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THE ADAMS-NOVIKOV SPECTRAL SEQUENCE FOR THE SPHERES

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The Adams spectral sequence has been an important tool in research on the stable homotopy of the spheres. In this note we outline new information about a variant of the Adams sequence which was introduced by Novikov [7]. We develop simplified techniques of computation which allow us to discover vanishing lines and periodicity near the edge of the E_2-term, interesting elements in $E_2^{2,*}$, and a counterexample to one of Novikov's conjectures. In this way we obtain independently the values of many low-dimensional stems up to group extension. The new methods stem from a deeper understanding of the Brown-Peterson cohomology theory, due largely to Quillen [8]; see also [4]. Details will appear elsewhere; or see [11].

When p is odd, the p-primary part of the Novikov sequence behaves nicely in comparison with the ordinary Adams sequence. Computing the E_2-term seems to be as easy, and the Novikov sequence has many fewer nonzero differentials (in stems $\leqq 45$, at least, if $p=3$), and periodicity near the edge. The case $p=2$ is sharply different. Computing E_2 is more difficult. There are also hordes of nonzero differentials d_3, but they form a regular pattern, and no nonzero differentials outside the pattern have been found. Thus the diagram of E_4 ($=E_\infty$ in dimensions $\leqq 17$) suggests a vanishing line for E_∞ much lower than that of E_2 of the classical Adams spectral sequence [3].

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1. The spectral sequence. The construction of the classical Adams spectral sequence for the spheres [1] works equally well if the spec-

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[1] Most of the results announced in this paper were in the author's doctoral dissertation, written under an NSF graduate fellowship at the University of Chicago.

trum $K(Z_p)$ representing ordinary cohomology is replaced by an arbitrary ring spectrum X. If X satisfies certain conditions, the E_2-term of the resulting sequence will be isomorphic to

$$\mathrm{Ext}_{A^X}(\Lambda^X, \Lambda^X),$$

where $A^X = X^*(X)$ is the algebra of operations in X-cohomology theory and $\Lambda^X = \pi_*(X)$ is the coefficient ring. Novikov showed [7] that if $X = MU$ (the spectrum representing complex cobordism) this multiplicative spectral sequence converges to the stable homotopy ring π_*^S:

$$E_\infty^{s,t} \cong F^s \pi_{t-s}^S / F^{s+1} \pi_{t-s}^S,$$

where F^* is a filtration of π_*^S. Furthermore, if $X' = BP_p$, the Brown-Peterson spectrum [4] for the prime p, the resulting spectral sequence $\{{}_pE_r, {}_pd_r\}$ is exactly the p-primary part $\{E_r \otimes Q_p, d_r \otimes Q_p\}$ of the MU spectral sequence (Q_p is the ring of rational numbers with denominators prime to p.)

Not much is known about the MU spectral sequence, because even limited computations of E_2 have been difficult. This is regrettable, since what *is* known indicates that the Novikov sequence has certain a priori advantages over the usual one. The nonzero terms are sparse, for example: ${}_pE_2^{s,t} = 0$ if $t \not\equiv 0 \bmod 2(p-1)$. Furthermore, almost all of the image of the J-homomorphism [2], [9] lies on the line $s = 1$, in the following sense. According to Novikov, $E_2^{1,2t} = Z_{m(t)}\langle \alpha_t \rangle$, a cyclic group with generator α_t, isomorphic to the image of J in dimension $2t-1$ (isomorphic to Z_2 if $2t-1 \equiv 5 \bmod 8$). There is a map $q_1 : \pi_n^S \to E_2^{1,n+1}$ such that an element of $E_2^{1,n+1}$ survives to E_∞ iff it belongs to im q_1. Furthermore, if $\tilde{q}_1$ denotes the restriction of q_1 to im J, then [7, Chapters 10 and 11]

(1) if $n = 8k+1$, $E_\infty^{1,n+1} = E_\infty^{1,n+1} = Z_2$;

(2) if $n = 8k+3$ $(k > 0)$, then im $q_1 =$ im $\tilde{q}_1$ has index 2 in $E_2^{1,n+1} = Z_{m(4k+2)}$, and $\tilde{q}_1$ has kernel Z_2; in fact, $d_3\alpha_{4k+2} = h^3\alpha_{4k} \neq 0$;

(3) if $n = 8k+5$, $E_2^{1,n+1} = Z_2$ does not survive to E_∞; in fact, $d_3\alpha_{4k+3} = h^3\alpha_{4k+1} \neq 0$;

(4) if $n = 8k+7$, im $\tilde{q}_1 = Z_{m(4k+4)} = E_2^{1,n+1} = E_\infty^{1,n+1}$.

