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The adiabatic stability of stars containing magnetic fields – VI. The influence of rotation

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Summary. An attempt is made to study the effect of uniform rotation on the instabilities discussed in previous papers of this series. Although rotation exerts some stabilizing influence, it does not seem likely that it removes most of the instabilities. In some cases complete stability may be produced but the rotation speed required is unrealistically high. In other cases the instability growth rate may be reduced but instabilities are still able to grow in a time which is very short compared to any time important in stellar evolution. All of these results are obtained from approximate treatments of the problems and they cannot be regarded as rigorously established.

1 Introduction

In previous papers of this series and in related papers (Tayler 1973, 1980, 1982; Markey & Tayler 1973, 1974; Goossens & Tayler 1980; Goossens, Biront & Tayler 1981; Van Assche, Goossens & Tayler 1982) it has been shown that both toroidal and poloidal magnetic fields in stars are likely to be unstable to adiabatic perturbations. The occurrence of the instability is determined mainly by the shape of the field rather than by its strength and, because the instabilities are dynamical, their growth times are short even if the field is weak enough not to have any significant influence on the overall structure of the star. Fewer results have been obtained for the case of fields with both poloidal and toroidal components, although Markey & Tayler (1973), Wright (1973) and Tayler (1980) showed that some configurations are unstable and that stable fields are likely to require the toroidal and poloidal components to have comparable strength in some global sense.

Two things are lacking in the discussion to date. The non-linear behaviour of the instabilities has not been investigated and there has been no serious discussion of the interaction of rotation with the instabilities. Before we attempt to decide whether the instabilities develop in such a way as to lead to a destruction of magnetic flux and to a restriction on the possible structure of stellar magnetic fields that are not being maintained by dynamo action, it seems desirable to ask whether the instabilities survive in the presence of rotation. It seems clear that they will occur in a rotating star in which the rotation speed is small compared to the hydromagnetic speed in the region of instability, because then the instability could grow significantly in one rotation period. This suggests that very strong magnetic fields in stars will be unstable in agreement with our previous results. However, in most stars in which both magnetic fields and rotation are actually observed, the rotation speed exceeds the hydromagnetic speed. This means that the influence of rotation on the instabilities must be studied.

Consideration will be restricted to the case of uniformly rotating stars. We continue to make the approximation that the material has infinite electrical conductivity, as we are studying instabilities whose growth times are very much less than the decay times of magnetic fields. We also neglect other non-adiabatic effects, although we comment on that approximation in the final section. In the approximation of infinite electrical conductivity, equilibrium is only possible in a non-uniformly rotating system if all points on a field line have the same angular velocity. Such a rotation law is implausible if the field has a poloidal component. Although differential rotation is possible for a purely toroidal field in which the rotation axis and the axis of symmetry of the field coincide, this is also a very implausible configuration. The restriction to uniform rotation means that we are not concerned with the role of differential rotation on hydromagnetic instability (see e.g. Proctor & Fearn 1983a, b; Fearn 1983) and with instabilities caused by differential rotation, such as have recently been discussed by Papaloizou & Pringle (1984) in the case of thick accretion discs.

In this paper we shall assume that equilibrium configurations of rotating magnetic stars exist. In doing so, we are ignoring the slow meridional circulation which is in general driven by both magnetic fields and rotation, because any instability which we find will grow in a time very much less than the circulation time. It has been our aim in this series of papers to try to obtain general results rather than to discuss the stability of particular configurations in great detail. The problem of the stability of a rotating star is, however, considerably more difficult than that of a non-rotating star because of the absence of the simple energy principle (Bernstein *et al.* 1958) which has been used in the previous papers. Although modifications of the energy principle applicable to rotating systems have been discussed, for example, by Frieman & Rotenberg (1960), it does not seem easy to obtain general results using them. In consequence, in this paper we are reduced to solving model problems rather than general ones.

The equations for adiabatic perturbations of a uniformly rotating star in a frame which is itself rotating with the uniform angular velocity $\mathbf{\Omega}$ of the star are:

$$\varrho\left[\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+2\mathbf{\Omega}\times\mathbf{v}+\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})\right]=-\nabla P+\mathbf{j}\times\mathbf{B}+\varrho\nabla\Phi, \tag{1.1}$$

$$\frac{\partial\varrho}{\partial t}+\nabla\cdot\varrho\mathbf{v}=0, \tag{1.2}$$

$$\frac{\partial P}{\partial t}+\mathbf{v}\cdot\nabla P=\frac{\gamma P}{\varrho}\left[\frac{\partial\varrho}{\partial t}+\mathbf{v}\cdot\nabla\varrho\right], \tag{1.3}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B}), \tag{1.4}$$

where $\mathbf{v}$ is the velocity relative to uniform rotation, P, ϱ, Φ, $\mathbf{B}$, $\mathbf{j}$ and γ are pressure, density, gravitational potential, magnetic induction, electric current density and ratio of specific heats, and $\mathbf{r}$ is the vector displacement from the rotation axis. We use electromagnetic units in which $\mathbf{j}=\nabla\times\mathbf{B}$. Rotation enters the problem in two ways. The equilibrium configuration is modified by the centrifugal term $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$; this means that Ω enters implicitly in (1.2) to (1.4) through its influence on the equilibrium P, ϱ and $\mathbf{B}$, although as in previous papers of this series we shall specify simple forms for $\mathbf{B}$. As the centrifugal force is the gradient of a scalar, it is possible to absorb it in an apparent gravitational potential $\Phi'\equiv\Phi+\varpi^2\Omega^2/2$, where ϖ is the distance from the axis of rotation; for realistic angular velocities Φ' must deviate only slightly from Φ. When linearized departures from equilibrium are studied, the equations can be written in such a way that the only explicit dependence on Ω is through the Coriolis term $\mathbf{\Omega}\times\mathbf{v}$ but the implicit dependence remains. We shall neglect changes in the gravitational field produced by the perturbation and, as a result, we do not list Poisson's equation; this is likely to be valid if perturbation wavelengths are sufficiently small.

We are mainly concerned with situations in which the influence of magnetic fields and rotation on the structure of the star is small compared to the rôle of gravity; thus stars differ only slightly from spherical shape. In addition we are studying regions that are stable to convection so that any instabilities must be caused by these perturbing influences. If this is so, it seems clear that motions in the direction of the gravitational field must be significantly inhibited. This immediately gives us an idea about which configurations are most likely to have their stability affected by rotation. Suppose first that $\mathbf{\Omega}$ and $\nabla\Phi$ are parallel. In that case the Coriolis force will be perpendicular to the gravitational force and its action will not be inhibited. In such a case we expect rotation to have its maximum influence. Suppose, in contrast, that $\mathbf{\Omega}$ and $\nabla\Phi$ are perpendicular. In that case one component of $\mathbf{\Omega}\times\mathbf{v}$ is in the direction of gravity and we expect it to be relatively ineffective, while the other term involves the component of $\mathbf{v}$ in the direction of gravity, which we expect to be small. As a result we expect the influence of rotation to be minimized. It is obviously easiest to consider problems in which the rotation axis and the axis of symmetry of the field coincide, but there is good observational evidence for the existence of oblique rotators so other cases should also be studied.

