

## The adjoint of a positive semigroup

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### 0. Introduction

Let  $\{T(t)\}_{t \geq 0}$  be a  $C_0$ -semigroup of bounded linear operators on a (real or complex) Banach space  $X$ . By defining  $T^*(t) := (T(t))^*$  for each  $t$ , one obtains a semigroup  $\{T^*(t)\}_{t \geq 0}$  on the dual space  $X^*$ . Throughout this paper, we will denote the semigroups  $\{T(t)\}_{t \geq 0}$  and  $\{T^*(t)\}_{t \geq 0}$  by  $T(t)$  and  $T^*(t)$ , respectively, and it will be clear from the context when we mean the semigroup or the single operator.

The adjoint semigroup  $T^*(t)$  fails in general to be strongly continuous again. Therefore it makes sense to define

$$X^\odot = \left\{ x^* \in X^* : \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0 \right\}.$$

This is the maximal subspace of  $X^*$  on which  $T^*(t)$  acts in a strongly continuous way. The space  $X^\odot$  was introduced by Phillips in 1955 [Ph]. Recently, this space has been studied extensively by various authors (e.g., [Ne], [NP], [P]), in particular in connection with applications to certain evolution equations (e.g., [Cl]).

The purpose of this paper is to study the properties of  $E^\odot$  in case  $E$  is a Banach lattice and  $T(t)$  is a positive  $C_0$ -semigroup. Virtually nothing is known about the Banach lattice properties of  $E^\odot$  and one of the most obvious questions, viz. under what conditions  $E^\odot$  is a sublattice of  $E^*$ , is wide open. If  $T^*(t)$  is a lattice semigroup, in particular if  $T(t)$  extends to a positive group, then  $E^\odot$  is a sublattice [Cl, Part IV]; this follows from

$$|T^*(t)(x^\odot)^+ - (x^\odot)^+| = |(T^*(t)x^\odot)^+ - (x^\odot)^+| \leq |T^*(t)x^\odot - x^\odot|$$

and the lattice property of the norm. Recently, Grabosch and Nagel [GN] constructed a positive  $C_0$ -semigroup on an AL-space  $E$  for which  $E^\odot$  is not a sublattice of  $E^*$ . In fact, in this example the space  $E^\odot$ , with the order inherited from  $E^*$ , even fails to be a Banach lattice in its own right.

In order to motivate our main results, we start by considering in some detail the translation group  $T(t)$  on  $C_0(\mathbb{R})$ , given by  $T(t)f(s) = f(t + s)$ . This semi-group has some features which turn out to be representative for the abstract situation.

**THEOREM 0.1.** *Let  $T(t)$  be the translation group on  $E = C_0(\mathbb{R})$ .*

- (i) ([P1])  $\mu \in E^\ominus$  if and only if  $\mu$  is absolutely continuous with respect to the Lebesgue measure  $m$ .
- (ii) ([MG], [WY]) If  $\mu \in E^*$  is singular with respect to  $m$ , then  $T(t)\mu \perp \mu$  for almost all  $t \in \mathbb{R}$ . In particular, for any  $v \in E^*$  we have  $\limsup_{t \downarrow 0} \|T^*(t)v - v\| = 2\|v_s\|$ , where  $v_s$  is the singular part of  $v$ .
- (iii) The space of singular measures is  $T^*(t)$ -invariant.

Note that  $T^*(t)v$  is just the translate in the opposite direction of  $v$  in the sense that for a measurable set  $G$  we have  $(T^*(t)v)(G) = v(G + t)$ . Also, by (i) it is clear that a measure  $\mu$  is singular if and only if  $\mu \perp E^\ominus$  in the Banach lattice sense. Versions of Theorem 0.1 for commutative locally compact groups (instead of  $\mathbb{R}$ ) can be found in [GM, Chapter 8]. In [Pa2], the Wiener-Young theorem ((ii) above) has been analysed in detail in the context of adjoint semigroups. There, extensions have been obtained for the adjoints of positive semigroups essentially on  $C(K)$ -spaces. In the present paper, most of the results in [Pa2] will be extended to positive semigroups on arbitrary Banach lattices. For the convenience of the reader, we include full proofs. Although several proofs are completely different, this causes a small overlap with [Pa2].

We will prove the following Banach lattice versions of (i)-(iii). Let  $T(t)$  be a positive  $C_0$ -semigroup on a Banach lattice  $E$ . Then:

- (i)  $E^\ominus$  is a projection band if  $E^*$  has order continuous norm (Theorem 2.1).

The most important class of (non-reflexive) Banach lattices whose duals have order continuous norm is the class of AM-spaces. This class contains  $C_0(\mathbb{R})$ . In contrast, note that the dual of an AL-space does not have order continuous norm unless  $E$  is finite-dimensional.

- (ii) Suppose  $x^* \perp E^\ominus$ . Then we have  $\limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\| \geq 2\|x^*\|$  (Theorem 4.4). If moreover  $E^*$  has order continuous norm or  $E$  has a quasi-interior point, then  $T^*(t)x^* \perp x^*$  for almost all  $t \geq 0$  (Corollary 3.4).
- (iii) The disjoint complement of  $E^\ominus$  is  $T^*(t)$ -invariant if  $T^*(t)$  is a lattice semigroup (Corollary 4.8).

We use (iii) to show that if  $T^*(t)$  is a lattice semigroup, then the quotient  $E^*/(E^\ominus)^{dd}$  is either zero or else ‘very large’ (Theorem 4.10). Here  $(E^\ominus)^{dd}$  is the band generated by  $E^\ominus$ .

We assume the reader to be familiar with some standard theory of Banach lattices. For more information as well as the terminology we refer to [M], [AB], [S], [Z]. Throughout this paper, all Banach spaces and lattices may be either real or complex.

**1. Some preliminary information**

In this section we recall some of the basic facts about adjoint semigroups which will be used in the sequel. Proofs can be found e.g. in [BB].

Let  $T(t)$  be a  $C_0$ -semigroup (i.e., a strongly continuous semigroup) on a Banach space  $X$ . Its generator will be denoted by  $A$  with domain  $D(A)$ . Considering the adjoint semigroup  $T^*(t)$  on the dual space  $X^*$ , we define

$$X^\odot = \left\{ x^* \in X^* : \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0 \right\},$$

the domain of strong continuity of  $T^*(t)$ . Then  $X^\odot$  is a  $T^*(t)$ -invariant, norm closed, weak\*-dense subspace of  $X^*$  (hence  $X^\odot = X^*$  if  $X$  is reflexive). The space  $X^\odot$  is precisely the norm closure of  $D(A^*)$ , the domain of the adjoint of  $A$ . In particular, for  $\lambda \in \rho(A) = \rho(A^*)$  we have  $R(\lambda, A^*)x^* \in X^\odot$  for all  $x^* \in X^*$ , where  $R(\lambda, A^*) = R(\lambda, A)^* = (\lambda - A^*)^{-1}$  is the resolvent. For all  $x^* \in X^*$  we have  $\lim_{\lambda \rightarrow \infty} \lambda R(\lambda, A^*)x^* = x^*$ , where the limit is in the weak\*-sense. An alternative description of  $X^\odot$  is given by

$$X^\odot = \left\{ x^* \in X^* : \lim_{\lambda \rightarrow \infty} \|\lambda R(\lambda, A^*)x^* - x^*\| = 0 \right\}.$$

If  $T(t)$  extends to a  $C_0$ -group, then the space  $X^\odot$  with respect to the semigroup  $\{T(t)\}_{t \geq 0}$  is equal to the domain of strong continuity of the group  $\{T(t)\}_{t \in \mathbb{R}}$ .

