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THE ADIABATIC STABILITY OF STARS CONTAINING MAGNETIC FIELDS—III

ADDITIONAL RESULTS FOR POLOIDAL FIELDS

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SUMMARY

In a previous paper it has been shown that a particular poloidal magnetic field configuration, with closed magnetic field lines inside a star, is unstable and that the instabilities grow in a short time compared with time scales important in stellar evolution. In that paper it was suggested that the results obtained depended only on the field topology and not on its precise structure. In the present paper three further types of poloidal field configuration are studied and they are all shown to be unstable. It thus appears very probable that all poloidal configurations with closed field lines are unstable.

1. INTRODUCTION

In a previous paper (Markey & Tayler 1973, Paper II), we have shown that, if a star contains a purely poloidal magnetic field with closed field lines, it is likely to suffer from hydromagnetic instabilities in the region of the closed field lines. A similar result has been obtained by Wright (1973). In both of these papers the explicit demonstration of instability has been restricted to the case in which the closed field lines are concentric circles and in which the magnetic field configuration has a centre of symmetry, which is also the centre of the star. It was conjectured both by Markey & Tayler and by Wright that the results obtained in their papers could be generalized to other configurations with the same magnetic field topology but this was not demonstrated. The purpose of this paper is to describe briefly some additional results which we have obtained which tend to confirm the conjectures of Wright and ourselves.

We have studied the stability of three additional types of poloidal field structure. In each case, as in Paper II, we have not produced a consistent model of a star containing a magnetic field. As the instabilities studied depend only on the shape of the magnetic field and not on its strength, we have supposed that the field is sufficiently weak that the star can be assumed spherical, so that the pressure, P, and the density ρ are functions of radius alone. In addition, as our aim is merely to demonstrate that instability occurs rather than to show that the worst instability has been found, we once again consider only the sub-class of perturbations that are incompressible and are parallel to surfaces of constant gravitational potential. The first assumption is made mainly for mathematical convenience but the second recognizes that instabilities involving changes in the gravitational energy of the star are unlikely to occur if the magnetic energy density is very much less than the gravitational energy density.

The three configurations that we have studied are discussed briefly in the following three sections. In Section 2 we study a configuration in which the field lines are still circular about the magnetic axis* but in which the circles form a coaxial set instead of being concentric. With such field lines it is possible to define a magnetic field which is well behaved throughout the volume of the star. In Section 3 the configuration has field lines which are concentric ellipses in the neighbourhood of the magnetic axis. Finally in Section 4 the magnetic field configuration is identical with that studied in Paper II but the centre of symmetry of the magnetic field is no longer the centre of the star.

In the first two cases it is easy to demonstrate that the system is unstable. In the case of the off-centred field, the system is no longer axially symmetric and it has not proved possible to obtain an expression for an unstable perturbation in a closed form. As a result it is given by an infinite series and a rigorous proof of instability would require a proof of the convergence of the expression for the change of potential energy. The occurrence of instability, although not rigorously proved, has been made extremely plausible.

As a result of the calculations which will be described in the next three sections of this paper, we are confirmed in our view that instabilities will occur whenever the topology of the field is such that it is purely poloidal and possesses closed field lines. We have not, of course, shown that there will be no instabilities, if all of the field lines are open. It now seems likely that nothing is to be gained by trying to study even more general poloidal field configurations including those in which the field is no longer axially symmetric. It is much more important to attempt to study the stability of stars in which there is linked poloidal and toroidal flux, to investigate the effect of rotation on the instabilities and to discuss their large scale development.

In this paper the existence of convection zones in stars has been neglected. The results obtained should apply to stars containing poloidal magnetic fields with closed field lines in a radiative zone.

2. CONFIGURATIONS WITH MAGNETIC SURFACES WHICH ARE COAXIAL TORUSES

We suppose that a star contains a poloidal magnetic field such that the field lines in a meridional plane form a system of coaxial circles, as shown in Fig. 1. The advantage of this configuration over that studied in Paper II is that a field can be defined which is well behaved throughout the star. However, the field lines which pass near to the axis of symmetry will certainly close outside the star and the considerations of this paper may not be immediately applicable to these field lines.