Frieman & Rotenberg (1960) demonstrated that rotation would not have a significant effect on hydromagnetic instabilities unless the rotation velocity was at least comparable to the hydromagnetic velocity; although their discussion did not include a gravitational field, it can easily be extended. Our purpose is to try to determine whether uniform rotation usually acts to stabilize magnetic instabilities and, if so, whether speeds comparable with the hydromagnetic speed or very much in excess of it are required. Because the realistic problem is extremely difficult, we proceed via model problems which, we hope, contain the essentials of the true problem.

In Section 2 we study the effect of rotation on hydromagnetic instabilities in the absence of gravitation. In doing so, we concentrate on the potentially stabilizing effect of rotation and avoid discussing, for example, instabilities of free boundaries which can be produced by rapid rotation in the absence of gravitation. In Section 3 we introduce gravitation in an approximate way which we hope gives a reasonable clue to the true joint behaviour of rotation and gravitation. Finally, we discuss our results in Section 4.

2 Model problems excluding gravity

Many of the instabilities discussed in earlier papers resemble those of a pinched gas discharge. In those papers we were able to demonstrate instability in some cases by restricting our attention to perturbations which were everywhere perpendicular to gravity. The power of the energy principle was such that the existence of any perturbation which lowered the potential energy of the system was enough to ensure that there would also exist an unstable normal mode perturbation. When rotation is introduced we have to solve the normal mode equations and in this Section, to obtain a first idea of the influence of rotation, we discuss the stability of pinches in which there is no gravitational field. In the following Section we discuss the possible influence of gravity. In this Section we also consider only an incompressible fluid. In the absence of rotation (and gravitation), the stability properties of compressible and incompressible fluids are very similar but the mathematics is much simpler for incompressible fluids; the main difference arises when incompressible fluids are marginally stable when compressible fluids may be unstable. To keep the paper within manageable length we omit most of the detailed algebra. To make the model problems soluble, they are highly idealized, but it is hoped that some clues can be obtained to more general behaviour.

We first consider the problem of an incompressible fluid of uniform density ϱ_0 which contains a magnetic field of the form

$$\mathbf{B}_0=(0,\ B_0\varpi/\varpi_0,\ B_0 b) \tag{2.1}$$

in cylindrical polar coordinates ϖ, ϕ, z, related to the axis of the field. B_0, ϖ_0 and b are constants. We can now let the configuration rotate either about the z axis or about an axis in the (ϖ, ϕ) plane. The perturbed equations are

$$\varrho_0\left[\frac{\partial \mathbf{v}_1}{\partial t}+2\boldsymbol{\Omega}\times\mathbf{v}_1\right]=-\nabla P_1+[(\nabla\times\mathbf{B})\times\mathbf{B}]_1, \tag{2.2}$$

$$\nabla\cdot\mathbf{v}_1=0, \tag{2.3}$$

$$\frac{\partial \mathbf{B}_1}{\partial t}=\nabla\times(\mathbf{v}_1\times\mathbf{B}_0), \tag{2.4}$$

where the suffix 1 indicates perturbed quantities. Note that our assumption of an incompressible fluid of uniform density means that $\boldsymbol{\Omega}$ only enters these equations through the Coriolis term. In addition only the direction of the axis of rotation is relevant and not its position, although the equilibrium pressure distribution is affected by the centrifugal term. In the absence of rotation, this problem was solved by Tayler (1957a, b); it was also shown in Tayler (1960) that the problem could be solved readily if $\boldsymbol{\Omega}$ was parallel to the z axis.

In this case equations (2.2) to (2.4) can be solved for normal mode disturbances proportional to $\exp\,(im\phi+ikz+i\omega t)$. The ϖ component of v_1 can be shown to be

$$v_{1\varpi}\propto\frac{m\lambda}{(\lambda^2-k^2)\varpi}\,J_m\{(\lambda^2-k^2)^{1/2}\varpi\}+\frac{k}{(\lambda^2-k^2)^{1/2}}\,J_m'\{(\lambda^2-k^2)^{1/2}\varpi\}, \tag{2.5}$$

where J_m is the Bessel function of order m and the prime denotes differentiation with respect to its argument and

$$\lambda=\frac{2k\Omega\omega-2k(m+bk\varpi_0)B_0^2/\varpi_0^2\varrho_0}{\omega^2-(m+bk\varpi_0)^2B_0^2/\varpi_0^2\varrho_0}\,. \tag{2.6}$$

Consider first the case $b=0$. In that case it can be shown that in the absence of rotation there exists an instability for $m=1$ regardless of the boundary conditions that we apply. If the fluid has a free surface, there is instability for all k; if there is a rigid boundary at $\varpi=\varpi_0$, say, there is instability for k greater than some k_{crit}. In either case the maximum growth rate occurs as $k\to\infty$ and the value of ω is the same, $\omega^2=-B_0^2/\varpi_0^2\varrho_0$. We will consider the case of a rigid boundary and rotation. This boundary condition seems appropriate for the discussion of instabilities localized to a region of a star where (2.1) is an approximate form of the field. In addition, in the presence of rotation a free boundary might experience Rayleigh–Taylor type instabilities which would be removed by a realistic gravitational force. The boundary condition on the rigid surface is $v_{1\varpi}=0$. The system is regarded as infinite in the z direction, although for large wavenumbers the wavelength in the z direction is very small so that instabilities can also be localized in that direction.

It can first be noted that the resulting equation (for $m=1$)

$$0=\frac{\lambda}{(\lambda^2-k^2)\varpi_0}\,J_1\{(\lambda^2-k^2)^{1/2}\varpi_0\}+\frac{k}{(\lambda^2-k^2)^{1/2}}\,J_1'\{(\lambda^2-k^2)^{1/2}\varpi_0\} \tag{2.7}$$

Table. 1. The solution of equation (2.7) for λ/k in terms of $k\varpi_0$.

λ/k	1.0	1.1	1.2	1.4	1.6	1.8
$k\varpi_0$	∞	5.3	3.8	2.5	2.1	1.8
λ/k	2.0	3.0	4.0	6.0	10.0	∞
$k\varpi_0$	1.6	1.1	0.83	0.56	0.35	0.00

can be solved for λ as a function of k without using equation (2.6); thus the λ, k relationship is independent of b and Ω. Once this is known, (2.6) can be used to give a value for ω. The λ, k relationship is shown in Table 1. If $\Omega=b=0$, instability occurs for $k\leqslant\lambda\leqslant 2k$. If we now introduce Ω but not b and write $\lambda/k=\beta$, equation (2.6) can easily be solved for ω to give

$$\omega=\frac{\Omega}{\beta}\pm\left(\frac{\Omega^2}{\beta^2}-\frac{2\Omega_B^2}{\beta}+\Omega_B^2\right)^{1/2}, \tag{2.8}$$

where

$$\Omega_B^2=B_0^2/\varpi_0^2\varrho_0. \tag{2.9}$$

The roots of (2.8) are real and the system is stable if

$$\Omega^2>\beta(2-\beta)\Omega_B^2, \tag{2.10}$$

and complete stability is assured if

$$\Omega^2>\Omega_B^2. \tag{2.11}$$

This corresponds to $\beta=1$ showing that the most difficult mode to stabilize is that with $k=\infty$. The system is therefore stable if the rotation frequency exceeds the hydromagnetic frequency.

