Tôhoku Math. J.
46 (1994), 27–34

THE ADJOINTNESS OF THE MAPS BETWEEN THE SPACE OF JACOBI FORMS AND THE SPACE OF MODULAR FORMS

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(Received August 11, 1992, revised November 6, 1992)

Abstract. M. Eichler and D. Zagier constructed a map from a space of Jacobi forms to a space of elliptic modular forms. On the other hand, T. Satoh constructed a map from a space of cusp forms to a space of Jacobi cusp forms. In this paper, we prove a conjecture of N. P. Skoruppa to the effect that these maps are, up to constant, adjoint with respect to the Petersson products.

Introduction. Eichler and Zagier [2] constructed a map $D_{2\nu}$ ($\nu \geq 0$) from the space $J_{k,m}$ of Jacobi forms of weight k and index m with respect to Γ^J (where $\Gamma = SL_2(\boldsymbol{Z})$ and $\Gamma^J = \Gamma \ltimes \boldsymbol{Z}^2$) to the space $M_{k+2\nu}$ of modular forms of weight $k+2\nu$ with respect to Γ. Satoh [6] constructed a map $\{\ \}_{k,m}^{\nu}$ ($k \geq 2$, $m \geq 1$, $\nu \geq 0$) from the space $S_{k+2\nu}$ of cusp forms of weight $k+2\nu$ with respect to Γ to the space $J_{k,m}^{\mathrm{cusp}}$ of Jacobi cusp forms of weight k and index m with respect to Γ^J. The purpose of this paper is to prove that the map $D_{2\nu}$, up to constant, is the adjoint of $\{\ \}_{k,m}^{\nu}$ with respect to the natural Petersson products.

Kohnen [4] stated that N. P. Skoruppa was probably the first to notice this adjointness, which could follow from the existence of Jacobi Poincaré series. Neither Skoruppa nor Kohnen, however, proved the conjecture nor determined the constant. In this paper, we determine the constant explicitly. From this adjointness, we can obtain a relation between $\{\ \}_{k,m}^{\nu}$ and $\{\ \}_{k,1}^{\nu}$. We also investigate the property of a certain Dirichlet series.

The author would like to express his gratitude to Dr. Koichi Takase and Professor Yasuo Morita for helpful advice.

1. Statement of the results.

Let $(\ ,\ )$ and $\langle\ ,\ \rangle$ be the usual Petersson product on S_k and $J_{k,m}^{\mathrm{cusp}}$, respectively. For any non-negative integers λ, μ, we put

$$\lambda^{(\mu)} = \begin{cases} \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+\mu-1) & (\mu > 0) \\ 1 & (\mu = 0) . \end{cases}$$

THEOREM 1.1. *Let $k \geq 3$, $\nu \geq 0$ and $m \geq 1$ be integers. If $f \in S_{k+2\nu}$ and $\phi \in J_{k,m}^{\mathrm{cusp}}$, then*

$$(D_{2\nu}(\phi), f) = C_{\nu,m} \langle \phi, \{f\}_{k,m}^{\nu} \rangle ,$$

1991 *Mathematics Subject Classification*. Primary 11F99; Secondary 11F11, 11F55.

where

$$C_{\nu,m}=\frac{(2\nu)!\,\Gamma(k+2\nu-1)m^{-k+\nu+2}}{2^{2k+4\nu-3}\pi^{2\nu+1/2}(k-1)^{(\nu)}\Gamma(k-3/2)}\,.$$

For any positive integer m, we have linear operators V_m: $J_{k,1}^{\mathrm{cusp}}\to J_{k,m}^{\mathrm{cusp}}$, V_m^*: $J_{k,m}^{\mathrm{cusp}}\to J_{k,1}^{\mathrm{cusp}}$ (see [5, p. 549] and [2, p. 41]). V_m^* is the adjoint of V_m with respect to the Petersson product.

COROLLARY 1.2. *Let k, ν, m, f be as in Theorem* 1.1. *If f is a common eigenform and $f\,|\,T(m)=\lambda_m f$, then*

$$\{f\}_{k,m}^{\nu}\,|\,V_m^*=\lambda_m m^{k-\nu-2}\{f\}_{k,1}^{\nu}\,,$$

where $T(m)$ is the Hecke operator acting on $M_{k+2\nu}$.

