

THE AGE OF THE UNIVERSE AND THE COSMOLOGICAL CONSTANT DETERMINED FROM COSMIC MICROWAVE BACKGROUND ANISOTROPY MEASUREMENTS

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ABSTRACT

If $\Omega_{\text{tot}} = 1$ and structure formed from adiabatic initial conditions, then the age of the universe, as constrained by measurements of the cosmic microwave background (CMB), is $t_0 = 14.0 \pm 0.5$ Gyr. The uncertainty is surprisingly small given that CMB data alone do not significantly constrain either h or Ω_{Λ} . This small uncertainty is due to the tight (and accidental) correlation in flat adiabatic models of the age with the angle subtended by the sound horizon on the last-scattering surface and, thus, with the well-determined acoustic peak locations. If we assume either the *Hubble Space Telescope* Key Project result $h = 0.72 \pm 0.08$ or simply that $h > 0.55$, we find $\Omega_{\Lambda} > 0.4$ at 95% confidence—another argument for dark energy, independent of supernovae observations. Our analysis is greatly simplified by the Monte Carlo Markov chain approach to Bayesian inference combined with a fast method for calculating angular power spectra.

Subject headings: cosmic microwave background — cosmological parameters — cosmology: observations — cosmology: theory — distance scale — methods: data analysis — methods: statistical

1. INTRODUCTION

Determining the expansion age of the universe has been a major goal of cosmology ever since Hubble discovered the expansion. Compatibility with determinations of stellar ages is an important consistency check of cosmological models. Traditional methods of determining the expansion age rely on Hubble constant measurements, which either are highly imprecise or have error budgets dominated by systematics. In addition, one must determine Ω_{Λ} (or more generally the mean density of the various components), since it affects how the expansion rate has changed over time. In this Letter we present highly precise age determinations from cosmic microwave background (CMB) data that completely bypass the need for independent determinations of H_0 and Ω_{Λ} .

There are a handful of cosmological parameters that can be determined from measurements of the CMB angular power spectrum to percent-level accuracy, such as ω_b , ω_m , and Ω_{tot} (where $\omega_i \equiv \Omega_i h^2$ and $H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$; e.g., Eisenstein, Hu, & Tegmark 1999). Other parameters cannot be well determined and require the addition of complementary observations. For example Ω_{Λ} is poorly determined by the CMB alone (e.g., Efsthathiou & Bond 1999) but well determined when supernovae observations are included (e.g., Netterfield et al. 2001).

It has been pointed out (Ferrerias, Melchiorri, & Silk 2001) and demonstrated (Netterfield et al. 2001) that the CMB can be used to place tight constraints on the age of the universe. This is due to the high degree of correlation between the angle subtended by the sound horizon on the last-scattering surface, θ_s , and age in flat adiabatic models, also noticed by Hu et al. (2001), who used it to place an *upper* bound on the age. Here we extend the previous work by including additional data, by taking the flatness assumption seriously, and by the use of a new analysis technique that has advantages as described below. We also demonstrate the accidental nature of the age–sound-horizon correlation by showing that its tightness depends on

where we are in the (Ω_{Λ}, h) parameter space. We are fortunate that the correlation is tightest near the “concordance” values of $h = 0.72$ and $\Omega_{\Lambda} = 0.65$.

Our age determination is model-dependent and the model (adiabatic cold dark matter [CDM]) has many parameters. We take them to be the amplitude and power-law spectral index of the primordial matter power spectrum, A and n , the baryon and dark matter densities, ω_b and ω_d , the cosmological constant divided by the critical density, Ω_{Λ} , and the redshift of reionization of the intergalactic medium, z_{ri} . The Hubble constant and age are derived parameters, given in terms of the others by $h^2 = (\omega_b + \omega_d)/(1 - \Omega_{\Lambda})$ and $t_0 = 6.52 \text{ Gyr } (\Omega_{\Lambda} h^2)^{-1/2} \ln \{ [1 + (\Omega_{\Lambda})^{1/2}] / (1 - \Omega_{\Lambda})^{1/2} \}$.

