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# THE AFFINE STRUCTURES ON THE REAL TWO-TORUS. I

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We wish to complete the study of the affine structures on the real affine 2-tori $T^2$, following N. H. Kuiper [2], J. P. Benzecri [1] and others. The category of the affine manifolds is defined, as usual, by the manifolds equipped with maximal atlas whose coordinate transformations are affine transformations $y^i = \Sigma_j a^i_j x^j + b^i$, $a^i_j$, $b^i \in R$, in the cartesian space $R^n$, and by the maps which are expressed locally with affine transformations in terms of the affine charts.

Our main result asserts that the affine structures on $T^2$ are completely determined by the holonomy groups, in which, however, the concept of the holonomy group requires a slight modification as follows.

Given an affine manifold $M$, its universal covering manifold $M^\sim$ with the induced affine structure is immersed equidimensionally into $R^n$ by an affine map $d$. The map $d$ gives rise to a homomorphism $\eta : \pi_1(M) \to A(R^n)$ of the fundamental group into the affine group $A(R^n)$ in such a way that $d$ is $\pi_1(M)$-equivariant with respect to the action of $\pi_1(M)$ on $R^n$ through $\eta$. The image of $\eta$ is called the holonomy group $H$ of $M$, which is unique up to an inner automorphism of $A(R^n)$. Here $A(M)$, in general, denotes the affine automorphism group of the affine manifold $M$. When the image $dM^\sim$ is not simply connected, we switch to its universal covering $(dM^\sim)^\sim$ from $R^n$; that is, we construct an affine immersion: $d^* : M^\sim \to (dM^\sim)^\sim$ which covers $d$ and a homomorphism $\eta^* : \pi_1 M \to A((dM^\sim)^\sim)$ accordingly. Now the modified holonomy group $H^*$ of $M$ is by definition the image $\eta^*(\pi_1 M)$. When $dM^\sim$ is simply connected, we simply put $H^* = H$. At any rate $H^*$ can be regarded as a subgroup of the universal covering group $A(R^2)^\sim$ of $A(R^2)$.

THEOREM 1. *Two affine structures on $T^2$ are isomorphic if and only if the modified holonomy groups are conjugate in $A(R^2)^\sim$.*

The difficulty in the proof lies in establishing that $d$ is a covering map onto $dM^\sim$. The difficulty may be illustrated by the fact that a surjective immersion of $R^2$ onto itself is not always a diffeomorphism. In any case, that $d$ is a covering implies that $T^2$ is affine isomorphic with $(dM^\sim)^\sim/H^*$. In order to describe the classification of $H^*$ it is convenient to state the following theorem.

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**THEOREM 2.** *For any affine torus $T^2$, the affine group $A(T^2)$ admits nonempty open orbits.*

In the transitive case, $H^*$ is characterized as a lattice subgroup $\cong Z^2$ of a maximal connected abelian subgroup $G^* \cong R^2$ of $A(R^2)^\sim$. The projection $G = \pi(G^*)$ of $G^*$ in $A(R^2)$ is listed below. Since $G^*$ acts on the affine plane $R^2$ almost effectively, $G^*$ has the induced affine structure, and so $G^*/H^*$ becomes an affine torus naturally. In the intransitive case, the situation is more complicated; the affine 2-torus $T^2$ is then partitioned into several, say $n$, isomorphic open cylinders and their boundaries (which are closed geodesics in one and the same homotopy class $\alpha$ in $\pi_1(T^2)$; those cylinders together constitute the open orbit of $A(T^2)$). To be more precise, $T^2$ has a cylinder $R \times S^1$ as an affine (regular) covering space which admits the affine transformations $\beta(k):(x, y) \to (x + k, y)$, $k \in Z$, and the covering group is generated by $\beta(n)$. $H^*$ is contained in a 2-dimensional abelian subgroup $G^*$ of $A(R^2)^\sim$ which is saturated (viz. $G^* = \pi^{-1}\pi(G^*)$) with respect to the projection $\pi : A(R^2)^\sim \to A(R^2)$ and whose image under $\pi$ has the identity component $G$ of type (I-1) or (I-2) in the list below. In particular $\pi(G^*)$ is a linear transformation group having no translation part. $\pi(G)$ is generated by $G$ and the reflection, $-1$, with respect to the fixed point of $G$. $G^*$ is isomorphic with $\operatorname{Ker} \pi \times \pi(G^*) \cong Z \times G$. Now $H^*$ is generated by two members $\alpha^*$, $\beta^*$ such that we have $\alpha^* = (0, \alpha)$ and $\beta^* = \beta(n) = (n, \beta)$ in the above correspondence, and that $\alpha$ is expanding (viz. the eigenvalues of the linear map $\alpha$ are greater than one and this is a characterization of $H^*$).

A question yet to be answered would be: What is the whole picture of all the affine structure of $T^2$? We intend to answer this question in a forthcoming paper.

Finally we list the conjugate classes of the maximal abelian connected subgroups $G$ of $A(R^2)$, writing $\left(\begin{smallmatrix} a & b & p \\ c & d & q \end{smallmatrix}\right)$ for the affine transformation $(x, y) \to (ax + by + p, cx + dy + q)$. $G$ consists of

(I-1): $\begin{pmatrix} a & b & 0 \\ 0 & a & 0 \end{pmatrix}$, (III-1): $\begin{pmatrix} 1 & b & p \\ 0 & 1 & b \end{pmatrix}$,

(I-2): $\begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \end{pmatrix}$, (III-2): $\begin{pmatrix} 1 & 0 & p \\ 0 & 1 & q \end{pmatrix}$,

(I-3): $\begin{pmatrix} u & v & 0 \\ -v & u & 0 \end{pmatrix}$, (III-3): $\begin{pmatrix} 1 & b & p \\ 0 & 1 & 0 \end{pmatrix}$,

(II): $\begin{pmatrix} 1 & 0 & p \\ 0 & d & 0 \end{pmatrix}$, where $a > 0$, $d > 0$, $(u, v) \neq (0, 0)$ and the others are arbitrary real numbers.

## References

1. J. P. Benzecri, *Variétés localement affines*, Séminaire Ehresmann, May 1959.

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