For any $\phi\in J_{k,1}^{\mathrm{cusp}}$ and $f\in S_{k+2\nu}$, we define

$$L_{f,\phi}(s)=\sum_{m=1}^{\infty}\langle\{f\}_{k,m}^{\nu},\,\phi\,|\,V_m\rangle m^{-s}\,.$$

We use the notation $e(\tau)=\exp(2\pi i\tau)$.

COROLLARY 1.3. *For integers $k\geq 3$ and $\nu\geq 0$, let $f(\tau)=\sum_{n=1}^{\infty}a(n;f)e(n\tau)\in S_{k+2\nu}$ be a common eigenform and $\phi\in J_{k,1}^{\mathrm{cusp}}$.*

(i) *If $\langle\{f\}_{k,1}^{\nu},\phi\rangle=0$, then $L_{f,\phi}(s)$ vanishes.*

(ii) *If $\langle\{f\}_{k,1}^{\nu},\phi\rangle\neq 0$, then $L_{f,\phi}(s)$ is absolutely convergent for* $\mathrm{Re}(s)>3k/2-1$. *Put $L_{f,\phi}^*(s)=(2\pi)^{-s}\Gamma(s)L_{f,\phi}(s+k-\nu-2)$. Then $L_{f,\phi}^*(s)$ has a holomorphic continuation to $\boldsymbol{C}$ and satisfies a functional equation*

$$L_{f,\phi}^*(s)=i^{k+2\nu}L_{f,\phi}^*(k+2\nu-s)\,.$$

Further, suppose $a(1;f)=1$, l and h are two positive integers less than $k+2\nu$ such that $l\equiv h$ (mod 2) *and $L_f(h)\neq 0$. Then we have*

$$L_{f,\phi}^*(h)\neq 0 \quad and \quad \frac{L_{f,\phi}^*(l)}{L_{f,\phi}^*(h)}\in\boldsymbol{Q}(f)\,,$$

where $L_f(s)=\sum_{n=1}^{\infty}a(n;f)n^{-s}$ and $\boldsymbol{Q}(f)$ is the field generated over the rational number field $\boldsymbol{Q}$ by the coefficients $a(n;f)$ for all n.

REMARK 1.4. By [6, p. 477], for $\Delta(\tau)=e(\tau)\prod_{n=1}^{\infty}(1-e(n\tau))^{24}$, we have $\{\Delta\}_{12,1}^{0}\neq 0$. Hence, $\langle\{\Delta\}_{12,1}^{0},\{\Delta\}_{12,1}^{0}\rangle$ does not vanish.

2. The proofs. In order to prove Theorem 1.1, we use Poincaré series. For any positive integers n, k, we define

$$P_n^k(\tau)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{-k}e\left(\frac{n(a\tau+b)}{c\tau+d}\right),$$

where

$$\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \quad\text{and}\quad \Gamma_\infty=\left\{\pm\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\middle|\, h\in\boldsymbol{Z}\right\}.$$

The following lemma is well-known (e.g. [1, Theorems 2.6.9 and 2.6.10]).

LEMMA 2.1. *Let $k\geq 3$, and let $P_n^k(\tau)$ be as above. Then the Poincaré series $P_n^k(\tau)$ is a cusp form belonging to S_k. For any $f(\tau)=\sum_{l=1}^\infty a(l;f)e(l\tau)\in S_k$, the value of the Petersson product of f and $P_n^k(\tau)$ is given by*

$$(f, P_n^k)=\frac{a(n;f)}{(4\pi n)^{k-1}}\Gamma(k-1).$$

For any integers m, n, r such that $n>0$ and $4mn-r^2>0$, we define

$$P_{n,r}(\tau, z)=\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} e(n\tau+rz)|_{k,m}\gamma,$$

where the operation $|_{k,m}$ is defined as in [2] and

$$\Gamma_\infty^J=\{\gamma\in\Gamma^J \mid 1|_{k,m}\gamma=1\}=\left\{\left(\pm\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}, (0,\mu)\right)\middle|\, h, \mu\in\boldsymbol{Z}\right\}.$$

The following lemma is a special case of [9, Prop. 2, p. 76]. See also [3, p. 520].