We do not consider models with $\Omega_{\text{tot}} \neq 1$ or dark energy models other than the limiting case of a cosmological constant with $w \equiv P/\rho = -1$. The first we justify on the grounds of simplicity: CMB observations indicate the mean curvature is close to zero and generally agree well with inflation. If we did allow the curvature to vary, our age result would become significantly less precise. We expect that allowing w to vary will have little effect, as we discuss below.

We explore the likelihood in a 10-dimensional parameter space (six cosmological parameters plus four experimental parameters) by Monte Carlo generation of a Markov chain of parameter values as described in Christensen et al. (2001). From the chain one can rapidly calculate marginalized one-dimensional or two-dimensional probability distributions for chain parameters, or derived parameters, with or without additional priors. Generating a sufficiently long chain in a reasonable amount of time requires a fast means of calculating the angular power spectrum for a given model. We describe this fast method briefly below and more thoroughly in M. Kaplinghat, L. Knox, and C. Skordis (2001, in preparation).

Supernovae observations constrain the combination $H_0 t_0$ better than either parameter by itself. Perlmutter et al. (1999a)

find for flat universes that $t_0 = 13.0_{-1.0}^{+1.2}(0.72/h)$ Gyr. Combining this result with our t_0 determination leads to $h = 0.67_{-0.06}^{+0.07}$, in agreement with the *HST* result. Riess et al. (1998) find for arbitrary Ω_{tot} that $t_0 = 14.2 \pm 1.7$ Gyr.

L. Krauss and B. Chaboyer (2001, in preparation) estimate the age of 17 metal-poor globular clusters to be $t_{\text{GC}} = 12.5$ Gyr with a 95% lower bound of 10.5 Gyr and a 68% upper bound of 14.4 Gyr. The minimum requirement for consistency, that $t_f \equiv t_0 - t_{\text{GC}} > 0$, is easily satisfied with a few Gyr to spare. Unfortunately, the upper bound on t_{GC} is not sufficiently restrictive to set an interesting lower bound on t_f .

Below we tabulate our constraints on all the model parameters and emphasize not only t_0 but also Ω_Λ . With the inclusion of prior information on H_0 , the CMB data provide strong evidence for $\Omega_\Lambda > 0$. The same conclusion can be reached by, instead, combining the CMB data with observations of large-scale structure (Efstathiou et al. 2001) or clusters of galaxies (Dodelson & Knox 2000).

2. METHOD

Our first step in exploring the high-dimensional parameter space is the creation of an array of parameter values called a chain, in which each element of the array, θ , is a location in the n -dimensional parameter space. The chain has the useful property once it has converged so that $P(\theta \in R) = N(\theta \in R)/N$, where the left-hand side is the posterior probability that θ is in the region R , N is the total number of chain elements, and $N(\theta \in R)$ is the number of chain elements with θ in the region R . Once the chain is generated, one can rapidly explore one-dimensional or two-dimensional marginalizations in either the original parameters or the derived parameters, such as t_0 . Calculating the marginalized posterior distributions is simply a matter of histogramming the chain.

2.1. Generating the Chain

The chain we generate is a Monte Carlo Markov chain (MCMC) produced via the Metropolis-Hastings algorithm described by Christensen et al. (2001). The candidate-generating function for an initial run was a normal distribution for each parameter. Subsequent runs used a multivariate-normal distribution with cross-correlations between cosmological parameters equal to those of the posterior as calculated from the initial run.

All of our results are based on MCMC runs consisting of 2×10^5 iterations. For the ‘‘burn-in’’ the initial 2.5×10^4 samples were discarded, and the remaining set was thinned by accepting every 25th iteration. We used the CODA software (Best, Cowles, & Vines 1995) to confirm that all chains passed the Referty-Lewis convergence diagnostics and the Heidelberg-Welch stationarity test.

