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**THE ALGEBRA OF BOUNDED CONTINUOUS FUNCTIONS
INTO A NONARCHIMEDEAN FIELD**

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Let S be a topological space, F a complete nonarchimedean rank 1 valued field, and $C^*(S, F)$ the Banach algebra of bounded, continuous, F -valued functions on S . Various topological conditions on S and/or F are shown to be equivalent, respectively, to each of the following: every maximal ideal of $C^*(S, F)$ is fixed; the only quotient field of $C^*(S, F)$ is F itself; every homomorphism of $C^*(S, F)$ into F is an evaluation at a point of S ; the Stone-Weierstrass theorem holds for $C^*(S, F)$. It is also shown that a certain topological space derived from S may be embedded in the space of maximal ideals of $C^*(S, F)$ with Gelfand topology, or in the space of homomorphisms of $C^*(S, F)$ into F .

0. Introduction. Throughout this paper, $C^*(S, F)$ denotes the Banach algebra of bounded, continuous functions on a topological space S into a complete nonarchimedean rank 1 valued field F . We introduce several stronger-than-usual topological separation properties, such as ultrahausdorff, ultraregular, and ultranormal; and several weaker-than-usual compactness properties, such as mildly compact, mildly countably compact, and mildly Lindelof. We then show that several key implications involving $C^*(S, F)$ become equivalences when the new topological properties replace their conventional counterparts.

In §1, we define and discuss these new topological properties, and relate them to the cofilters ("ouf-filtres") of van der Put [13]. In §2, we obtain a result on the metric structure of non-locally compact nonarchimedean Banach spaces.

In §3, we show that all maximal ideals of $C^*(S, F)$ are fixed if and only if S is mildly compact (Theorem 15); and that F is the only quotient field of $C^*(S, F)$ if and only if F is locally compact or S is mildly countably compact (Theorem 19). Using the result of §2, we also give necessary and/or sufficient conditions for the only homomorphisms of $C^*(S, F)$ into F to be evaluations at points of S (Theorems 20 and 21). We also show that the set of quasicomponents of S , appropriately topologized, is homeomorphic to the space of fixed maximal ideals of $C^*(S, F)$, with either of the Gelfand topologies defined by Shilkret [14] (Theorems 10 and 12).

In §4, we extend results of Kaplansky [7] and Chernoff, Rasala, and Waterhouse [3]: we introduce two versions of the Stone-Weierstrass property, and show that the stronger version in $C^*(S, F)$ is equivalent to mild compactness of S , and the weaker version is sufficient for mild

countable compactness of S (Theorems 22 and 23).

It is interesting to note that many of our results involve properties of S only, and are independent of the choice of the nonarchimedean field F .

1. **Topology.** A *quasicomponent* of S is a minimal nonempty intersection of sets clopen in S . The quasicomponents form a partition of S into closed sets. Each quasicomponent is a union of components; if S is compact and Hausdorff, the quasicomponents and components are identical [6].

Distinct points or sets in S will be called *ultraseparated* if they are contained in disjoint clopen sets. S will be called *ultrahausdorff*, or *UT2*, if distinct points are ultraseparated; equivalently, if every quasicomponent is a singleton. After Ellis [4], S will be called *ultraregular*, or *UR*, if disjoint points and closed sets are ultraseparated; equivalently, if S has a basis consisting of clopen sets. S will be called *ultranormal*, or *UN*, if disjoint closed sets are ultraseparated.

S is *totally disconnected*, or *TD*, if every component is a singleton. Hence, if S is ultrahausdorff, it is totally disconnected; and if S is compact, Hausdorff, and totally disconnected, then it is ultrahausdorff. We also note that, for a T_1 space, ultranormality implies ultraregularity, and ultraregularity implies the ultrachusdorff property. For a compact space, the ultrahausdorff property implies ultraregularity, and ultraregularity implies ultranormality.

LEMMA 1. *In an ultraregular space, every open or closed set is a union of quasicomponents.*

Proof. If S is ultraregular, then every open set is a union of clopen sets and hence a union of quasicomponents. It follows that every closed set, being the complement of an open set, is also a union of quasicomponents.