Here $h = \alpha_1$.

2. **Quillen's algebra.** Novikov knew that, given a prime p, the algebra $A^{BP} = BP^*(BP)$ was much simpler than $A^{MU} \otimes Q_p$, but he did not have complete information about A^{BP}. Later, Quillen [8] discovered an idempotent ϵ, which split the spectrum MUQ_p into a sum of suspensions of the spectrum BP_p [4]. Now

$$\pi_*(BP) = Q_p[k_1, k_2, \cdots], \qquad H_*(BP) = Q_p[m_1, m_2, \cdots],$$

with $|k_i| = -|m_i| = -2(p^i-1)$. We can take $m_i = (1/p^i)h\epsilon[CP^{p^i-1}]$; the Hurewicz homomorphism h is monic, and may be computed using Quillen's formal-group techniques [11] or standard methods. Thanks to the idempotent ϵ, Quillen and Adams were able to write down explicit formulas for the Hopf-algebra structure of the algebra of operations A^{BP} ($=A$, for short).

First, there is a coalgebra R of operations, free as a Q_p-module on generators r_E, where E runs over all finitely nonzero sequences $(e_1, e_2, \cdots)$ of nonnegative integers and $|r_E| = 2(\sum (p^i-1)e_i)$. The diagonal map is given by $\phi^* r_E = \sum_{E'+E''=E} r_{E'} \otimes r_{E''}$. Then $\Lambda' = \pi_*(BP)$ is an algebra over the coalgebra R, with action given (via the Hurewicz map) by $r_E m_n = m_{n-i}$ if $e_i = p^{n-i}$ and all other e_j are zero, and $r_E m_n = 0$ otherwise. Moreover, multiplication by an element λ of Λ' is also a BP-cohomology operation, and in fact every operation can be written as a (possibly infinite) sum $\sum \lambda_i r_{E_i}$ in which the degree of each $\lambda_i r_{E_i}$ is a constant independent of i. Unfortunately, the composition $r_E r_F$ of two operations in R does not usually lie in R; however, it can be written uniquely as a finite sum $r_E r_F = \sum_K c_K r_K$ with $c_K \in \Lambda'$, using the methods of [11] or those of [4]. This enables us to express compositions $(\lambda r_E)(\lambda' r_F)$ in the form $\sum \lambda_i r_{E_i}$. Thus the algebra A of all operations is the completed tensor product $\Lambda' \hat{\otimes} R$.

Proposition 1. *Let $\bar{\Lambda}$ be the two-sided ideal in A generated by all elements of Λ of negative degree. Let $\mathcal{A}_p/(Q_0)$ be the algebra of reduced Steenrod pth powers* [6]. *Then there is an isomorphism $f: A/\bar{\Lambda} \cong \mathcal{A}_p/(Q_0)$.*

Proof. Let $\mathrm{Th}: BP_p \to K(Z_p)$ be the Z_p Thom class. Then

$$\begin{array}{ccc} \bar{f} = \mathrm{Th}_*: [BP, BP] & \to & [BP, K(Z_p)] \\ \| & & \| \\ A & & H^*(BP; Z_p) \\ & & \| \\ & & \mathcal{A}_p/(Q_0) \end{array}$$

satisfies

$$\begin{aligned} \bar{f}(k^E r_F) &= c(\mathcal{P}^F), && E = 0 \; [6]; \\ &= 0, && \text{otherwise;} \end{aligned}$$

where c is the canonical antiautomorphism. The map $\bar{f}$ induces the required f on $A/\bar{\Lambda}$.

A generator r_E is *indecomposable* if it cannot be expressed as a finite sum $r_E=\sum \lambda_i R_i R_i'$, where $\lambda_i \in \Lambda'$; R_i, $R_i' \in R$; and $|R_i|$, $|R_i'|>0$.