While generating the chain we always restrict our sampling to the $h > 0.4$ and $5.8 < z_{\text{ri}} < 6.3$ region of parameter space. The former is a very conservative lower bound on h , and the latter is a simple interpretation of the spectra of quasars at very high redshift (Becker et al. 2001; Djorgovski et al. 2001). For some of our results we assume an ‘‘*HST* prior,’’ which means $h = 0.72 \pm 0.08$ (Freedman et al. 2001) with a normal distribution.

2.2. C_l Calculation

We calculate C_l rapidly with a preliminary version of the Davis anisotropy shortcut (DASH; M. Kaplinghat, L. Knox, and C. Skordis 2001, in preparation). We first calculate the Fourier and Legendre transformed photon temperature perturbation, $\Delta_l(k)$, on

a grid over parameters ω_b and ω_d at fixed values of $\Omega_k \equiv 1 - \Omega_{\text{tot}} = \Omega_k^*$, $\Omega_\Lambda = \Omega_\Lambda^*$, and $\tau = 0$ using CMBFAST (Seljak & Zaldarriaga 1996). From this grid, we get C_l for any ω_b , ω_d , and the primordial power spectrum $P(k) = A(k/0.05 \text{ Mpc}^{-1})^n$ by performing multilinear interpolation on the grid of $\Delta_l(k)$ and then the following integral:

$$C_l \equiv \frac{l(l+1)C_l}{2\pi} = 8\pi l(l+1) \int k^2 dk \Delta_l^2(k) P(k). \quad (1)$$

We can get any C_l in the entire model space of $[\omega_b, \omega_d, \tau, \Omega_\Lambda, \Omega_k, P(k)]$ by the use of analytic relations between the $\Delta_l(k)$ for different models. For varying Ω_Λ and Ω_k , $C_l = C_l$ where $l/l = \theta_s(\Omega_k^*, \Omega_\Lambda^*)/\theta_s(\Omega_k, \Omega_\Lambda)$ and θ_s is the angle subtended by the sound horizon at the last-scattering surface. For $\Omega_k = 0$, $\theta_s = s/\eta_0$, where η_0 is the conformal time today (or, equivalently, the comoving distance to the horizon) and s is the comoving sound horizon at the last-scattering surface.

Altering θ_s is not the only effect of varying Ω_k and Ω_Λ . Varying Ω_k changes the eigenvalues of the Laplacian on very large scales, and hence the power spectrum at the last-scattering surface and both Ω_k and Ω_Λ affect the late-time evolution of the gravitational potential. Both of these effects only affect C_l at $l \ll 100$. We, therefore, make an additional grid over the parameters ω_b , ω_d , Ω_k , and Ω_Λ but with smaller maximum l - and k -values than the lower dimensional high- l grid. The low- l grid used for the calculations presented here has ranges of $0.01 < \omega_b < 0.03$, $0.05 < \omega_d < 0.25$, and $0 < \Omega_\Lambda < 0.85$ with four, four, and eight uniformly spaced samplings of the range, respectively. For the present application we have fixed $\Omega_k = 0$. For the high- l grid, the ranges for ω_b and ω_d are the same but with twice as many samples, and $\Omega_\Lambda^* = 0.6$.

The split into a low- l grid and a high- l grid has been used by others, although for grids of C_l , not $\Delta_l(k)$. We follow Tegmark, Zaldarriaga, & Hamilton (2001) in joining our grids with a smooth k -space kernel, $g(k) = 2/[1 + \exp(2k/k_s)]^4$, where $k_s = 1.5/s$; in the integrand of equation 1 $P(k)$ is replaced with $g(k)P(k)$ for the low- l grid and $[1 - g(k)]P(k)$ for the high- l grid and $C_l = C_l^{\text{low}} + C_l^{\text{high}}$. Finally, we allow for nonzero z_{ri} by sending $C_l \rightarrow \mathcal{R}_l(z_{\text{ri}})C_l$, where $\mathcal{R}_l(z_{\text{ri}})$ is given by the fitting formula of Hu & White (1996).