LEMMA 2. *Let G be a family of functions on a set A into a topological space B , and let A be topologized with the weak- G topology. Then:*

(1) *If B is ultraregular, A is ultraregular.*

(2) *If B is ultrahausdorff, and G separates points of A , then A is ultrahausdorff.*

Proof. (1) If B is ultraregular, it has a clopen basis. The preimages, under the members of G , of these clopen sets form a clopen subbasis for A . Hence A is ultraregular.

(2) Let p and q be distinct points of A . If G separates points, then $g(p) \neq g(q)$ for some g in G . If B is ultrahausdorff, then $g(p)$ and $g(q)$ are contained in disjoint clopen sets V and W of B . Hence $g^{-1}(V)$ and $g^{-1}(W)$ are disjoint clopen neighborhoods of p and q in A . Thus, A is ultrahausdorff.

THEOREM 1. *S is ultraregular if and only if the topology on S is the weak- $C^*(S, F)$ topology.*

Proof. If S is ultraregular, it has a clopen basis. Since $C^*(S, F)$ contains all characteristic functions of clopen sets, it follows that these basis sets are weak- $C^*(S, F)$ clopen as well. Hence the two topologies are identical.

To prove the converse, we apply Lemma 2, part (1), setting $A = S$, $B = F$, and $G = C^*(S, F)$. Since F is ultraregular, it follows that the weak- $C^*(S, F)$ topology on S is ultraregular.

We will call S *mildly compact*, or *MC*, if every clopen cover of S has a finite subcover; *mildly countably compact* if every countable clopen cover has a finite subcover; and *mildly Lindelof* if every clopen cover has a countable subcover.

We mention several examples. The closed interval $[0, 1]$, with the points $1, 1/2, 1/3, \dots$ deleted, is mildly compact but not compact. A countably infinite set with discrete topology is mildly Lindelof, but not mildly countably compact. The space of all countable ordinals is mildly countably compact, but not mildly Lindelof [5].

LEMMA 3. *S is mildly countably compact if and only if every partition of S into clopen sets is finite.*

Proof. If S is not mildly countably compact, it has a clopen cover $\{A_i: i = 1, 2, 3, \dots\}$ with no finite subcover. For each positive integer n , let $B_n = A_n - \bigcup \{A_i: 1 \leq i < n\}$. Then the nonempty members of the family $\{B_n: n = 1, 2, 3, \dots\}$ form an infinite clopen partition of S . The proof of the converse is direct.

THEOREM 2. (1) *An ultraregular, mildly compact space is compact.*
 (2) *An ultraregular, mildly Lindelof space is Lindelof.*

Proof. If S is ultraregular, it has a clopen basis. If S is also mildly compact, then every covering of S by members of this basis has a finite subcover. This last condition is sufficient for compactness [8]. The proof for mildly Lindelof spaces is similar.

The following diagrams of implications summarize some of our results. In these diagrams, *COMP* denotes "compact".

DIAGRAM 1. For all spaces,

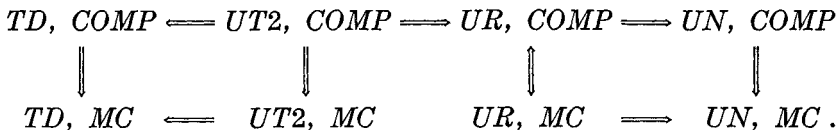
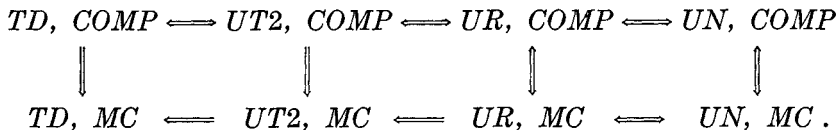


DIAGRAM 2. For Hausdorff spaces,



We note that all the implications of Diagram 2, except $TD, COMP \rightarrow UT_2, COMP$, hold for T_1 spaces as well.