An orthogonal curvilinear coordinate system (toroidal coordinates) exists in which the toruses are the surfaces σ = constant, ϕ is the azimuthal angle about the axis of symmetry and the surfaces χ = constant are spheres orthogonal to the toruses. The relation of the toroidal coordinates to Cartesian coordinates is

$$\left.\begin{aligned} x &= R\sinh\sigma\cos\phi/(\cosh\sigma-\cos\chi),\\ y &= R\sinh\sigma\sin\phi/(\cosh\sigma-\cos\chi),\\ z &= R\sin\chi/(\cosh\sigma-\cos\chi), \end{aligned}\right\} \quad (2.1)$$

* The magnetic axis is the curve (a circle in all cases considered in this paper) on which the poloidal field vanishes.

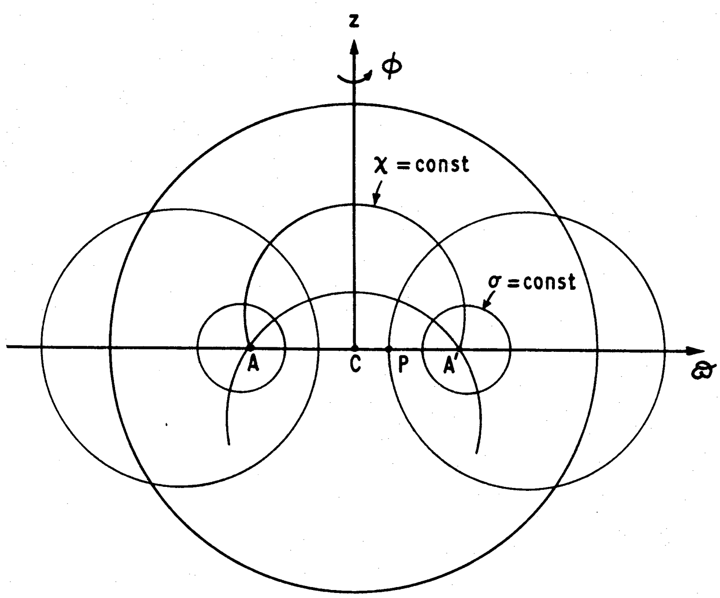

FIG. 1. *Sketch of configuration in which magnetic surfaces are coaxial toruses. Cylindrical polar coordinates ϖ, ϕ, z and toroidal coordinates σ, ϕ, χ are marked, as is the surface of the star. The magnetic axis passes through AA', C is the centre of the star and the distance CA is R. If the last unstable torus passes through P, CP is ϖ_c.*

where R is the distance from the magnetic axis to the centre of the star and the magnetic axis is $\sigma = \infty$. The coordinate system has line element

$$ds^2 = \frac{R^2}{(\cosh\sigma - \cos\chi)^2}(d\sigma^2 + d\chi^2) + \frac{R^2\sinh^2\sigma}{(\cosh\sigma - \cos\chi)^2}\,d\phi^2. \tag{2.2}$$

The magnetic field has only a χ component and application of the condition div $\mathbf{B} = 0$ which reduces to

$$\frac{\partial}{\partial\chi}\left\{\frac{\sinh\sigma}{(\cosh\sigma - \cos\chi)^2}B_\chi\right\} = 0 \tag{2.3}$$

and the requirement of axial symmetry shows that

$$B_\chi = B_0 f(\sigma)(\cosh\sigma - \cos\chi)^2. \tag{2.4}$$

We consider cases in which

$$f(\sigma) = \frac{a_1\sinh^p\sigma + a_2\cosh^q\sigma}{\cosh^r\sigma}. \tag{2.5}$$

Using expressions (2.4), (2.5) for B, it is possible to calculate the current density required to produce the field. The current density is finite everywhere provided that

$$r \geqslant 3 + \max(p, q). \tag{2.6}$$

We shall assume that this is so.

The change in potential energy of such a system consequent upon a perturbation $\boldsymbol{\xi}$ which satisfies

$$\text{div}\,\boldsymbol{\xi} = 0 \tag{2.7}$$

and

$$\boldsymbol{\xi}.\mathbf{g} = 0, \tag{2.8}$$

where **g** is the equilibrium gravitational field, has been calculated. As in Paper II, $\boldsymbol{\xi}$ can be Fourier analysed in the form

$$\boldsymbol{\xi} = \sum_m \boldsymbol{\xi}_m \exp im\phi \tag{2.9}$$

and the different m modes can be considered separately. Instability is most likely for $m \to \infty$. As before a separate stability criterion can be obtained on each surface σ = constant. A particular choice of $\boldsymbol{\xi}$ has been taken in which

$$\xi_\sigma = \text{constant} \times \sin\chi(\cosh\sigma - \cos\chi)/B_\chi \sinh\sigma. \tag{2.10}$$

This is closely analogous to the perturbation considered in Paper II.