This result is not in fact changed if the axial magnetic field is introduced, because the system will be stable if

$$\Omega^2>\Omega_B^2(1+bk\varpi_0)\beta[2-(1+bk\varpi_0)\beta]. \tag{2.12}$$

The right-hand side of (2.12) only differs from that of (2.10) in having $(1+bk\varpi_0)\beta$ instead of β. Its maximum value is still Ω_B^2 but it now occurs when $(1+bk\varpi_0)\beta=1$. As the relevant values of β exceed 1, the hardest mode to stabilize has negative bk, a result which is to be expected from previous investigations without rotation (Tayler 1957b). Thus stability is ensured if $\Omega^2>\Omega_B^2$. Although this detailed discussion has been given for $m=1$, it is easy to see that the result is also true for $m>1$.

We next suppose that the rotation axis is in the (ϖ, ϕ) plane. If the rotation is about the y axis of cartesian coordinates, we have in cylindrical polar coordinates:

$$\boldsymbol{\Omega}=(\Omega\sin\phi, \Omega\cos\phi, 0). \tag{2.13}$$

It is then clear there can no longer be a simple decomposition into normal modes in ϕ. In fact all of the ϕ modes are coupled so that we obtain an infinite set of coupled differential equations. Obviously we cannot hope to find an exact solution to this system of equations. What we can do is to consider how the $m=1$ mode is modified by a small rotation velocity. In addition we consider only the case $b=0$. If we express all of the perturbed quantities in terms of exp $(ikz+i\omega t)$, the linearized equations are (with the suffix 1 on perturbed quantities dropped):

$$i\omega\varrho_0 v_\varpi+\Omega\varrho_0\{\exp(i\phi)+\exp(-i\phi)\}v_z=-\frac{\partial P}{\partial\varpi}-\frac{2B_0}{\varpi_0}B_\phi-\frac{B_0}{\varpi}\frac{\partial}{\partial\varpi}(\varpi B_\phi)+\frac{B_0}{\varpi_0}\frac{\partial B_\varpi}{\partial\phi}, \tag{2.14}$$

$$i\omega\varrho_0 v_\phi + i\Omega\varrho_0\{\exp(i\phi) - \exp(-i\phi)\}v_z = -\frac{1}{\varpi}\frac{\partial P}{\partial\phi} + \frac{2B_0}{\varpi_0}B_\varpi, \tag{2.15}$$

$$i\omega\varrho_0 v_z - i\Omega\varrho_0\{\exp(i\phi) - \exp(-i\phi)\}v_\phi - \Omega\varrho_0\{\exp(i\phi) + \exp(-i\phi)\}v'_{\varpi_0}$$
$$= -ikP + \frac{B_0}{\varpi}\frac{\partial B_z}{\partial\phi} - \frac{ikB_0\varpi}{\varpi_0}B_\phi, \tag{2.16}$$

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi v_\varpi) + \frac{1}{\varpi}\frac{\partial v_\phi}{\partial\phi} + ikv_z = 0, \tag{2.17}$$

$$i\omega B_\varpi = \frac{B_0}{\varpi_0}\frac{\partial v_\varpi}{\partial\phi}, \quad i\omega B_\phi = \frac{B_0}{\varpi_0}\frac{\partial v_\phi}{\partial\phi}, \quad i\omega B_z = \frac{B_0}{\varpi_0}\frac{\partial v_z}{\partial\phi}. \tag{2.18}$$

We now consider how an unstable $m=1$ mode is modified by rotation. Working to zero order in Ω, we have the equations for $b=\Omega=0$, whose solution has been discussed above. To first order in Ω, $m=0$ and $m=2$ modes are introduced. To second order in Ω, the $m=1$ mode is modified by the $m=0$ and $m=2$ modes and, if

$$\omega = \omega_0 + \omega_2, \tag{2.19}$$

where $\omega_2 = O(\Omega^2)$, the boundary condition $v_\varpi = 0$ then determines ω_2.

To this order (and presumably to higher orders as well), the equations can be solved completely in terms of Bessel functions although the algebra is very lengthy and complicated. We omit the details of the derivation and we simply state the relevant results. We first introduce some special cases of (2.6) for $\Omega=b=0$ and $m=1, 2$:

$$\lambda_1 = -2k\Omega_B^2/(\omega_0^2 - \Omega_B^2), \tag{2.20}$$

$$\lambda_2 = -4k\Omega_B^2/(\omega_0^2 - 4\Omega_B^2), \tag{2.21}$$

where $\omega_0^2 < 0$ for unstable modes and where in such a case $\lambda_1/k > 1$ and $\lambda_2/k < 1$. The terms in the solutions of the equations for v_ϖ of first order in Ω are then:

$$v_{\varpi 1}^0 = (A_0 k/\omega_0\varrho_0)I_1(k\varpi) - (\Omega k/\omega_0\lambda_1)J_1\{(\lambda_1^2 - k^2)^{1/2}\varpi\}, \tag{2.22}$$

$$v_{\varpi 1}^2 = [-2\lambda_2 A_2/\varpi(k^2 - \lambda_2^2)]I_2\{(k^2 - \lambda_2^2)^{1/2}\varpi\} - [kA_2/(k^2 - \lambda_2^2)^{1/2}]I_2'\{(k^2 - \lambda_2^2)^{1/2}\varpi\}$$
$$-\left[\Omega'\omega_0(\lambda_2 - \lambda_1)\left(1 - \frac{4\Omega_B^2}{\omega_0^2}\right)\right][\{2(\lambda_1 - k)^{1/2}/(\lambda_1 + k)^{1/2}\varpi\}J_2\{(\lambda_1^2 - k^2)^{1/2}\varpi\} + kJ_1\{(\lambda_1^2 - k^2)^{1/2}\varpi\}], \tag{2.23}$$

where the superfixes denote the value of m and the suffixes the order in Ω, the factor exp $im\phi$ being omitted. I_1 and I_2 are modified Bessel functions and the primes denote derivatives with respect to the argument. The constant of proportionality in (2.5) has been taken as unity and the constants A_0 and A_2 are determined by requiring that $v_{\varpi 1}^0$ and $v_{\varpi 1}^2$ both vanish at $\varpi=\varpi_0$. The expression for A_0 is simple

$$A_0 = \Omega\varrho_0 J_1\{(\lambda_1^2 - k^2)^{1/2}\varpi_0\}/\lambda_1 I_1(k\varpi_0), \tag{2.24}$$

and the more complicated expression for A_2 can be obtained from (2.23).

