The Active Disturbance Rejection Control to Stabilization for Multi-Dimensional Wave Equation With Boundary Control Matched Disturbance

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Abstract—In this paper, we consider boundary stabilization for a multi-dimensional wave equation with boundary control matched disturbance that depends on both time and spatial variables. The active disturbance rejection control (ADRC) approach is adopted in investigation. An extended state observer is designed to estimate the disturbance based on an infinite number of ordinary differential equations obtained from the original multi-dimensional system by infinitely many test functions. The disturbance is canceled in the feedback loop together with a collocated stabilizing controller. All subsystems in the closed-loop are shown to be asymptotically stable. In particular, the time varying high gain is first time applied to a system described by the partial differential equation for complete disturbance rejection purpose and the peaking value reduction caused by the constant high gain in literature. The overall picture of the ADRC in dealing with the disturbance for multi-dimensional partial differential equation is presented through this system. The numerical experiments are carried out to illustrate the convergence and effect of peaking value reduction.

Index Terms—Boundary control, disturbance rejection, stabilization, wave equation.

I. INTRODUCTION

ESSENTIALLY speaking, if there is no uncertainty in systems, the feedback control is not necessary in many situations. In the past three decades, many different approaches have been developed to deal with disturbance such as the internal model principle for output regulation, the robust control for systems with uncertainties from both internal and external disturbance, the adaptive control for systems with unknown parameters, to name just a few. Most of these approaches however, focus on the worst case scenario which makes the controller designed conservative. The active disturbance rejection control

(ADRC), as an unconventional design strategy similar to the external model principle ([22]), was first proposed by Han in [17]. The uncertainties dealt with by the ADRC are much more complicated. It can be the coupling of the external disturbances, the system un-modeled dynamics, and the superadded unknown part of control input. One of the remarkable features of ADRC is that the disturbance is estimated in real time through an extended state observer ([8]) and is canceled in the feedback loop which reduces the control energy significantly in practice ([29]). The convergence of the ADRC for general nonlinear lumped parameter systems is only available recently in [9]. The generalization of the ADRC to the systems described by one-dimensional partial differential equations (PDEs) are also available in our recent works [10]–[12] but the ADRC for multi-dimensional PDEs has not yet been studied.

It should be pointed out that many control methods aforementioned have also been applied to deal with uncertainties in PDEs in literature. The sliding mode control (SMC) that is inherently robust is the most popular one for infinite-dimensional systems but most often, it requires that the input and output operators to be bounded which is not the case for boundary control of PDEs ([24]). Very recently, the boundary SMC controllers are designed for one-dimensional heat, wave, Euler-Bernoulli, and Schrödinger equations with boundary input disturbance in [3], [5], [10]–[12]. In [14], [15], and [19] the adaptive controls are designed for one-dimensional wave equations in which the uncertainties are the unknown parameters in disturbance. Another powerful method in dealing with uncertainties is based on the Lyapunov functional approach. In [7], a boundary control is designed by the Lyapunov method for one-dimensional Euler-Bernoulli beam equation with spatial and boundary disturbance. The internal model principle is also generalized to infinite-dimensional systems [18], [25]. However, there are not many works, to the best of our knowledge, on stabilization for multi-dimensional PDEs with disturbance.

In this paper, we are concerned with stabilization for a multi-dimensional wave equation with Neumann boundary control and control matched external disturbance. The system is governed by the following partial differential equation:

$$\begin{cases} w_{tt}(x,t) = \Delta w(x,t), x \in \Omega, t > 0, \\ w(x,t)|_{\Gamma_0} = 0, t \geq 0, \\ \frac{\partial w}{\partial \nu}|_{\Gamma_1} = v(x,t) + d(x,t), t \geq 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), x \in \Omega \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is an open bounded domain with a smooth C^2-boundary $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$ where Γ_0 and Γ_1 are relatively open sets of Γ, $\text{int}(\Gamma_0) \neq \emptyset$, $\text{int}(\Gamma_1) \neq \emptyset$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and ν is the unit normal vector of Γ pointing the exterior of Ω; v is the control input, d is the unknown external disturbance which is supposed to satisfy

$$d \in L^\infty \left(0, \infty; C(\Gamma_1)\right) \cap C\left(0, \infty; C(\Gamma_1)\right),$$
$$d_t \in L^\infty \left(0, \infty; C(\Gamma_1)\right). \tag{2}$$

In what follows, we use w_t or $\dot{w}$ to denote the derivative of w respect to t which is clear from the context.

The wave equation is perhaps one the most important second-order linear partial differential equations for the description of waves occurred in physics such as sound waves, light waves, and water waves. It arises also in the fields like acoustics, electromagnetics, and fluid dynamics. Owing to its hyperbolic nature, the wave system (1) represents an essential infinite-dimensional system for which there are infinitely many unstable poles on the imaginary axis if there are not control and disturbance on the boundary. This is sharp contrast to the linear heat equation which has at most a finite number of unstable poles. Among many applications of system (1), one of them is in the vibration control of membranes in industry like traveling belts. The boundary disturbance happens typically in combustion process of automotive engine where the knocking phenomena can be described by the wave (1) driven by a periodic unknown signal, for which we refer to [15] and [4].

It is well known that when there is no disturbance, the collocated feedback control

$$v(x, t) = -kw_t(x, t), x \in \Gamma_1, t \geq 0, k > 0 \tag{3}$$

exponentially stabilizes system (1) provided that there exists a coercive smooth vector field h on Γ, that is, the following condition is satisfied ([20, p. 668]):

$$\begin{cases} \text{(i). } h \cdot \nu \leq 0 \text{ on } \Gamma_0. \\ \text{(ii). } h \text{ is parallel to } \nu \text{ on } \Gamma_1, h(\sigma) = \ell(\sigma)\nu(\sigma) \text{ for} \\ \quad \text{a smooth } \ell, \sigma \in \Gamma_1. \\ \text{(iii). For some constant } \rho > 0 \text{ and all vectors} \\ \quad y \in \left(L^2(\Omega)\right)^n : \\ \quad \int_\Omega H(x)y(x) \cdot y(x)dx \geq \rho \int_\Omega |y(x)|^2\, dx \\ \quad \text{where } H(x) = \left\{\frac{\partial h_i}{\partial x_j}\right\}_{i,j=1}^n. \end{cases} \tag{4}$$

The assumption (4) is satisfied if the Ω is the "star-complemented-star-shaped" ([2]), that is, there exists a point $x_0 \in \mathbb{R}^n$ such that

$$\begin{cases} (x - x_0) \cdot \nu \leq 0 \text{ on } \Gamma_0, \\ \quad (\Gamma_0 \text{ is star complemented with respect to } x_0), \\ (x - x_0) \cdot \nu > 0 \text{ on } \Gamma_1, \\ \quad (\Gamma_1 \text{ is star shaped with respect to } x_0) \end{cases}$$

by setting $H(x) = I_{n \times n}$, $\rho = 1$, and $h(x) = x - x_0$, where $I_{n \times n}$ stands for n-dimensional identity matrix.

However, the stabilizing controller (3) is not robust to the external disturbance, which is seen from the following example.

Example 1.1: Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 | 1 < x_1^2 + x_2^2 < 4\}$ be a two-dimensional annulus. Let $\Gamma_0 = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$, $\Gamma_1 = \Gamma \setminus \Gamma_0$, and $d(x, t) \equiv d$ be a constant. Then, the condition (4) is satisfied with $h(x) = x$. However, system (1) under the feedback (3) admits a solution $(w, w_t) = (d \ln(x_1^2 + x_2^2), 0)$.

From example 1.1, we see that in the presence of disturbance, the control must be re-designed.

We consider system (1) in the energy Hilbert state space $\mathcal{H} = H_{\Gamma_0}^1(\Omega) \times L^2(\Omega)$ where $H_{\Gamma_0}^1(\Omega) = \{f \in H^1(\Omega) | f = 0 \text{ on } \Gamma_0\}$ with the usual inner product given by

$$\langle (f_1, g_1)^\top, (f_2, g_2)^\top \rangle = \int_\Omega [\nabla f_1(x)\overline{\nabla f_2(x)} + g_1(x)\overline{g_2(x)}]dx, \, \forall (f, g)^\top \in \mathcal{H}$$

and the control space $U = L^2(\Gamma_1)$. Define the operator $\mathcal{A}$ as follows:

$$\begin{cases} \mathcal{A}(f, g)^\top = (g, \Delta f)^\top, \, \forall (f, g)^\top \in D(\mathcal{A}), \\ D(\mathcal{A}) = \left\{(f, g)^\top \in \mathcal{H} \cap (H^2(\Omega) \times H^1(\Omega)) \right. \\ \left. \quad \mid \frac{\partial f}{\partial \nu}\Big|_{\Gamma_1} = g|_{\Gamma_0} = 0\right\}. \end{cases} \tag{5}$$

Then it is easy to verify that $\mathcal{A}^* = -\mathcal{A}$ in $\mathcal{H}$.

Let $A = -\Delta$ be the usual Laplacian with $D(A) = \{f | f \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega), (\partial f/\partial \nu)|_{\Gamma_1} = 0\}$, which is a positive definite unbounded operator in $L^2(\Omega)$. It is easily shown (see e.g., [13]) that $D(A^{1/2}) = H_{\Gamma_0}^1(\Omega)$ and $A^{1/2}$ is a canonical isomorphism from $H_{\Gamma_0}^1(\Omega)$ onto $L^2(\Omega)$. We consider $L^2(\Omega)$ as the pivot space. Then, the following Gelfand triple inclusions are valid:

$$\left[D(A^{\frac{1}{2}})\right] \hookrightarrow L^2(\Omega) = \left(L^2(\Omega)\right)' \hookrightarrow \left[D(A^{\frac{1}{2}})\right]'$$

where $[D(A^{1/2})]'$ is the dual space of $[D(A^{1/2})]$ with the pivot space $L^2(\Omega)$. An extension $\tilde{A} \in \mathcal{L}([D(A^{1/2})], [D(A^{1/2})]')$ of A is defined by

$$\langle \tilde{A}f, g \rangle_{\left[D(A^{\frac{1}{2}})\right]' \times \left[D(A^{\frac{1}{2}})\right]} = \langle A^{\frac{1}{2}}f, A^{\frac{1}{2}}g \rangle_{L^2(\Omega)}, \, \forall f, g \in D(A^{\frac{1}{2}}) = H_{\Gamma_0}^1(\Omega).$$

Define the Neumann map $\Upsilon \in \mathcal{L}(H^s(\Gamma_1), H^{3/2+s}(\Omega))$ ([20, p. 668]), i.e., $\Upsilon(u + d) = v$ if and only if

$$\begin{cases} \Delta v = 0 \text{ in } \Omega, \\ v|_{\Gamma_0} = 0, \frac{\partial v}{\partial \nu}|_{\Gamma_1} = u + d. \end{cases} \tag{6}$$

Using the Neumann map, one can write (1) in $[D(A^{1/2})]'$ as

$$\ddot{w} + \tilde{A}(w - \Upsilon(u + d)) = 0 \tag{7}$$

which is further written as

$$\ddot{w} = -\tilde{A}w + B(u + d) \tag{8}$$

where $B \in \mathcal{L}(U, [D(A^{1/2})]')$ is given by

$$Bu_0 = \tilde{A}\Upsilon u_0, \, \forall u_0 \in U. \tag{9}$$

Define $B^* \in \mathcal{L}([D(A^{1/2})], U)$, the adjoint of B, by

$$\langle B^* f, u_0 \rangle_U = \langle f, Bu_0 \rangle_{\left[D(A^{\frac{1}{2}})\right] \times \left[D(A^{\frac{1}{2}})\right]'},$$

$$\forall f \in D(A^{\frac{1}{2}}), u_0 \in U.$$

Then, for any $f \in D(A)$ and $u \in C_0^\infty(\Gamma_1)$, By Green's formula

$$\langle f, Bu_0 \rangle_{\left[D(A^{\frac{1}{2}})\right] \times \left[D(A^{\frac{1}{2}})\right]'} = \langle Af, \tilde{A}^{-1} Bu_0 \rangle_{L^2(\Omega)}$$

$$= \langle Af, \Upsilon u_0 \rangle_{L^2(\Omega)} = \langle Af, v_0 \rangle_{L^2(\Omega)} = -\int_\Omega \Delta f(x) v_0(x) dx$$

$$= -\int_\Omega f \Delta v_0(x) dx - \int_{\Gamma_0 \cup \Gamma_1} \frac{\partial f(x)}{\partial \nu} v_0(x) dx$$

$$+ \int_{\Gamma_0 \cup \Gamma_1} f(x) \frac{\partial v_0(x)}{\partial \nu} dx = -\int_{\Gamma_1} f(x) u_0(x) dx$$

$$= \langle f, u_0 \rangle_U$$

where $v_0 = \Upsilon u_0$. Since $C_0^\infty(\Gamma_1)$ is dense in $L^2(\Gamma_1)$, we obtain

$$B^* f = f|_{\Gamma_1}. \tag{10}$$

Therefore, system (1) can be written as

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathcal{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathcal{B}\left[u(x,t) + d(x,t)\right] \tag{11}$$

where $\mathcal{B} = (0, B^*)^\top$

$$\mathcal{B}^*(f, g)^\top = g|_{\Gamma_1}, \ \forall (f, g) \in (H_{\Gamma_0}^1(\Omega))^2. \tag{12}$$