2.3. Likelihood Calculation

To calculate the likelihood we use the offset lognormal approximation of Bond, Jaffe, & Knox (2000), which is a better approximation to the likelihood function than a normal distribution. We include bandpower data from BOOMERANG, the Degree Angular Scale Interferometer (DASI; Halverson et al. 2001), MAXIMA (Lee et al. 2001), and the *Cosmic Background Explorer* (COBE; Bennet et al. 1996). The weight matrices, band powers, and window functions for DASI are available in Leitch et al. (2001), Halverson et al. (2001), and Pryke et al. (2001). For COBE we approximate the window functions as top-hat bands; all other information is available in Bond et al. (2000) and in electronic form at RadPack (Radical Compression Data Analysis Package, L. Knox 2001).¹ For BOOMERANG and MAXIMA we approximate the window functions as top-hat bands, the weight matrices as diagonal, and the lognormal offsets, x , as zero. The BOOMERANG team report the uncertainty in their beam full width at half-maximum (FWHM) as $12.9 \pm$

¹ Available at <http://bubba.ucdavis.edu/~knox/radpack.html>.

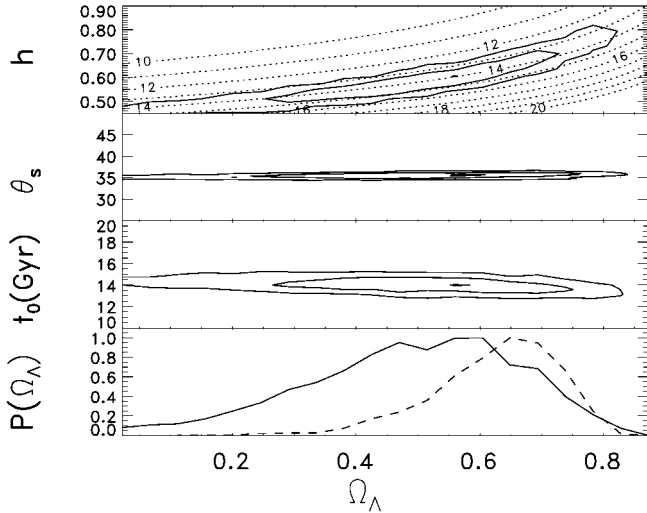


FIG. 1.—Posterior probability density of Ω_Λ (lowest panel) and contours of equal probability density in the (Ω_Λ, t_0) -plane (lower middle panel), $(\Omega_\Lambda, \theta_s)$ -plane (upper middle panel), and (Ω_Λ, h) -plane (top panel). Contour levels are at $e^{-6.17/2}$, $e^{-2.3/2}$, and 0.95 of maximum. The lowest panel curves are for $h > 0.4$ (solid curve) and $h = 0.72 \pm 0.08$ (dashed curve). Top panel dotted lines are at constant t_0 .

1/4. We follow them in modeling the departure from the nominal (non-Gaussian) beam shape as a Gaussian. For the BOOMERANG Z'_i , C_i is therefore actually $C_i \exp(-l^2 b^2)$. Our prior for b is uniform, bounded such that the FWHM is always between 11/5 and 14/3. Calibration parameters, for example, $u_{\text{DASI}} = 1 \pm 0.04$, are taken to have normal prior distributions, and we alter model angular power spectra via $C_l \rightarrow u_{\text{DASI}}^2 C_l$ prior to comparison with the reported band powers. To reduce our sensitivity to beam errors, we use only bands with maximum l -values less than 1000. Thus, we use all nine DASI bands, the first 11 MAXIMA bands, and all but the last BOOMERANG band.

Five of the 24 DASI fields are completely within the area of sky analyzed by BOOMERANG, and three partially overlap this area. We expect the resulting DASI-BOOMERANG band-power error correlations (which we neglect) to be small and to have negligible effect on our results.