We next need an expression for $v_{\varpi 2}^1$. If we introduce the vectors

$$\mathbf{R} = (-i\Omega\varrho_0, \Omega\varrho_0, 0)\exp i\phi, \tag{2.25}$$

$$\mathbf{T}=\mathbf{R}\times\mathbf{v}_1^0, \tag{2.26}$$

$$\mathbf{U}=\mathbf{R}^*\times\mathbf{v}_1^2, \tag{2.27}$$

where the asterisk denotes the complete conjugate, the expression for $\mathbf{v}_{\varpi 2}^1$ can be shown to be

$$(\lambda_1^2-k^2)v_{\varpi 2}^1=(i\lambda_1/\varpi)v_{z2}^1+ik(dv_{z2}^1/d\varpi)+2\omega_2\omega_0\lambda_1(\lambda_1 v_{\varpi 0}^1-ikv_{\phi 0}^1)/(\omega_0^2-\Omega_B^2)$$
$$-i(\omega_0/\varrho_0)[\lambda_1\{\nabla\times(\mathbf{T}+\mathbf{U})\}_\varpi-ik\{\nabla\times(\mathbf{T}+\mathbf{U})\}_\phi]/(\omega_0^2-\Omega_B^2), \tag{2.28}$$

where

$$iv_{z2}^1=A_1J_1\{(\lambda_1^2-k^2)^{1/2}\varpi\}+(a/\lambda_1^2)I_1(k\varpi)-\{b\varpi/2(\lambda_1^2-k^2)^{1/2}\}J_1'\{(\lambda_1^2-k^2)^{1/2}\varpi\}$$
$$+\{c/(\lambda_1^2-k^2)\}I_1\{(k^2-\lambda_2^2)^{1/2}\varpi\}, \tag{2.29}$$

$$a=-\Omega^2k^2J_1\{(\lambda_1^2-k^2)^{1/2}\varpi_0\}/(\omega_0^2-\Omega_B^2)I_1(k\varpi_0), \tag{2.30}$$

$$b=\frac{4\omega_2\omega_0\lambda_1^2}{(\omega_0^2-\Omega_B^2)}-\frac{2\Omega^2(\lambda_1^2-k^2)}{(\omega_0^2-\Omega_B^2)}+\frac{2\Omega^2\omega_0^2\lambda_1(\lambda_1^2-k^2)}{(\lambda_2-\lambda_1)(\omega_0^2-\Omega_B^2)(\omega_0^2-4\Omega_B^2)}, \tag{2.31}$$

$$c=-\Omega\omega_0(\lambda_1+\lambda_2)(k^2-\lambda_2^2)^{1/2}A_2/(\omega_0^2-\Omega_B^2), \tag{2.32}$$

exp $i\phi$ is omitted from (2.29), A_1 is constant and all the other expressions can be obtained in a straightforward manner.

As explained earlier, the value of ω_2 is obtained by putting $v_{\varpi 2}^1=0$ at $\varpi=\varpi_0$. Obviously the general solution for ω_2 is very complex. It is not, however, difficult to see what happens for $k\to\infty$, which is the most unstable case in the absence of rotation. In this limit we have $\lambda_1\to k$, $\lambda_2\to 4k/5$, $\omega_0^2\to-\Omega_B^2$. We put $\lambda_1=k(1+\varepsilon)$ and work to lowest order in ε. The zero order (in Ω) boundary condition can be rewritten as

$$J_1\{(2\varepsilon)^{1/2}k\varpi_0\}=-(2\varepsilon)^{1/2}k\varpi_0 J_1'\{(2\varepsilon)^{1/2}k\varpi_0\} \tag{2.33}$$

to lowest order in ε and as $k\to\infty$

$$(2\varepsilon)^{1/2}k\varpi_0=j_0, \tag{2.34}$$

where $j_0\approx 2.4$ is the first zero of $J_0(x)$. When only leading order terms in ε are retained in the equation for ω_2, the result finally obtained is

$$\omega_2/\omega_0=2\Omega^2\varepsilon/\omega_0^2=\Omega^2j_0^2/\omega_0^2k^2\varpi_0^2. \tag{2.35}$$

Two things can be seen from this result. Since ω_0^2 is negative, ω_2 has the opposite sign to ω_0; thus the effect of rotation is to reduce the growth rate of the instability just as when the axes of rotation and magnetic field are parallel. However, the effect of rotation is minimized for large values of $k\varpi_0$ through the occurrence of the factor $k^2\varpi_0^2$ in (2.35). This indicates that for short wavelengths in the z direction the effect of rotation is much less stabilizing when the axes of rotation and magnetic field are perpendicular than when they are parallel. We shall discuss the possible significance of this result later. Note that, although this is a similar effect to the one suggested in the previous section as a result of the joint action of rotation and gravity, the present discussion does not include gravity.

We now return to the case of the axes of rotation and magnetic field parallel. We have so far considered only a very special configuration, which is incompressible and of uniform density and in which the ratio of the values of rotation speed and azimuthal magnetic field is independent of position. It is important to ask whether the result that stability occurs if the rotation frequency exceeds the hydromagnetic frequency is crucially dependent on these assumptions. We therefore consider a more general azimuthal field

$$\mathbf{B}_0=(0,\ B_0(\varpi/\varpi_0)^{n+1},\ 0). \tag{2.36}$$

Tayler (1957a) has shown that in the absence of rotation this field is stable if

$$m^2>2n+4. \tag{2.37}$$

If we introduce uniform rotation, the equations for the perturbed quantities can be written:

$$D(\varpi\psi)-m^2\beta\psi+(m^2+k^2\varpi^2)\chi=0, \tag{2.38}$$

$$D(\varpi\chi)+\chi\left[m^2\beta+\frac{2nm^2(\varpi/\varpi_0)^{2n}}{m^2(\varpi/\varpi_0)^{2n}-(\omega^2/\Omega_B^2)}\right]$$
$$+\psi\left[1-m^2\beta^2+\frac{2n(1-m^2\beta)(\varpi/\varpi_0)^{2n}}{m^2(\varpi/\varpi_0)^{2n}-(\omega^2/\Omega_B^2)}+\frac{4n\{\omega(\omega-m\Omega)/\Omega_B^2\}(\varpi/\varpi_0)^{2n}}{\{m^2(\varpi/\varpi_0)^{2n}-(\omega^2/\Omega_B^2)\}^2}\right]=0 \tag{2.39}$$

where

$$\beta=2[(\varpi/\varpi_0)^{2n}-(\Omega\omega/m\Omega_B^2)]/[m^2(\varpi/\varpi_0)^{2n}-(\omega^2/\Omega_B^2)], \tag{2.40}$$

$$\psi=mv_\varpi, \quad \chi=\beta mv_\varpi+iv_\phi, \tag{2.41}$$

D is $d/d\varpi$ and ω is as before the frequency in the rotating frame.