Examples of spaces  $X^\odot$  for various semigroups can be found in [BB], [Ne], [NP]. In particular we mention that if  $T(t)$  is the translation group on  $C_0(\mathbb{R})$  or  $L^1(\mathbb{R})$ , then  $X^\odot$  can be identified canonically with  $L^1(\mathbb{R})$  and  $BC(\mathbb{R})$  (the space of all bounded, uniformly continuous functions on  $\mathbb{R}$ ), respectively.

We will have the occasion to use the so-called weak\*-integrals (or Gelfand integrals) of  $X^*$ -valued functions. Let  $[a, b] \subset \mathbb{R}$  and  $f: [a, b] \rightarrow X^*$  a weak\*-continuous function (or, more generally, a weak\*-measurable function such that  $t \mapsto \langle f(t), x \rangle \in L^1[a, b]$  for all  $x \in X$ ). The weak\*-integral  $\text{weak}^* \int_a^b f(t) dt \in X^*$  is then defined by the formula

$$\left\langle \text{weak}^* \int_a^b f(t) dt, x \right\rangle = \int_a^b \langle f(t), x \rangle dt, \quad \forall x \in X.$$

In this situation, the function  $t \mapsto \|f(t)\|$  is a Borel function on  $[a, b]$  and we have the estimate

$$\left\| \text{weak}^* \int_a^b f(t) dt \right\| \leq \int_a^b \|f(t)\| dt.$$

If  $T(t)$  is a  $C_0$ -semigroup on  $X$ , then for each  $x^* \in X^*$  the map  $t \mapsto T^*(t)x^*$  is weak\*-continuous on  $[0, \infty)$  and for all  $0 \leq a < b \in \mathbb{R}$  we have

$$\text{weak}^* \int_a^b T^*(t)x^* dt \in D(A^*) \subset X^\odot.$$

Finally we say a few words about the Banach lattice situation. Let  $E$  be a Banach lattice and  $T(t)$  a positive  $C_0$ -semigroup on  $E$ . Suppose that  $M, \omega$  are such that  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . If  $\lambda \in \mathbb{R}$  is such that  $\lambda > \omega$ , then  $\lambda \in \rho(A)$  and  $R(\lambda, A) \geq 0$  (for the basic theory of positive semigroups we refer to [Na]). As mentioned in the introduction,  $E^\odot$  need not be a sublattice of  $E^*$ . As usual, we denote by  $(E^\odot)^d$  the disjoint complement of  $E^\odot$  in  $E^*$ , i.e.,

$$(E^\odot)^d = \{x^* \in E^* : x^* \perp y^\odot \text{ for all } y^\odot \in E^\odot\}.$$

Here  $x^* \perp y^\odot$  means that  $|x^*| \wedge |y^\odot| = 0$ . Then  $(E^\odot)^{dd}$ , the disjoint complement of  $(E^\odot)^d$ , is equal to the band generated by  $E^\odot$ . Since  $E^\odot = \overline{D(A^*)}$ , it is clear that  $(E^\odot)^{dd} = (D(A^*))^{dd}$ . In general,  $(E^\odot)^d$  is not  $T^*(t)$ -invariant (see Example 3.7). However, the subspace  $(E^\odot)^{dd}$  is always  $T^*(t)$ -invariant. Indeed, if  $x^* \in E^*$  is such that  $|x^*| \leq |R(\lambda, A^*)y^*|$  for some  $y^* \in E^*$  and  $\lambda > \omega$ , then  $|T^*(t)x^*| \leq R(\lambda, A^*)T^*(t)|y^*|$ . This shows that the (order) ideal generated by  $R(\lambda, A^*)(E^*) = D(A^*)$  is  $T^*(t)$ -invariant. Since  $T^*(t)$ , being the adjoint of a positive operator, is order continuous, this implies that the band  $(D(A^*))^{dd} = (E^\odot)^{dd}$  is  $T^*(t)$ -invariant as well.

## 2. The structure of $E^\odot$

In this section we will assume that  $T(t)$  is a positive  $C_0$ -semigroup on a Banach lattice  $E$ .

**THEOREM 2.1.** *If  $E^\odot$  is contained in a sublattice of  $E^*$  with order continuous norm, then  $E^\odot$  is an ideal in  $E^*$ . In particular, if  $E^*$  has order continuous norm, then  $E^\odot$  is a projection band.*

*Proof.* Let  $F$  be a sublattice of  $E^*$  with order continuous norm, containing  $E^\odot$ .

Step 1. First let  $0 \leq |x^*| \leq y^*$  with  $y^* \in E^\odot$ . We will show that  $x^* \in E^\odot$ . Choose  $\lambda_0 > 0$  such that  $R(\lambda, A) \geq 0$  for  $\lambda \geq \lambda_0$ . Put

$$G := \{\lambda R(\lambda, A)^*y^* : \lambda \geq \lambda_0\}.$$

Since  $y^* \in E^\odot$ , this set is relatively compact subset of  $F$ , hence certainly relatively weakly compact in  $F$ . Let

$$\text{sol}_F G := \{f \in F: \exists g \in G \text{ with } |f| \leq g\}$$

be the solid hull of  $G$  in  $F$ . Since  $F$  has order continuous norm,  $\text{sol}_F G$  is relatively weakly compact in  $F$  [M, Prop. 2.5.12 (iv)]. Since  $E^\odot \subset F$  and  $0 \leq |\lambda R(\lambda, A)^* x^*| \leq R(\lambda, A)^* |x^*| \leq \lambda R(\lambda, A)^* y^*$  for all  $\lambda \geq \lambda_0$ , it is clear that

$$H := \{\lambda R(\lambda, A)^* x^*: \lambda \geq \lambda_0\} \subset \text{sol}_F G.$$

In particular,  $H$  is relatively weakly compact in  $F$ . Let  $z^*$  be any  $\sigma(F, F^*)$ -accumulation point of  $H$  as  $\lambda \rightarrow \infty$ . Then  $z^*$  is also a weak- and hence a weak\*-accumulation point of  $H$ . But on the other hand,  $\text{weak}^* \lim_{\lambda \rightarrow \infty} \lambda R(\lambda, A)^* x^* = x^*$ . Therefore necessarily  $z^* = x^*$ . Since  $\lambda R(\lambda, A)^* x^* \in E^\odot$  for each  $\lambda \geq \lambda_0$ , it follows that  $x^*$  belongs to the weak closure of  $E^\odot$ . Hence  $x^* \in E^\odot$ .

*Step 2.* Suppose  $|x^*| \leq |y^*|$  with  $y^* \in E^\odot$ . We will show that  $x^* \in E^\odot$ . By Step 1 it suffices to show that  $|x^*| \in E^\odot$ . Therefore we may assume that  $x^* \geq 0$ . For  $\lambda \geq \lambda_0$  put

$$z_\lambda^* := |\lambda R(\lambda, A)^* y^*| \wedge x^*.$$

Then, since  $x^* \geq 0$  and  $\lambda R(\lambda, A)^* \geq 0$ ,

$$0 \leq z_\lambda^* \leq |\lambda R(\lambda, A)^* y^*| \leq \lambda R(\lambda, A)^* |y^*|,$$

and since  $\lambda R(\lambda, A)^* |y^*|$  is a positive element in  $E^\odot$ , it follows from Step 1 that  $z_\lambda^* \in E^\odot$ . But since  $y^* \in E^\odot$  we have  $\lim_{\lambda \rightarrow \infty} |\lambda R(\lambda, A)^* y^*| = |y^*|$ , and therefore

$$\lim_{\lambda \rightarrow \infty} z_\lambda^* = \lim_{\lambda \rightarrow \infty} |\lambda R(\lambda, A)^* y^*| \wedge x^* = |y^*| \wedge x^* = x^*.$$

Since  $E^\odot$  is closed it follows that  $x^* \in E^\odot$ . This proves that  $E^\odot$  is an ideal.