From (12), we see why the controller (3) is collocated. However, since $\mathcal{B}$ is not admissible for the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ on $\mathcal{H}$ (see [28] and [20, p. 669]), (11) does not always admit a unique solution in $\mathcal{H}$ for general $v \in L_{loc}^2(0, \infty, U)$. To overcome this difficulty, we first introduce a damping on the control boundary by designing

$$v(x,t) = -kw_t(x,t) + u(x,t), k > 0, \ \forall x \in \Gamma_1, t \geq 0 \tag{13}$$

under which, system (1) becomes

$$\begin{cases} w_{tt}(x,t) = \Delta w(x,t), x \in \Omega, t > 0, \\ w(x,t)|_{\Gamma_0} = 0, t \geq 0, \\ \frac{\partial w}{\partial \nu}|_{\Gamma_1} = -kw_t(x,t) + u(x,t) + d(x,t), t \geq 0, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), x \in \Omega. \end{cases} \tag{14}$$

Exactly the same as from (1)to (8), we can write (14) as

$$\ddot{w} = -\tilde{A}w - kBB^*\dot{w} - B(u+d) \text{ in } [D(A^{\frac{1}{2}})]' \tag{15}$$

or in the first order form

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathbb{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathbb{B}(u+d)$$

$$\text{in } \left[D(A^{\frac{1}{2}})\right] \times \left[D(A^{\frac{1}{2}})\right]' \tag{16}$$

where

$$\begin{cases} \mathbb{A}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ -\tilde{A}f - kBB^*g \end{pmatrix}, \quad \forall \begin{pmatrix} f \\ g \end{pmatrix} \in D(\mathbb{A}), \\ D(\mathbb{A}) = \left\{ (f, g)^\top | f, g \in D\left(A^{\frac{1}{2}}\right), \\ \qquad \tilde{A}f + kBB^*g \in L^2(\Omega) \right\}, \\ \mathbb{B} = (0, -B)^\top. \end{cases} \tag{17}$$

Proposition 1.1: The operator $\mathbb{A}$ defined in (17) generates a C_0-semigroup of contractions $e^{\mathbb{A}t}$ on $\mathcal{H}$ and $\mathbb{B}$ is admissible to $e^{\mathbb{A}t}$. Therefore, for any initial value $(w(\cdot, 0), \dot{w}(\cdot, 0))^\top \in \mathcal{H}$ and control input $u \in L_{loc}^2(0, \infty, U)$, (14) admits a unique solution $(w, \dot{w})^\top \in \mathcal{H}$.

Proof: We first show the first assertion. Actually, for any $(f, g)^\top \in D(\mathbb{A})$

$$\text{Re } \langle \mathbb{A}(f,g)^\top, (f,g)^\top \rangle_{\mathcal{H}} = \text{Re } \langle A^{\frac{1}{2}}g, A^{\frac{1}{2}}f \rangle_{L^2(\Omega)}$$

$$- \text{Re } \langle \tilde{A}f + kBB^*g, g \rangle_{L^2(\Omega)}$$

$$= \text{Re } \left\langle A^{\frac{1}{2}}g, A^{\frac{1}{2}}f \right\rangle_{L^2(\Omega)}$$

$$- \text{Re } \langle \tilde{A}f + kBB^*g, g \rangle_{\left[D(A^{\frac{1}{2}})\right]', \left[D(A^{\frac{1}{2}})\right]}$$

$$= \text{Re } \langle \tilde{A}f, g \rangle_{\left[D(A^{\frac{1}{2}})\right]', \left[D(A^{\frac{1}{2}})\right]}$$

$$- \text{Re } \langle \tilde{A}f + kBB^*g, g \rangle_{\left[D(A^{\frac{1}{2}})\right]', \left[D(A^{\frac{1}{2}})\right]}$$

$$= -k\text{Re } \langle BB^*g, g \rangle_{\left[D(A^{\frac{1}{2}})\right]', \left[D(A^{\frac{1}{2}})\right]}$$

$$= -k\text{Re}\langle B^*g, B^*g \rangle_U = -k\|B^*g\|_U^2 \leq 0. \tag{18}$$

This shows that $\mathbb{A}$ is dissipative. Now we show that $\mathbb{A}^{-1} \in \mathcal{L}(\mathcal{H})$. Solve the equation

$$\mathbb{A}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ -\tilde{A}f - kBB^*g \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{H}$$

to obtain $g = \phi \in D(A^{1/2})$, $-\tilde{A}f - kBB^*g = \psi$. The latter is equivalent to

$$\tilde{A}f = -kBB^*\phi - \psi \in \left[D(A^{\frac{1}{2}})\right]'.$$

Since $\tilde{A}$ is isometric from $[D(A^{1/2})]$ to $[D(A^{1/2})]'$, we find that

$$f = \tilde{A}^{-1}(-kBB^*\phi - \psi) \in D(A^{\frac{1}{2}}).$$

Hence

$$\mathbb{A}^{-1}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \tilde{A}^{-1}(-kBB^*\phi - \psi) \\ \phi \end{pmatrix}. \tag{19}$$

By the Lumer-Phillips theorem [23, Theorem 1.4.3], $\mathbb{A}$ generates a C_0-semigroup of contractions $e^{\mathbb{A}t}$ on $\mathcal{H}$. To prove the second assertion, we consider the following system:

$$\frac{d}{dt}\begin{pmatrix} p \\ \dot{p} \end{pmatrix} = \mathbb{A}\begin{pmatrix} p \\ \dot{p} \end{pmatrix}. \tag{20}$$

Since $\mathbb{A}$ generates a C_0-semigroup on $\mathcal{H}$ that is justified by the first assertion, for any $(p(\cdot,0),\dot{p}(\cdot,0))^\top \in D(\mathbb{A})$, the solution to (20) satisfies $(p,\dot{p})^\top \in D(\mathbb{A})$. Take the inner product on both sides of (20) with $(p,\dot{p})^\top$ and take (18) into account to obtain

$$\mathrm{Re}\langle \ddot{p},\dot{p}\rangle + \mathrm{Re}\langle A^{\frac{1}{2}}p, A^{\frac{1}{2}}\dot{p}\rangle = -k\|B^*\dot{p}\|^2$$

that is

$$\dot{F}(t) = -k\|B^*\dot{p}\|_U^2, F(t) = \frac{1}{2}\left[\left\|A^{\frac{1}{2}}p\right\|_{L^2(\Omega)}^2 + \|\dot{p}\|_{L^2(\Omega)}^2\right].$$

Therefore

$$k\int_0^T \|B^*\dot{p}\|_U^2 dt = F(0) - F(T) \le F(0).$$

This shows that the operator $\mathbb{B}$ is admissible to $e^{\mathbb{A}t}$ ([28]).   ∎

By proposition 1.1, the (weak) solution of (14) is understood in the sense of

$$\frac{d}{dt}\left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, \mathbb{A}^*\begin{pmatrix} f \\ g \end{pmatrix}\right\rangle_{\mathcal{H}}$$

$$+ \int_{\Gamma_1}[u(x,t) + d(x,t)]g(x)dx, \ \forall (f,g)^\top \in D(\mathbb{A}^*). \quad (21)$$

A simple computation shows that

$$\begin{cases} \mathbb{A}^*(f,g) = -(g,\Delta f)^\top, \\ D(\mathbb{A}^*) = \Big\{(f,g)^\top \in \left(H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \mid \\ \qquad \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_1} = kg|_{\Gamma_1}\Big\}. \end{cases} \quad (22)$$

Equation (21) demonstrates clearly the infinite-dimensional nature of PDEs where $(f,g)^\top \in D(\mathbb{A}^*)$ is called the (smooth) test function. The aim of the abstract formulation from (5) to (21) is to deduce (21) because (21) is a system of infinitely many ODEs for which the ADRC can be applied like that in [8], [9] for lumped parameter systems.

Let us briefly indicate the main contributions of this paper. First of all, we reduce (14) into an infinite number of ordinary differential equations by appropriately chosen time varying test functions in (21) from which the disturbance is estimated through an extended state observer. Secondly, the disturbance is canceled in the feedback loop together with the collocated stabilizing controller (3). Finally, all internal subsystems in the closed-loop are shown to be asymptotically stable. In particular, the time varying high gain is first applied to the extended state observer for a system described by the partial differential equation for complete disturbance rejection purpose and the peaking value reduction caused by the constant high gain in literature. The overall picture of the ADRC in dealing with the disturbance for multi-dimensional partial differential equation is demonstrated through system (14). The numerical experiments are carried out to illustrate the convergence and the effect of peaking value reduction.

We process as follows. In Section II, we state the main results. Section III is devoted to the proof of the main results. Some numerical simulations for Example 1.1 are presented in Section IV for illustration.

II. THE MAIN RESULTS

In addition to (2), we suppose further that $d(\cdot,t)$ is Hölder continuous with index $\alpha \in (0,1]$, that is, there exists a positive nondecreasing differentiable continuous function K such that

$$|d(x_1,t)-d(x_2,t)| \le K(t)|x_1-x_2|^\alpha, \ \forall x_1, x_2 \in \Gamma_1, t\ge 0. \quad (23)$$

The examples of such kinds of disturbances include all spatial independent disturbance $d(x,t) = d(t)$, all finite sum of harmonic disturbances like $d(x,t) = \sin(xt)$ with $K(t) = t+1$, $\alpha = 1$ and $d(x,t) = \sin x \sin t$ with $K = \alpha = 1$, and all periodic disturbances with respect to t.

Let ε be a continuous function such that

$$\varepsilon(t) \in (0,1], \dot{\varepsilon}(t) < 0, \lim_{t\to\infty}\varepsilon(t) = 0. \quad (24)$$

In addition, suppose that we can choose ε appropriately so that

$$\delta(t) = \left(\frac{\varepsilon(t)}{K(t)}\right)^{\frac{1}{\alpha}} \text{ satisfies } \sup_{t>0}\left|\dot{\delta}(t)\delta^{n-2}(t)\right| < \infty \quad (25)$$

where $r \in C(\bar{\mathbb{R}}^+,\mathbb{R}^+)$ is a time varying gain to be required to satisfy

$$\dot{r}(t) > 0, \lim_{t\to\infty} r(t) = \infty,$$

$$\frac{\dot{r}(t)}{r(t)} \le \overline{M}, \overline{M} > 0, \lim_{t\to\infty} r(t)\delta^{n-1}(t) = \infty. \quad (26)$$

To deal with the disturbance that depends on the spatial variable, we need the time varying covers of Γ_1 so that the measures of these covers are uniformly bounded with respect to time.

Lemma 2.1: Let δ be defined by (25). Then, there exists $\{x^{(i)}\}_{i=1}^\infty \subset \overline{\Gamma_1}$ so that a finite number of time varying covers $\{\overline{\Gamma_1}\cap U(x^{(i)},\delta(t))\}_{i=1}^{N(t)}$ of $\overline{\Gamma_1}$ satisfies

$$\overline{\Gamma_1} = \left\{\overline{\Gamma_1}\cap U\left(x^{(i)},\delta(t)\right)\right\}_{i=1}^{N(t)},$$

$$\sum_{i=1}^{N(t)} \mathrm{meas}\left(\overline{\Gamma_1}\cap U\left(x^{(i)},\delta(t)\right)\right)$$

$$\le C_0(\Gamma_1)(n-1)^{\frac{n-1}{2}}2^{n-1}\,\mathrm{meas}\left(\overline{\Gamma_1}\right) \quad (27)$$

where $U(x^{(i)},\delta(t))$ denotes the ball in $\mathbb{R}^n$ centered at $x^{(i)} \in \overline{\Gamma_1}$ with radius $\delta(t)$, the boundary measure is the Lebesgue measure in $\mathbb{R}^{(n-1)}$ space, $C_0(\Gamma_1)$ is a positive constant depending on Γ_1 only, the time dependent integer N depends on δ directly for which we denote by $N(t) = N^\delta(t)$ and $\lim_{t\to\infty} N(t) = +\infty$.