For the field configuration (2.4), (2.5) and the trial perturbation (2.10), it is found that instability always occurs close to the magnetic axis, in agreement with the expectation expressed in Paper II. In general, field lines which pass close to the axis of symmetry of the star are not unstable to perturbation (2.10), but this does not prove that instability cannot occur for more general perturbations.

Numerical results have been obtained for three particular cases:

(i) $a_1 = 0$, $a_2 \neq 0$, various values of r–q,

(ii) $a_1 \neq 0$, $a_2 = 0$, various values of p, r,

(iii) $p = q = 2$, various values of r, a_1/a_2.

In the first case the magnitude of the magnetic field increases from zero on the magnetic axis to a maximum value on the axis of symmetry. In the second case the field vanishes both on the magnetic axis and on the axis of symmetry. In the third case it is finite on the axis of symmetry but the maximum value of the field is at an intermediate point. Table I shows, for a number of examples of each case, which is the last magnetic surface for which instability exists; the quantity ϖ_c tabulated is the distance from the centre of the star in the equatorial plane to the last unstable magnetic surface.

In case (i) the value of σ on the critical surface (σ_c) is given by

$$\sinh^2\sigma_c = 1/2(r-q-2) \tag{2.11}$$

TABLE I

Distance, ϖ_c to last unstable magnetic surface from centre of star

(a) Case (i)

$r-q$	3	4	5	6	7	∞
ϖ_c/R	0·318	0·236	0·196	0·172	0·154	0·000

(b) Case (ii) ϖ_c/R is shown as a function of $r-p$ and p

$r-p$ / p	3	4	5	6	7	∞
0	0·318	0·236	0·196	0·172	9·154	0·000
2	0·551	0·447	0·388	0·348	0·318	0·000
4	0·634	0·535	0·475	0·432	0·399	0·000

(c) Case (iii) ϖ_c/R is shown as a function of r and a_1/a_2

r / a_1/a_2	5	6	7	10	13	∞
0·5	0·395	0·280	0·224	0·153	0·123	0·000
2·0	0·485	0·364	0·294	0·189	0·144	0·000
10·0	0·535	0·427	0·364	0·263	0·210	0·000

and

$$\varpi_c/R = \sinh \sigma_c/(\cosh \sigma_c + 1). \tag{2.12}$$

From the table it can be seen that the maximum value of ϖ_c is 0·318 R and that, as $r - q \to \infty$ all magnetic surfaces become unstable to the type of perturbation being considered. This corresponds to a field which decreases very rapidly away from the axis of the star. As mentioned earlier, the results for those field lines which pass near to the centre of the star and close outside the star really need further investigation in case the assumption of infinite electrical conductivity breaks down in these outer regions.

In case (ii) we have, instead of (2.11),

$$\sinh^2 \sigma_c = (2p+1)/2(r-p-2). \tag{2.13}$$

If $p = 0$, the results are the same in case (i) with p replacing q. For $p \neq 0$, the region of instability for given $r - p$ is less but instability occurs for all magnetic surfaces for $r/p \to \infty$. In case (iii) results have been obtained for a variety of values of a_1/a_2. For each value instability is more likely for a large value of r; for given r greater instability occurs for small values of a_1/a_2.

We have demonstrated that instability is common for field configurations of the type discussed in this section. Two further points can be made. The first is to repeat that we have not tried to discover the most unstable perturbation. The second is to remark that, provided the magnetic axis is placed near enough to the centre of the star, it is possible for the majority of the magnetic flux to be buried within the star. Thus in case (iii) with $a_1/a_2 = 1$, $r = 5$ and R equal to $r_s/10$, where r_s is the stellar radius, more than 97 per cent of the flux does not cross the surface. We have not, of course, produced a self-consistent model of a star with such a large trapped flux. Models of *rotating* magnetic stars with strong poloidal fields have, however, been obtained by Wright (1969).

3. CONFIGURATIONS WITH MAGNETIC SURFACES WHICH ARE CONCENTRIC ELLIPTICAL TORUSES

In this section the field configuration of Paper II is replaced by one in which the field lines are elliptical instead of circular; the ellipses are concentric with the same eccentricity. The configuration is shown in Fig. 2. Although we have no reason for believing that it will not be unstable, we study it because the orthogonal trajectories of the magnetic surfaces have a singular behaviour near to the magnetic axis and it seems desirable to check that the circular field lines are not a singular case as far as stability properties are concerned.

