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THE C^*-ALGEBRA GENERATED BY AN ISOMETRY[1]

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1. Introduction. In this paper, I determine the structure of any C^*-algebra generated by an isometry. Using a theorem of Halmos [3], the problem is reduced to the study of C^*-algebras $\mathcal{A}(A)$ generated by A and A^* where (i) A is unitary, (ii) $A=S_\alpha$ with S_α the shift of multiplicity α, and (iii) $A=W\oplus S_\alpha$ with W unitary. In case (i), the resulting algebra is isometrically *-isomorphic to the algebra $C(\sigma(A))$ of all complex-valued continuous functions on the spectrum of A and nothing more need be said. In cases (ii) and (iii), it turns out that $\mathcal{A}(A)$ is isometrically *-isomorphic to $\mathcal{A}(S_1)$ so that $\mathcal{A}(A)$ is independent of W and α. In each of these cases, there is a unique minimal closed two-sided ideal $\mathcal{I}(A)$ such that $\mathcal{A}(A)/\mathcal{I}(A)$ is isometrically *-isomorphic to $C(T)$, where T is the perimeter of the unit circle. The ideal $\mathcal{I}(A)$ is determined spatially in the cases $A=S_1$ and $A=W\oplus S_1$.

We begin with the notation. For our purposes, all Hilbert spaces are complex and all ideals are closed and two-sided. If $\{e_n: n=0, 1, 2, \cdots\}$ is an orthonormal basis for a Hilbert space H then the shift $S=S_1$ is defined by $Se_n=e_{n+1}$. By a shift of multiplicity α is meant the α-fold direct sum $S\oplus S\oplus \cdots \oplus S$ acting on $H\oplus H\oplus \cdots \oplus H$. The orthogonal projection onto the one-dimensional subspace of H spanned by e_n is denoted by P_n.

If H (or H_i) is a Hilbert space then $\mathcal{B}(H)$ (or $\mathcal{B}(H_i)$) denotes the algebra of all bounded operators with the usual norm topology and $\mathcal{K}$ (or $\mathcal{K}_i$) denotes the ideal of all compact operators. The natural quotient map from $\mathcal{B}(H)$ to $\mathcal{B}(H)/\mathcal{K}$ ($\mathcal{B}(H_i)$ to $\mathcal{B}(H_i)/\mathcal{K}_i$) is given by

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π (π_i). If A is an operator in $\mathcal{B}(H)$, the C^*-algebra generated by A will be named $\mathcal{A}(A)$ or just $\mathcal{A}$ when there is no possible doubt about A. An operator A is called a Fredholm operator if $\pi(A)$ is invertible. The set of all Fredholm operators in $\mathcal{B}(H)$ is denoted by $\mathcal{F}$. It is known [1] that A is in $\mathcal{F}$ if and only if A has closed range and finite-dimensional null and defect spaces.

2. **The algebra** $\mathcal{A}(S)$. Our first object is to determine the ideals of $\mathcal{A}(S)$. For vectors y and z in H, we define the operator $T_{y,z}$ by

$$T_{y,z}(x) = (x, y)z.$$

It is well known that the smallest closed subspace of $\mathcal{B}(H)$ containing all $T_{y,z}$ is just $\mathcal{K}$.

THEOREM 1. *The algebra* $\mathcal{A}(S)$ *contains the full ideal of compact operators* $\mathcal{K}$ *and* $\mathcal{K}\subset\mathcal{J}$ *for every nontrivial ideal* $\mathcal{J}$ *in* $\mathcal{A}(S)$.

PROOF. Since $1-SS^*=P_0$ is in $\mathcal{K}$, we see that $\mathcal{A}\cap\mathcal{K}$ is a nontrivial ideal in $\mathcal{A}$. Now suppose that $\mathcal{J}$ is any nontrivial ideal in $\mathcal{A}$. If $A\neq 0$ is in $\mathcal{J}$ then A^*A is also in $\mathcal{J}$. For some $N\geqq 0$ we have $\|Ae_N\|\neq 0$. Since $S^mP_0S^{m*}=P_m$, we see that P_m is in $\mathcal{A}$ for all $m\geqq 0$. Hence $P_NA^*AP_N$ is in $\mathcal{J}$. But

$$\begin{aligned} P_NA^*\,AP_Nx &= (A^*\,AP_Nx, e_N)e_N \\ &= (x, P_NA^*\,Ae_N)e_N = \|Ae_N\|^2P_Nx; \end{aligned}$$

so P_N is in $\mathcal{J}$ and thus $S^{*N}P_NS^N=P_0$ is in $\mathcal{J}$.