The next step is to construct an extended state observer to estimate the disturbance by the constructed time varying covers in Lemma 2.1. To this purpose, let $(f_i^t, g_i^t)^\top \in D(\mathbb{A}^*)$, $i = 1, 2, \cdots$, so that

$$\begin{cases} \Delta f_i^t = 0, f_i^t\Big|_{\Gamma_0} = 0, \left.\frac{\partial f_i^t}{\partial \nu}\right|_{\Gamma_1} = kg_i^t, \\ g_i^t|_{\Gamma_0} = 0, g_i^t|_{\Gamma_1\backslash U(x^{(i)},\delta(t))} = 0, g_i^t|_{\Gamma_1\cap U(x^{(i)},\frac{1}{2}\delta(t))} = 1, \\ 0 \le g_i^t|_{\Gamma_1\cap U(x^{(i)},\delta(t))} \le 1, |g_{it}^t|_{\Gamma_1\cap U(x^{(i)},\delta(t))}| \le \pi\left|\frac{\dot{\delta}(t)}{\delta(t)}\right|, \\ \left|\nabla g_i^t|_{\Gamma_1\cap\left(U(x^{(i)},\delta(t))\backslash U(x^{(i)},\frac{1}{2}\delta(t))\right)}\right| \le \frac{\pi}{\delta(t)}, \\ \left|\nabla g_i^t|_{\Gamma_1\backslash\left(U(x^{(i)},\delta(t))\backslash U(x^{(i)},\frac{1}{2}\delta(t))\right)}\right| = 0. \end{cases}$$

$$\quad (28)$$

It is seen from (6) that $f_i^t = k\Upsilon g_i^t$ and hence $f_{it}^t = k\Upsilon g_{it}^t$, and the function g_i^t can be constructed analytically as follows (there are many ways to choose different g_i^t):

$$g_i^t(x) = \begin{cases} 1, & |x - x^{(i)}| < \frac{\delta(t)}{2}, \\ -\frac{1}{2}\sin\left(\frac{2\pi|x-x^{(i)}|}{\delta(t)} - \frac{3\pi}{2}\right) \\ \quad +\frac{1}{2}, & \frac{\delta(t)}{2} \le |x-x^{(i)}| < \delta(t), \\ 0, & |x-x^{(i)}| \ge \delta(t). \end{cases} \tag{29}$$

It is found that $g_i^t(x)$ is continuously differentiable with respect to t and

$$g_{it}^t(x) = \frac{\partial g_i^t(x)}{\partial t}$$
$$= \begin{cases} 0, |x - x^{(i)}| < \frac{\delta(t)}{2} \text{ or } |x-x^{(i)}| \ge \delta(t), \\ -\pi\,|x-x^{(i)}|\,\frac{\dot\delta(t)}{\delta^2(t)}\cos\left(\frac{2\pi|x-x^{(i)}|}{\delta(t)} - \frac{3\pi}{2}\right), \\ \frac{\delta(t)}{2} \le |x-x^{(i)}| < \delta(t). \end{cases} \tag{30}$$

Moreover, it is also found that $\nabla g_i^t(x)$ is continuous as well, and

$$\nabla g_i^t(x) = \begin{cases} 0, |x - x^{(i)}| < \frac{\delta(t)}{2} \text{ or } |x-x^{(i)}| \ge \delta(t), \\ -\pi\,\frac{x-x^{(i)}}{|x-x^{(i)}|}\,\frac{1}{\delta(t)}\cos\left(\frac{2\pi|x-x^{(i)}|}{\delta(t)} - \frac{3\pi}{2}\right), \\ \frac{\delta(t)}{2} \le |x-x^{(i)}| < \delta(t). \end{cases} \tag{31}$$

Substitute $(f_i^t, g_i^t)^\top$ into (21) to obtain

$$\frac{d}{dt}\int_\Omega \left[\nabla w(x,t)\nabla f_i^t(x) + w_t(x,t)g_i^t(x)\right]dx$$
$$= \int_\Omega \left[\nabla w(x,t)\nabla f_{it}^t(x) + w_t(x,t)g_{it}^t(x)\right]dx$$
$$\quad - \int_\Omega [\nabla w(x,t)\nabla g_i^t(x)dx + \int_{\Gamma_1}[u(x,t)+d(x,t)]\,g_i^t(x)dx$$
$$= \int_\Omega \left[\nabla w(x,t)\nabla f_{it}^t(x) + w_t(x,t)g_{it}^t(x)\right]dx$$
$$\quad - \int_\Omega \nabla w(x,t)\nabla g_i^t(x)dx + \int_{\Gamma_1} u(x,t)g_i^t(x)dx$$
$$\quad + d(\xi_i(t),t)\int_{\Gamma_1} g_i^t(x)dx \tag{32}$$

where $\xi_i : [0,\infty) \to \Gamma_1 \cap U(x^{(i)}, \delta(t))$ satisfies

$$\begin{cases} d(\xi_i(t),t) \\ \quad = \dfrac{\int_{\Gamma_1} d(x,t)g_i^t(x)dx}{\int_{\Gamma_1} g_i^t(x)dx}, \\ \dfrac{d}{dt}\left(d(\xi_i(t),t)\int_{\Gamma_1} g_i^t(x)dx\right) \\ \quad = \int_{\Gamma_1} d_t(x,t)g_i^t(x)dx + \int_{\Gamma_1} d(x,t)g_{it}^t(x)dx, \\ \left|\int_{\Gamma_1} d(x,t)g_{it}^t(x)dx\right| \\ \quad \le C'(\Gamma_1)\|d\|_{L^\infty(0,\infty;C(\Gamma_1))}\left|\dot\delta(t)\right|\delta^{n-2}(t) \end{cases} \tag{33}$$

where $C'(\Gamma_1) > 0$ is a constant. Let M be a constant such that $|d(x,t)| \le M$, $|d_t(x,t)| \le M$ for all $x \in \Gamma_1$ and $t \ge 0$. By (25), it has, for all $t \ge 0$, that

$$\begin{cases} |d(\xi_i(t),t)| \le M, \\ \left|\dfrac{d}{dt}\left(d(\xi_i(t),t)\int_{\Gamma_1} g_i^t(x)dx\right)\right| \\ \quad \le M + C'(\Gamma_1)\|d\|_{L^\infty(0,\infty;C(\Gamma_1))} \\ \sup_{t>0}\left(\left|\dot\delta(t)\right|\delta^{n-2}(t)\right) < \infty. \end{cases} \tag{34}$$

Let

$$\begin{cases} y_i(t) = \int_\Omega \left[\nabla w(x,t)\nabla f_i^t(x) + w_t(x,t)g_i^t(x)\right]dx, \\ y_{2i}(t) = \int_\Omega \left[\nabla w(x,t)\nabla f_{it}^t(x) + w_t(x,t)g_{it}^t(x)\right]dx \\ \quad - \int_\Omega \nabla w(x,t)\nabla g_i^t(x)dx, i = 1,2,\dots. \end{cases} \tag{35}$$

Then

$$\dot y_i(t) = y_{2i}(t) + \int_{\Gamma_1} u(x,t)g_i^t(x)dx$$
$$+ d(\xi_i(t),t)\int_{\Gamma_1} g_i^t(x)dx, i = 1,2,\dots. \tag{36}$$

The system (36), as an infinite number of ordinary differential equations, is our starting point to estimate the general disturbance $d(x,t)$ motivated from the ADRC to lumped parameter systems ([8]). To this purpose, we design a time varying high gain extended state observer as follows:

$$\begin{cases} \dot{\hat y}_i(t) = y_{2i}(t) + \int_{\Gamma_1} u(x,t)g_i^t(x)dx \\ \qquad + \hat d_i(t)\int_{\Gamma_1} g_i^t(x)dx - r(t)\left[\hat y_i(t) - y_i(t)\right], \\ \dfrac{d}{dt}\left(\hat d_i(t)\int_{\Gamma_1} g_i^t(x)dx\right) = -r^2(t)\left[\hat y_i(t) - y_i(t)\right], \\ \qquad\qquad i = 1,2,\dots \end{cases} \tag{37}$$

where r is defined in (26). We regard $\hat d_i(t)$ as an approximation of $d(\xi_i(t),t)$ which is confirmed by the succeeding Lemma 2.2.

Lemma 2.2: Let $\{x^{(i)}\}$ be defined in Lemma 2.1, g_i^t be defined by (29), ξ_i and $d(\xi_i(t),t)$ be defined by (33), and y_i and y_{2i} by (35). Then, under the conditions (24), (25), and (26), the solution of (37) satisfies

$$\lim_{t\to\infty}\left|\hat d_i(t) - d(\xi_i(t),t)\right| = 0, \lim_{t\to\infty}|\hat y_i(t) - y_i(t)| = 0 \tag{38}$$

uniformly for all $i = 1,2,\cdots$.

Now, we define

$$\hat d(x,t) = \begin{cases} \hat d_1(t), & x \in \Gamma_1 \cap U(x^{(1)},\delta(t)), \\ \hat d_2(t), & x \in \Gamma_1 \cap \left(U(x^{(2)},\delta(t))\setminus U(x^{(1)},\delta(t))\right), \\ \cdots, \\ \hat d_i(t), & x \in \Gamma_1 \cap (U(x^{(i)},\delta(t))\setminus\cup_{j=1}^{i-1}U(x^{(j)},\delta(t))), \\ \cdots, \\ \hat d_{N(t)}(t), & x \in \Gamma_1 \\ & \cap\left(U(x^{N(t)},\delta(t))\setminus\cup_{j=1}^{N(t)-1}U(x^{(j)},\delta(t))\right) \end{cases} \tag{39}$$

where $x^{(i)}$ is defined in Lemma 2.1. The following Lemma 2.3 shows that $\hat d$ can be regarded as an approximation of d.

Lemma 2.3: Let $\hat{d}$ be defined by (39). Then, under the conditions of Lemma 2.2

$$\lim_{t\to\infty} \left\| \hat{d}(\cdot,t) - d(\cdot,t) \right\|_{L^2(\Gamma_1)} = 0. \tag{40}$$

The last step is to design the feedback control. By (40), we design naturally a collocated like state feedback controller to (14) as follows:

$$u(x,t) = \hat{d}(x,t). \tag{41}$$

Compared with (3), it is seen that in (41), the $\hat{d}$ is used to cancel the effect of the disturbance which is just the essence of the ADRC in dealing with the disturbance. Under the feedbacks (13) and (41), the closed-loop system of (14) becomes

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) = 0, x \in \Omega, t > 0, \\ w(x,t)|_{\Gamma_0} = 0, \\ \frac{\partial w}{\partial \nu}(x,t)|_{\Gamma_1} = -kw_t(x,t) - \hat{d}(x,t) + d(x,t), t \geq 0, \\ \dot{\hat{y}}_i(t) = y_{2i}(t) - k\int_{\Gamma_1} g_i^t(x)w_t(x,t)dx - r(t)[\hat{y}_i(t) - y_i(t)], \\ \frac{d}{dt}\left(\hat{d}_i(t) \int_{\Gamma_1} g_i^t(x)dx \right) = -r^2(t)[\hat{y}_i(t) - y_i(t)], i = 1, 2, \ldots. \end{cases} \tag{42}$$

Now we are in a position to state the convergence of the ADRC for system (14), the main result of this paper.

Theorem 2.1: Let $\{x^{(i)}\}$ be defined in Lemma 2.1, g_i^t be defined by (29), $\hat{d}$ be defined by (39), and y_i and y_{2i} by (35). Then, for any initial value $(w(\cdot,0), w_t(\cdot,0))^\top \in \mathcal{H}$, the closed-loop system (42) admits a unique solution $(w(\cdot,t), w_t(\cdot,t))^\top \in C(0,\infty,\mathcal{H})$; $\{\hat{y}_i, \hat{d}_i\}_{i=1}^\infty \in C(0,\infty)$. Moreover, under the conditions (2), (4), (23), (24), (25), and (26), system (42) is asymptotically stable in the sense of

$$\lim_{t\to\infty} E_i(t) = 0 \tag{43}$$

uniformly for $i = 1, 2, \ldots$, where

$$E_i(t) = \int_\Omega \left[|\nabla w(x,t)|^2 + |w_t(x,t)|^2 \right] dx + |\hat{y}_i(t)| + \int_{\Gamma_1} |\hat{d}(x,t) - d(x,t)|^2 dx. \tag{44}$$

Remark 2.1: In the disturbance estimator (37), if we replace the time varying high gain $r(t)$ by the constant high gain $r(t) \equiv 1/\kappa$, then we have

$$\begin{cases} \dot{\hat{y}}_i(t) = y_{2i}(t) + \int_{\Gamma_1} u(x,t)g_i^t(x)dx \\ \qquad + \hat{d}_i(t) \int_{\Gamma_1} g_i^t(x)dx - \frac{1}{\kappa}[\hat{y}_i(t) - y_i(t)], \\ \frac{d}{dt}\left(\hat{d}_i(t) \int_{\Gamma_1} g_i^t(x)dx \right) = -\frac{1}{\kappa^2}[\hat{y}_i(t) - y_i(t)], \\ \qquad\qquad i = 1, 2, \ldots \end{cases} \tag{45}$$

and the convergence (43) becomes

$$\limsup_{t\to\infty} E_i(t) \leq \widehat{C}\kappa \tag{46}$$

where $\widehat{C} > 0$ is a constant independent of κ and i.