3. RESULTS

In the top panel of Figure 1 we see that neither h nor Ω_Λ can be constrained well by CMB data alone. The shape of the contour “banana” in the top panel is determined by the dependence of the well-determined θ_s on h and Ω_Λ ; lines of constant θ_s run along the ridge of high likelihood. Since θ_s correlates well with the age, as seen in Figure 2, the ridge of high likelihood is also at nearly constant age.

Although Ω_Λ is poorly determined by CMB data alone, we find that the addition of the *HST* prior allows one to set a 95% lower limit of $\Omega_\Lambda > 0.4$. This same lower bound can be achieved by simply rejecting models with $h < 0.55$.

We can understand the t_0 - θ_s correlation with the aid of an approximate analytic expression given by Hu et al. (2001) from which we derive

$$\frac{\Delta\theta_s}{\theta_s} = 0.060 \left(2.9 \frac{\Delta\omega_m}{\omega_m} + 1.0 \frac{\Delta\omega_\Lambda}{\omega_\Lambda} - 1.14 \frac{\Delta\omega_b}{\omega_b} \right) \quad (2)$$

for the fiducial values $\omega_m = 0.15$, $\omega_\Lambda = 0.3$, and $\omega_b = 0.02$.

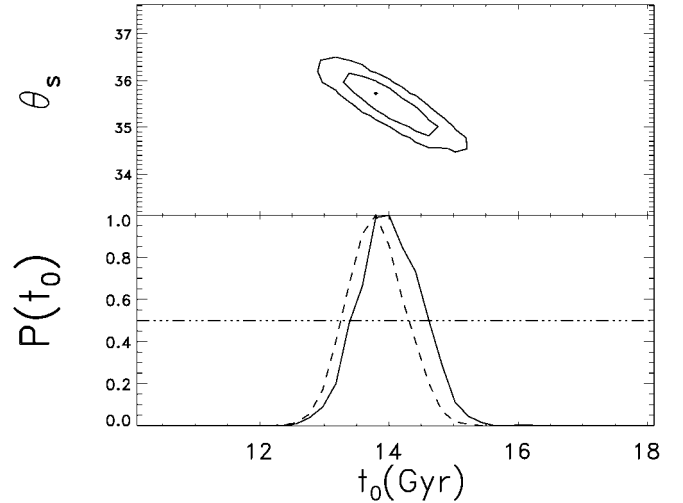


FIG. 2.—Posterior probability density of the age of the universe (lower panel) and contours of equal probability density (as in Fig. 1) in the (θ_s, t_0) -plane (upper panel). The lower panel curves are for $h > 0.4$ (solid curve) and $h = 0.72 \pm 0.08$ (dashed curve).

Expanding t_0 about our fiducial values we find that

$$\frac{\Delta t_0}{t_0} = -0.12 \left(3.0 \frac{\Delta\omega_m}{\omega_m} + 1.2 \frac{\Delta\omega_\Lambda}{\omega_\Lambda} \right). \quad (3)$$

Thus, a change from the fiducial values by $\Delta\omega_m$ and $\Delta\omega_\Lambda$ that keeps θ_s fixed will leave the age nearly unchanged.

In general, the parameter controlling this correlation is the ratio of ratios:

$$R = \frac{(\partial \ln \theta_s / \partial \ln \omega_m) / (\partial \ln \theta_s / \partial \ln \omega_\Lambda)}{(\partial \ln t_0 / \partial \ln \omega_m) / (\partial \ln t_0 / \partial \ln \omega_\Lambda)}, \quad (4)$$

and the correlation is tightest when $R = 1$. Ratio R has little dependence on ω_b . For what used to be called standard CDM, $R = 0.75$. The value of R can be as small as 0.53 for $\Omega_m = 1$ and $h = 0.72$ and as large as 2.0 for $\Omega_\Lambda = 0.83$ and $h = 0.72$. At the maximum of the likelihood, $R = 0.88$ (1.08) with (without) the *HST* prior.