Now given any $\epsilon>0$ and y in H there is a polynomial $p(x)$ so that $\|p(S)e_0-y\|<\epsilon$. It follows that the operator T_{y,e_0} has the property that $\|P_0[p(S)]^*-T_{y,e_0}\|<\epsilon$. Thus, T_{y,e_0} is in $\mathcal{J}$. Similarly, if z is in H then there is a polynomial $q(x)$ with $\|q(S)e_0-z\|<\epsilon$ and $\|q(S)T_{y,e_0}-T_{y,z}\|<\epsilon\|y\|$ so that for all y, z, $T_{y,z}$ is in $\mathcal{J}$. It follows that $\mathcal{J}$ contains all finite rank operators and hence $\mathcal{K}\subset\mathcal{J}$. □

As immediate consequences of Theorem 1 we have two well-known results.

COROLLARY 1.1. *The algebra* $\mathcal{A}(S)$ *is dense in* $\mathcal{B}(H)$ *with the strong topology.*

PROOF. $\mathcal{K}$ is strongly dense in $\mathcal{B}(H)$. □

COROLLARY 1.2. *The shift* S *has no reducing subspaces except the trivial ones* (0) *and* H.

PROOF. Otherwise, by Corollary 1.1 there would be a proper subspace invariant under all the operators in $\mathcal{B}(H)$. □

We can now complete the ideal theory for $\mathcal{A}(S)$.

THEOREM 2. *The algebra* $\mathcal{A}(S)/\mathcal{K}$ *is* $*$*-isomorphic and isometric to* $C(T)$.

PROOF. Since $S^*S-SS^*=P_0$ is in $\mathcal{K}$, it is apparent that $\mathcal{A}/\mathcal{K}$ is an abelian C^*-algebra. Hence $\mathcal{A}/\mathcal{K}$ is $*$-isomorphic and isometric to $C(X)$ where X is the maximal ideal space of $\mathcal{A}/\mathcal{K}$. Now $\mathcal{A}/\mathcal{K}$ is generated by $\pi(S)$ and $\pi(S^*)$ so X is homeomorphic to the spectrum of $\pi(S)$ in $\mathcal{A}/\mathcal{K}$. By a theorem in [2], the spectrum of $\pi(S)$ in $\mathcal{A}/\mathcal{K}$ is the set $\{\lambda: S-\lambda$ is not in $\mathfrak{F}\}$ and an elementary computation shows that this set is just the perimeter of the unit circle T. □

Theorems 1 and 2 determine the structure of the ideals of $\mathcal{A}(S)$ since the ideal theory for $C(T)$ is well known.

3. **The algebra** $\mathcal{A}(W\oplus S)$. The next part of the program is to determine the structure of $\mathcal{A}(W\oplus S)$ where W is a unitary operator on H_1 and S is the shift on H_2 with $H_1\oplus H_2=H$. We require a Lemma which may be of some intrinsic interest.

LEMMA. *If* $A\oplus B$ *is in* $\mathcal{A}(W\oplus S)$ *then* $\|A\|\leqq\|\pi_2(B)\|\leqq\|B\|$.

PROOF. There is a sequence of "polynomials" in two noncommuting "indeterminates,"

$$p_n(x, y)=\sum a^{(n)}_{i_1i_2i_3\ldots i_k}x^{i_1}y^{i_2}x^{i_3}\cdots y^{i_k},$$

such that $p_n(W, W^*)\to A$ and $p_n(S, S^*)\to B$ in the operator norm topology. Thus

$$p_n(\pi_2(S), \pi_2(S^*))\to\pi_2(B)$$

since π_2 is norm-decreasing. Now applying the Gelfand transform to the abelian C^*-algebra generated by $\pi_2(S)$, we see that $\sup_{\lambda\in T}|p_n(\lambda, \bar{\lambda})|\to\|\pi_2(B)\|$ since the spectrum of $\pi_2(S)$ in $\mathcal{A}(S)/\mathcal{K}_2$ is T and the Gelfand transform is an isometry. On the other hand, applying the Gelfand transform to the C^*-algebra generated by W, we see that $\sup_{\lambda\in\sigma(W)}|p_n(\lambda, \bar{\lambda})|\to\|A\|$. Since $\sigma(W)\subset T$, the desired result follows. □

THEOREM 3. *The algebra* $\mathcal{A}(W\oplus S)$ *is isometrically* $*$*-isomorphic to* $\mathcal{A}(S)$ *under the mapping* $W\oplus S\leftrightarrow S$.