The constant high gain (45) shares the advantage of many high gain controls that the high frequency noise can be filtered yet brings the peaking value problem ([27]). Recommended control strategy is to use the time varying gain first to reduce the peaking value in the initial stage to a reasonable level and then apply the constant high gain. This will be explained numerically in Section IV.

At the end of this section, we point out that in Theorem 2.1, both $d(\cdot,t)$ and $d_t(\cdot,t)$ are supposed to be uniformly bounded in time t in $L^\infty(\Gamma_1)$. The boundedness of d is necessary because this ensures that the controller (41) is bounded, which is the basic requirement for ADRC due to its estimation/cancellation nature as well as for many other methods even the sliding mode control ([10]–[12]). However, the boundedness of $d_t(\cdot,t)$ with respect to time t is not necessary since otherwise, some disturbance like $d(x,t) = \sin(xt^2)$ will be excluded. From the proof of Theorem 2.1 in next section (see (74)), we see that the boundedness of $d_t(\cdot,t)$ is to guarantee that

$$\frac{d}{dt}\int_{\Gamma_1} d(x,t)g_i^t(x)dx$$

$$= \int_{\Gamma_1} d_t(x,t)g_i^t(x)dx + \int_{\Gamma_1} d(x,t)g_{it}^t(x)dx \tag{47}$$

is uniformly bounded with respect to t and i. Now, by (25) and (30), the second term on the right-hand side of (47) is estimated as

$$\left| \int_{\Gamma_1} d(x,t)g_{it}^t(x)dx \right| \leq C\|d\|_{L^\infty} \sup_{t\geq 0}\left(\left|\dot{\delta}(t)\right| \delta^{n-2} \right) < \infty \tag{48}$$

for some constant $C > 0$. So the boundedness of $(d/dt)\int_{\Gamma_1} d(x,t)g_i^t(x)dx$ is guaranteed if the first term on the right-hand side of (47) is uniformly bounded with respect to t and i. However, by the construction of g_i^t in (29)

$$\left| \int_{\Gamma_1} d_t(x,t)g_i^t(x)dx \right| \leq C_1 \|d_t(\cdot,t)\|_{L^\infty(\Gamma_1)} \delta^{n-1}(t), \ \forall t \geq 0 \tag{49}$$

where $C_1 > 0$ is a constant and δ is defined by (25). Since $\lim_{t\to\infty}\delta(t) = 0$, $\|d_t(\cdot,t)\|_{L^\infty(\Gamma_1)}$ can be relaxed to grow slowly than $1/\delta^{n-1}(t)$. However, since from (25), there exists $t_0 > 0$ such that $1/\delta^{n-1}(t) < r(t)$ for all $t \geq t_0$ and (26) limits the exponential growth rate of r, $\|d_t(\cdot,t)\|_{L^\infty(\Gamma_1)}$ can grow at most exponentially. For instance for $d = \sin(e^{\lambda t}x)$ where $\lambda > 0$, we have $K(t) = e^{\lambda t}$, $\alpha = 1$ in (23). Hence for this special example, (24), (25), and (26) are satisfied by choosing $\varepsilon(t) = e^{-\lambda t}$ and $r(t) = e^{2n\lambda t}$ but $\|d_t(\cdot,t)\|_{L^\infty(\Gamma_1)}$ grows exponentially at the growth rate λ and satisfies (49). This relaxes the limitation of d_t in large extent. However, it should be noticed that (49) is only true for $n \geq 2$. For the case of $n = 1$, since the disturbance $d(x,t) = d(t)$ is independent of x, we need only one test function (29). We give a sketch for this special case and details are left as an exercise for reader. Let $U(x^*, \delta_0)$ be the ball centered at x^* with radius $\delta_0 > 0$ where $x^* \in \Gamma_1$ and

$\delta_0 = (1/2) \inf_{y \in \Gamma_0} |y - x^*| > 0$, we construct $g \in C^\infty(\Omega)$ as an mollifier given by ([1, p. 36])

$$g(x) = \begin{cases} \exp(-1/\left(1 - |x - x^*|^2\right)), |x - x^*| < \delta_0, \\ 0, |x - x^*| \geq \delta_0. \end{cases} \quad (50)$$

Then

$$g|_{\Gamma_0} = 0, \; g|_{\Gamma_1} \geq 0, \; \int_{\Gamma_1} g(x) dx > 0. \quad (51)$$

Let f be a solution of the following elliptic boundary problem:

$$\begin{cases} \Delta f = 0, x \in \Omega, \\ f|_{\Gamma_0} = 0, \; \frac{\partial f}{\partial \nu}|_{\Gamma_1} = kg|_{\Gamma_1}. \end{cases}$$

Then, by the embedding theorem (see, e.g., [21]), there exists a constant $C > 0$ such that

$$\|f\|_{H^2(\Omega)} \leq C\|g\|_{H^{\frac{1}{2}}(\Gamma_1)} \leq C\|g\|_{H^1(\Gamma_1)}.$$

Substitute $(f, g)^\top \in D(\mathcal{A}^*)$ into (21) to obtain

$$\frac{d}{dt} \int_\Omega [\nabla w(x,t) \nabla f(x) + w_t(x,t) g(x)] \, dx$$

$$= - \int_\Omega \nabla w(x,t) \nabla g(x) dx + \int_{\Gamma_1} [u(x,t) + d(x,t)] \, g(x) dx$$

$$= - \int_\Omega \nabla w(x,t) \nabla g(x) dx + \int_{\Gamma_1} u(x,t) g(x) dx$$

$$+ d(t) \int_{\Gamma_1} g(x) dx. (52)$$

Set

$$y(t) = \int_\Omega [\nabla w(x,t) \nabla f(x) + w_t(x,t) g(x)] \, dx,$$

$$y_0(t) = - \int_\Omega \nabla w(x,t) \nabla g(x) dx. \quad (53)$$

Then, (52) shows that

$$\dot{y}(t) = y_0(t) + \int_{\Gamma_1} u(x,t) g(x) dx + d(t) \int_{\Gamma_1} g(x) dx. \quad (54)$$

It is seen that (54) is an ordinary differential equation where the disturbance $d(x,t) = d(t)$ appears on the right side. It is the counterpart of (36) for the spatial variable dependent disturbance, similar to one-dimensional PDEs in [10]–[12].

Along the same way, we can show that d_t can be relaxed to be growing exponentially at any growth rate which almost removes limitation for the boundedness of d_t for one-dimensional PDEs in [10]–[12].

III. PROOF OF THE MAIN RESULTS

Proof of Lemma 2.1: Since $\Omega \subset \mathbb{R}^n$ is an open bounded set and its boundary parts Γ_0 and Γ_1 are of C^2-class, for any $x^0 = (x_1, x_2, \ldots, x_n) \in \Gamma$, there exists a C^2-function ψ and a neighborhood $U_{x^0} \subset \Gamma$ of x^0 such that $x_n = \psi(x_1, x_2, \ldots, x_{n-1})$ holds in U_{x^0} (see, e.g., [6, p. 626]). Since $\overline{\Gamma_1}$ is compact in $\mathbb{R}^{n-1}$, by finite covering theorem, we may assume without loss of generality that $\overline{\Gamma_1}$ can be described by $x_n = \psi(x_1, x_2, \ldots, x_{n-1}) \in C^2(\Omega_{n-1})$ for some hypercube $\Omega_{n-1} \subset \mathbb{R}^{n-1}$.

Let Ω_c be an $(n-1)$-hypercube in $\mathbb{R}^{n-1}$ space. Suppose that each side of Ω_c parallels the corresponding orthogonal coordinate axis of $\mathbb{R}^{n-1}$ so that $\Omega_{n-1} \subset \Omega_c$. Let

$$C_2(\Gamma_1) = \left\| \sqrt{1 + \psi_{x_1}^2 + \psi_{x_2}^2 + \ldots + \psi_{x_{n-1}}^2}^{\,2} \right\|_{C(\Omega_{n-1})},$$

$$\delta_1(t) = \frac{\delta(t)}{C_2(\Gamma_1)}. \quad (55)$$

We suppose that $\Omega_c = \cup_{j=1}^{k_0} \overline{U}_{rect}(y_0^{(j)}, \delta_1(0)/2\sqrt{n-1})$ and $U_{rect}(y_0^{(p)}, \delta_1(0)/2\sqrt{n-1}) \cap U_{rect}(y_0^{(q)}, \delta_1(0)/2\sqrt{n-1}) = \emptyset$ for $1 \leq p \neq q \leq k_0 \geq 1$, where $U_{rect}(y^*, r) = \{y \in \mathbb{R}^{n-1} | \; |y_i - y_i^*| < r, i = 1, 2, \ldots, n-1\}$ denotes a hypercube of $\mathbb{R}^{n-1}$ where y_i is the i-th component of y and so is y_i^* for y^*.

Before processing the covers, we state a simple fact on geometry of $\mathbb{R}^{n-1}$ space. Let $\rho > 0$ and $S = \{z_h = (\rho i_1, \ldots, \rho i_{n-1}) | i_j \in \mathbb{Z}, j = 1, 2 \ldots, n-1\}$. Let $\{U_{rect}(z_h, \rho) | z_h \in S\}$ be a set of hypercubes of $\mathbb{R}^{n-1}$. Then the length of boundary of $U_{rect}(z_h, \rho)$ along any axis of the coordinate of $\mathbb{R}^{n-1}$ is just 2ρ. Let F be the set of points in $\mathbb{R}^{n-1}$ that starts from any $z_h \in S$ pointing to one direction of a fixed axis of $\mathbb{R}^{n-1}$. Then it is seen that any point in F belongs to at most two hypercubes of $\{U_{rect}(z_h, \rho)\}$. Since $\mathbb{R}^{n-1}$ has $n-1$ number of axes, any point of $\mathbb{R}^{n-1}$ belongs to at most 2^{n-1} number of hypercubes of $\{U_{rect}(z_h, \rho)\}$.