The second statement is a consequence of the fact that every closed ideal in a Banach lattice with order continuous norm is a projection band.

In [NP] we observed that if  $E$  is a  $\sigma$ -Dedekind complete Banach lattice, then the band generated by  $E^\odot$  is the whole  $E^*$ . In fact, this follows from  $\text{weak}^* \lim_{\lambda \rightarrow \infty} \lambda R(\lambda, A)^* x^* = x^*$  and the fact that every band projection in the dual of a  $\sigma$ -Dedekind complete Banach lattice is weak\*-sequentially continuous [AB, Thm. 13.14] (consider the band projection onto the band generated by  $E^\odot$ ).

**COROLLARY 2.2.** *If  $E$  is a  $\sigma$ -Dedekind complete Banach lattice whose dual has order continuous norm, then  $E^\odot = E^*$ .*

An example of such a Banach lattice is  $E = c_0$ .

The following corollary is a converse of Theorem 2.1 in case  $R(\lambda, A)$  is weakly compact for some  $\lambda \in \rho(A)$  (hence for all  $\lambda \in \rho(A)$ ). This is the case if and only if  $E$  is  $\odot$ -reflexive with respect to  $T(t)$ ; see [Pa1].

**COROLLARY 2.3.** *If  $R(\lambda, A)$  is weakly compact, then the following assertions are equivalent:*

- (i)  $E^\odot$  is an ideal;
- (ii)  $E^\odot$  is contained in a sublattice with order continuous norm;
- (iii)  $E^\odot$  is a  $\sigma$ -Dedekind complete sublattice.

*Proof.* (iii)  $\Rightarrow$  (ii): If  $E^\odot$  is  $\sigma$ -Dedekind complete then, by the weak compactness of  $R(\lambda, A)$ ,  $E^\odot$  actually has order continuous norm [NP]. (ii)  $\Rightarrow$  (i) follows from Theorem 2.1 and (i)  $\Rightarrow$  (iii) follows from the fact that the dual of a Banach lattice is always Dedekind complete.

### 3. Disjointness almost everywhere

Throughout this section, let  $T(t)$  be a positive  $C_0$ -semigroup on a Banach lattice  $E$ . Fix a real  $\lambda \in \rho(A)$  with  $\lambda > \omega$ , with  $\omega \in \mathbb{R}$  such that  $\|T(t)\| \leq M e^{\omega t}$  for a suitable constant  $M \geq 1$ .

We start with the simple observation that  $x \in \{R(\lambda, A)x\}^{dd}$  for all  $0 \leq x \in E$ . Indeed, suppose  $y \in E$  such that  $y \wedge R(\lambda, A)x = 0$ . Since  $0 \leq R(\mu, A)x \leq R(\lambda, A)x$  for all  $\mu \geq \lambda$ , this implies that

$$y \wedge (\mu R(\mu, A)x) = 0.$$

Now it follows from  $\lim_{\mu \rightarrow \infty} \mu R(\mu, A)x = x$  that  $y \wedge x = 0$ . This shows that

$$\{R(\lambda, A)x\}^d \subset \{x\}^d,$$

and hence  $x \in \{R(\lambda, A)x\}^{dd}$ .

For the adjoint semigroup the situation is different. It can happen that  $x^* \perp R(\lambda, A^*)x^*$  for some  $0 \leq x^* \in X^*$ . For example, let  $T(t)$  be the translation group on  $E = C_0(\mathbb{R})$  and let  $x^*$  be a measure which is singular with respect to the Lebesgue measure. Then  $x^* \perp L^1(\mathbb{R})$ , here identifying absolutely continuous measures with their  $L^1$ -densities. But  $R(\lambda, A^*)x^* \in E^\odot = L^1(\mathbb{R})$ , so indeed  $x^* \perp R(\lambda, A^*)x^*$ .

As one of the results of this section we will characterize these functionals  $x^*$

as the elements of  $(E^\odot)^d$ . The following lemma is a first step towards this characterization.

We will use repeatedly the formula

$$\langle x^* \wedge y^*, x \rangle = \inf \{ \langle x^*, u \rangle + \langle y^*, v \rangle : u, v \in [0, x], u + v = x \}, \quad (*)$$

valid for arbitrary  $x^*, y^* \in E^*$  and  $0 \leq x \in E$  (see e.g. [Z], Theorem 83.6).

**LEMMA 3.1.** *Suppose  $0 \leq x \in E$ ,  $0 \leq x^* \in E^*$  and  $0 \leq y^* \in E^*$  satisfy  $\langle R(\lambda, A^*)x^* \wedge y^*, x \rangle = 0$ . Then, for almost all  $t \geq 0$  (with respect to the Lebesgue measure) we have  $\langle T^*(t)x^* \wedge y^*, x \rangle = 0$ .*

*Proof.* The formula (\*) applied to  $T^*(t)x^* \wedge y^*$  shows that for  $x \geq 0$  the function  $f(t) := \langle T^*(t)x^* \wedge y^*, x \rangle$  is measurable, being the infimum of continuous functions. We must show that  $f = 0$  a.e. Fix  $\varepsilon > 0$ . By (\*), applied to  $R(\lambda, A^*)x^* \wedge x^*$ , it is possible to choose  $u, v \in [0, x]$  such that  $u + v = x$  and

$$\langle R(\lambda, A^*)x^*, u \rangle < \varepsilon, \langle y^*, v \rangle < \varepsilon.$$

Then

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \langle T^*(t)x^* \wedge y^*, x \rangle dt &\leq \int_0^\infty e^{-\lambda t} \langle T^*(t)x^*, u \rangle dt + \int_0^\infty e^{-\lambda t} \langle y^*, v \rangle dt \\ &= \langle R(\lambda, A^*)x^*, u \rangle + \lambda^{-1} \langle y^*, v \rangle \leq (1 + \lambda^{-1})\varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary it follows that

$$\int_0^\infty e^{-\lambda t} \langle T^*(t)x^* \wedge y^*, x \rangle dt = 0.$$

The lemma now follows from the fact that the integrand is a positive function.

Thus, if  $R(\lambda, A^*)x^* \wedge y^* = 0$ , then by the lemma for all  $x \geq 0$  we have  $\langle T^*(t)x^* \wedge y^*, x \rangle = 0$ , except for  $t$  belonging to a set of measure zero. This exceptional set, however, may vary with  $x$  and therefore one cannot conclude that  $T^*(t)x^* \wedge y^* = 0$  for almost all  $t$ . The following example shows that indeed this need not be the case.

**EXAMPLE 3.2.** Let  $T$  be the unit circle in the complex plane, which will be identified with the interval  $[0, 2\pi)$ , and let  $C(T)$  denote the Banach lattice of continuous functions on  $T$ . Let  $E = l^1([0, 2\pi]; C(T))$ . With the pointwise order,  $E$  is a Banach lattice. Note that  $E^* = l^\infty([0, 2\pi]; M(T))$ , where  $M(T) = C(T)^*$  is the space of bounded Borel measures on  $T$ . Define an element  $x^* \in E^*$  by

$x^*(z) = \delta_0 + \delta_z$ , where  $\delta_z$  is the Dirac measure concentrated at  $z$ . Let  $R(t)$  be the rotation group on  $C(T)$  and define a positive  $C_0$ -group  $T(t)$  on  $E$  by

$$(T(t)x)(z) := R(t)(x(z)).$$

Then, using the fact that the lattice operations on  $E$  are defined pointwise, for any  $t \in [0, \pi)$  we have

$$\begin{aligned} \|T^*(t)x^* \wedge x^*\| &\geq \|(T^*(t)x^* \wedge x^*)(t)\| = \|R(t)(x^*(t)) \wedge x^*(t)\| \\ &= \|(\delta_t + \delta_{2t}) \wedge (\delta_0 + \delta_t)\| = \|\delta_t\| = 1. \end{aligned}$$