THEOREM 2. *The generator r_E* of *R is indecomposable if and only if $E=(p^i, 0, 0, \cdots), i\geqq 0$. Moreover, $pr_{(p^i,0,0,\ldots)}$ is decomposable.*

The proof is obtained by noticing certain pleasant properties of the multiplication table for R and applying them in the proper sequence.

3. **Resolutions over** A. To compute Ext we must construct resolutions over A, which seems difficult at first glance since R is not an algebra, A is not connected, and the ground ring Q_p is not a field. The next proposition shows how to circumvent some of these difficulties. Define the filtrations $F^s\Lambda'=\sum_{i\leq 2s}(\Lambda')^i$, $F^sA=F^s\Lambda'\hat{\otimes}A$, and $F^sM=(F^sA)M$ if M is an A-module. We have

$$0\to F^1M\xrightarrow{i} M\xrightarrow{j}\operatorname{cok} i\to 0.$$

Write JM for cok i; then J is easily made into a functor on the category of A-modules.

PROPOSITION 3. *There exist complexes*

$$C:\ \cdots\to C_i\xrightarrow{d_i}C_{i-1}\to\cdots\to C_1\xrightarrow{d_1}C_0 = A\to\Lambda'\to 0$$

satisfying

(1) $C_1=\sum Au_j$ *with* $d_1u_j=r_{(p^j,0,0,\ldots)}$;

(2) $C_i=\prod_j Aw_j^{(i)}$ *is locally finitely generated as an A-module*, $i>1$;

(3) $\ker(Jd_i)\subset j(\operatorname{im} d_{i+1})$ *in* JC_i *for all* $i, n\geqq 0$.

Any such C is an A-projective resolution of Λ'.

The proof is straightforward. Notice that the infinite direct product $\prod Aw_j^{(i)}$ is not necessarily free over A; it is projective, however. As a further aid to computation there is

LEMMA 4. *If $\{C_i, d_i\}$ is any A-projective resolution of Λ', write $C_i^*=\operatorname{Hom}_A^*(C_i,\Lambda')$, $d_i^*=\operatorname{Hom}_A^*(d_i,\Lambda')$. Then*

$$\operatorname{Ext}_A^{s,t}(\Lambda',\Lambda') = \operatorname{Tors}(\operatorname{cok}(d_s^*)^t),\qquad (s,t)\neq(0,0).$$

PROOF. This follows from the fact that $\operatorname{Ext}_{A'}^{s,t}$ is finite for $(s,t)\neq(0,0)$ [7, Corollary 2.1].

Thus in determining Ext we need know just the boundaries, and not the cycles too. In fact we can even work over Z_{p^f} for suitable f.

Now we can prove

PROPOSITION 5. $\mathrm{Ext}^{0,t}=0$ *unless* $t=0$; $\mathrm{Ext}^{0,0}=Z$.

THEOREM 6. $\mathrm{Ext}^{2,t}$ *contains a direct summand isomorphic to* Z_p *for* $t=2p^i(p-1)$ $(i\geqq 1)$ *and* $t=2(p^i+1)(p-1)$ $(i>1)$.

THEOREM 7. *For* $p=2$, *the element of* $\mathrm{Ext}^{2,2^i}$ *found in Theorem* 6 *maps to the Arf-invariant element* h_i^2 *of the classical Adams spectral sequence* [5].

PROOF. Apply the Thom map (Proposition 1) to a suitable A-resolution.

PROPOSITION 8. *The two-primary part* ${}_2\mathrm{Ext}^{s,t}$ *has the following "edge" values*:

$$
\begin{aligned}
{}_2\mathrm{Ext}^{n,2(n+k)} &= 0, && k<0;\\
&= Z_2, && k=0,\ n\geqq 1 \ \text{(generated by } h^n);\\
&= 0, && k=1,\ n\geqq 2;\\
&= Z_2, && 2\leqq k\leqq 5,\ n\geqq 4 \ \text{(generated by } h^{n-1}\alpha_{k+1}).
\end{aligned}
$$

Further computations of the additive structure of ${}_2\mathrm{Ext}^{*,*}$ in low dimensions are given in Figure 1. Thanks to Proposition 8, the first three nonzero Novikov differentials $d_3\alpha_i=h^3\alpha_{i-1}$, $i=3, 6, 7$, give rise to infinite towers of nonzero d_3's. Moreover, every other differential in the range $t-s\leqq 17$ must be zero for dimensional reasons. Finally, ${}_2E_\infty$ has a vanishing line considerably lower than that of the E_∞-term of the classical Adams spectral sequence in this range of dimensions. We conjecture that the preceding four sentences are also true without restriction on the dimensions.