It is clear that there is no simple solution to equations (2.38) and (2.39). We attempt to obtain an approximate solution as follows. Consider $m=1$ and suppose that

$$|\omega^2/\Omega_B^2|\gg 1, \tag{2.42}$$

$$|\omega\Omega/\Omega_B^2|\gg 1. \tag{2.43}$$

If these inequalities are true and if n is not very large, equations (2.38), (2.39) then take the approximate form

$$D(\varpi\psi)-\beta_0\psi+(1+k^2\varpi^2)\chi=0, \tag{2.44}$$

$$D(\varpi\chi)+\beta_0\chi+(1-\beta_0^2)\psi=0, \tag{2.45}$$

where

$$\beta_0=2\Omega/\omega. \tag{2.46}$$

Equations (2.44), (2.45) combine to give Bessel's equation with the solutions involving $J_1\{(\beta_0^2-1)^{1/2}k\varpi\}$ which we have found earlier. Solutions which satisfy the boundary condition at $\varpi=\varpi_0$ only exist if $\beta_0^2>1$. For $k\to\infty$ we have $\beta_0\to 1$ and $\omega^2\to 4\Omega^2$, which is a stable solution. This solution is consistent with our assumptions (2.42) and (2.43) if

$$\Omega^2\gg\Omega_B^2. \tag{2.47}$$

It is not clear from this approximate discussion just how much the ratio Ω/Ω_B must exceed unity in order to give stability. If n is large, we shall require

$$\Omega^2\gg n\Omega_B^2 \tag{2.48}$$

to guarantee that a similar approximation is valid.

It is tempting to deduce from this result that, if the field gradient is high, stability will be produced provided that the rotation speed is also sufficiently high. It is not, however, obvious that this is correct. Although we have found some solutions of the dispersion relation satisfying inequalities (2.42) and (2.43), it was pointed out to us by Dr H. C. Spruit that there may be other solutions which do not satisfy these inequalities and which correspond to instabilities with growth rates $\ll\Omega_B$. An approach to these is to consider local stability criteria obtained by assuming that perturbation wavelengths in both the ϖ and z directions are very small compared to ϖ_0 and by replacing the differential equations by algebraic equations. Such local criteria have been obtained for much more general toroidal field configurations and including both gravitation and dissipative processes by Acheson (1978), and Spruit (private communication) has studied such a generalization of our present problem. Although it is not certain that an instability predicted by a local criterion does exist, it can certainly be highly suggestive. Instead of obtaining the result that we require from the general criterion of Acheson, we deduce it directly from equations (2.38) and (2.39).

If we write $D=ik_\varpi$ and assume that $k_\varpi\varpi_0$, $k\varpi_0\gg 1$ and also write $(\varpi/\varpi_0)^{2n}=\Lambda<1$, the approximate forms of the two equations become

$$ik_\varpi\varpi\psi+k^2\varpi^2\chi=0, \tag{2.49}$$

$$ik_\varpi\varpi\chi+\psi\left[1-\frac{4m^2(\Lambda-\Omega\omega/m\Omega_B^2)^2}{(m^2\Lambda-\omega^2/\Omega_B^2)^2}\right.$$
$$\left.+\frac{2n\Lambda}{(m^2\Lambda-\omega^2/\Omega_B^2)}\left\{1-\frac{2m^2(\Lambda-\Omega\omega/m\Omega_B^2)}{(m^2\Lambda-\omega^2/\Omega_B^2)}\right\}+\frac{4n\Lambda(\omega^2-m\omega\Omega)/\Omega_B^2}{(m^2\Lambda-\omega^2/\Omega_B^2)^2}\right]=0. \tag{2.50}$$

This gives the dispersion relation

$$k_\varpi^2(m^2\Lambda-\omega^2/\Omega_B^2)^2+k^2[m^2(m^2-2n-4)\Lambda^2+2(n-m^2)\Lambda\omega^2/\Omega_B^2$$
$$+\omega^4/\Omega_B^4+8m\Lambda\Omega\omega/\Omega_B^2-4\Omega^2\omega^2/\Omega_B^4]=0. \tag{2.51}$$

It is obvious from this equation that, if $\Omega=0$, there are no negative roots for ω^2/Ω_B^2, if $m^2<2n+4$, in agreement with the result of Tayler (1957a). It is less easy to obtain general results when $\Omega\neq 0$ so we will once again consider special cases. In fact, as there is no instability if $k_\varpi\gg k$, we will consider $k\gg k_\varpi$. In this case the dispersion relation takes the approximate form

$$[(m^2-n)\Lambda-\omega^2/\Omega_B^2]^2=4[m\Lambda-\Omega\omega/\Omega_B^2]^2+n^2\Lambda^2. \tag{2.52}$$

As this equation is unchanged if the signs of both m and ω are changed, there is instability if there is a complex root for ω.

We now consider a limiting case of this equation. Suppose that $m^2\ll n$, $n\gg 1$, so that there is instability in the case $\Omega=0$. Then the equation becomes

$$(\omega^4/\Omega_B^4)+2n\Lambda(\omega^2/\Omega_B^2)-4(\Omega^2\omega^2/\Omega_B^4)+8\Lambda m(\omega\Omega/\Omega_B^2)-2m^2n\Lambda^2=0. \tag{2.53}$$

This has two possible types of root for large Ω. The first has $\omega=O(\Omega)\gg\Omega_B$. In this case, if we write $\omega=c_1\Omega$, we can see that c_1 will be real and the system stable if

$$\Omega^2>n\Omega_B^2/2, \tag{2.54}$$

which is a refinement of inequality (2.48). However, there is a second root which has $\omega=O(\Omega_B^2/\Omega)\ll\Omega_B$. In this case we write $\omega=c_2\Omega_B^2/\Omega$ and we find that c_2 is complex and the system is unstable if

$$\Omega^2\geqslant n\Omega_B^2/2. \tag{2.55}$$

Inequalities (2.54) and (2.55) are almost exactly the same and they become the same in the limit $n\to\infty$. Provided that the local criterion is a good approximation to the true global criterion, we have the following conclusion. If the gradient of the magnetic induction is large, a greater rotation speed is needed to stabilize the instabilities that are present in the absence of rotation than is required for a field with a linear profile; this is a slightly imprecise statement because it has not been defined exactly what is held fixed. However, at essentially the same rotation speed as stabilizes the original instabilities, rotation introduces a new instability. Although its growth rate may be significantly smaller than the instability in the absence of rotation, for realistic values of

Ω_B and Ω the growth time can still be very short compared to any time of relevance in stellar evolution. It thus appears that, at least in the absence of gravitation, rotation is unlikely to lead to complete stability of general magnetic field configurations. While our discussion relates only to incompressible fluids, it seems likely that compressibility allows further freedom and more instability rather than less.

After this discussion of the stability of rotating pinches without gravitation, we try to consider the rôle of gravitation. We expect this to exert an additional stabilizing influence if, as we assume, the star is stable against convection.

3 The influence of gravity

In Tayler (1980) we discussed the stability of the field configuration (2.1), but without the added assumption that the fluid was incompressible and of uniform density, in the case that the axis of the magnetic field passed through the centre of a non-rotating star. What we showed was that unstable perturbations existed which involved no motion in the direction of gravity. To be precise, we studied perturbations near the axis of symmetry which had no z component, and we neglected the ϖ component of the gravitational field. Implicit in this was the assumption that the result obtained would be the same as if we considered a perturbation ξ such that $\xi\cdot\nabla\Phi=0$. Before we consider the rotating case, we demonstrate that this is true and we obtain an estimate of the growth rate of the instability. What was shown in Tayler (1980) was that, if the fluid is compressible, there are always instabilities when $m+bk\varpi_0=0$; in order to use the expression exp (ikz), it is being assumed that the wavelength in the z direction is small compared to the distance in which any of the physical variables changes significantly.