**THEOREM 3.3.** *Suppose that  $E$  has a quasi-interior point, or that  $E^*$  has order continuous norm. Then  $R(\lambda, A^*)x^* \wedge y^* = 0$  ( $0 \leq x^*, y^* \in E^*$ ) implies that  $T^*(t)x^* \wedge y^* = 0$  for almost all  $t \geq 0$ .*

*Proof.* Suppose first that  $u > 0$  is quasi-interior. We have by Lemma 3.1 that

$$\langle T^*(t)x^* \wedge y^*, u \rangle = 0, \quad \text{a.a. } t \geq 0.$$

Since  $u$  is a quasi-interior point, this implies that

$$T^*(t)x^* \wedge y^* = 0, \quad \text{a.a. } t \geq 0.$$

If  $E^*$  has order continuous norm, then for all  $z^* \in E^*$  the closed unit ball  $B_E$  is approximately  $z^*$ -order bounded [M, Prop. 2.3.2], i.e. for all  $\varepsilon > 0$  and  $z^* \in E^*$  there is an  $x \geq 0$  such that

$$B_E \subset [-x, x] + \varepsilon B_{z^*}.$$

Here  $B_{z^*}$  is the closed unit ball of the seminorm  $p_{z^*}$  defined by  $p_{z^*}(x) = \langle |z^*|, |x| \rangle$ . Choose  $x_n \geq 0$  such that  $B_E \subset [-x_n, x_n] + n^{-1}B_{y^*}$ . By Lemma 3.1, there is a set  $F_n \subset \mathbb{R}_{>0}$  of full measure such that for all  $t \in F_n$  we have  $\langle T^*(t)x^* \wedge y^*, x_n \rangle = 0$ . Fix any  $t \in F_n$ . Let  $y \in B_E$  arbitrary. Write  $y = y_1 + y_2$  with  $y_1 \in [-x_n, x_n]$ ,  $y_2 \in n^{-1}B_{y^*}$ . Then

$$\begin{aligned} \langle T^*(t)x^* \wedge y^*, |y| \rangle &\leq \langle T^*(t)x^* \wedge y^*, |y_1| \rangle + \langle T^*(t)x^* \wedge y^*, |y_2| \rangle \\ &\leq 0 + \langle y^*, |y_2| \rangle \leq \frac{1}{n}. \end{aligned}$$

It follows that  $\langle T^*(t)x^* \wedge y^*, |y| \rangle = 0$  for all  $t \in F := \bigcap_n F_n$ . Since  $y$  is arbitrary and the  $F_n$  do not depend on  $y$ , it follows that  $T^*(t)x^* \wedge y^* = 0$  for  $t \in F$ .

**COROLLARY 3.4.** *Suppose  $x^* \in E^*$ ,  $y^* \in (E^\odot)^d$  and either  $E^*$  has order continu-*



ous norm or  $E$  has a quasi-interior point. Then  $T^*(t)x^* \perp y^*$  for almost all  $t \geq 0$ .

*Proof.*  $y^* \perp E^\ominus$  implies  $|y^*| \perp E^\ominus$ , so in particular  $R(\lambda, A^*)|x^*| \wedge |y^*| = 0$ . Therefore  $T^*(t)|x^*| \wedge |y^*| = 0$  for almost all  $t$ . But  $|T^*(t)x^*| \leq T^*(t)|x^*|$ , hence for almost all  $t$  also  $|T^*(t)x^*| \wedge |y^*| = 0$ .

The following theorem gives the characterization of functionals in  $(E^\ominus)^d$ , mentioned at the beginning of this section.

**THEOREM 3.5.** *For  $0 \leq x^* \in E^*$  the following statements are equivalent:*

- (i)  $x^* \in (E^\ominus)^d$ ;
- (ii)  $R(\lambda, A^*)x^* \wedge x^* = 0$ ;
- (iii) For all  $0 \leq x \in E$  we have  $\langle T^*(t)x^* \wedge x^*, x \rangle = 0$  for almost all  $t \geq 0$ ;
- (iv) For all  $0 \leq x \in E$  we have  $\liminf_{t \downarrow 0} \langle T^*(t)x^* \wedge x^*, x \rangle = 0$ .

*Proof.* The implications (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (iv) are trivial, and (ii)  $\Rightarrow$  (iii) follows from Lemma 3.1. So only (iv)  $\Rightarrow$  (i) needs proof. Take  $0 \leq x^* \in E^*$  satisfying (iv). Since  $E^\ominus = \overline{D(A^*)} = \overline{R(\lambda, A^*)E^*}$ , it is sufficient to prove that  $x^* \perp R(\lambda, A^*)y^*$  for all  $y^* \in E^*$ . Moreover, since

$$|R(\lambda, A^*)y^*| \leq R(\lambda, A^*)|y^*| \in E^\ominus,$$

all we have to show is that  $x^* \wedge z^\ominus = 0$  for all  $0 \leq z^\ominus \in E^\ominus$ . To this end, fix  $0 \leq z^\ominus \in E^\ominus$  and let  $x_1^* \in E^*$  be any vector such that  $0 \leq x_1^* \leq nx^* \wedge z^\ominus$  for some number  $n \in \mathbb{N}$ . It follows from  $0 \leq x_1^* \leq nx^*$  that

$$0 \leq T^*(t)x_1^* \wedge x_1^* \leq T^*(t)nx^* \wedge nx^* = n(T^*(t)x^* \wedge x^*).$$

so  $x_1^*$  satisfies (iv) as well. Fix  $\varepsilon > 0$  and  $0 \leq x \in E$  with  $\|x\| = 1$ . There exists  $\delta > 0$  such that  $\|T^*(t)z^\ominus - z^\ominus\| < \varepsilon$  for all  $0 \leq t < \delta$ . Furthermore, since  $\liminf_{t \downarrow 0} \langle T^*(t)x_1^* \wedge x_1^*, x \rangle = 0$ , there exists  $0 < t_0 < \delta$  such that

$$0 \leq \langle T^*(t_0)x_1^* \wedge x_1^*, x \rangle < \varepsilon.$$

By the formula (\*), there exist  $0 \leq u, v \in E$  such that  $u + v = x$  and

$$\langle T(t_0)^*x_1^*, u \rangle < \varepsilon, \langle x_1^*, v \rangle < \varepsilon.$$

Then

$$\langle x_1^*, u \rangle = \langle x_1^*, x \rangle - \langle x_1^*, v \rangle > \langle x_1^*, x \rangle - \varepsilon$$

and

$$\langle T^*(t_0)x_1^*, v \rangle = \langle x_1^*, x \rangle + \langle T^*(t_0)x_1^* - x_1^*, x \rangle - \langle T^*(t_0)x_1^*, u \rangle > \langle x_1^*, x \rangle - 2\varepsilon.$$

This implies that

$$\begin{aligned} \langle z^\ominus, v \rangle &= \langle T^*(t_0)z^\ominus, v \rangle - \langle T^*(t_0)z^\ominus - z^\ominus, v \rangle \\ &\geq \langle T^*(t_0)x_1^*, v \rangle - \|T^*(t_0)z^\ominus - z^\ominus\| \|v\| \\ &> (\langle x_1^*, x \rangle - 2\varepsilon) - \varepsilon \|v\| \geq \langle x_1^*, x \rangle - 3\varepsilon. \end{aligned}$$

Hence

$$\langle z^\ominus, x \rangle = \langle z^\ominus, u \rangle + \langle z^\ominus, v \rangle > \langle x_1^*, u \rangle + (\langle x_1^*, x \rangle - 3\varepsilon) > \langle 2x_1^*, x \rangle - 4\varepsilon.$$

Since  $\varepsilon$  is arbitrary it follows that  $\langle z^\ominus, x \rangle \geq \langle 2x_1^*, x \rangle$  for all  $x \geq 0$ , i.e.  $0 \leq 2x_1^* \leq z^\ominus$ . Hence,  $0 \leq 2x_1^* \leq 2nx^* \wedge z^\ominus$  and we can repeat the above argument. After doing so  $k$  times we find that  $0 \leq 2^k x_1^* \leq z^\ominus$ . Hence this holds for all  $k \in \mathbb{N}$ , so  $x_1^* = 0$ . In particular, letting  $x_1^* = x^* \wedge z^\ominus$ , it follows that  $x^* \wedge z^\ominus = 0$ . This completes the proof.