LEMMA 2.2. *Let $k\geq 3$, and let $P_{n,r}(\tau, z)$ be as above. Then $P_{n,r}(\tau, z)\in J_{k,m}^{\text{cusp}}$. For any $\phi(\tau, z)=\sum_{4mn-r^2>0}c(n, r;\phi)e(n\tau+rz)\in J_{k,m}^{\text{cusp}}$, we have*

$$\langle\phi, P_{n,r}\rangle=\frac{c(n, r;\phi)m^{k-2}}{(4mn-r^2)^{k-3/2}2\pi^{k-3/2}}\Gamma\left(k-\frac{3}{2}\right).$$

LEMMA 2.3. *Let the notation be as above. Then we have*

$$D_{2\nu}(P_{n,r})(\tau)$$
$$=\sum_{\mu=0}^{\nu}\frac{(-1)^\mu(k+2\nu-\mu-2)!(2\nu)!}{(k+\nu-2)!\mu!(2\nu-2\mu)!}\sum_{\lambda\in\boldsymbol{Z}}(2m\lambda+r)^{2\nu-2\mu}(m(m\lambda^2+r\lambda+n))^\mu P_{m\lambda^2+r\lambda+n}^{k+2\nu}(\tau).$$

PROOF. By [2, p. 31, (7) and p. 32, (9)], if $\phi(\tau, z)=\sum_{\lambda=0}^\infty \chi_\lambda(\tau)z^\lambda\in J_{k,m}$, then

$$D_{2\nu}(\phi)(\tau)=(2\pi i)^{-2\nu}\frac{(2\nu)!}{(k+\nu-2)!}\sum_{\mu=0}^{\nu}(-2\pi im)^\mu\frac{(k+2\nu-\mu-2)!}{\mu!}\frac{d^\mu}{d\tau^\mu}(\chi_{2\nu-2\mu}(\tau)).$$

We can expand $P_{n,r}(\tau, z)$ as

$$P_{n,r}(\tau, z)=\sum_{\lambda\in\boldsymbol{Z}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{-k}e\left((m\lambda^2+r\lambda+n)\frac{a\tau+b}{c\tau+d}+\frac{(2m\lambda+r)z}{c\tau+d}-\frac{mcz^2}{c\tau+d}\right)$$

where we denote

$$\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in the rest of the proof (see [3, p. 520]). Since

$$\frac{d^{2q}}{dz^{2q}}(\exp(bz^2))\bigg|_{z=0}=\frac{b^q(2q)!}{q!} \quad\text{and}\quad \frac{d^{2q+1}}{dz^{2q+1}}(\exp(bz^2))\bigg|_{z=0}=0\,,$$

we obtain

$$\frac{d^{2q}}{dz^{2q}}(\exp(az+bz^2))\bigg|_{z=0}=\sum_{h=0}^{q}\frac{(2q)!}{(2h)!(q-h)!}a^{2h}b^{q-h}\,.$$

Thus, if we write $P_{n,r}(\tau, z)=\sum_{\lambda=0}^{\infty}\tilde{\chi}_\lambda(\tau)z^\lambda$, then we have

$$\begin{aligned}\tilde{\chi}_{2q}(\tau)&=\frac{1}{(2q)!}\frac{\partial^{2q}}{\partial z^{2q}}(P_{n,r}(\tau, z))\bigg|_{z=0}\\&=\sum_{h=0}^{q}(2\pi i)^{h+q}\frac{1}{(2h)!(q-h)!}\sum_{\lambda\in\boldsymbol{Z}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(-mc)^{q-h}(2m\lambda+r)^{2h}\\&\quad\times(c\tau+d)^{-k-q-h}e\left((m\lambda^2+r\lambda+n)\frac{a\tau+b}{c\tau+d}\right).\end{aligned}$$

Put

$$\tilde{P}(\tau)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(-mc)^{q-h}(c\tau+d)^{-k-q-h}e\left((m\lambda^2+r\lambda+n)\frac{a\tau+b}{c\tau+d}\right).$$