We once again name our coordinate system σ, ϕ, χ, although σ and χ must not be confused with those of Section 2. σ is constant on an ellipse of equation

$$\lambda(R-\varpi)^2 + z^2 = \sigma^2, \tag{3.1}$$

where ϖ, ϕ, z are cylindrical polar coordinates. If the major axis of the ellipse is parallel to the axis of symmetry, as shown in the figure,

$$\lambda = 1/(1-e^2), \tag{3.2}$$

where e is the eccentricity. If the major axis is perpendicular to the axis of symmetry,

$$\lambda = 1 - e^2. \tag{3.3}$$

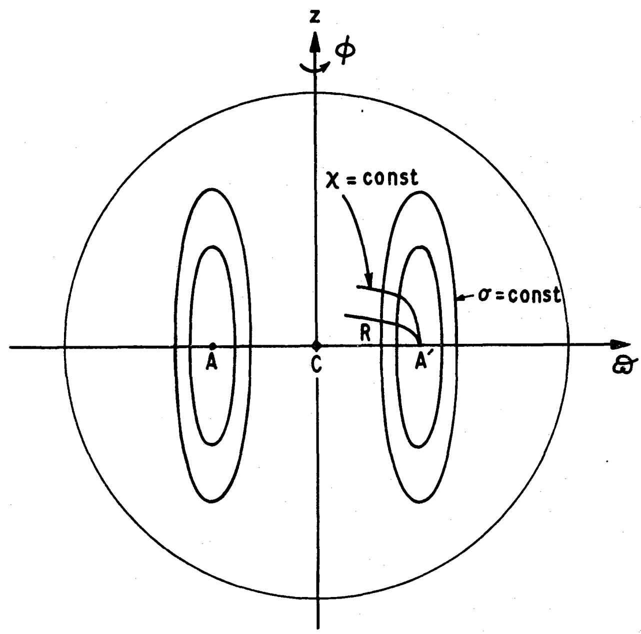

FIG. 2. *Sketch of configuration in which magnetic surfaces are elliptical toruses. The elliptical coordinate system σ, φ, χ is marked and other details are as in Fig.* 1.

The coordinate χ is defined by

$$\chi = (\varpi - R)/z^{\lambda}. \tag{3.4}$$

The line element of the coordinate system is

$$ds^2 = \frac{\sigma^2}{\lambda^2(\varpi-R)^2+z^2}\,d\sigma^2 + \frac{z^{2\lambda+2}}{\lambda^2(\varpi-R)^2+z^2}\,d\chi^2 + \varpi^2\,d\phi^2. \tag{3.5}$$

The condition div $\mathbf{B} = 0$ combined with axial symmetry then gives

$$B_\chi = B_0 f(\sigma)[\lambda^2(\varpi-R)^2+z^2]^{1/2}/\varpi, \tag{3.6}$$

where $f(\sigma)$ is an arbitrary function of σ. In order that this field reduces to the one studied in Paper II if the field lines are circular, we must put $f(\sigma) \equiv 1$ and we study this case. As before the stability of the system is studied with respect to perturbations which are incompressible and parallel to surfaces of constant gravitational potential. The form chosen for ξ_σ is

$$\xi_\sigma \propto z[\lambda(R-\varpi)+\varpi]/\sigma\varpi B, \tag{3.7}$$

which reduces to the form used in Paper II when $\lambda = 1$.

The change of potential energy has been calculated for such a perturbation and it has been found to be negative for sufficiently small elliptical surfaces whatever their eccentricity and regardless of whether the major axis is parallel to or perpendicular to the axis of symmetry. The numerical results are given in Table II, which shows in terms of R both the major axis, a_c, of the last unstable ellipse and the distance from the axis of the star to the last unstable ellipse. It can be seen that, for major axis parallel, almost all ellipses are unstable and the region of instability increases as the ellipses become more eccentric; in this case the field configuration is likely to cease to make sense before a stable ellipse is reached. For major axis

TABLE II

Parameters for limiting stable ellipses

(a) Major axis parallel to axis of star

e	0·0	0·2	0·4	0·6	0·8	0·9	1·0
a_c/R	0·963	0·988	1·069	1·240	1·663	2·293	∞
ϖ_c/R	0·037	0·031	0·019	0·008	0·002	0·000	0·000

(b) Major axis perpendicular to axis of star

e	0·0	0·2	0·4	0·6	0·8	0·9	1·0
a_c/R	0·963	0·957	0·929	0·826	0·569	0·384	0·000
ϖ_c/R	0·037	0·043	0·071	0·174	0·431	0·616	1·000

perpendicular, the region of instability decreases with increasing eccentricity and instability disappears, for the chosen perturbation, when the ellipse is infinitely flattened. It is not possible to tell without much more detailed study whether this result is entirely dependent on the choice of perturbation or whether the perpendicular case is genuinely more stable.