Next we will study the behaviour of  $T^*(t)$  on the disjoint complement  $(E^\ominus)^d$ . In general,  $(E^\ominus)^d$  need not be  $T^*(t)$ -invariant. It may even happen that  $T^*(t)E^* \subset E^\ominus$  for all  $t > 0$ , e.g. if  $T(t)$  is an analytic semigroup. Using the above theorem we obtain the following result.

**COROLLARY 3.6.** *If  $T^*(t)$  is a lattice semigroup, then  $(E^\ominus)^d$  is  $T^*(t)$ -invariant.*

*Proof.* If  $0 \leq x^* \in (E^\ominus)^d$ , then  $R(\lambda, A^*)x^* \wedge x^* = 0$ . Hence also

$$R(\lambda, A^*)T^*(t)x^* \wedge T^*(t)x^* = T^*(t)(R(\lambda, A^*)x^* \wedge x^*) = 0,$$

so  $T^*(t)x^* \in (E^\ominus)^d$  by Theorem 3.5.

We note that, in particular, if  $T(t)$  extends to a positive group, then  $T^*(t)$  is a lattice semigroup and the above corollary applies. Furthermore we note that, as observed before, if  $T^*(t)$  is a lattice semigroup, then  $E^\ominus$  is a sublattice of  $E^*$ .

The following example shows that Corollary 3.6 (and some results to follow) fail if  $T^*(t)$  is not a lattice semigroup.

**EXAMPLE 3.7.** Let  $T(t)$  be the semigroup on  $E = C[0, 1]$  defined by

$$T(t)f(s) = \begin{cases} f(t+s), & t+s \leq 1; \\ f(1), & \text{else.} \end{cases}$$

Then one easily verifies the following facts:

- (i)  $E^\ominus = L^1[0, 1] \oplus \mathbb{R}\delta_1$ ;
- (ii)  $\delta_0 \perp E^\ominus$  and  $T^*(t)\delta_0 = \delta_1 \in E^\ominus$  for all  $t \geq 1$ .

In view of Corollary 3.6 we will restrict our attention in the last part of

this section mainly to the situation in which  $T^*(t)$  is a lattice semigroup. We will study the occurrence of mutually disjoint elements in the orbits  $\{T^*(t)x^*: t \geq 0\}$ , where  $0 \leq x^* \in (E^\odot)^d$ . The first result in this direction is a simple consequence of Theorem 3.3.

**PROPOSITION 3.8.** *Assume that  $E^*$  has order continuous norm, or that  $E$  has a quasi-interior point. Furthermore, assume that  $T^*(t)$  is a lattice semigroup. Then for  $0 \leq x^* \in (E^\odot)^d$  we have:*

- (i) *If  $s \geq 0$  is fixed, then  $T^*(t)x^* \perp T^*(s)x^*$  for almost all  $t \geq 0$ ;*
- (ii)  *$T^*(t)x^* \perp T^*(s)x^*$  for almost all pairs  $(t, s) \geq 0$  (with respect to the Lebesgue measure on  $\mathbb{R}_+ \times \mathbb{R}_+$ ).*

*Proof.* (i) Take  $s \geq 0$ . It follows from Corollary 3.6 that  $T^*(s)x^* \in (E^\odot)^d$ . Now the result follows from Theorem 3.3 (with  $y^* = T^*(s)x^*$ ).

(ii) This follows via Fubini's theorem from (i).

Suppose that  $(E^\odot)^d \neq \{0\}$  and let  $0 < x^* \in (E^\odot)^d$  be fixed. We define

$$t_0 := \inf\{t > 0: T^*(t)x^* = 0\}.$$

If  $T^*(t)x^* \neq 0$  for all  $t \geq 0$  we put  $t_0 = \infty$ . If  $t_0 < \infty$ , it follows from the weak\*-continuity of  $t \mapsto T^*(t)x^*$  that  $T^*(t_0)x^* = 0$ ; in particular  $t_0 > 0$ . Hence  $T^*(t)x^* > 0$  for all  $0 \leq t < t_0$  and  $T^*(t)x^* = 0$  for all  $t \geq t_0$ .

We will say that a set  $H \subset [0, t_0)$  supports a disjoint system (for  $x^*$ ) if  $\{T^*(t)x^*: t \in H\}$  is a disjoint system in  $E^*$ , i.e.  $T^*(t)x^* \perp T^*(s)x^*$  for any two  $t \neq s \in H$ . In view of Proposition 3.8 one might ask whether there exist 'large' sets supporting a disjoint system. Observe already that, by Zorn's lemma, any set supporting a disjoint system is contained in a maximal one.

Let  $m^*$  denote the outer Lebesgue measure.

**LEMMA 3.9.** *Suppose that  $E^*$  has order continuous norm, or  $E$  has a quasi-interior point. Suppose  $T^*(t)$  is a lattice semigroup and let  $x^* \in (E^\odot)^d$ .*

- (i) *If  $H \subset [0, t_0)$  is a countable set supporting a disjoint system, and if  $J \subset [0, t_0)$  is an open interval, then there exists  $s \in J \setminus H$  such that  $H \cup \{s\}$  supports a disjoint system.*
- (ii) *If  $H \subset [0, t_0)$  is a maximal set supporting a disjoint system, then  $H$  is uncountable.*
- (iii) *Let  $H \subset [0, t_0)$  support a disjoint system. If  $T^*(t)x^* \wedge T^*(s)x^* > 0$  for some  $0 < s < t$ , then  $m^*([0, t] \setminus H) \geq \frac{1}{2}s$ .*

*Proof.* (i) For  $t \in H$  let

$$F_t = \{h \geq 0: T^*(h)x^* \wedge T^*(t)x^* = 0\}.$$

By Proposition 3.8(i) we know that  $m(\mathbb{R}_+ \setminus F_t) = 0$ . Since  $H$  is countable, the set

$F = \cap \{F_t; t \in H\}$  satisfies  $m(\mathbb{R}_+ \setminus F) = 0$  as well, and hence  $F \cap J \neq \emptyset$ . Now take any  $s \in F \cap J$ .

- (ii) Follows immediately from (i).
- (iii) Since  $|T^*(t)x^*| \wedge |T^*(s)x^*| > 0$ , also  $|T^*(t-s+h)x^*| \wedge |T^*(h)x^*| > 0$  for all  $0 \leq h \leq s$ . Hence, if  $h \in H \cap [0, s]$ , then  $h + t - s \notin H$ , i.e.

$$([0, s] \cap H) + t - s \subset [0, t] \setminus H,$$

so  $m^*([0, s] \cap H) \leq m^*([0, t] \setminus H)$ . Now

$$s \leq m^*([0, s] \cap H) + m^*([0, s] \setminus H) \leq 2m^*([0, t] \setminus H),$$

$$\text{so } m^*([0, t] \setminus H) \geq \frac{1}{2}s.$$

We do not know whether a maximal set supporting a disjoint system must be measurable. This is the reason for taking the outer Lebesgue measure rather than the Lebesgue measure.