Similar computations for $p=3$ disclose striking edge properties like Proposition 8, but many fewer differentials. Contrary to Novikov's conjecture [7], there is a nonzero differential $d_5: E_2^{2,36}\to E_2^{7,40}$ for $p=3$. This differential, whose existence is inferred from Toda's result [10], also gives rise to an infinite family of nonzero differentials. It is encouraging that there is only one nonzero differential in the range $t-s\leqq 40$, as compared to 17 in the classical 3-primary Adams spectral sequence.

REFERENCES

1. J. F. Adams, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. 32 (1958), 180–214. MR 20 #2711.

2. ———, *On the groups J(X).* II, Topology 3 (1965), 137–173; IV, ibid. 5 (1966), 21–71. MR 33 #6626; 6628.

3. ———, *A periodicity theorem in homological algebra*, Proc. Cambridge Philos. Soc. **62** (1966), 365–377. MR **33** #2696.

4. ———, *Quillen's work on formal group laws and complex cobordism*, University of Chicago Lecture Notes Series, 1970.

5. W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, Ann. of Math. (2) **90** (1969), 157–186.

6. J. W. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171. MR **20** #6092.

7. S. P. Novikov, *The method of algebraic topology from the point of view of cobordism theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 855–951 = Math. USSR Izv. **1** (1967), 827–913. MR **36** #4561.

8. D. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.

9. ———, *The Adams conjecture* (to appear).

10. H. Toda, *An important relation in homotopy groups of spheres*, Proc. Japan Acad. **43** (1967), 839–842. MR **37** #5872.

11. R. Zahler, *The Adams-Novikov spectral sequence for the spheres*, Thesis, University of Chicago, Chicago, Ill., 1970.

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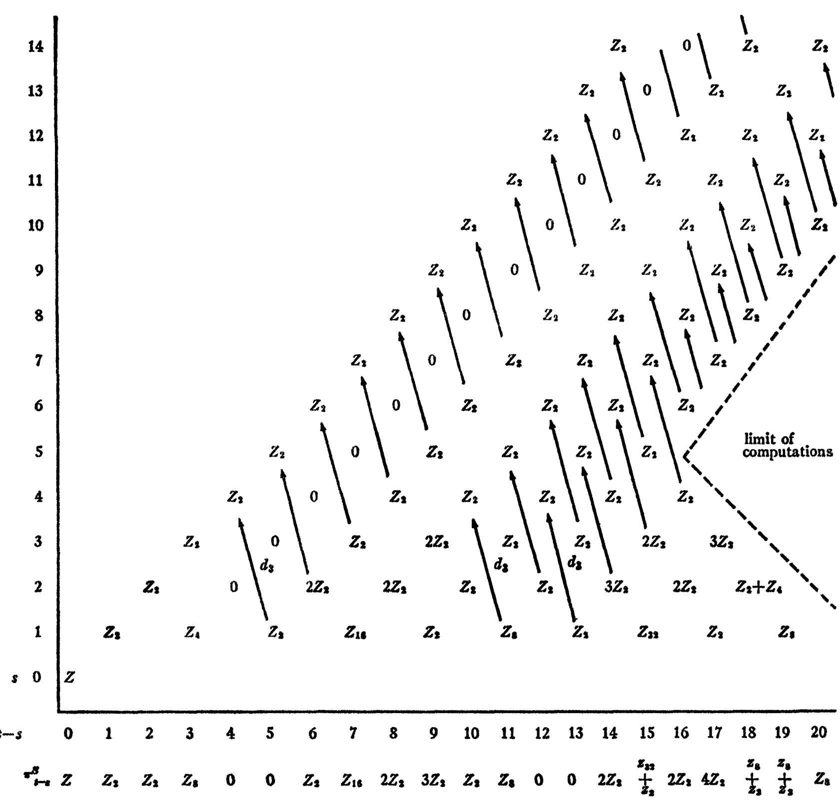

FIGURE 1. ${}_2\mathrm{Ext}^{s,t}$ for the Novikov sequence.