4. OFF-CENTRED MAGNETIC FIELDS

We suppose that a star contains a magnetic field which is axially symmetric and which has concentric circular field lines near to the magnetic axis. The centre of symmetry of the field is displaced from the centre of the star. This is illustrated in Fig. 3, which shows that cross-section of the star which passes through both the centre of symmetry of the field and the centre of the star. As in the other sections of this paper, we suppose that the field is weak enough that the star can be assumed to be spherical, so that the gravitational field always acts towards the centre of the star.

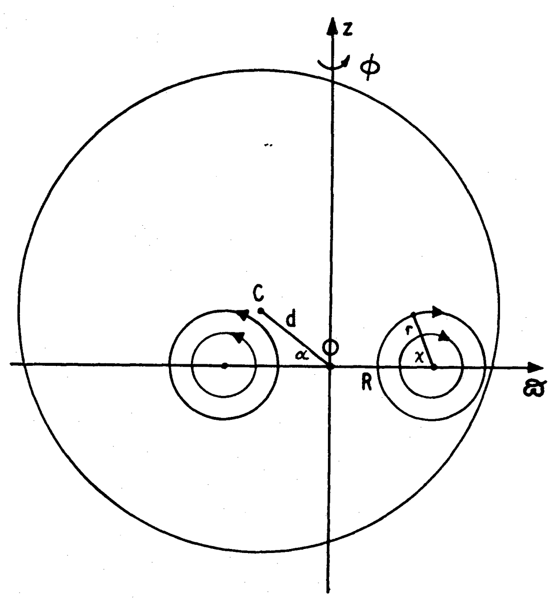

FIG. 3. *Sketch of configuration in which the magnetic field has a centre of symmetry which is displaced from the centre of the star. C is the centre of the star and O the centre of symmetry of the field and the cross-section through both C and O is shown.*

To discuss the stability of the system we use a coordinate system ψ, ϕ, χ based on the magnetic field; this is the system used in Paper II. We once again suppose that the perturbation can be Fourier analysed, as in equation (2.9). However, when we apply conditions (2.7) and (2.8), the different m modes do not completely decouple because **g** depends on ϕ, except when the angle α shown in Fig. 3 is $\pi/2$. Thus the expressions for the components of **g** are

$$\left.\begin{aligned} g_\psi &\propto r - R\cos\chi - d\sin\alpha\sin\chi - d\cos\alpha\cos\chi\cos\phi, \\ g_\phi &\propto d\cos\alpha\sin\phi, \\ g_\chi &\propto d\sin\alpha\cos\chi - (R + d\cos\alpha\cos\phi)\sin\chi, \end{aligned}\right\} \quad (4.1)$$

where, as shown in Fig. 3, d is the distance of the centre of symmetry of the field from the centre of the star.

The study of the stability of configurations with $\alpha = \pi/2$ for which no coupling of m modes occurs, is a simple generalization of the problem studied in Paper II and it is easy to show that field lines near to the magnetic axis are always unstable. For the case of $\alpha \neq \pi/2$ we first write

$$\left.\begin{aligned} \xi_\psi &= \sum_m X_m \cos m\phi/\varpi B, \\ \xi_\phi &= \sum \varpi Y_m \sin m\phi/m, \\ \xi_\chi &= \sum BZ_m \cos m\phi, \end{aligned}\right\} \quad (4.2)$$

where X_m, Y_m, Z_m are functions of ψ and χ. The equation div $\boldsymbol{\xi} = 0$ gives a relation between X_m, Y_m and Z_m for all m. In contrast the condition $\boldsymbol{\xi}.\mathbf{g} = 0$ now couples the different m modes.