Let $\mathcal{X}(0) = \{y_0^{(j)} | U_{rect}(y_0^{(j)}, \delta_1(0)/2\sqrt{n-1}) \cap \Omega_{n-1} \neq \emptyset\}$. Obviously, $\{U_{rect}(y_0^{(j)}, \delta_1(0)/\sqrt{n-1}) | y_0^{(j)} \in \mathcal{X}(0)\}$ is a cover of Ω_{n-1}. Taking $\rho = \delta_1(0)/\sqrt{n-1}$ as that in above paragraph, we see that there are at most 2^{n-1} number of such hypercubes such that

$$y \in \bigcap_{j=1}^{2^{n-1}} U_{rect}\left(y_0^{(j)}, \frac{\delta_1(0)}{\sqrt{n-1}}\right), y_0^{(j)} \in \mathcal{X}(0), \; \forall y \in \Omega_{n-1} \quad (56)$$

and hence

$$\sum_{j=1}^{N(0)} \text{meas}\left(\Omega_{n-1} \cap U_{rect}\left(y_0^{(j)}, \frac{\delta_1(0)}{\sqrt{n-1}}\right)\right)$$

$$\leq 2^{n-1} \text{meas}(\Omega_{n-1}) \quad (57)$$

where $N(0) = \#\mathcal{X}(0)$. By the continuity and non-increasing property of δ_1, for all sufficiently small $t > 0$

$$\sum_{j=1}^{N(0)} \text{meas}\left(\Omega_{n-1} \cap U_{rect}\left(y_0^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}}\right)\right)$$

$$\leq 2^{n-1} \text{meas}(\Omega_{n-1}).$$

Since $\lim_{t\to\infty}\delta(t)=0$, there exists a $t^*>0$ such that for all $t>t^*$, $\{\Omega_{n-1}\cap U_{rect}(y_0^{(j)},\delta_1(t)/\sqrt{n-1})\}_{j=1}^{N(0)}$ cannot cover Ω_{n-1}. Let

$$t_1=\inf\left\{t>0\left|\left\{\Omega_{n-1}\cap U_{rect}\left(y_0^{(j)},\frac{\delta_1(t)}{\sqrt{n-1}}\right)\right\}_{j=1}^{N(0)}\right.\right.$$
$$\left.\text{cannot cover }\Omega_{n-1}\right\}.$$

Note that the boundary of each $U_{rect}(y_0^{(j)},\delta_1(t_1)/\sqrt{n-1})$ $(0\le j\le N(0))$ consists of $2(n-1)$ number of boundary hypercubes $\{U_{rect}^{j,k}\}_{k=1}^{2(n-1)}\subset\mathbb{R}^{n-2}$. Along any oriented coordinate axis direction of $\mathbb{R}^{n-2}$, the length of each hypercube $U_{rect}^{j,k}$ is $2\delta_1(t_1)/\sqrt{n-1}$. We partition symmetrically each hypercube $U_{rect}^{j,k}$ into 2^{n-2} number of hypercubes $\{U_{rect}^{j,k,l}\}_{l=1}^{2^{n-2}}\subset\mathbb{R}^{n-2}$ so that the boundary length of each $U_{rect}^{j,k,l}$ along any oriented coordinate axis direction of $\mathbb{R}^{n-2}$ is just $\delta_1(t_1)/\sqrt{n-1}$. Let $\{y_1^{(j)}\}_{j=N(0)+1}^{N(t_1)}$ be all vertices of all $U_{rect}^{j,k,l}$ for $0\le j\le N(0)$, $1\le k\le 2(n-2)$, $1\le l\le 2^{n-2}$. There vertices are considered as points of $\mathbb{R}^{n-1}$.

In this way, we have

$$\Gamma_1\subset\left\{U_{rect}\left(y_1^{(j)},\frac{\delta_1(t_1)}{\sqrt{n-1}}\right)\right\}_{j=N(0)+1}^{N(t_1)}$$
$$\bigcup\left\{U_{rect}\left(y_0^{(j)},\frac{\delta_1(t_1)}{\sqrt{n-1}}\right)\right\}_{j=1}^{N(0)}. \quad (58)$$

For notation simplicity, we still denote by $\{y^{(j)}\}_{j=1}^{N(t_1)}=\{y_0^{(j)}\}_{j=1}^{N(0)}\cup\{y_1^{(j)}\}_{j=N(0)+1}^{N(t_1)}$. Same as (56) and (57), we have

$$y\in\bigcap_{j=1}^{2^{n-1}}U_{rect}\left(y^{(j)},\frac{\delta_1(t_1)}{\sqrt{n-1}}\right),\,y^{(j)}\in\mathcal{X}(t_1),$$
$$\forall\,y\in\Omega_{n-1}\quad (59)$$

and

$$\sum_{j=1}^{N(t_1)}\text{meas}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(j)},\frac{\delta_1(t_1)}{\sqrt{n-1}}\right)\right)$$
$$\le 2^{n-1}\,\text{meas}\,(\Omega_{n-1})\quad (60)$$

where $\mathcal{X}(t_1)=\mathcal{X}(0)\cup\{y^{(j)}\}_{j=N(0)+1}^{N(t_1)}$, $N(t_1)=\#\mathcal{X}(t_1)$. By induction, there exist $\{t_i\}_{i=2}^{\infty}$ and $\{y^{(j)}\}_{j=1}^{N(t_i)}$ such that

$$y\in\bigcap_{j=1}^{2^{n-1}}U_{rect}\left(y^{(j)},\frac{\delta_1(t_i)}{\sqrt{n-1}}\right),\,y^{(j)}\in\mathcal{X}(t_i),\,\forall\,y\in\Omega_{n-1}\quad (61)$$

and

$$\sum_{j=1}^{N(t_i)}\text{meas}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(j)},\frac{\delta_1(t_i)}{\sqrt{n-1}}\right)\right)$$
$$\le 2^{n-1}\,\text{meas}\,(\Omega_{n-1})\quad (62)$$

where $\mathcal{X}(t_i)$ is defined iteratively by

$$\mathcal{X}(t_{i+1})=\mathcal{X}(t_i)\cup\left\{y^{(j)}\right\}_{j=N(t_i)+1}^{N(t_{i+1})},\,t_0=0,\,i=0,1,2,\cdots.\quad (63)$$

By this construction, we see that the bounded measure cover

$$\bigcup_{j=1}^{N(t_i)}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(j)},\frac{\delta_1(t_i)}{\sqrt{n-1}}\right)\right)=\Omega_{n-1}\quad (64)$$

is a discrete series of cover which is independent of time t. Now, we relate this cover with time t by setting

$$\mathcal{X}(t):=\mathcal{X}(t_i),t\in[t_i,t_{i+1}),\,N(t)=\#\mathcal{X}(t),$$
$$\lim_{t\to\infty}N(t)=\infty,\,i=0,1,2,\cdots.\quad (65)$$

Then, we obtain from (62) that for all $t\ge 0$

$$\bigcup_{j=1}^{N(t)}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(j)},\frac{\delta_1(t)}{\sqrt{n-1}}\right)\right)=\Omega_{n-1}\quad (66)$$

and

$$\sum_{j=1}^{N(t)}\text{meas}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(j)},\frac{\delta_1(t)}{\sqrt{n-1}}\right)\right)$$
$$\le 2^{n-1}\,\text{meas}\,(\Omega_{n-1}).\quad (67)$$

Let $x^{(i)}=(y^{(i)},\psi(y^{(i)}))\in\Gamma_1$ for $i=1,2,\ldots$ Then,

$$x\in\bigcup_{i=1}^{N(t)}U\left(x^{(i)},\delta(t)\right),\,\forall\,x\in\Gamma_1,t\ge 0\quad (68)$$

and by (67)

$$\sum_{i=1}^{N(t)}\text{meas}\left(\overline{\Gamma_1}\cap U\left(x^{(i)},\delta(t)\right)\right)$$
$$=\sum_{i=1}^{N(t)}\int_{\Omega_{n-1}\cap U((y^{(i)},0),\delta(t))}$$
$$\sqrt{1+\psi_{x_1}^2+\psi_{x_2}^2+\ldots+\psi_{x_{n-1}}^2}\,dx_1 dx_2\ldots dx_{n-1}$$
$$\le\sum_{i=1}^{N(t)}\int_{\Omega_{n-1}\cap U_{rect}(y^{(i)},\delta(t))}$$
$$\sqrt{1+\psi_{x_1}^2+\psi_{x_2}^2+\ldots+\psi_{x_{n-1}}^2}\,dx_1 dx_2\ldots dx_{n-1}$$
$$\le\sum_{i=1}^{N(t)}C_2(\Gamma_1)\text{meas}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(i)},\delta(t)\right)\right)$$
$$\le\sum_{i=1}^{N(t)}C_2^n(\Gamma_1)(n-1)^{\frac{n-1}{2}}$$
$$\times\text{meas}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(i)},\frac{\delta_1(t)}{\sqrt{n-1}}\right)\right)$$
$$\le C_2^n(\Gamma_1)(n-1)^{\frac{n-1}{2}}2^{n-1}\,\text{meas}\,(\Omega_{n-1})$$
$$\le C_2^n(\Gamma_1)(n-1)^{\frac{n-1}{2}}2^{n-1}$$
$$\times\int_{\Omega_{n-1}}\sqrt{1+\psi_{x_1}^2+\psi_{x_2}^2+\ldots+\psi_{x_{n-1}}^2}\,dx_1 dx_2\ldots dx_{n-1}$$
$$=C_0(\Gamma_1)(n-1)^{\frac{n-1}{2}}2^{n-1}\,\text{meas}\,(\overline{\Gamma_1}),\,\forall\,t\ge 0\quad (69)$$

where

$$C_0(\Gamma_1)=C_2^n(\Gamma_1)$$
$$=\left\|\sqrt{1+\psi_{x_1}^2+\psi_{x_2}^2+\ldots+\psi_{x_{n-1}}^2}\right\|_{C(\Omega_{n-1})}^n.$$

In the derivation of (69), we used a trivial fact in the space $\mathbb{R}^{n-1}$ that

$$\text{meas}\left(U_{rect}\left(y^{(i)}, \delta(t)\right)\right)$$

$$= C_2^{n-1}(\Gamma_1)(n-1)^{\frac{n-1}{2}}\text{meas}\left(U_{rect}\left(y^{(i)}, \frac{\delta_1(t)}{\sqrt{n-1}}\right)\right)$$

and hence

$$\text{meas}\left(\Omega_{n-1} \cap U_{rect}\left(y^{(i)}, \delta(t)\right)\right)$$

$$\leq C_2^{n-1}(\Gamma)(n-1)^{\frac{n-1}{2}}\text{meas}\left(\Omega_{n-1} \cap U_{rect}\left(y^{(i)}, \frac{\delta_1(t)}{\sqrt{n-1}}\right)\right)$$

since Ω_{n-1} is supposed to be a hypercube of $\mathbb{R}^{n-1}$. Another fact that we used in the derivation of (69) is that for a $(n-1)$-dimensional surface $S = \psi(U)$, $U \subset \mathbb{R}^{n-1}$, $\psi(x) = (x', \psi^0(x'))$, $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $\psi^0 \in C^1(\mathbb{R}^{n-1})$,

$$\text{meas}(S) = \int_U \sqrt{1 + |D\psi^0|^2}dx_1 dx_2 \ldots dx_{n-1}$$

where $D\psi^0 = (\psi_{x_1}^0, \psi_{x_2}^0, \ldots, \psi_{x_{n-1}}^0)$ ([16, p. 101–102]). Combining (68) and (69) gives the required result.   ∎

Proof of Lemma 2.2: Let

$$\tilde{y}_i(t) = r(t)\left[\hat{y}_i(t) - y_i(t)\right],$$

$$\tilde{d}_i(t) = \hat{d}_i(t) - d\left(\xi_i(t), t\right), \quad i = 1, 2, \ldots \quad (70)$$

be the errors. Then, by (36) and (37), $(\tilde{y}, \tilde{d})$ satisfies

$$\frac{d}{dt}\begin{pmatrix} \tilde{y}_i(t) \\ \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}$$

$$= r(t)\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{y}_i(t) \\ \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{r'(t)}{r(t)}\tilde{y}_i(t) \\ -\frac{d}{dt}\left(d\left(\xi_i(t),t\right)\int_{\Gamma_1} g_i^t(x)dx\right) \end{pmatrix}, \quad i = 1, 2, \ldots. \quad (71)$$