**EXAMPLE 3.10.** Let  $T(t)$  be the rotation group on  $E = C(T)$ . Identifying the unit circle  $T$  with  $[0, 2\pi)$ , we let  $x^* = \delta_0 + \delta_\pi$ . Then  $H = [0, \pi)$  is a maximal set supporting a disjoint system for  $x^*$ . This shows that the constant  $\frac{1}{2}$  in Lemma 3.9(iii) is optimal.

**THEOREM 3.11.** *Suppose that  $E^*$  has order continuous norm, or  $E$  has a quasi-interior point. Suppose  $T^*(t)$  is a lattice semigroup and let  $x^* \in (E^\odot)^d$ .*

- (i) *There exists an uncountable dense set  $H \subset [0, t_0)$  supporting a disjoint system.*
- (ii) *If  $T(t)$  extends to a positive group, then either the orbit  $\{T^*(t)x^*; t \in \mathbb{R}\}$  is a disjoint system, or  $m^*(\mathbb{R} \setminus H) = \infty$  for each set  $H \subset \mathbb{R}$  supporting a disjoint system.*

*Proof.* (i) Let  $(J_n)_{n=1}^\infty$  be an enumeration of the open intervals in  $[0, t_0)$  with rational endpoints. Using Lemma 3.9(i) we inductively construct a sequence  $(t_n)_{n=1}^\infty$  supporting a disjoint system with  $t_n \in J_n$  for all  $n$ . This sequence  $(t_n)$  is contained in some maximal  $H$  supporting a disjoint system. Clearly  $H$  is dense in  $[0, t_0)$ , and by Lemma 3.9(ii)  $H$  is uncountable.

(ii) Now assume in addition that  $T(t)$  extends to a positive group, and that  $H \subset \mathbb{R}$  supports a disjoint system with  $m^*(\mathbb{R}_+ \setminus H) = K < \infty$ . Then also  $H_+ := H \cap \mathbb{R}_+$  supports a disjoint system and  $m^*(\mathbb{R}_+ \setminus H_+) \leq K$ . It follows from Lemma 3.9(iii) that  $T^*(t)x^* \wedge T^*(s)x^* = 0$  for all  $s \neq t > 2K$ . Therefore, if  $s \neq t$  in  $\mathbb{R}$ , then for  $n$  so large that  $s + n > 2K$ ,  $t + n > 2K$  we have

$$T^*(n)(T^*(t)x^* \wedge T^*(s)x^*) = T^*(t+n)x^* \wedge T^*(s+n)x^* = 0.$$

Since  $T^*(n)$  is injective, this implies that  $T^*(t)x^* \wedge T^*(s)x^* = 0$ .

In the situation of Theorem 3.11, it is clear from (i) that  $(E^\odot)^d$  is not norm separable. So in this situation we have either  $(E^\odot)^d = \{0\}$  or  $(E^\odot)^d$  is non-separable. In this direction we can prove more, under much weaker assumptions, using a different method of proof. This is what we will do next.

First we recall some facts. Let  $E$  be a Banach lattice and  $J \subset E$  an ideal. The annihilator  $J^\perp = \{x^* \in E^* : \langle x^*, x \rangle = 0, \forall x \in J\}$  is a band in  $E^*$ , and hence we have the band decomposition  $E^* = J^\perp \oplus (J^\perp)^d$ . Let  $P_J$  be the band projection in  $E^*$  onto  $(J^\perp)^d$ .

LEMMA 3.12. *Let  $J \subset E$  be an ideal and  $0 \leq T : E \rightarrow E$  be a positive operator such that  $T(J) \subset J$ . Then  $P_J T^* \leq T^* P_J$ .*

*Proof.* Since  $T(J) \subset J$  implies that  $T^*(J^\perp) \subset J^\perp$ , it follows that

$$T^*(I - P_J) = (I - P_J)T^*(I - P_J),$$

and so  $P_J T^* P_J = P_J T^*$ . Hence  $P_J T^* = P_J T^* P_J \leq T^* P_J$ .

In the following theorem,  $T(t)$  is any positive  $C_0$ -semigroup on  $E$ . We do not assume that  $T^*(t)$  be a lattice semigroup.

THEOREM 3.13. *If  $(E^\odot)^d$  contains a weak order unit, then  $T^*(t)(E^*) \subset (E^\odot)^{dd}$  for all  $t > 0$ .*

*Proof.* Let  $0 \leq w^* \in (E^\odot)^d$  be a weak order unit. Fix  $0 \leq x^* \in E^*$  and  $0 \leq x \in E$ . Let  $J$  be the closed ideal in  $E$  generated by the orbit  $\{T(t)x : t \geq 0\}$ . Then  $J$  is  $T(t)$ -invariant and has a quasi-interior point  $0 \leq u \in J$ . By Lemma 3.1,  $0 \leq w^* \in (E^\odot)^d$  implies that  $\langle T^*(t)x^* \wedge w^*, u \rangle = 0$  for almost all  $t \geq 0$ . Since

$$0 \leq P_J(T^*(t)x^*) \wedge w^* \leq T^*(t)x^* \wedge w^*,$$

it follows that  $\langle P_J(T^*(t)x^*) \wedge w^*, u \rangle = 0$  a.e., and hence  $P_J(T^*(t)x^*) \wedge w^* \in J^\perp$  a.e. But also  $P_J(T^*(t)x^*) \wedge w^* \in (J^\perp)^d$ , so  $P_J(T^*(t)x^*) \wedge w^* = 0$  a.e., hence  $P_J(T^*(t)x^*) \in (E^\odot)^{dd}$  a.e. Now observe that, if  $t \geq 0$  is such that  $P_J(T^*(t)x^*) \in (E^\odot)^{dd}$ , then by Lemma 3.12,

$$P_J(T^*(t+s)x^*) = P_J(T^*(s)T^*(t)x^*) \leq T^*(s)P_J(T^*(t)x^*).$$

Also, as observed in Section 1,  $(E^\odot)^{dd}$  is  $T^*(t)$ -invariant. Combining these facts, we conclude that  $P_J(T^*(t)x^*) \in (E^\odot)^{dd}$  for all  $t > 0$ . Therefore,  $P_J(T^*(t)x^* \wedge w^*) = 0$ , i.e.,  $T^*(t)x^* \wedge w^* \in J^\perp$  for all  $t > 0$ , which implies in particular that  $\langle T^*(t)x^* \wedge w^*, x \rangle = 0$  for all  $t > 0$ . Since  $0 \leq x \in E$  was arbitrary, it follows that  $T^*(t)x^* \wedge w^* = 0$  for all  $t > 0$ , i.e.,  $T^*(t)x^* \in (E^\odot)^{dd}$  for all  $t > 0$ .

Together with Theorem 2.1 this implies:

**COROLLARY 3.14.** *Suppose  $E^*$  has order continuous norm. If  $(E^\odot)^d$  contains a weak order unit, then  $T^*(t)(E^*) \subset E^\odot$  for all  $t > 0$ , i.e.  $T^*(t)$  is strongly continuous for  $t > 0$ .*

**COROLLARY 3.15.** *Suppose  $E^*$  has order continuous norm and suppose  $T(t)$  extends to a (not necessarily positive) group. Then either  $E^* = E^\odot$  or  $(E^\odot)^d$  does not contain a weak order unit.*

**COROLLARY 3.16.** *Suppose  $T^*(t)$  is a lattice semigroup. Then either  $(E^\odot)^d = \{0\}$  or  $(E^\odot)^d$  does not contain a weak order unit.*

*Proof.* Suppose  $(E^\odot)^d$  contains a weak order unit. By Theorem 3.13,  $T^*(t)(E^*) \subset (E^\odot)^{dd}$  for all  $t > 0$ . It follows from Corollary 3.6 that  $(E^\odot)^d$  is  $T^*(t)$ -invariant, and hence  $T^*(t)((E^\odot)^d) = \{0\}$  for all  $t > 0$ . From the weak\*-continuity of  $t \mapsto T^*(t)x^*$  it now follows that  $(E^\odot)^d = \{0\}$ .