By induction on g, we obtain

$$\begin{aligned}\frac{d^g}{d\tau^g}\tilde{P}(\tau)&=\sum_{j=0}^{g}(-1)^{g-j}(2\pi i)^j(k+q+h+j)^{(g-j)}\frac{g!}{j!(g-j)!}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(-mc)^{q-h}c^{g-j}\\&\quad\times(m\lambda^2+r\lambda+n)^j(c\tau+d)^{-k-q-h-g-j}e\left((m\lambda^2+r\lambda+n)\frac{a\tau+b}{c\tau+d}\right).\end{aligned}$$

Thus we obtain

$$D_{2\nu}(P_{n,r})(\tau)$$

$$= \sum_{\lambda\in\mathbf{Z}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \sum_{\mu=0}^{\nu} \sum_{h=0}^{\nu-\mu} \sum_{j=0}^{\mu} (-1)^{\nu+\mu-h-j} \frac{(2\nu)!(k+2\nu-\mu-2)!}{(2h)!(\nu-\mu-h)!j!(\mu-j)!(k+\nu-2)!}$$

$$\times (2\pi i)^{h+j-\nu}(k+\nu-\mu+h+j)^{(\mu-j)}(2m\lambda+r)^{2h}(m\lambda^2+r\lambda+n)^j m^{\nu-h} c^{\nu-h-j}$$

$$\times (c\tau+d)^{-k-\nu-h-j} e\left((m\lambda^2+r\lambda+n)\frac{a\tau+b}{c\tau+d}\right)$$

$$= \sum_{j=0}^{\nu} \sum_{h=0}^{\nu-j} \left\{ \sum_{\mu=j}^{\nu-h} (-1)^\mu \frac{(k+2\nu-\mu-2)!}{(\nu-h-\mu)!(\mu-j)!(k+\nu-\mu+h+j-1)!} \right\} \frac{(2\nu)!(k+\nu+h-1)!}{(2h)!j!(k+\nu-2)!}$$

$$\times \sum_{\lambda\in\mathbf{Z}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (-1)^{\nu-j-h}(2\pi i)^{h+j-\nu} m^{\nu-j-h} c^{\nu-h-j}(2m\lambda+r)^{2h}(m(m\lambda^2+r\lambda+n))^j$$

$$\times (c\tau+d)^{-k-\nu-h-j} e\left((m\lambda^2+r\lambda+n)\frac{a\tau+b}{c\tau+d}\right).$$

Here we have used the notation

$$(k+\nu-\mu+h+j)^{(\mu-j)} = \frac{(k+\nu+h-1)!}{(k+\nu-\mu+h+j-1)!}.$$

By induction on t we can prove that

$$\sum_{\mu=j}^{j+t} (-1)^\mu \frac{(k+2\nu-\mu-2)!}{(\nu-h-\mu)!(\mu-j)!(k+\nu-\mu+h+j-1)!}$$

$$= (-1)^{j+t} \frac{(k+2\nu-j-t-2)!}{(\nu-j-h-t-1)!(k+\nu+h-t-2)!t!(\nu-j-h)(k+\nu+h-1)}$$

for $t=0, 1, 2, \ldots, \nu-h-j-1$. Since $\nu-h-1=j+(\nu-h-j-1)$, we have for $\nu-h>j$

$$\sum_{\mu=j}^{\nu-h} (-1)^\mu \frac{(k+2\nu-\mu-2)!}{(\nu-h-\mu)!(\mu-j)!(k+\nu-\mu+h+j-1)!}$$

$$= \sum_{\mu=j}^{\nu-h-1} (-1)^\mu \frac{(k+2\nu-\mu-2)!}{(\nu-h-\mu)!(\mu-j)!(k+\nu-\mu+h+j-1)!}$$

$$+ (-1)^{\nu-h} \frac{(k+\nu+h-2)!}{(\nu-h-j)!(k+2h+j-1)!}$$

$$= (-1)^{\nu-h-1} \frac{(k+\nu+h-1)!}{(k+2h+j-1)!(\nu-h-j-1)!(\nu-h-j)(k+\nu+h-1)}$$

$$+ (-1)^{\nu-h} \frac{(k+\nu+h-2)!}{(\nu-h-j)!(k+2h+j-1)!} = 0.$$

Hence we obtain

$$\sum_{\mu=j}^{\nu-h}(-1)^{\mu}\frac{(k+2\nu-\mu-2)!}{(\nu-h-\mu)!(\mu-j)!(k+\nu-\mu+h+j-1)!}=\begin{cases}(-1)^{j}/(k+2\nu-j-1) & (\nu-h=j)\\ 0 & (\nu-h>j)\,.\end{cases}$$