We denote by

$$(V_i(t) := \begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}$$

$$\times P\begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}^\top, \quad i = 1, 2, \ldots \quad (72)$$

where $\{\tilde{y}_i, \tilde{d}_i\}$ is the solution of (71) and the positive definite matrix P is the solution of the following Lyapunov equation:

$$F^\top P + PF = -I_{2\times2}, \ F = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is easy to verify that

$$\lambda_{\min}(P)\|(y_1, y_2)\|^2 \leq V(y_1, y_2) = (y_1, y_2)P(y_1, y_2)^\top$$
$$\leq \lambda_{\max}(P)\|(y_1, y_2)\|^2, \ \forall (y_1, y_2) \in \mathbb{R}^2 \quad (73)$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of P, respectively. Finding the derivative of V_i along the solution of (71) to yield

$$\dot{V}_i(t) = \begin{pmatrix} \dot{\tilde{y}}_i(t), \dot{\tilde{d}}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}$$

$$\times P\begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}^\top$$

$$+ \begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}$$

$$\times P\begin{pmatrix} \dot{\tilde{y}}_i(t), \dot{\tilde{d}}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}^\top$$

$$= r(t)\begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}$$

$$\times [PF + F^\top P]\begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}^\top$$

$$+ \begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}$$

$$\times P\begin{pmatrix} \frac{\dot{r}(t)}{r(t)}\tilde{y}_i(t), -\frac{d}{dt}\left(d\left(\xi_i(t),t\right)\int_{\Gamma_1} g_i^t(x)dx\right) \end{pmatrix}^\top$$

$$+ \begin{pmatrix} \frac{\dot{r}(t)}{r(t)}\tilde{y}_i(t), -\frac{d}{dt}\left(d\left(\xi_i(t),t\right)\int_{\Gamma_1} g_i^t(x)dx\right) \end{pmatrix}$$

$$\times P\begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}^\top$$

$$\leq -r(t)\left\|\begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}\right\|^2$$

$$+ N_1\left\|\begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}\right\|^2$$

$$+ N_2\left\|\begin{pmatrix} \tilde{y}_i(t), \tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \end{pmatrix}\right\| \quad (74)$$

where N_1 and N_2 are two positive constants. In the last step of (74), (34) was used. This together with (73) gives

$$\frac{dV_i(t)}{dt} \leq -\frac{r(t)}{\lambda_{\max}(P)}V_i(t) + \frac{N_1}{\lambda_{\min}(P)}V_i(t)$$

$$+ \frac{N_2}{\sqrt{\lambda_{\min}(P)}}\sqrt{V_i(t)}. \quad (75)$$

Since $\lim_{t\to\infty} r(t) = +\infty$, there exists $t_0 > 0$ such that $r(t) > (2\lambda_{\max}(P)/\lambda_{\min}(P))N_1$ for all $t \geq t_0$. This together with (75) shows that

$$\frac{d\sqrt{V_i(t)}}{dt} \leq -\frac{1}{4\lambda_{\max}(P)} r(t)\sqrt{V_i(t)}$$
$$+ \frac{N_2}{2\sqrt{\lambda_{\min}(P)}}, \ \forall t \geq t_0 \quad (76)$$

and hence yields

$$0 \leq \lim_{t\to\infty} \sqrt{V_i(t)}$$

$$\leq \lim_{t\to\infty} \left(e^{-\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma} \sqrt{V_i(t_0)} \right.$$

$$+ \frac{N_2}{2\sqrt{\lambda_{\min}(P)}} \frac{\int_{t_0}^{t} e^{\int_{t_0}^{s} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma} ds}{e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}} \right)$$

$$= \lim_{t\to\infty} \frac{2N_2\lambda_{\max}(P)}{\sqrt{\lambda_{\min}(P)}} \frac{e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}}{r(t) e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}} = 0. \quad (77)$$

In the last step of (77), the L'Hospital rule and assumption (26) were used. Furthermore, by (33), (35), and the fact $\|\nabla f_i^t\|_{L^2(\Omega)} \leq C\|f\|_{H^2(\Omega)} \leq C\|g_i^t\|_{H^{1/2}(\Gamma_1)} \leq C\|g_i^t\|_{H^1(\Gamma_1)} \leq C\delta^{n-2}(t) \leq C\delta^{n-2}(0)$ by the construction (29) ([21]) for some constant $C > 0$, we may suppose $|\sqrt{V_i(t_0)}| \leq C$ for all $i \geq 1$ for some constant $C > 0$ that is independent of i. Then, we obtain

$$\lim_{t\to\infty} V_i(t) = 0 \quad (78)$$

uniformly for $i = 1, 2, \cdots$. This together with (73), (70), (72), and the assumption (26) leads to the second limit in (38). Furthermore, using (73) once again, one obtains

$$\left| \tilde{d}_i(t) \int_{\Gamma_1} g_i^t(x)dx \right| \leq \frac{\sqrt{V_i(t)}}{\sqrt{\lambda_{\min}(P)}}$$

$$\leq \frac{C}{\sqrt{\lambda_{\min}(P)} e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}}$$

$$+ \frac{N_2}{\lambda_{\min}(P)} \frac{\int_{t_0}^{t} e^{\int_{t_0}^{s} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma} ds}{e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}}, \ \forall t \geq t_0,$$
$$i = 1, 2, \ldots. \quad (79)$$

Since

$$\frac{1}{2^{n-1}C_1(\Gamma_1)} \Pi_{n-1}\delta^{n-1}(t) \leq \int_{\Gamma_1} g_i^t(x)dx$$

$$\leq C_1(\Gamma_1)\Pi_{n-1}\delta^{n-1}(t) \quad (80)$$

where Π_{n-1} is the unit volume of $\mathbb{R}^{n-1}$ and $C_1(\Gamma_1)$ is a positive constant which depends on Γ_1 only, we have, for all $t \geq t_0$ and $i = 1, 2, \ldots$, that

$$\left| \tilde{d}_i(t) \right| \leq 2^{n-1} C_1(\Gamma_1) \left(\frac{C}{\sqrt{\lambda_{\min}(P)}\delta^{n-1}(t) e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}} \right.$$

$$\left. + \frac{N_2}{\lambda_{\min}(P)} \frac{\int_{t_0}^{t} e^{\int_{t_0}^{s} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma} ds}{\delta^{n-1}(t) e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}} \right). \quad (81)$$

We claim that $\tilde{d}_i(t) \to 0$ as $t \to \infty$. To this purpose, it suffices to show the convergence of the second term of the right-hand side of (81) since the first term is less than the second term up to a constant as $t \to \infty$. Using the L'Hospital rule and assumption (25), we have

$$\lim_{t\to\infty} \frac{\int_{t_0}^{t} e^{\int_{t_0}^{s} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma} ds}{\delta^{n-1}(t) e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}}$$

$$= \lim_{t\to\infty} \frac{e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}}{\left(\frac{r(t)\delta^{n-1}(t)}{4\lambda_{\max}(P)} + (n-1)\delta^{n-2}\dot{\delta}(t) \right) e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}}$$

$$= \lim_{t\to\infty} \frac{1}{\frac{r(t)\delta^{n-1}(t)}{4\lambda_{\max}(P)} + (n-1)\delta^{n-2}\dot{\delta}(t)} = 0.$$

This together with (81) proves the first limit of (38). The proof is complete. ∎

Proof of Lemma 2.3: Define

$$\tilde{d}(x,t) = \hat{d}(x,t) - d(x,t), \ \forall x \in \Gamma_1, t \geq 0. \quad (82)$$

Since $d(\cdot, t)$ is Hölder continuous with index $\alpha \in (0,1]$ and satisfies $|d(x',t) - d(x'',t)| \leq K(t)|x' - x''|^\alpha$. For the given $\varepsilon(t) > 0$, since $\delta(t) = (\varepsilon(t)/K(t))^{1/\alpha} > 0$, we have $|d(x',t) - d(x'',t)| \leq 2^\alpha \varepsilon(t)$ as long as $|x' - x''| \leq 2\delta(t)$. Moreover, since $\xi_i : [0,\infty) \to \Gamma_1 \cap U(x^{(i)}, \delta(t))$ which is defined in (33), we have

$$\|d(\xi_i(t),t) - d(\cdot,t)\|_{L^2(\Gamma_1 \cap U(x_i, \delta(t)))}$$
$$\leq \text{meas}\left(\Gamma_1 \cap U\left(x^{(i)}, \delta(t)\right)\right) 2^\alpha \varepsilon(t), \ \forall t \geq 0. \quad (83)$$

This together with (81) yields

$$\left\| \tilde{d}(\cdot,t) \right\|_{L^2(\Gamma_1)}$$

$$\leq \sum_{i=1}^{N(t)} \left[\left\| \hat{d}(\cdot,t) - d(\xi_i(t),t) \right\|_{L^2(\Gamma_1 \cap U(x^{(i)}, \delta(t)))} \right.$$

$$\left. + \|d(\xi_i(t),t) - d(\cdot,t)\|_{L^2(\Gamma_1 \cap U(x^{(i)}, \delta(t)))} \right]$$

$$\leq \text{meas}(\Gamma_1) 2^{n-1} C_1(\Gamma_1)$$

$$\times \left(\frac{C}{\sqrt{\lambda_{\min}(P)}\delta^{n-1}(t) e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}} \right.$$

$$\left. + \frac{N_2}{\lambda_{\min}(P)} \frac{\int_{t_0}^{t} e^{\int_{t_0}^{s} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma} ds}{\delta^{n-1}(t) e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma}} \right)$$

$$+ C_0(\Gamma_1)(n-1)^{\frac{n-1}{2}} 2^{n-1} \text{meas}(\Gamma_1) 2^\alpha \varepsilon(t), \ \forall t \geq t_0 \quad (84)$$

where we used (69) and (81). Since $\lim_{t\to\infty}\varepsilon(t)=0$, applying the L'Hospital rule and assumption (25) to (84), we obtain

$$\lim_{t\to\infty}\left\|\tilde{d}(\cdot,t)\right\|_{L^2(\Gamma_1)}=0. \qquad (85)$$

This is just (40) by (82). The proof is complete. $\blacksquare$

Proof of Theorem 2.1: Using the error variables $(\tilde{y}_i,\tilde{d}_i)$ defined in (70), we can write the equivalent system of (42) as follows:

$$\begin{cases} w_{tt}(x,t)-\Delta w(x,t)=0, x\in\Omega, t>0, \\ w(x,t)|_{\Gamma_0}=0, \\ \frac{\partial w}{\partial\nu}(x,t)|_{\Gamma_1}=-kw_t(x,t)-\hat{d}(x,t)+d(x,t),\ t\geq0 \\ \dot{\tilde{y}}_i(t)=-r(t)\tilde{y}_i(t)+r(t)\tilde{d}_i(t)\int_{\Gamma_1}g_i(x)dx+\frac{\dot{r}(t)}{r(t)}\tilde{y}_i(t), \\ \frac{d}{dt}\left(\tilde{d}_i(t)\int_{\Gamma_1}g_i(x)dx\right)=-r(t)\tilde{y}_i(t) \\ \quad -\frac{d}{dt}\left(d(\xi_i(t),t)\int_{\Gamma_1}g_i(x)dx\right),\ i=1,2,\ldots. \end{cases} \qquad (86)$$

The "ODE part" in (86) has been shown in (38) and (40)through (70) to tend to zero as $t\to\infty$. Now we only need to consider the "w part" of system (86) which is rewritten as

$$\begin{cases} w_{tt}(x,t)-\Delta w(x,t)=0,\ x\in\Omega,\ t>0, \\ w(x,t)|_{\Gamma_0}=0,\ t\geq0, \\ \frac{\partial w}{\partial\nu}(x,t)|_{\Gamma_1}=-kw_t(x,t)-\tilde{d}(x,t),t\geq0. \end{cases} \qquad (87)$$

Exactly to (15), we write (87) as

$$\ddot{w}=-\tilde{A}w-kBB^*\dot{w}-B\tilde{d}\text{ in }\left[D(A^{\frac{1}{2}})\right]'. \qquad (88)$$

Owing to (40), for any given $\sigma>0$, we may suppose that $\|\tilde{d}(\cdot,t)\|_{L^2(\Gamma_1)}\leq\sigma$ for all $t>t_0$ for some $t_0>0$. Now, we write the solution of (88) as

$$\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}$$

$$=e^{\mathbb{A}t}\begin{pmatrix} w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix}+\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds$$

$$=e^{\mathbb{A}t}\begin{pmatrix} w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix}+e^{\mathbb{A}(t-t_0)}\int_0^{t_0} e^{\mathbb{A}(t_0-s)}\mathbb{B}\tilde{d}(s)ds$$

$$+\int_{t_0}^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds \qquad (89)$$

where $\mathbb{A}$ and $\mathbb{B}$ are defined in (17). The admissibility of $\mathbb{B}$ proved in Proposition 1.1 implies that

$$\left\|\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds\right\|_{\mathcal{H}}^2\leq C_t\|\tilde{d}\|_{L^2(0,t;L^2(\Gamma_1))}^2$$

$$\leq C_t t^2\|\tilde{d}\|_{L^\infty(0,\infty;L^2(\Gamma_1))}^2,\ \forall t>0$$

for some constant C_t that is independent of $\tilde{d}$. On the other hand, under the assumption of the theorem which is (4), it

is known that $e^{\mathbb{A}t}$ is exponentially stable ([20, p. 668]). By Remark 2.6 of [28], we have

$$\left\|\int_{t_0}^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds\right\|_{\mathcal{H}}\leq\left\|\int_{t_0}^t e^{\mathbb{A}(t-s)}\mathbb{B}(0\underset{t_0}{\diamond}\tilde{d})(s)ds\right\|_{\mathcal{H}}$$

$$\leq L\|\tilde{d}\|_{L^\infty(t_0,\infty;L^2(\Gamma_1))}\leq L\sigma$$

where L is a constant that is independent of $\tilde{d}$, and

$$(u\underset{\tau}{\diamond}v)(t)=\begin{cases} u(t),\ & 0\leq t\leq\tau, \\ v(t),\ & t>\tau. \end{cases}$$