As a test, we have estimated the age via a direct grid-based evaluation of the likelihood given DASI and DMR data using Code for Anisotropies in the Microwave Background (CAMB; Lewis, Challinor, & Lasenby 2001) to calculate C_l -values. Taking the grid parameters to be t_0 , ω_d , n , and A and fixing $u_{\text{DASI}} - 1 = \omega_b - 1 = \Omega_k = z_{\text{ri}} = 0$, we find $t_0 = 13.6 \pm 0.6$ Gyr. This agrees very well with our MCMC + DASH results when the same assumptions and data selection are made: $t_0 = 13.7 \pm 0.6$ Gyr. We can also reproduce the Netterfield et al. (2001) result, finding for BOOMERANG and DMR data (although ignoring the highest l BOOMERANG bandpower) $t_0 = 14.32 \pm 0.68$ Gyr.

In Table 1 we show means and standard deviations for our 10 original parameters and a number of derived parameters. Particularly noteworthy is θ_s , determined with an error of less than 3%. The agreement between the data sets on this number is also remarkable. From DASI $\theta_s = 0^\circ 60 \pm 0^\circ 01$, and from BOOMERANG $\theta_s = 0^\circ 59 \pm 0^\circ 01$.

For studying the early evolution of structure, it is useful to know the age at redshifts in the matter-dominated era. For $1 \ll z < 100$, $t_z \times (1+z)^{3/2} = 6.52/\sqrt{\omega_m}$ Gyr = 16 ± 1 Gyr,

TABLE 1
PARAMETER BOUNDS

| Parameter | Mean | Standard Deviation |
|--|-------|--------------------|
| Chain Parameters | | |
| ω_b | 0.021 | 0.002 |
| ω_d | 0.145 | 0.021 |
| Ω_Λ^a | 0.49 | 0.17 |
| z_{ri}^a | 6.0 | 0.14 |
| A | 6.7 | 0.56 |
| n | 0.96 | 0.04 |
| u_{DASI} | 1.00 | 0.03 |
| u_{BOOM} | 1.07 | 0.03 |
| u_{MAXIMA} | 1.00 | 0.03 |
| $\text{FWHM}_{\text{BOOM}}^a$ (arcmin) | 13.9 | 0.3 |
| Derived Parameters | | |
| t_0 (Gyr) | 14.0 | 0.48 |
| h^a | 0.59 | 0.07 |
| Ω_m^a | 0.51 | 0.17 |
| ω_Λ^a | 0.19 | 0.11 |
| ω_m | 0.166 | 0.021 |
| $c\eta_0$ (Gpc) | 13.95 | 0.56 |
| θ_s (arcmin) | 35.5 | 0.43 |
| l_{eq} | 168 | 15 |
| l_d | 1392 | 18 |
| H_2 | 0.481 | 0.024 |
| H_3 | 0.486 | 0.030 |
| $t_z \times (1+z)^{3/2}$ | 16.0 | 1.0 |

NOTES.—The mean and standard deviations for the 10 chain parameters (*top*) plus derived parameters (*bottom*). For these results we use all the data with our weakest prior assumptions. Units of A are arbitrary. See Hu et al. 2001 for the definition of l_{eq} , l_d , H_2 , and H_3 .

^a The uncertainties are not well described by a mean and standard deviation. For example, the posterior probability distribution for z_{ri} is not significantly different from the prior one we assumed, which is uniform between 5.8 and 6.3.

where t_z is the age at redshift z . Since *Microwave Anisotropy Probe* (*MAP*)² and *Planck*³ will determine ω_m to 10% and 2%,

² Additional information is available at <http://map.gsfc.nasa.gov>.

³ Additional information is available at <http://astro.estec.esa.nl/Planck>.

respectively (Eisenstein et al. 1999), they will determine $t_z \times (1+z)^{3/2}$ to 5% and 1%, respectively.