The preceding results can be regarded as lattice versions of the following result proved in [Ne]: If  $T(t)$  is a  $C_0$ -semigroup on a Banach space  $X$  such that  $X^*/X^\odot$  is separable, then  $T(t)(X^*) \subset X^\odot$  for all  $t > 0$ , i.e.  $T^*(t)$  is strongly continuous for  $t > 0$ . In particular, if  $T(t)$  extends to a group, then either  $X^\odot = X^*$  or  $X^*/X^\odot$  is non-separable.

In the setting of Corollary 3.15, one might wonder when exactly one has  $E^\odot = E^*$ . In this direction, we can prove:

**PROPOSITION 3.17.** *Let  $E = C_0(\Omega)$  with  $\Omega$  locally compact Hausdorff, and let  $T(t)$  be a positive  $C_0$ -group on  $E$ . If  $E^\odot = E^*$  then  $T(t)$  is a multiplication group.*

*Proof.* Since each operator  $T^*(t)$  is a lattice isomorphism, atoms in  $M(\Omega) = (C_0(\Omega))^*$  are mapped to atoms. Hence, for each  $\omega \in \Omega$  we have  $T^*(t)\delta_\omega = \phi_\omega(t)\delta_{\omega(t)}$ , say. Here  $\delta_\omega$  is the Dirac measure at  $\omega$ . By the strong continuity of  $t \mapsto T^*(t)\delta_\omega$ , we must have that  $\omega(t) = \omega$ , so  $T^*(t)\delta_\omega = \phi_\omega(t)\delta_\omega$ . For  $f \in C_0(\Omega)$  one then has

$$(T(t)f)(\omega) = \langle \delta_\omega, T(t)f \rangle = \phi_\omega(t) \langle f, \delta_\omega \rangle = \phi_\omega(t)f(\omega).$$

Every multiplication group on a real Banach lattice  $E$  has a bounded generator [Na, Proposition. C-II.5.16]. If  $E$  is complex, then a positive semigroup leaves invariant the real part of  $E$ . Therefore, both in the real and complex case, from the above results we conclude:

**COROLLARY 3.18.** *Let  $T(t)$  be a positive  $C_0$ -group with unbounded generator on the Banach lattice  $E = C_0(\Omega)$ . Then  $(E^\odot)^d$  does not contain a weak order unit.*

**4. Limes superior estimates**

We start in this section with an arbitrary  $C_0$ -semigroup  $T(t)$  on a Banach space  $X$ . We choose  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|T(t)\| \leq Me^{\omega t}$ . It is our objective to study the quantity  $\|T^*(t)x^* - x^*\|$  as  $t \downarrow 0$  for  $x^* \in X^*$ . Our first results are general limes superior estimates, which we will improve later in the context of positive semigroups.

For  $x^* \in X^*$  define

$$\rho(x^*) := \limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\|.$$

It is clear that  $\rho$  defines a seminorm on  $X^*$ . Note that  $\rho(x^* + x^\ominus) = \rho(x^*)$  for all  $x^\ominus \in X^\ominus$  and  $x^* \in X^*$ . In particular,  $\rho(x^*) = 0$  if and only if  $x^* \in X^\ominus$ . Furthermore,

$$\rho(x^*) \leq \limsup_{t \downarrow 0} (\|T^*(t)\| + 1)\|x^*\| \leq (M + 1)\|x^*\|$$

for all  $x^* \in X^*$ .

Since  $X^\ominus$  is a closed subspace of  $X^*$ , the quotient space  $X^*/X^\ominus$  is a Banach space. Let  $q: X^* \rightarrow X^*/X^\ominus$  be the quotient map. The following result shows that the seminorm  $\rho$  is actually equivalent to the quotient norm on  $X^*/X^\ominus$ .

**THEOREM 4.1.** *For all  $x^* \in X^*$  we have  $\|qx^*\| \leq \rho(x^*) \leq (M + 1)\|qx^*\|$ .*

*Proof.* For arbitrary  $x^* \in X^*$  and  $x^\ominus \in X^\ominus$  we have

$$\rho(x^*) = \rho(x^* + x^\ominus) \leq (M + 1)\|x^* + x^\ominus\|.$$

By taking the infimum over all  $x^\ominus \in X^\ominus$  we obtain  $\rho(x^*) \leq (M + 1)\|qx^*\|$ .

For the converse, we recall that for any  $\tau > 0$  we have  $\text{weak}^* \int_0^\tau T^*(t)x^* dt \in X^\ominus$ . Therefore,

$$\begin{aligned} \|qx^*\| &\leq \left\| \frac{1}{\tau} \text{weak}^* \int_0^\tau T^*(t)x^* dt - x^* \right\| = \frac{1}{\tau} \left\| \text{weak}^* \int_0^\tau (T^*(t)x^* - x^*) dt \right\| \\ &\leq \frac{1}{\tau} \int_0^\tau \|T^*(t)x^* - x^*\| dt \leq \sup_{0 \leq t \leq \tau} \|T^*(t)x^* - x^*\|. \end{aligned}$$

Hence,

$$\|qx^*\| \leq \inf_{\tau > 0} \left( \sup_{0 \leq t \leq \tau} \|T^*(t)x^* - x^*\| \right) = \rho(x^*).$$

We mention an immediate consequence of the above theorem.

**COROLLARY 4.2.** *Let  $X^\ominus \subset Y$ , with  $Y$  a complemented subspace of  $X^*$ , say  $X^* = Y \oplus Z$ . Then there is a constant  $C > 0$  such that for all  $x^* \in Z$  we have*

$$\limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\| \geq C\|x^*\|.$$

*Proof.* Since  $X^\ominus \subset Y$ , the formula  $\|x^*\| := \|qx^*\|$  defines a norm on  $Z$  which satisfies  $\|x^*\| = \inf_{x^\ominus \in X^\ominus} \|x^* - x^\ominus\| \geq \inf_{y \in Y} \|x^* - y\|$ . But  $X^*/Y \simeq Z$  and consequently  $\|x^*\| \geq C\|x^*\|$ . Now we can apply Theorem 4.1.

On the quotient space  $X^*/X^\ominus$  we can define a quotient semigroup  $T_q^*(t)$  via the formula

$$T_q^*(t)qx^* := q(T^*(t)x^*).$$

Using the equivalence in Theorem 4.1, we can investigate some properties of this quotient semigroup via the seminorm  $\rho$ . For this purpose, the following result turns out to be useful.

**LEMMA 4.3.** *Let  $[a, b] \subset \mathbb{R}$  be a closed interval and  $f: [a, b] \rightarrow X^*$  a weak\*-continuous function. Then  $t \mapsto \rho(f(t))$  is a bounded Borel function on  $[a, b]$  and*

$$\rho\left(\text{weak}^* \int_a^b f(t) dt\right) \leq \int_a^b \rho(f(t)) dt.$$

*Proof.* For  $n \in \mathbb{N}$ ,  $n > 0$ , define

$$\rho_n(x^*) := \sup_{0 \leq t \leq 1/n} \|T^*(t)x^* - x^*\|, \quad x^* \in X^*.$$

Each  $\rho_n$  is a seminorm on  $X^*$  and  $\rho_n(x^*) \downarrow \rho(x^*)$  for all  $x^* \in X^*$ . Note that

$$\begin{aligned} \rho_n(x^*) &= \sup_{0 \leq t \leq 1/n} \left( \sup_{\|x\| \leq 1} |\langle T^*(t)x^* - x^*, x \rangle| \right) \\ &= \sup_{0 \leq t \leq 1/n} \left( \sup_{\|x\| \leq 1} |\langle x^*, (T(t) - I)x \rangle| \right) \\ &= \sup \{ |\langle x^*, y \rangle| : y \in D_n \}, \end{aligned}$$

where  $D_n = \bigcup_{0 \leq t \leq 1/n} (T(t) - I)B_X$ ,  $B_X$  being the closed unit ball of  $X$ . Hence,  $\rho_n(f(t)) = \sup_{x \in D_n} |\langle f(t), x \rangle|$  for all  $a \leq t \leq b$ . Being the pointwise supremum of continuous functions,  $\rho_n(f(\cdot))$  is lower semi-continuous. Since  $\rho_n(f(t)) \downarrow \rho(f(t))$  for all  $a \leq t \leq b$ , it follows that  $\rho(f(\cdot))$  is a Borel function.