The energy principle of Bernstein *et al.* (1958) tells us that the non-rotating system is unstable if there is any perturbation ξ which reduces the potential energy of the system. The change of potential energy is

$$\delta W=\frac{1}{2}\int d\tau[\mathbf{Q}^2-\mathbf{j}\cdot\mathbf{Q}\times\xi+\gamma P(\nabla\cdot\xi)^2+\xi\cdot\nabla P\ \nabla\cdot\xi+\xi\cdot\nabla\Phi\ \nabla\cdot\varrho\xi]^{\star}, \tag{3.1}$$

where

$$\mathbf{Q}=\nabla\times(\xi\times\mathbf{B}), \tag{3.2}$$

and the integral is over the volume of the system. If we consider only perturbations which satisfy

$$\xi\cdot\nabla\Phi=0 \tag{3.3}$$

and if we use the equilibrium equation

$$\nabla P=\mathbf{j}\times\mathbf{B}+\varrho\nabla\Phi, \tag{3.4}$$

$$\delta W=\frac{1}{2}\int\ d\tau[\mathbf{Q}^2-\mathbf{j}\cdot\mathbf{Q}\times\xi+\gamma P(\nabla\cdot\xi)^2+\xi\cdot\mathbf{j}\times\mathbf{B}\ \nabla\cdot\xi]. \tag{3.5}$$

Here it can be seen that, if the structure of the field is specified, P only enters through the term $\gamma P(\nabla\cdot\xi)^2$, while the density does not appear in (3.5).

If we now assume a cylindrical field $B_\phi(\varpi)$, $B_z(\varpi)$, and perturbations of the form exp $(im\phi+ikz)$, we get

$$\begin{aligned}\delta W\propto\int\varpi\ d\varpi\Bigg\{\left(\frac{mB_\phi}{\varpi}+kB_z\right)^2\xi\cdot\xi^*+(B_\phi^2+B_z^2+\gamma P)\ \nabla\cdot\xi\ \nabla\cdot\xi^*-2B_\phi\ D(B_\phi/\varpi)\xi_\varpi\xi^*_\varpi\\ -(2B_\phi^2/\varpi)(\xi_\varpi\nabla\cdot\xi^*+\xi^*_\varpi\nabla\cdot\xi)+i\left(\frac{mB_\phi}{\varpi}+kB_z\right)\bigg[\frac{2B_\phi}{\varpi}(\xi^*_\varpi\xi_\phi-\xi^*_\phi\xi_\varpi)\\ -B_\phi(\xi_\phi\nabla\cdot\xi^*-\xi^*_\phi\nabla\cdot\xi)-B_z\,(\xi_z\nabla\cdot\xi^*-\xi^*_z\nabla\cdot\xi)\bigg]\Bigg\},\end{aligned} \tag{3.6}$$

$\star$As we have taken the gravitational force to be $+\varrho\nabla\Phi$, the sign of the term in Φ differs from that in Bernstein *et al.*

where the asterisk denotes complex conjugate. In the special case of the field (2.1) and of $m+bk\varpi_0=0$, this becomes

$$\delta W \propto \int \varpi \, d\varpi \, [(B_\phi^2+B_z^2+\gamma P)\, \nabla\cdot\xi \, \nabla\cdot\xi^* - (2B_\phi^2/\varpi)(\xi_\varpi \nabla\cdot\xi^* + \xi_\varpi^* \nabla\cdot\xi)]$$

$$= \int \varpi \, d\varpi \left[(B_\phi^2+B_z^2+\gamma P) \left(\nabla\cdot\xi - \frac{(2B_\phi^2/\varpi)\xi_\varpi}{B_\phi^2+B_z^2+\gamma P}\right)\left(\nabla\cdot\xi^* - \frac{(2B_\phi^2/\varpi)\xi_\varpi^*}{B_\phi^2+B_z^2+\gamma P}\right) - \frac{(4B_\phi^4/\varpi^2)\xi_\varpi \xi_\varpi^*}{B_\phi^2+B_z^2+\gamma P}\right]. \tag{3.7}$$

The system can now be seen to be unstable because δW is certainly negative for perturbations which satisfy

$$\nabla\cdot\xi = \frac{1}{\varpi} D(\varpi\xi_\varpi) + \frac{im}{\varpi}\xi_\phi + ik\xi_z = \frac{(2B_\phi^2/\varpi)}{B_\phi^2+B_z^2+\gamma P}\xi_\varpi \tag{3.8}$$

as well as (3.3). This is precisely the same result as was obtained in Tayler (1980) by putting $\xi_z=0$ and neglecting g_ϖ.

It is possible to obtain a rough estimate of the growth rate of the instability as follows. The kinetic energy associated with the disturbance, if it behaves like exp $(i\omega t)$, is

$$\delta T = -\frac{1}{4}\omega^2 \int d\tau \varrho \xi\cdot\xi^*. \tag{3.9}$$

Noting that in a realistic situation $\gamma P \gg B_\phi^2$, B_z^2 and that ξ_ϕ and ξ_ϖ must be of the same order, and equating the change of kinetic energy to the release of potential energy, we obtain

$$-\omega^2 = O(B_0^4/\gamma P \varrho \varpi_0^2) = O(\Omega_B^4/\Omega_s^2), \tag{3.10}$$

where Ω_B^2 is as defined earlier and $\Omega_s^2 = \gamma P_0/\varrho_0\varpi_0^2$, P_0 and ϱ_0 being the values of P and ϱ on the axis of symmetry. Although in general $\Omega_s \gg \Omega_B$, the resulting growth time is very small compared to any time important in stellar evolution even for quite weak fields.

When we introduce rotation we do something which is analogous to putting $\xi_z=0$ and neglecting g_ϖ. It is not clear that this is a valid approach as will become obvious in our discussion; apart from anything else the relation between the axis of rotation and the gravitational field changes as ϖ increases. When we introduce rotation we cannot use the energy principle but must solve normal-mode equations. In fact we study a constrained problem in which motion in the z direction, which differs only slightly from the direction of gravity close to the axis of symmetry, is impossible. We enforce this by not using the z component of the equation of motion and at the same time we put v_z equal to zero in the other equations. In addition we neglect the centrifugal force and g_ϖ. It hoped that the properties of the resulting normal modes are similar to those of a real problem in which z motion is not impossible but is strongly inhibited. It is necessary to allow for compressibility, so that the equations are more complicated than in the previous Section.