Therefore

$$\begin{aligned}&D_{2\nu}(P_{n,r})(\tau)\\ &=\sum_{j=0}^{\nu}\sum_{h=0}^{\nu-j}\left\{\sum_{\mu=j}^{\nu-h}(-1)^{\mu}\frac{(k+2\nu-\mu-2)!}{(\nu-h-\mu)!(\mu-j)!(k+\nu-\mu+h+j-1)!}\right\}\frac{(2\nu)!(k+\nu+h-1)!}{(2h)!j!(k+\nu-2)!}\\ &\quad\times\sum_{\lambda\in\boldsymbol{Z}}\sum_{\gamma\in\Gamma_{\infty}\backslash\Gamma}(-1)^{\nu-j-h}(2\pi i)^{h+j-\nu}m^{\nu-j-h}c^{\nu-h-j}(2m\lambda+r)^{2h}(m(m\lambda^{2}+r\lambda+n))^{j}\\ &\quad\times(c\tau+d)^{-k-\nu-h-j}e\left((m\lambda^{2}+r\lambda+n)\frac{a\tau+b}{c\tau+d}\right)\\ &=\sum_{j=0}^{\nu}\frac{(-1)^{j}(2\nu)!(k+2\nu-j-2)!}{(k+\nu-2)!j!(2\nu-2j)!}\sum_{\lambda\in\boldsymbol{Z}}(2m\lambda+r)^{2\nu-2j}(m(m\lambda^{2}+r\lambda+n))^{j}\\ &\quad\times\sum_{\gamma\in\Gamma_{\infty}\backslash\Gamma}(c\tau+d)^{-k-2\nu}e\left((m\lambda^{2}+r\lambda+n)\frac{a\tau+b}{c\tau+d}\right),\end{aligned}$$

because $\nu-h=j$ is equivalent to $h=\nu-j$. q.e.d.

PROOF OF THEOREM 1.1. Let $D_{2\nu}^{*}\colon S_{k+2\nu}\to J_{k,m}^{\mathrm{cusp}}$ be the adjoint map of the restriction $D_{2\nu|J_{k,m}^{\mathrm{cusp}}}$ with respect to the Petersson product. By Lemma 2.2, we have

$$\langle D_{2\nu}^{*}(f),P_{n,r}\rangle=\frac{c(n,r;D_{2\nu}^{*}(f))m^{k-2}}{(4mn-r^{2})^{k-3/2}2\pi^{k-3/2}}\Gamma\left(k-\frac{3}{2}\right).$$

On the other hand, by Lemmas 2.3 and 2.1,

$$\begin{aligned}&\langle D_{2\nu}^{*}(f),P_{n,r}\rangle=(f,D_{2\nu}(P_{n,r}))\\ &=\sum_{\mu=0}^{\nu}\sum_{\lambda\in\boldsymbol{Z}}\frac{(-1)^{\mu}(k+2\nu-\mu-2)!(2\nu)!}{(k+\nu-2)!\mu!(2\nu-2\mu)!}(2m\lambda+r)^{2\nu-2\mu}(m(m\lambda^{2}+r\lambda+n))^{\mu}(f,P_{m\lambda^{2}+r\lambda+n}^{k+2\nu})\\ &=\sum_{\mu=0}^{\nu}\sum_{\lambda\in\boldsymbol{Z}}\frac{(-1)^{\mu}(k-1)^{(2\nu-\mu)}(2\nu)!}{(k-1)^{(\nu)}\mu!(2\nu-2\mu)!}(2m\lambda+r)^{2\nu-2\mu}(m(m\lambda^{2}+r\lambda+n))^{\mu}\\ &\quad\times\frac{a(m\lambda^{2}+r\lambda+n;f)}{(4\pi(m\lambda^{2}+r\lambda+n))^{k+2\nu-1}}\Gamma(k+2\nu-1)\\ &=\frac{(2\nu)!m^{\nu}\Gamma(k+2\nu-1)}{(4\pi)^{k+2\nu-1}(k-1)^{(\nu)}}\sum_{\lambda\in\boldsymbol{Z}}\frac{a(m\lambda^{2}+r\lambda+n;f)}{(m\lambda^{2}+r\lambda+n)^{k+\nu-1}}\end{aligned}$$

