cdots + X_n).$$

Again, let $F_n(x)$ denote the distribution function of S_n. The characteristic function of $\bar{F}(x)$ (c.f. [2] page 94 ff) is

$$\begin{aligned} \bar{\varphi}(t) &= 1-|t| \quad \text{or} \quad |t| \leqq 1,. \\ &= 0 \qquad \text{for} \quad |t| > 1 \end{aligned}$$

and hence the characteristic function of $F_n(x)$ is

$$\begin{aligned} \varphi_n(t) &= (1-n^{-1}|t|)^n \quad \text{for} \quad |t| \leqq n, \\ &= 0 \qquad\qquad \text{for} \quad |t| > n. \end{aligned}$$

Clearly $\varphi_n(t)$ converges to $\varphi(t) = e^{-|t|}$ which is the characteristic function of the well-known Cauchy distribution function

$$F(x) = \pi^{-1}(\tfrac{1}{2}\pi + \arctan x).$$

We shall establish that the rate of convergence of $\sup_{-\infty < x < +\infty} |F_n(x) - F(x)|$ to zero is bounded by $C/n^{1/15}$ where C is a constant.

As is well known, $F(x)$ is stable with (c.f. [9] Section 4) the constants in (5.2) and (2.2) given by

$$\begin{aligned} \alpha &= 1, \\ c_1 &= c_2 = \pi^{-1} \qquad \text{and} \\ \gamma &= 0. \end{aligned} \tag{6.1}$$

To apply the result of Theorems 1 or 3, we put $X_{nk} = n^{-1}X_k$. We have

$$\begin{aligned} \mu_{nk}(a) &= 0 = \mu_n(a), \\ \sigma_{nk}^2(a) &= \frac{1}{n^2}\int_{-na}^{na} x^2 \frac{1}{\pi}\frac{1-\cos x}{x^2}\,dx \\ &= \frac{2}{\pi n^2}(na - \sin na), \qquad \text{and} \\ \sigma_n^{\,2}(a) &= \frac{2}{\pi n}(na - \sin na). \end{aligned}$$

For $-a < v < a$ we have

$$K_n{}^a(v) = \frac{n}{n^2}\int_{-na}^{nv} u^2 \frac{1}{\pi}\frac{1-\cos u}{u^2}\,du$$

$$= \frac{1}{n\pi}[n(v+a)-(\sin nv+\sin na)].$$

Hence

$$\begin{aligned} K_n{}^a(v) &= 0 \quad \text{for} \quad v < -a, \\ &= \frac{1}{n\pi}[n(v+a)-(\sin nv-\sin na)] \quad \text{for} \quad -a \leqq v < a, \\ &= \frac{2}{n\pi}[na-\sin na] \quad \text{for} \quad a \leqq v. \end{aligned}$$

From (6.1) and (5.3) we have $dG(u) = \pi^{-1}(1+u^2)^{-1}du$. Applying Lemma 7 we have

$$\mu(a) = 0, \qquad \sigma^2(a) = 2a/\pi, \qquad \text{and}$$

$$\begin{aligned} K^a(v) &= 0 \quad \text{for} \quad v < -a, \\ &= (a+v)/\pi \quad \text{for} \quad -a \leqq v < a, \\ &= 2a/\pi \quad \text{for} \quad a \leqq v. \end{aligned}$$

For simplicity we let $a = A = \delta^{-\frac{1}{2}}$. Then we have

$$\left\{\frac{1}{3}\sigma_n{}^2(a)\max\sigma_{nk}^2(a)\right\}^{1/5} \leqq \left\{\frac{16}{3\pi^2 n\delta}\right\}^{1/5}, \tag{6.2}$$

$$\left\{\frac{5}{6}\delta(\sigma_n{}^2(a)+\sigma^2(a))\right\}^{1/4} \leqq \left\{\frac{5}{\pi}\sqrt{\delta}\right\}^{1/4}, \quad \text{and} \tag{6.3}$$

$$\begin{aligned} \left\{\frac{1}{2}\sum_{i=0}^{m}\left|K_n{}^a(x_i)-K^a(x_i)\right|\right\}^{1/3} &= \left\{\frac{1}{2}\sum_{i=0}^{m}\frac{|\sin nx_i-\sin na|}{n\pi}\right\}^{1/3} \\ &\leqq \left\{\frac{1}{2}(m+1)\frac{2}{n\pi}\right\}^{1/3} \leqq \left\{\frac{4}{n\pi\delta\sqrt{\delta}}\right\}^{1/3}, \end{aligned} \tag{6.4}$$

from the definition of m. Furthermore,

$$\{4A^{-1}(K_n{}^a(+\infty)-K_n{}^a(A)+K^a(+\infty)-K^a(A)+K_n{}^a(-A)+K^a(-A)) + 2|\mu_n(a)-\mu(a)|\}^{1/2} = 0. \tag{6.5}$$

Applying Lemma 8, and (6.1)–(6.5) to $g^a(n, m(A, \delta), r)$ as given in (2.9) we have

$$g^a(n, m(A,\delta), r) \leqq \left\{\frac{16}{3\pi^2 n\delta}\right\}^{1/5} + \left\{\frac{5}{\pi}\sqrt{\delta}\right\}^{1/4} + \left\{\frac{4}{\pi n\,\delta\sqrt{\delta}}\right\}^{1/3} + \left\{\frac{16}{\pi r(1-r)}\right\}^{(1+r)^{-1}} \{\sqrt{\delta}\}^{(1-r)/(1+r)}.$$

Applying (5.13) we have

$$\sum_{k=1}^{k_n} \{F_{nk}(-a)+1-F_{nk}(a)\} = n\{\bar{F}(-an)+1-\bar{F}(an)\}$$

$$= n\frac{2}{\pi}\int_{an}^{+\infty} \frac{1-\cos x}{x^2}\,dx \leqq \frac{4}{\pi a} = \frac{4}{\pi}\sqrt{\delta}.$$

From Theorem 1 or 3 we have

$$\sup_{-\infty<x<+\infty} |F_n(x)-F(x)| \leqq k\left\{\frac{16}{3\pi^2 n\delta}\right\}^{1/5} + \left\{\frac{5}{\pi}\sqrt{\delta}\right\}^{1/4}$$

$$+\left\{\frac{4}{\pi n\delta\sqrt{\delta}}\right\}^{1/3} + \left\{\frac{16}{\pi r(1-r)}\right\}^{(r+1)^{-1}} \{\sqrt{\delta}\}^{(1-r)/(1+r)} + \frac{4}{\pi}\sqrt{\delta}.$$

Taking for example $\delta = 1/n^{8/15}$, $r = \frac{3}{5}$, we find that

$$\sup_{-\infty<x<+\infty} |F_n(x)-F(x)| \leqq C/n^{1/15},$$

where C is a constant.

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