We first consider the field (2.1), ϱ=constant$=\varrho_0$, and $P=P_0-B_0^2\varpi^2/\varpi_0^2$; here our neglect of g_ϖ and of the centrifugal force is explicit. If we were not neglecting g_ϖ, this density profile would be unstable to convection, but that instability is suppressed in our discussion. If we write down the perturbed equations and consider only those perturbations for which $m+bk\varpi_0=0$, after very considerable algebra the following equation is obtained for v_ϖ:

$$D\left[\frac{(\bar{\Omega}^2/\omega^2)}{(1-m^2\bar{\Omega}^2/\omega^2)}\varpi D(\varpi v_\varpi)\right] + v_\varpi\left[1 - \frac{A^2}{(1-m^2\bar{\Omega}^2/\omega^2)} - mA\varpi D\left\{\frac{\bar{\Omega}^2/\omega^2}{1-m^2\bar{\Omega}^2/\omega^2}\right\}\right] = 0, \tag{3.11}$$

where

$$\bar{\Omega}^2 = \frac{\gamma P}{\varrho_0 \varpi^2} + \frac{B_0^2}{\varrho_0 \varpi_0^2} + \frac{B_0^2 b^2}{\varrho_0 \varpi^2} \tag{3.12}$$

and

$$A = \frac{2\Omega}{\omega} + \frac{2mB_0^2}{\varrho_0 \omega^2 \varpi_0^2}. \tag{3.13}$$

This full equation is very complicated but it can be simplified if we note that we are usually interested in fields such that

$$P_0 \gg B_0^2. \tag{3.14}$$

If this is so,

$$\bar{\Omega}^2 \approx \gamma P_0 / \varrho_0 \varpi^2 \tag{3.15}$$

and

$$\bar{\Omega}^2/\omega^2(1 - m^2\bar{\Omega}^2/\omega^2) \approx 1/(\varrho_0 \omega^2 \varpi^2/\gamma P_0 - m^2). \tag{3.16}$$

In addition, provided $\gamma P_0 \gg |\varrho_0 \omega^2 \varpi^2|$ so that any instability growth rates are small compared to the sound frequency,

$$\bar{\Omega}^2/\omega^2(1 - m^2\Omega^2/\omega^2) \approx -1/m^2. \tag{3.17}$$

With these approximations the equation for v_ϖ can be written in the form

$$D[\varpi D\,(\varpi v_\varpi)] = m^2 \lambda v_\varpi, \tag{3.18}$$

where

$$\lambda = 1 - A^2/(1 - m^2\bar{\Omega}/\omega^2) - mA\varpi D[\bar{\Omega}^2/\omega^2(1 - m^2\bar{\Omega}^2/\omega^2)]. \tag{3.19}$$

If only the first approximation in λ is kept, equation (3.18) can be further rewritten

$$D[\varpi D(\varpi v_\varpi)] = (m^2 + \mu^2 \varpi^2) v_\varpi, \tag{3.20}$$

where

$$\mu^2 = \frac{\varrho_0 \omega^2}{\gamma P_0}\left[A^2 + \frac{2A}{m}\right], \tag{3.21}$$

which is a form of Bessel's equation.

We now require v_ϖ to vanish at $\varpi = \varpi_0$; we are once again considering disturbances which are confined within a radius which can in turn be taken to have an arbitrary value. The solution for v_ϖ is

$$v_\varpi \propto J_m(i\mu\varpi)/\varpi \tag{3.22}$$

and the boundary condition is satisfied if

$$\mu^2 \varpi_0^2 + j_m^2 = 0, \tag{3.23}$$

where j_m is a zero of $J_m(x)$. It is now possible to ask what is the condition for all of the roots of (3.23) for ϖ to be real so that the system is stable. After some considerable algebra it can be shown that the system is only stabilized if $\Omega^2 > \Omega_s^2 \equiv \gamma P_0/\varrho_0 \varpi_0^2$. It is not, however, realistic to suppose that the rotational energy exceeds the thermal energy, which is implied by this inequality. For any reasonable rotation speed there is an instability whose growth rate can be shown to be

$$\omega \approx -2im\Omega_B^2/j_m\Omega_s. \tag{3.24}$$

This is essentially the result given in equation (3.10) and in Tayler (1980). It also satisfies the inequality leading to (3.17). If our approximate treatment of gravity is valid, the extra freedom allowed by compressibility means that rotation is not generally stabilizing.

Suppose next that we consider the case of rotation about the y axis but assume that the gravitational force close to the axis of symmetry of the magnetic field enforces $v_z=0$ and that we do not use the z component of the equation of motion. If we look at the left-hand sides of equations (2.14) and (2.15) and note that the equivalent equation to (2.16) is being disregarded, it can be seen that the assumption $v_z=0$ removes the Coriolis force from the equations. We are thus left with the non-rotating equations and with no apparent stabilizing effect due to rotation. It should be noted that the Coriolis force drops out in this approximation for any magnetic field in which B_ϕ and B_z are functions of ϖ alone. It thus appears that stabilization is even less effective when the axes of rotation and of the magnetic field are perpendicular than when they are parallel. There are, however, all of the uncertainties which there were in the previous problem relating to the treatment of centrifugal force and gravity, and in addition the axis of rotation and gravity are only precisely perpendicular on the axis of symmetry of the magnetic field. Despite this it seems very likely that the qualitative result is correct.

In Section 2, our discussion without gravity indicated that a toroidal field with a high value of $dB/d\varpi$ (high n) would be less readily stabilized by rotation than a low-n field. In earlier papers in which the effect of gravity was included (Tayler 1973, 1980) we have indicated that instability is unlikely to occur close to the axis of symmetry of a star if n is high and it may therefore appear that the high-n case is irrelevant. In fact our previous discussion, which was concerned with the region close to $\varpi=0$ where $dB/d\varpi$ is small, was incomplete as can be seen easily (see also more general discussions of the stability of toroidal fields in Goossens & Tayler 1980 and Goossens *et al.* 1981). It is convenient to consider first the $m=0$ criterion for interchanges in the ϖ direction, in which case it is also simple to include rotation. In the case of uniform rotation (Tayler 1973; Acheson 1978; Chanmugam 1979) the general criterion for instability is

$$\frac{\partial\varrho}{\partial\varpi}(g_\varpi+\Omega^2\varpi)-\left(\frac{2B^2}{\varpi}-\varrho\Omega^2\varpi-\varrho g_\varpi\right)^2\Big/(B^2+\gamma P)-\frac{2B}{\varpi}\frac{\partial B}{\partial\varpi}+\frac{2B^2}{\varpi^2}+4\varrho\Omega^2<0. \tag{3.25}$$

When $B=B_0(\varpi/\varpi_0)^{n+1}$ and n is large, it is clearly possible for any finite value of ϖ (say ϖ close to ϖ_0) to choose n large enough that the term $2B\partial B/\varpi\partial\varpi$ makes the system unstable, while at the same time the magnetic field is unimportant in the overall structure of the star. In addition, if Ω^2 is to be increased until stability is no longer possible, we must obviously have

$$\Omega^2 \gtrsim n\Omega_B^2/2 \tag{3.26}$$

as in some of the instabilities in Section 2. This condition for stability will only be compatible with $\Omega^2\varpi \ll g_\varpi$ if the potential magnetic instability occurs close enough to the axis of symmetry for the scale height of ϱ to be much greater than ϖ. If this is not the case, no acceptable rotation speed can remove the instability. The instability which we have just described appears to be similar to the magnetic buoyancy instability discussed by Parker (1979), for example. If the toroidal field is broken up into discrete flux ropes instead of having the continuous structure which we have considered, instability will always be predicted at the low ϖ side of the rope. If initially the configuration is continuous but unstable, instability may concentrate the flux into ropes and enhance the instability.