$$\times \sum_{\mu=0}^{\nu} \frac{(-1)^{\mu}(k-1)^{(2\nu-\mu)}}{\mu!(2\nu-2\mu)!}\left(\frac{(2m\lambda+r)^2}{m(m\lambda^2+r\lambda+n)}\right)^{\nu-\mu}.$$

Here we have used the relation

$$\frac{(k-1)^{(2\nu-\mu)}}{(k-1)^{(\nu)}}=\frac{(k+2\nu-\mu-2)!}{(k+\nu-2)!}.$$

Thus,

$$c(n, r; D_{2\nu}^*(f))=\frac{(2\nu)!\Gamma(k+2\nu-1)}{2^{2k+4\nu-3}\pi^{2\nu+1/2}m^{k-\nu-2}(k-1)^{(\nu)}\Gamma(k-3/2)}(4mn-r^2)^{k-3/2}$$

$$\times \sum_{\lambda\in \mathbf{Z}} \frac{a(m\lambda^2+r\lambda+n; f)}{(m\lambda^2+r\lambda+n)^{k+\nu-1}}$$

$$\times \sum_{\mu=0}^{\nu} \frac{(-1)^{\mu}(k-1)^{(2\nu-\mu)}}{\mu!(2\nu-2\mu)!}\left(\frac{(2m\lambda+r)^2}{m(m\lambda^2+r\lambda+n)}\right)^{\nu-\mu}.$$

By [6, Theorem 3.3, (3.12), p. 473], the right hand side of the above formula is exactly the Fourier coefficient $c(n, r; \{f\}_{k,m}^{\nu})$ times $C_{\nu,m}$. q.e.d.

PROOF OF COROLLARY 1.2. By Theorem 1.1 and [2, Corollary, p. 45], for any $\phi\in J_{k,1}^{\mathrm{cusp}}$, we have

$$\langle \phi, \{f\}_{k,m}^{\nu} \mid V_m^*\rangle=\langle \phi \mid V_m, \{f\}_{k,m}^{\nu}\rangle=\frac{1}{C_{\nu,m}}(D_{2\nu}(\phi \mid V_m), f)=\frac{1}{C_{\nu,m}}((D_{2\nu}(\phi)) \mid T(m), f)$$

$$=\frac{1}{C_{\nu,m}}(D_{2\nu}(\phi), f \mid T(m))=\frac{1}{C_{\nu,m}}(D_{2\nu}(\phi), \lambda_m f)=\frac{C_{\nu,1}}{C_{\nu,m}}\langle \phi, \lambda_m\{f\}_{k,1}^{\nu}\rangle$$

$$=\langle \phi, m^{k-\nu-2}\lambda_m\{f\}_{k,1}^{\nu}\rangle.$$

q.e.d.

PROOF OF COROLLARY 1.3. If $f \mid T(m)=\lambda_m f$, then $a(m; f)=\lambda_m a(1; f)$ and $a(1; f)\neq 0$ (e.g. [1, Lemmas 4.5.15 and 4.6.11]). By Corollary 1.2, we have

$$a(1; f)\langle\{f\}_{k,m}^{\nu}, \phi \mid V_m\rangle=a(1; f)\langle\{f\}_{k,m}^{\nu} \mid V_m^*, \phi\rangle$$

$$=\lambda_m a(1; f)\langle\{f\}_{k,1}^{\nu}, \phi\rangle m^{k-\nu-2}=a(m; f)\langle\{f\}_{k,1}^{\nu}, \phi\rangle m^{k-\nu-2}.$$

Hence,

$$L_{f,\phi}(s)=\frac{\langle\{f\}_{k,1}^{\nu}, \phi\rangle}{a(1; f)}L_f(s-k+\nu+2).$$

Thus we are done by a well-known theorem for $L_f(s)$ (e.g. [7, Theorem 3.66]) and by [8, Theorem 1, p. 784]. q.e.d.

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