For  $x \in D_n$  we have

$$\begin{aligned} \left| \langle \text{weak}^* \int_a^b f(t) dt, x \rangle \right| &= \left| \int_a^b \langle f(t), x \rangle dt \right| \\ &\leq \int_a^b |\langle f(t), x \rangle| dt \leq \int_a^b \rho_n(f(t)) dt, \end{aligned}$$

and so

$$\begin{aligned} \rho \left( \text{weak}^* \int_a^b f(t) dt \right) &\leq \rho_n \left( \text{weak}^* \int_a^b f(t) dt \right) \\ &= \sup_{x \in D_n} \left| \langle \text{weak}^* \int_a^b f(t) dt, x \rangle \right| \leq \int_a^b \rho_n(f(t)) dt. \end{aligned}$$

Finally, it follows from the monotone convergence theorem that

$$\int_a^b \rho_n(f(t)) dt \downarrow \int_a^b \rho(f(t)) dt.$$

This concludes the proof.

The above lemma can be used to prove the following property of the seminorm  $\rho$ .

**PROPOSITION 4.4.** *For all  $x^* \in X^*$  we have*

$$\rho(x^*) \leq \limsup_{t \downarrow 0} \rho(T^*(t)x^* - x^*).$$

*In particular, if  $x^* \in X^*$  is such that  $\lim_{t \downarrow 0} \rho(T^*(t)x^* - x^*) = 0$ , then  $\rho(x^*) = 0$ , i.e.,  $x^* \in X^\ominus$ .*

*Proof.* For all  $x^* \in X^*$  and  $\tau > 0$  we have, using Lemma 4.3,

$$\begin{aligned} \rho(x^*) &= \rho \left( \frac{1}{\tau} \text{weak}^* \int_0^\tau (T^*(t)x^* - x^*) dt \right) \\ &\leq \frac{1}{\tau} \int_0^\tau \rho(T^*(t)x^* - x^*) dt \leq \sup_{0 < t \leq \tau} \rho(T^*(t)x^* - x^*). \end{aligned}$$

A combination of this result with Theorem 4.1 yields the following:

**COROLLARY 4.5.** *If  $\lim_{t \downarrow 0} \|T_q^*(t)qx^* - qx^*\| = 0$ , then  $qx^* = 0$ .*

Thus the only element in  $X^*/X^\ominus$  whose  $T_q^*(t)$ -orbit is strongly continuous, is the zero element. This result was first proved in [Ne]. The (more complicated) proof given there shows that in fact the following stronger result is true: if the  $T_q^*(t)$ -orbit of some  $qx^*$  is norm-separable in  $X^*/X^\ominus$ , then it is identically zero for  $t > 0$ .

We now return to the Banach lattice case. In Theorem 0.1,  $E^\ominus$  is complemented in  $E^*$  and therefore we can already conclude from Theorem 4.1 that the limes superior estimate must hold with some constant  $C$ . In general  $E^\ominus$  is not complemented, but we always have a direct sum decomposition of  $E^*$  into the band generated by  $E^\ominus$  and the disjoint complement of  $E^\ominus$  (which of course may be  $\{0\}$ ). Therefore Corollary 4.2 can be applied and we get a constant  $C > 0$  such that for all  $x^* \perp E^\ominus$  we have

$$\limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\| \geq C\|x^*\|.$$

The following theorem shows that in fact we can achieve  $C = 2$ .

**THEOREM 4.6.** *If  $x^* \in (E^\ominus)^d$ , then  $\limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\| \geq 2\|x^*\|$ .*

*Proof.* First we observe that for  $x^* \in E^*$  and  $0 \leq x \in E$ ,

$$\liminf_{t \downarrow 0} \langle |T^*(t)x^*|, x \rangle \geq \langle |x^*|, x \rangle.$$

Indeed, if  $|y| \leq x$ , then

$$\liminf_{t \downarrow 0} \langle T^*(t)|x^*|, x \rangle \geq \liminf_{t \downarrow 0} \langle T^*(t)x^*, y \rangle = \lim_{t \downarrow 0} |\langle T^*(t)x^*, y \rangle| = |\langle x^*, y \rangle|,$$

and hence

$$\liminf_{t \downarrow 0} \langle |T^*(t)x^*|, x \rangle \geq \sup\{|\langle x^*, y \rangle| : |y| \leq x\} = \langle |x^*|, x \rangle.$$

Now take  $x^* \in (E^\ominus)^d$  and  $0 \leq x \in E$  with  $\|x\| = 1$ . From Lemma 3.1 we know that  $\langle T^*(t)|x^*| \wedge |x^*|, x \rangle = 0$  for almost all  $t \geq 0$ , and hence  $\langle |T^*(t)x^*| \wedge |x^*|, x \rangle = 0$  a.e. Using the lattice identity [AB, Theorem 1.4(4)]

$$2(|T^*(t)x^*| \wedge |x^*|) = |T^*(t)x^*| + |x^*| - \||T^*(t)x^*| - |x^*|\|,$$

we see that, for almost every  $t \geq 0$ ,

$$\begin{aligned} \|T^*(t)x^* - x^*\| &\geq \langle |T^*(t)x^* - x^*|, x \rangle \geq \langle \||T^*(t)x^*| - |x^*|\|, x \rangle \\ &= \langle |T^*(t)x^*|, x \rangle + \langle |x^*|, x \rangle. \end{aligned}$$

This implies that

$$\limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\| \geq \liminf_{t \downarrow 0} \langle |T^*(t)x^*|, x \rangle + \langle |x^*|, x \rangle \geq 2\langle |x^*|, x \rangle.$$

Since  $0 \leq x \in E$  of norm one is arbitrary, the result follows.

If  $E^*$  has order continuous norm, then by Theorem 2.1  $E^\odot$  is a projection band. Let  $\pi$  be the band projection onto its disjoint complement.

**COROLLARY 4.7.** *If  $E^*$  has order continuous norm, then*

$$2\|\pi x^*\| \leq \limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\| \leq (M + 1)\|\pi x^*\|.$$

*In particular, if  $M = 1$ , i.e., if  $\lim_{t \downarrow 0} \|T(t)\| = 1$ , then  $\limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 2\|\pi x^*\|$ .*

If  $x^*$  is contained in the band generated by  $E^\odot$  but not contained in  $E^\odot$  itself, then the limes superior can be anything between 0 and 2, as is shown by the following example.

**EXAMPLE 4.8.** Let  $E = L^1(\mathbb{R})$ ,  $T(t)$  the translation group on  $E$ . Let  $f \in C_0(\mathbb{R})$  be of norm one such that  $f = 0$  on  $[-1, 1]$ . Let  $0 \leq \alpha \leq 1$  and define  $g \in E^* = L^\infty(\mathbb{R})$  by

$$g(s) := \begin{cases} f(s), & |s| > 1; \\ \alpha, & s \in [0, 1]; \\ -\alpha, & s \in [-1, 0]. \end{cases}$$

Then  $\|g\| = 1$ ,  $g$  belongs to the band generated by  $E^\odot$ , and  $\limsup_{t \downarrow 0} \|T^*(t)g - g\| = 2\alpha$ .

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