PROOF. The mapping $W\oplus S\to S$ extends to the "polynomials" described in the Lemma. The extension is clearly a $*$-homorphism. If $p(x, y)$ is such a "polynomial" then

$$\|p(W, W^*)\oplus p(S, S^*)\|=\max(\|p(W, W^*)\|, \|p(S, S^*)\|).$$

But by the Lemma, $\|p(W, W^*)\| \leq \|p(S, S^*)\|$ so

$$\|p(W, W^*) \oplus p(S, S^*)\| = \|p(S, S^*)\|.$$

Hence, the mapping extends to an isometry from $\mathfrak{A}(W \oplus S)$ onto $\mathfrak{A}(S)$ which is also a *-isomorphism. □

COROLLARY 3.1. *The algebra* $\mathfrak{A}(W \oplus S)$ *has a unique minimal nontrivial ideal,* $\mathfrak{I}(W \oplus S)$, *and* $\mathfrak{A}(W \oplus S)/\mathfrak{I}(W \oplus S) \cong C(T)$.

PROOF. This follows from the properties of $\mathfrak{A}(S)$ established in Theorems 1 and 2. □

It is of some interest to determine the minimal ideal $\mathfrak{I}(W \oplus S)$ spatially. This can be done in a manner similar to Theorem 1.

THEOREM 4. *The minimal nontrivial ideal* $\mathfrak{I}(W \oplus S)$ *in* $\mathfrak{A}(W \oplus S)$ *is*

$$\mathfrak{I}(W \oplus S) = 0 \oplus \mathfrak{K}_2 = \mathfrak{K} \cap \mathfrak{A}(W \oplus S).$$

PROOF. Since

$$(W^* \oplus S^*)(W \oplus S) - (W \oplus S)(W^* \oplus S^*) = 0 \oplus P_0,$$

we see that $\mathfrak{K} \cap \mathfrak{A}$ is a nontrivial ideal in $\mathfrak{A}$. Now suppose $\mathfrak{I}$ is any nontrivial ideal. By the Lemma, if $C \oplus D$ is a nonzero element of $\mathfrak{I}$ then $D \neq 0$. Hence, for some e_N in the basis $\{e_n : n = 0, 1, 2, \cdots\}$ for H_2, we have $\|De_N\| \neq 0$. The argument that $0 \oplus \mathfrak{K}_2 \subset \mathfrak{I}$ now finishes as in the proof of Theorem 1. Further, if $C \oplus D$ is in $\mathfrak{K} \cap \mathfrak{A}$ then C is in $\mathfrak{K}_1$ and D is in $\mathfrak{K}_2$. It follows from the Lemma that $\|C\| = 0$ so that $0 \oplus \mathfrak{K}_2 = \mathfrak{K} \cap \mathfrak{A}$. □

4. The general case. For the case A an arbitrary isometry, the algebra $\mathfrak{A}(A)$ can now be determined. Using a decomposition due to Halmos [3], any isometry A on H is either (i) unitary, (ii) unitarily equivalent to a shift S_α of multiplicity α, or (iii) unitarily equivalent to a direct sum $W \oplus S_\alpha$ where W is unitary. In the first case, $\mathfrak{A}(A)$ is isometrically *-isomorphic to $C(\sigma(A))$. In case (ii), the mapping $S \leftrightarrow S_\alpha$ induces an isometric *-isomorphism between $\mathfrak{A}(A)$ and $\mathfrak{A}(S)$ so the theory of §2 carries over to $\mathfrak{A}(A)$. In case (iii), the mapping

$$W \oplus S \leftrightarrow W \oplus S_\alpha$$

induces an isometric *-isomorphism between $\mathfrak{A}(A)$ and $\mathfrak{A}(W \oplus S)$ so the theory of §3 carries over to $\mathfrak{A}(A)$. In cases (ii) and (iii), $\mathfrak{A}(A) \cong \mathfrak{A}(S)$ and there is a unique minimal ideal $\mathfrak{I}(A) \neq 0$ with $\mathfrak{A}(A)/\mathfrak{I}(A) \cong C(T)$. Thus the algebraic structure is independent of W and α.

One can hope that knowing the ideals of $\mathfrak{A}(A)$ makes possible a classification of the *-representations of $\mathcal{A}(A)$. In fact, the representation theory for $\mathcal{A}(S)$ can be handled by use of Theorem 1 and standard results on representations of $\mathcal{B}(H)$ and $\mathcal{K}$. In particular, using results from [4, p. 296] we see that every representation of $\mathcal{A}(S)$ is a direct sum of identity representations and representations of $C(T)$. Using the fact that for A an isometry, either $\mathcal{A}(A) \cong C(\sigma(A))$ or $\mathcal{A}(A) \cong \mathcal{A}(S)$, the *-representations for $\mathcal{A}(A)$ can now be determined.

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