We choose as a first approximation to our trial perturbation

$$\begin{aligned} X &\equiv \sum X_m \cos m\phi \\ &= X_0\left[\sin\chi - \frac{d}{R}\sin\alpha\cos\chi + \frac{d}{R}\cos\alpha\sin\chi\cos\phi\right]\cos M\phi, \\ Z &\equiv \sum Z_m \cos m\phi \\ &= \frac{-X_0}{\varpi B^2}\left[\cos\chi - \frac{r}{R} + \frac{d}{R}\sin\alpha\sin\chi + \frac{d}{R}\cos\alpha\cos\chi\cos\phi\right]\cos M\phi, \end{aligned} \quad (4.3)$$

so that both X and Z contain terms in $\cos(M-1)\phi$, $\cos M\phi$ and $\cos(M+1)\phi$ and where M is arbitrary at present, but will later be taken to be very large. From the equation div $\boldsymbol{\xi} = 0$ expressions can be found for the Y_m, which can be seen to be of the same order as the X_m and Z_m. The perturbation (4.3) satisfies the equation $\boldsymbol{\xi}.\mathbf{g} = 0$ only if the terms in Y_m are neglected. As these all contain a factor $1/m$, it should in principle be possible to choose small correction terms to X and Z which cause $\boldsymbol{\xi}.\mathbf{g}$ to vanish, provided M has been chosen to be very large. These correction terms $\delta_1 X$, $\delta_1 Z$ in turn lead to a correction $\delta_1 Y$ from the incompressibility condition. There is then a need for a further correction $\delta_2 X$, $\delta_2 Z$. In principle, instability can only be proved if all of the resulting series are shown to be convergent.

In fact, what has been demonstrated is the following. If the change of potential energy, δW, of the system is calculated for the trial function (4.3) with $M \gg 1$, it is always found to be negative near enough to the magnetic axis. Thus provided that the correction terms to δW are unimportant the system is unstable. The first approximations to the Y_m are then calculated and a choice must be made for the expressions $\delta_1 X$, $\delta_1 Z$. There *is* a choice because they are only connected by the one equation

$$\frac{\delta X}{\varpi B}[R\cos\chi - r + d\sin\alpha\sin\chi + d\cos\alpha\cos\phi\cos\chi]$$
$$+B\delta Z[R\sin\chi - d\sin\alpha\cos\chi + d\cos\alpha\cos\phi\sin\chi]$$
$$= \sum \frac{\varpi Y_m}{m}\sin m\phi\, d\cos\alpha\sin\phi. \tag{4.4}$$

The choice must be made to obtain convergent expressions for $\boldsymbol{\xi}$ and δW.

It has not proved possible to find appropriate expressions for $\delta_1 X$, $\delta_1 Z$ which certainly lead to convergence for all values of d and α but it appears that:
(i) if either $d\sin\alpha \gg r$ or $\alpha = 0$ the relation

$$\delta_1 X\cos\chi + \delta_1 Z\varpi B^2\sin\chi = 0 \tag{4.5}$$

leads to convergent expressions for large enough values of M;
(ii) if $R \gg r\cos\chi - d\cos\alpha\cos\phi$ or equivalently $R \gg r$ and $d\cos\alpha$,

$$\delta_1 X\sin\chi - \delta_1 Z\varpi B^2\cos\chi = 0 \tag{4.6}$$

is appropriate.

The first choice should give suitable expressions for $\delta_1 X$, $\delta_1 Z$ for all values of d and α for a range of values of r but if d is small the second choice may lead to more rapid convergence. Although a rigorous proof of convergence has not been obtained, instability of the general configuration has been made very plausible. It does not appear appropriate to include the full mathematical details in the present paper but we shall be happy to discuss them with any interested reader.

5. CONCLUDING REMARKS

The discussion in the previous three sections indicates that it is highly probable that all purely poloidal field configurations, which have closed field lines inside stars, are unstable. It is uncertain whether configurations, all of whose field lines are open, are unstable because they cannot be discussed without considering the physical conditions outside the star. In an earlier paper (Tayler 1973, Paper I), it has been shown that a wide class of toroidal field configurations are unstable, with the instability being located near to the axis of the star. In Wright (1973) and Paper II, it has been strongly suggested that the poloidal field instabilities near to the magnetic axis can be removed if the star contains a toroidal field of comparable strength to the poloidal field. At the same time, the poloidal field may reduce the instabilities of the toroidal field found in Paper I. In the absence of the gravitational field, it would be difficult to expect to find a completely stable configuration but, because of the constraint that the gravitational field exerts on allowed perturbations, it is possible that mixed toroidal/poloidal field configurations exist, which are stable throughout the star. We hope to consider that problem.

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