We can also consider the $m=1$ criterion but this time in the absence of rotation. From Tayler (1973) this is known to be

$$g_{\varpi}\frac{\partial \varrho}{\partial \varpi}-\frac{\varrho^2 g_{\varpi}^2}{\gamma P}-\frac{B^2}{\varpi^2}-\frac{2B}{\varpi}\frac{\partial B}{\partial \varpi}<0 \tag{3.27}$$

for instability. We have previously shown that close to $\varpi=0$ the toroidal field with $n=0$ must be unstable, that with $n=1$ instability may occur and that there is no such instability for $n>1$. This discussion once again relies on an expansion in powers of ϖ. It is, however, obvious that if there are strong enough field gradients some way from $\varpi=0$, there will be instability so that we expect isolated flux ropes to be unstable to non-axisymmetric as well as to axisymmetric perturbations. It has not proved possible to obtain a closed criterion analogous to (3.27) in the presence of uniform rotation, although Acheson (1978) has obtained local criteria from which we expect a high rotation speed to be necessary to suppress the instabilities which are present in the absence of rotation in a region of high field gradient, and we suspect that the new instabilities with lower but still significant growth rate, which we discussed in Section 2, will arise again.

4 Discussion of results

The first point to be made in this Section is that because we have been discussing model problems and because in several cases we have only been able to provide a very approximate discussion, the results which we have obtained are suggestive rather than rigorous. We will discuss our results in terms of the effect of rotation on particular instabilities which have been discussed in previous papers of this series.

Let us consider first the case of magnetic fields close to the axis of symmetry of a star. In Tayler (1973) it was shown that purely toroidal field produced by currents which did not vanish on the axis of symmetry were definitely unstable. In Tayler (1980) it was shown that this instability would persist even in the presence of a poloidal field which was to a first approximation uniform close to the axis of symmetry. It was also shown that the equations of equilibrium required that such a field must be confined to the radiative core of the star. Our discussion in Section 2, in which we neglected gravity, shows that the instability of a toroidal field and indeed of the mixed poloidal/toroidal field is removed if the rotation speed is equal to the hydromagnetic speed, if only incompressible disturbances are considered. It is obvious that a higher rotation speed would be necessary to stabilize compressible disturbances but we have not estimated the value. Although the toroidal field with uniform current density can be stabilized in this manner, our discussion of more general field configurations suggests that instabilities with smaller growth rates ($\approx\Omega_B^2/\Omega$) are likely to persist even when Ω is very high.

In reality, gravitation must also be taken into account. In a convectively stable region we expect it to exert an additional stabilizing influence and we know that in the absence of rotation only compressible instabilities remain. We have therefore given an approximate discussion of the stability of a compressible fluid in the presence of gravitation and rotation. Our results indicate that stability is only produced when rotation speeds are comparable with sound speeds, an unrealistically high value for an object that is recognizably a star. For lower rotation speeds an instability remains whose growth rate is $\approx\Omega_B^2/\Omega_s$, which is the growth rate of instabilities in the absence of rotation. Although this is less than Ω_B and even more so than Ω_s, it is still a high growth rate if the field is reasonably strong.

In Tayler (1980) we argued that a strong toroidal field in the centre of the Sun, such as had been suggested by Dicke (1979), would be unstable, whether or not it was accompanied by a weak poloidal field. In that discussion we made no estimate of the effect of rotation. We now believe that there will continue to be instabilities for any rotation speed which has been suggested for the centre of the Sun even though the instability growth rate may be $\approx\Omega_B^2/\Omega_s$ rather than $\approx\Omega_B$, as has been suggested earlier. If the field is to be strong enough to effect either the oblateness of the Sun or the solar neutrino problem, Ω_B cannot be very much less than Ω_s, so that the reduction of growth rate is not very great. This conclusion would apply to various other suggestions that the Sun might have a very strong internal field.

If the precise results which we have obtained are to be valid, non-adiabatic effects must be unimportant. In the Sun, radiative diffusion is more rapid than ohmic diffusion. The characteristic radiative diffusion time for the entire Sun is $\approx 10^7$ yr, so that for a fraction ε of the radius of the Sun it is $\approx \varepsilon^2 10^7$ yr. Our discussion requires perturbations to be confined fairly close to the axis of symmetry and that the wavelength in the direction of the axis of symmetry should be even smaller. Suppose for example we take $\varpi_0 \approx 0.1\ r_\odot$ and $2\pi/k \approx 0.01\ r_\odot$. We then require a growth time less than 10^3 yr for our treatment to be valid. Using values of the physical variables close to the centre of the Sun, our actual estimated growth time is $\approx 10^2(10^6\ \mathrm{G}/B)^2$ yr. This suggests that the instability which we have discussed would certainly occur if $B > 10^6$ G. Whether or not the field would survive such an instability depends on its non-linear behaviour, which has not yet been investigated. Furthermore, the fact that non-adiabatic effects would make our discussion invalid for weaker fields does not mean that instabilities would not occur; it is just that we have not investigated non-adiabatic effects, although Acheson (1978) and Spruit (private communication) have included them in their local criteria.

We next consider the same field configuration but suppose that the axis of rotation is perpendicular to the axis of the field. The discussion of Section 2 indicates that rotation alone is much less effective as a stabilizing agent in this case. Our discussion in Section 3, where the effect of gravity is also included, is much more sketchy but it also suggests that, if the rotation axis is perpendicular to the axis of symmetry of the field, instability will be much more likely. We think, however, that a more appropriate problem to discuss is that of the instabilities of a mainly poloidal field close to the magnetic axis, which have been studied by Markey & Tayler (1973, 1974) and Wright (1973).

Here, the field configuration near the magnetic axis resembles that of a toroidal pinched discharge; for short enough wavelengths in the direction around the torus axis, the configuration can be considered as essentially cylindrical. In the case of the purely poloidal field the worst instabilities are of short wavelength. If the star rotates about the axis of symmetry of the field, the rotation axis is everywhere perpendicular to the axis of the magnetic field. We therefore expect restricted stabilization by rotation, particularly in the case of the short-wavelength instabilities. This result has already been suggested in Markey & Tayler (1973) but it is now supported by the calculations in this paper. We might therefore expect these instabilities to be relevant to purely poloidal fields in the outer layers of early-type stars, provided that the fields are not being regenerated by any dynamo action in the convective core, that rotation is about the axis of symmetry of the field, and that the field is strong enough for non-adiabatic effects to be unimportant; as before, this final proviso is sufficient but may not be necessary.

We have evidence that many observed magnetic stars are oblique rotators and also that the field configurations are in some cases better approximated by off-centred dipoles than by centred dipoles. This means that we should consider a more general relation beween the magnetic field and the rotation axis. Suppose that in our simple model problem the rotation axis is perpendicular to the axis of symmetry of the field. In this case some of the field lines of the poloidal field are rotating about what is in effect a parallel axis, while others are rotating about a perpendicular axis. As a result, we can expect an increased degree of stabilization due to rotation, particularly if there is even a weak toroidal field connecting what would otherwise be independent poloidal field loops. Even if complete stability was not predicted, a reduced growth rate and consequent non-adiabatic effects might be equivalent to stability. We therefore have a rather weak suggestion that the field of the oblique rotator might be more stable than the field of the parallel rotator, particularly if the fields are strong enough.

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