4. DISCUSSION

Since our argument is model-dependent, it is worth pointing out that the model has been enormously successful on the relevant length scales (e.g., Wang, Tegmark, & Zaldarriaga 2001). Perhaps the weakest link is the dark energy equation of state since we have scant guidance from observations (e.g., Perlmutter, Turner, & White 1999b) and even less from theory. Fortunately, one can show that varying the equation of state away from $w = -1$ at fixed θ_s has very little effect on the age: at $\theta_s = 0.6$ and $\omega_m = \frac{1}{3}$, $\Delta t_0/t_0 = 0.05\Delta w$. Though we neglect the possibility of gravitational wave contributions to C_l (see Efstathiou 2001), we do not expect these to make much difference since θ_s is relatively unaffected by measurements at low l where gravitational waves are important.

Our age determination has the benefit of being derived from observations whose statistical properties can be predicted highly accurately using linear perturbation theory. We are encouraged that the observational errors are dominated by the reported statistical ones since nearly the same result can be derived from two independent data sets. We conclude that the best determination of t_0 now comes from CMB data. The prospects for improving the age determination are bright since the statistical errors (and any systematic ones too) will be greatly reduced by *MAP* data in the near future.

The MCMC chains we have generated are available via e-mail from the authors.

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REFERENCES

- Becker, R. H., et al. 2001, *AJ*, in press
 Bennet, C. L., et al. 1996, *ApJ*, 464, L1
 Best, N. G., Cowles, M. K., & Vines, S. K. 1995, CODA: Convergence, Diagnosis, and Output Software for Gibbs Sampler Output (version 0.30; Cambridge: MRC Biostatistics Unit)
 Bond, J. R., Jaffe, A. H., & Knox, L. 2000, *ApJ*, 533, 19
 Christensen, N., Meyer, R., Knox, L., & Luey, B. 2001, *Classical Quantum Gravity*, 18, 2677
 Dodelson, S., & Knox, L. 2000, *Phys. Rev. Lett.*, 84, 3523
 Djorgovski, S. G., Castro, S. M., Stern, D., & Mahabal, A. 2001, *ApJ*, 560, L5
 Efstathiou, G. 2001, preprint (astro-ph/0109151)
 Efstathiou, G., et al. 2001, *MNRAS*, submitted (astro-ph/0109152)
 Efstathiou, G., & Bond, J. R. 1999, *MNRAS*, 304, 75
 Eisenstein, D. J., Hu, W., & Tegmark, M. 1999, *ApJ*, 518, 2
 Ferreras, I., Melchiorri, A., & Silk, J. 2001, *MNRAS*, 327, L47
 Freedman, W. L., et al. 2001, *ApJ*, 553, 47
 Halverson, N., et al. 2001, preprint (astro-ph/014489)
 Hu, W., Fukugita, M., Zaldarriaga, M., & Tegmark, M. 2001, *ApJ*, 549, 669
 Hu, W., & White, M. 1997, *ApJ*, 479, 568
 Lee, A., et al. 2001, *ApJ*, 561, L1
 Leitch, E., et al. 2001, *ApJ*, submitted (astro-ph/0104488)
 Lewis, A., Challinor, A., & Lasenby, A. 2001, *ApJ*, 538, 473
 Netterfield, B., et al. 2001, *ApJ*, submitted (astro-ph/0104460)
 Perlmutter, S., et al. 1999a, *ApJ*, 517, 565
 Perlmutter, S., Turner, M., & White, M. 1999b, *Phys. Rev. Lett.*, 83, 670
 Pryke, C., et al. 2001, *ApJ*, submitted (astro-ph/0104490)
 Riess, A., et al. 1998, *AJ*, 116, 1009
 Seljak, U., & Zaldarriaga, M. 1996, *ApJ*, 469, 437
 Tegmark, M., Zaldarriaga, M., & Hamilton, A. J. S. 2001, *Phys. Rev. D*, 63, 043007
 Wang, M., Tegmark, M., & Zaldarriaga, M. 2001, submitted (astro-ph/0105091)13