Suppose that $\|e^{\mathbb{A}t}\|\leq L_0 e^{-\omega t}$ for some $L_0,\omega>0$, we have

$$\left\|\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}\right\|_{\mathcal{H}}\leq L_0 e^{-\omega t}\left\|\begin{pmatrix} w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix}\right\|_{\mathcal{H}}$$

$$+L_0 C_{t_0} t_0^2 e^{-\omega(t-t_0)}\|\tilde{d}\|_{L^\infty(0,t_0;L^2(\Gamma_1))}+L\sigma. \qquad (90)$$

Passing to the limit as $t\to\infty$, we finally obtain

$$\lim_{t\to\infty}\left\|\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}\right\|_{\mathcal{H}}\leq L\sigma. \qquad (91)$$

This proves that the solution of (87) satisfies

$$\lim_{t\to\infty}\left\|\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}\right\|_{\mathcal{H}}$$

$$=\lim_{t\to\infty}\int_\Omega\left[|\nabla w(x,t)|^2+|w_t(x,t)|^2\right]dx=0. \qquad (92)$$

Finally, it follows from (70) that

$$\hat{y}_i(t)=\frac{\tilde{y}_i(t)}{r(t)}+y_i(t),$$

$$y_i(t)=\int_\Omega\left[\nabla w(x,t)\nabla f_i^t(x)+w_t(x,t)g_i^t(x)\right]dx,\ i=1,2,\cdots. \qquad (93)$$

Since $\tilde{y}_i(t)/r(t)\to 0$ as $t\to\infty$ by (26), (38), and (70), it suffices to show the convergence of y_i in (93). This follows from:

$$\left|\int_\Omega w_t(x,t)g_i^t(x)dx\right|\leq\int_\Omega|w_t(x,t)|\,dx$$

$$\leq\|w_t(\cdot,t)\|_{L^2(\Omega)}\left(\text{meas}\left(\Omega\right)\right)^{\frac{1}{2}}\to 0\text{ as }t\to\infty$$

by $|g_i^t|\leq 1$ from (29) and (92), and

$$\left|\int_\Omega\nabla w(x,t)\nabla f_i^t(x)\right|$$

$$\leq\|\nabla w(\cdot,t)\|_{L^2(\Omega)}\left\|\nabla f_i^t(x)\right\|_{L^2(\Omega)}\to 0\text{ as }t\to\infty$$

from (92) and the fact $\|\nabla f_i^t\|_{L^2(\Omega)} \leq C\|f\|_{H^2(\Omega)} \leq C\|g_i^t\|_{H^{1/2}(\Gamma_1)} \leq C\|g_i^t\|_{H^1(\Gamma_1)} \leq C\delta^{n-2}(t) \leq C\delta^{n-2}(0)$ by the construction (29) ([21]) for some constant $C > 0$. Therefore

$$\lim_{t\to\infty} \hat{y}_i(t) = 0 \qquad (94)$$

uniformly for i. Combining (40), (92), and (94), we get (43). The proof is thus complete. ∎

Proof of Remark 2.1: We only give a sketch of the proof. From the proof of Theorem 2.1, we see that to arrive (45), it suffices to show that

$$\limsup_{t\to\infty} \left| \tilde{d}_i(t) \right| \leq \widetilde{C}\kappa \qquad (95)$$

where $\tilde{d}_i$ is defined by (70) and $\widetilde{C}$ is a constant independent of i. Similar to equations from (74) to (81), one can obtain the following estimation that is subtler than estimations (74) and (81):

$$
\begin{aligned}
&\left| \tilde{d}_i(t) \right| \\
&\leq \frac{2^{n-1}C_1(\Gamma_1)}{\Pi_{n-1}} \left(\frac{C}{\sqrt{\lambda_{\min}(P)}\delta^{n-1}(t)e^{\frac{t-t_0}{4\lambda_{\max}(P)\kappa}}} \right. \\
&\left. + \frac{N_2}{\lambda_{\min}(P)} \frac{\int_{t_0}^t \left| \frac{d}{ds}\left(d(\xi_i(s),s)\int_{\Gamma_1} g_i^s(x)dx \right) \right| e^{\frac{s-t_0}{4\lambda_{\max}(P)\kappa}}ds}{\delta^{n-1}(t)e^{\frac{t-t_0}{4\lambda_{\max}(P)\kappa}}} \right).
\end{aligned}
$$
$$ (96) $$

By condition (24), we can choose $\delta(t) = 1/(1+t)$. Since

$$(1+t)^{n-1}e^{-\frac{t-t_0}{4\lambda_{\max}(P)\kappa}} \leq C_1\kappa, \quad \forall t \geq t_0 \qquad (97)$$

for some constant $C_1 > 0$ independent of κ, it follows from (2), (29), and (33) that

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{\int_{t_0}^t \left| \frac{d}{ds}\left(d(\xi_i(s),s)\int_{\Gamma_1} g_i^s(x)dx \right) \right| e^{\frac{s-t_0}{4\lambda_{\max}(P)\kappa}}ds}{\delta^{n-1}(t)e^{\frac{t-t_0}{4\lambda_{\max}(P)\kappa}}} \\
&= \limsup_{t\to\infty} \frac{\left| \frac{d}{ds}\left(d(\xi_i(s),s)\int_{\Gamma_1} g_i^s(x)dx \right) \right|}{(n-1)\delta^{n-2}(t)\dot{\delta}(t) + \frac{\delta^{n-1}(t)}{4\lambda_{\max}(P)\kappa}} \\
&\leq C_2 \max\left\{ \|d\|_{L^\infty(0,\infty;C(\Gamma_1))}, \|d_t\|_{L^\infty(0,\infty;C(\Gamma_1))} \right\} \\
&\quad \times \limsup_{t\to\infty} \frac{\delta^{n-1} + \left|\dot{\delta}(t)\right|\delta^{n-2}(t)}{(n-1)\delta^{n-2}(t)\dot{\delta}(t) + \frac{\delta^{n-1}(t)}{4\lambda_{\max}(P)\kappa}} \\
&= C_2 \max\left\{ \|d\|_{L^\infty(0,\infty;C(\Gamma_1))}, \|d_t\|_{L^\infty(0,\infty;C(\Gamma_1))} \right\} \\
&\quad \times \limsup_{t\to\infty} \frac{1 + \frac{|\dot{\delta}(t)|}{\delta(t)}}{(n-1)\frac{\dot{\delta}(t)}{\delta(t)} + \frac{1}{4\lambda_{\max}(P)\kappa}} \\
&\leq C_2 \max\left\{ \|d\|_{L^\infty(0,\infty;C(\Gamma_1))}, \|d_t\|_{L^\infty(0,\infty;C(\Gamma_1))} \right\}\kappa \quad (98)
\end{aligned}
$$

where $C_2 > 0$ is independent of κ and d. Combining (96), (97), and (98) gives

$$
\begin{aligned}
&\limsup_{t\to\infty} \left| \tilde{d}_i(t) \right| \\
&\leq C \max\left\{ \|d\|_{L^\infty(0,\infty;C(\Gamma_1))}, \|d_t\|_{L^\infty(0,\infty;C(\Gamma_1))} \right\}\kappa. \quad (99)
\end{aligned}
$$

This gives (95). ∎

IV. NUMERICAL SIMULATION

In this section, we present some numerical simulations for Example 1.1 for illustration. The purpose is twofold. The first is to verify the theoretical results and the second is to look at the peaking value reduction by the time varying gain approach. For simplicity, we just take dimension $n = 2$. The system is described by

$$
\begin{cases}
w_{tt}(x,t) - \Delta w(x,t) = 0, \ x \in \Omega, \ t > 0, \\
w(x,t)|_{\Gamma_0} = 0, \\
\frac{\partial w}{\partial \nu}(x,t)|_{\Gamma_1} = -kw_t(x,t) - \hat{d}(x,t) + d(x,t), \\
\hat{y}_i(t) = y_{2i}(t) - k\int_{\Gamma_1} w_t(x,t)g_i^t(x)dx \\
\qquad\quad - r(t)\left[\hat{y}_i(t) - y_i(t)\right], \\
\frac{d}{dt}\left(\hat{d}_i(t) \int_{\Gamma_1} g_i^t(x)dx \right) = -r^2(t)\left[\hat{y}_i(t) - y_i(t)\right], \\
\qquad\qquad i = 1, 2, \ldots
\end{cases}
$$
$$ (100) $$

where same to Example 1.1, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 | 1 < x_1^2 + x_2^2 < 4\}$, $\Gamma_0 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$, $\Gamma_1 = \Gamma \setminus \Gamma_0$, g_i^t is defined by (29), y_i and y_{2i} are defined by (35), ξ_i is defined by (33), and $\hat{d}$ is defined by (39), r is the time varying gain in (26). For numerical computations, we take parameter $k = 3$, disturbance $d(x,t) = \sin(x_1 t)$, and the following initial values:

$$
\begin{cases}
w(x_1, x_2, 0) = \left(x_1^2 + x_2^2 - 1\right)^2 \\
\times \left(\frac{4x_1^3}{(x_1^2+x_2^2)^{1.5}} - \frac{3x_1}{\sqrt{x_1^2+x_2^2}} \right), \ 1 \leq x_1^2 + x_2^2 \leq 4, \\
w_t(x_1, x_2, 0) = 9\sin\left(2\sqrt{x_1^2+x_2^2} - 2\right) \\
\times \left(\frac{3x_2}{\sqrt{x_1^2+x_2^2}} - \frac{4x_2^3}{(x_1^2+x_2^2)^{1.5}} \right), \ 1 \leq x_1^2 + x_2^2 \leq 4, \\
\hat{y}_i(0) = y_i(0) + 2, \ \hat{d}_i(0) = d_i(0) + 1.5, \ i = 1, 2, \ldots.
\end{cases}
$$
$$ (101) $$

Since the spatial domain consists of a two-dimensional annulus, we can more easily solve (100) in the polar coordinate (γ, θ) and then convert back to the original coordinate for some figures if necessary. Under the polar coordinate, system (100) can be written as

$$
\begin{cases}
\frac{\partial^2 w(\gamma,\theta,t)}{\partial t^2} - \frac{\partial^2 w(\gamma,\theta,t)}{\partial \gamma^2} - \frac{1}{\gamma}\frac{\partial w(\gamma,\theta,t)}{\partial \gamma} \\
\quad - \frac{1}{\gamma^2}\frac{\partial^2 w(\gamma,\theta,t)}{\partial \theta^2} = 0, \ 1 < \gamma < 2, \ 0 < \theta < 2\pi, \ t > 0, \\
w(1, \theta, t) = 0, \ 0 \leq \theta \leq 2\pi, \ t \geq 0, \\
\frac{\partial w(2,\theta,t)}{\partial \gamma} = -k\frac{\partial w(2,\theta,t)}{\partial t} - \hat{d}(2\cos(\theta), 2\sin(\theta), t) \\
\quad + d(2\cos(\theta), 2\sin(\theta), t), \ 0 < \theta < 2\pi, \ t > 0, \\
\dot{\hat{y}}_i(t) = -r(t)\tilde{y}_i(t) + r(t)\tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx + \frac{\dot{r}(t)}{r(t)}\tilde{y}_i(t), \\
\frac{d}{dt}\left(\tilde{d}_i(t)\int_{\Gamma_1} g_i^t(x)dx \right) = -r(t)\tilde{y}_i(t) \\
\quad - \frac{d}{dt}\left(d(\xi_i(t),t)\int_{\Gamma_1} g_i^t(x)dx \right), \ i = 1, 2, \ldots
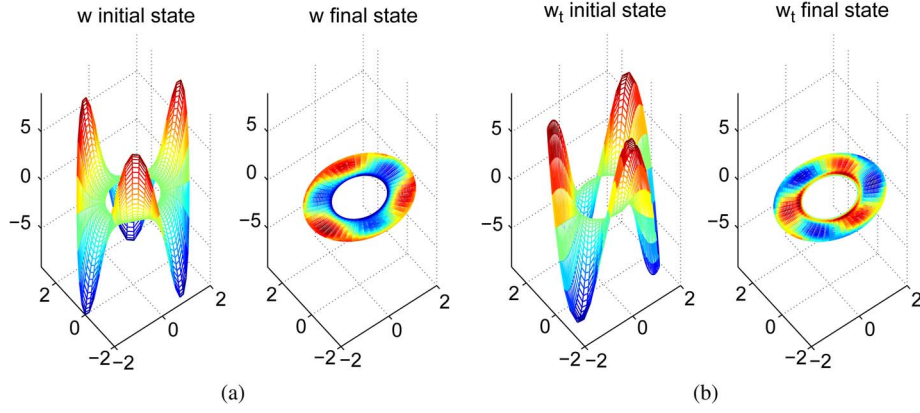\end{cases}
$$
$$ (102) $$

Fig. 1.   (a) Displacement w at initial time $t = 0$ and time $t = 15$. (b) Velocity w_t at initial time $t = 0$ and final time $t = 15$. The initial state and state at $t = 15$ of system (100) with $d(t) = \sin(x_1 t)$ (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).
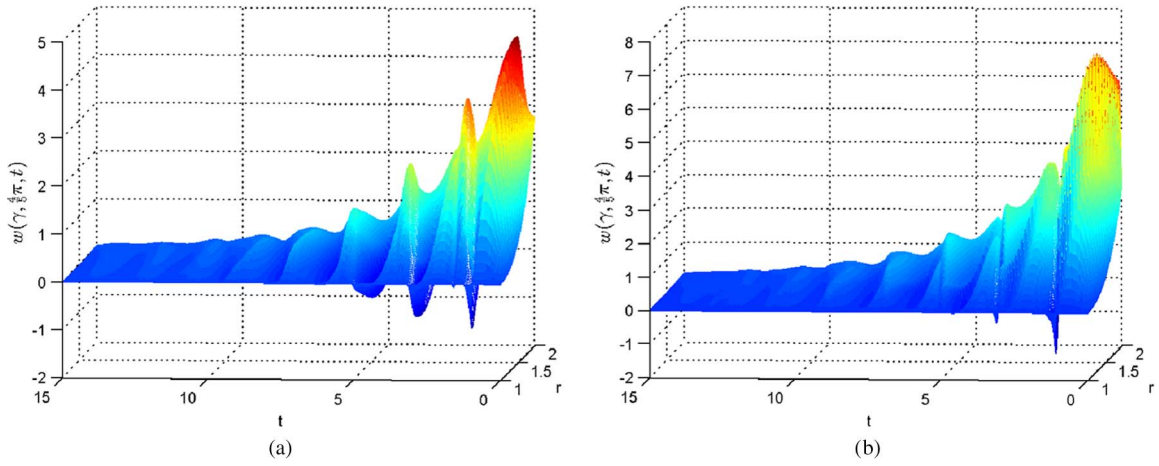

Fig. 2.   The evolution of $w(\gamma, (4/5)\pi, t)$ under the polar coordinate with both time varying gain and constant gain (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article). (a) The displacement $w(\gamma, (4/5)\pi, t)$ with the time varying gain. (b) The displacement $w(\gamma, (4/5)\pi, t)$ with the constant gain.

where we still use w to denote the state under the polar coordinate for notation simplicity, which is clear from the context, and $\tilde{y}_i$ and $\tilde{d}_i$ are defined by (70). The corresponding initial value (101) is transformed into

$$
\begin{cases}
w(\gamma, \theta, 0) = (\gamma^2 - 1)^2 \cos(3\theta), \ 1 \leq \gamma \leq 2, \ 0 \leq \theta \leq 2\pi, \\
w_t(\gamma, \theta, 0) = 9 \sin(2\gamma - 2) \sin(3\theta), \ 1 \leq \gamma \leq 2, \ 0 \leq \theta \leq 2\pi, \\
\tilde{y}_i(0) = 2r(0), \ \tilde{d}_i(0) = 1.5, \ i = 1, 2, \ldots .
\end{cases}
\tag{103}
$$

The backward Euler method in time and the Chebvshev spectral method for polar variables are used to discretize system (102). Here, we take the grid size $r_N = 30$ for γ, the grid size $\theta_N = 50$ for θ, and the time step $dt = 5 \times 10^{\sim -4}$. The time varying gain function r is taken as (see Remark 2.1)

$$
r(t) = \begin{cases}
e^{5t}, & t \leq \frac{\log(30)}{5}, \\
30, & t > \frac{\log(30)}{5}.
\end{cases}
\tag{104}
$$

It is seen that r grows slowly from the small value in the beginning to its maximum value $r = 30$ which is used as the constant gain in our numerical simulations. The numerical algorithm is programmed by Matlab ([26]) and the numerical results are plotted in Figs. 2–5.

Fig. 1(a) and (b) display the displacement w and the velocity w_t at the initial time $t = 0$ and the time $t = 15$, respectively. It is seen that the convergence for both w and w_t is very satisfactory.

To compare the effects of the time varying gain (104) and the constant gain $r = 30$, we plot $w(\gamma, (4/5)\pi, t)$ and $w_t(\gamma, (4/5)\pi, t)$ ($(4/5)\pi$ has no speciality. It can be any angle) in the polar coordinate for system (102) in Figs. 2 and 3, respectively, with both the time varying gain (Figs. 2(a) and 3(a)) and the constant gain (Figs. 2(b) and 3(b)). It is clearly seen that in both cases, the convergence is fast and satisfactory. The price is that the convergence with the time varying gain is slightly slower than the convergence with the constant gain. This is also observed in the succeeding Figs. 4 and 5 for the disturbance tracking.

Fig. 4 plots the tracking errors for the disturbance where Fig. 4(a) is with the time varying gain (104) and Fig. 4(b) is with the constant gain $r = 30$. It is clearly seen from these figures that the peaking value from Fig. 4(b) is dramatically reduced by the time varying gain in Fig. 4(a). This is the biggest advantage of the application of the time varying gain compared with the constant gain in existing literature [10]–[12]. This is also a remarkable property of the ADRC in dealing with the disturbance. We actually do not need much high gain for
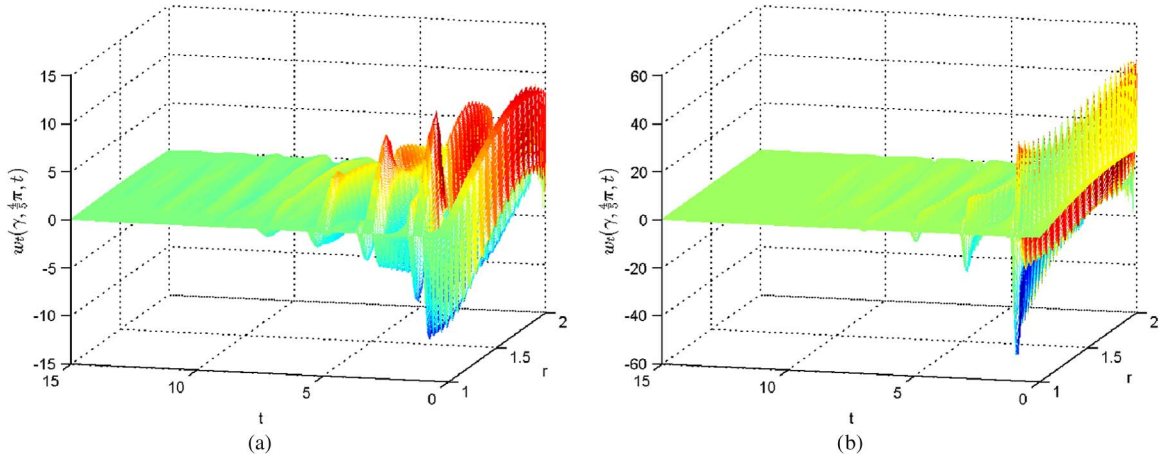
Fig. 3. The evolution of $w_t(\gamma, (4/5)\pi, t)$ under the polar coordinate with both time varying gain and constant gain (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article). (a) The velocity $w_t(\gamma, 4/5\pi, t)$ with time varying gain. (b) The velocity $w_t(\gamma, 4/5\pi, t)$ with constant gain.
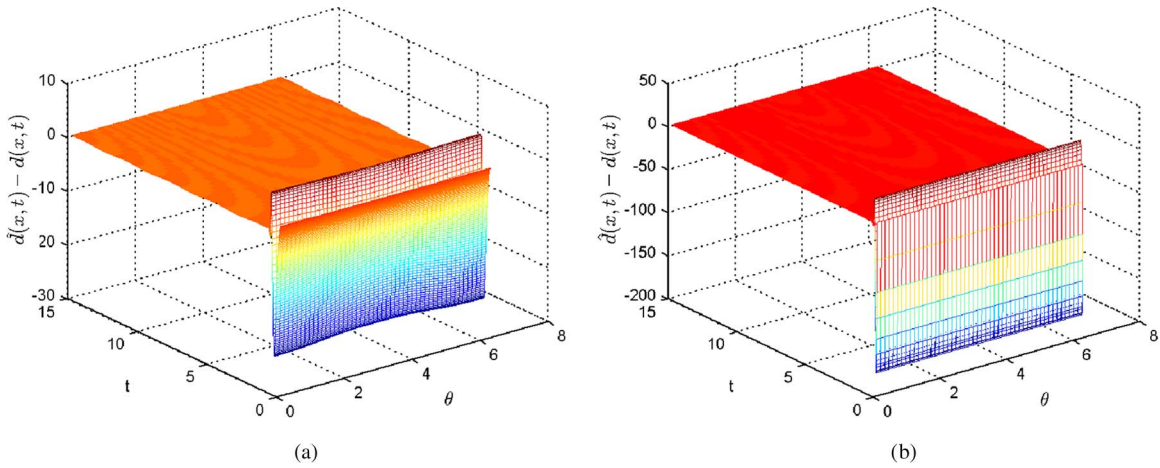

Fig. 4. (a) The error $\hat{d} - d$ under the time varying high gain. (b) The error $\hat{d} - d$ under the constant high gain (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).
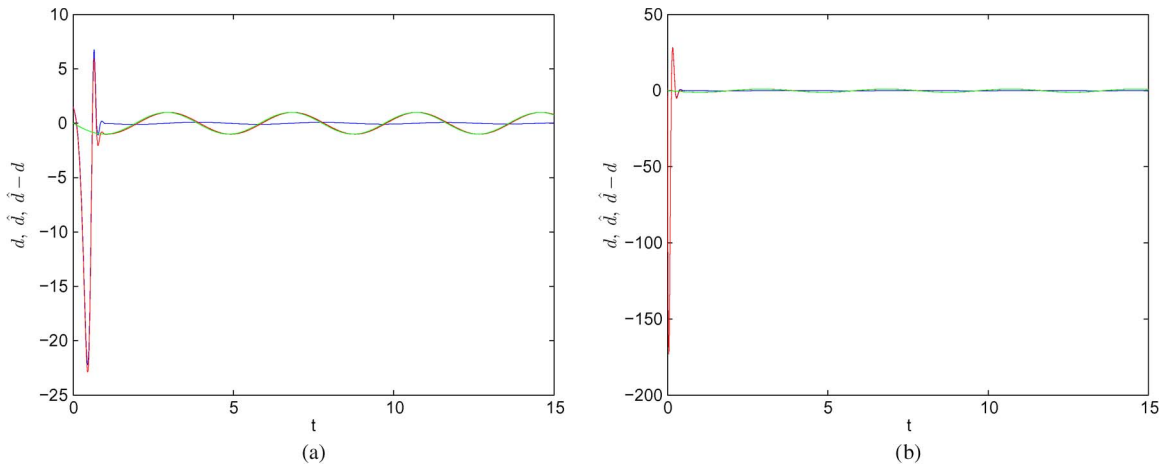

Fig. 5. The tracking effects of the disturbance in the radial direction of $\theta = (4/5)\pi$ with both time varying gain (a) and constant gain (b) (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article). (a) d (green), $\hat{d}$ (red), $\hat{d} - d$ (blue); (b) d(green), $\hat{d}$ (red), $\hat{d} - d$ (blue).

the convergence due to the nature of estimation/cancellation in ADRC although it is difficult to prove this fact theoretically. The convergence and peaking reduction are also clearly ob-served from the specific direction $\theta = 4\pi/5$ under the polar coordinate in Fig. 5 where Fig. 5(a) is with the time varying gain and Fig. 5(b) is with the constant gain.

V. Conclusion

In this paper, we present the active disturbance rejection control approach to boundary state feedback stabilization for a multi-dimensional wave equation with the disturbance suffered from the boundary. The time varying gain is first applied to the disturbance rejection control by the ADRC to PDEs which is contrast to the constant high gain in existing literature for one-dimensional ones. It is shown that for a quite general boundary disturbance with time dependent or both time and spatial dependent, the extended state observer can estimate effectively the disturbance. Having recovered the disturbance from the extended state observer, the disturbance is canceled in the feedback loop. The collocated feedback control is then applied to stabilize the overall system. The advantages of complete disturbance rejection are presented both theoretically and numerically. In particular, the numerical experiments show that the peaking value problem caused by the constant high gain in the extended state observer can be dramatically reduced through time varying gain which grows slowly to its constant maximum value by paying the price that it takes a little bit longer time to track the true values for both the state of the system and the disturbance compared with the constant gain.

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