The *quasicomponent quotient space* of S , denoted $Q(S)$, will be the space of quasicomponents of S with the quotient topology [9]. For each point s of S , $Q(s)$ will denote the quasicomponent containing s . If P is a clopen set in S , and hence a union of quasicomponents, then $Q(P)$ is clopen in $Q(S)$. If S is ultraregular, we also have, using Lemma 1: If P is open, then $Q(P)$ is open; and if P is closed, then $Q(P)$ is closed. The following theorem is now obvious:

- THEOREM 3.** (1) $Q(S)$ is ultrahausdorff. The quotient mapping $Q: S \rightarrow Q(S)$ is a homeomorphism if and only if S is ultrahausdorff.
 (2) If S is ultraregular, $Q(S)$ is ultraregular.
 (3) If S is compact, $Q(S)$ is compact.
 (4) $Q(S)$ is mildly compact, or mildly countably compact, or mildly Lindelof, if and only if S has the same property.

The *ultraregular kernel* of S , denoted $K(S)$, will be the space whose points are the points of S and whose topology is generated by the clopen sets of S . It is obvious that:

- THEOREM 4.** (1) The topology of $K(S)$ is the weak- $C^*(S, F)$ topology (for any nonarchimedean field F).
 (2) $K(S)$ is ultraregular. The topologies of S and $K(S)$ are identical if and only if S is ultraregular.
 (3) A subset of S is clopen in $K(S)$ if and only if it is clopen in S .
 (4) $K(S)$ is ultrahausdorff if and only if S is ultrahausdorff.
 (5) If S is compact, $K(S)$ is compact.
 (6) $K(S)$ is mildly compact, or mildly countably compact, or mildly Lindelof, if and only if S has the same property.

We now show that the mappings Q and K commute.

THEOREM 5. $K(Q(S))$ and $Q(K(S))$ are identical topological spaces.

Proof. The points of both $K(Q(S))$ and $Q(K(S))$ are the quasicomponents of S . To show that the topologies are identical, we note that the following statements are equivalent:

R is an open set in $K(Q(S))$.

R is a union of clopen sets of $Q(S)$.

$Q^{-1}(R)$ is a union of clopen sets of S .

$Q^{-1}(R)$ is open in $K(S)$.

R is open in $Q(K(S))$.

Henceforward, $QK(S)$ will denote the topological space of Theorem 5. We note that this space is both ultrahausdorff and ultraregular.

A filter on S with a base consisting of clopen sets will be called a *cofilter*, and a maximal cofilter will be called an *ultracofilter*. An arbitrary filter H will be called *fixed* (after van der Put, [13]) if it has nonempty intersection; if M is a cardinal number, H will be called *M -fixed* if every intersection of M members of H is nonempty. We will say that H *recognizes* a partition $\{A_i: i \in I\}$ of S if one of the sets A_i is in H . We note that a cofilter on S is an ultracofilter if and only if it recognizes all finite partitions of S into clopen sets. It is obvious that:

LEMMA 4. *If an ultracofilter on S is M -fixed, for some infinite cardinal M , then it recognizes all clopen partitions of S of cardinality M .*

A partial converse to Lemma 4 is:

LEMMA 5. *If a cofilter H on S recognizes all countable clopen partitions of S , then H is countably fixed.*

Proof. If H is not countably fixed, then it contains a family $\{A_i: i = 1, 2, 3, \dots\}$ of clopen sets with empty intersection. For every positive integer n , let $B_n = \bigcap \{A_i: 1 \leq i \leq n\}$; then $\{B_n: n = 1, 2, 3, \dots\}$ is a family of clopen sets in H , ordered by exclusion, with empty intersection. Let $C_1 = S - B_1$, and for $n > 1$, let $C_n = B_{n-1} - B_n$. Then the family $\{C_n: n = 1, 2, 3, \dots\}$ forms a clopen partition of S , but none of these sets is in H . Hence H does not recognize all clopen partitions of S .

Nonmeasurable cardinals [5] may be characterized as follows: every countably fixed ultrafilter on a set of nonmeasurable cardinality is fixed. It is known that the nonmeasurable cardinals include \aleph_0 , \aleph_1 , \aleph_2 , \dots ; they are closed under exponentiation, passage to a successor

or to any smaller cardinal, and the supremum operation over a nonmeasurable index. The conjecture that all cardinals are nonmeasurable remains unproved; however, it is known that it can never be disproved.

LEMMA 6. *An ultracofilter U on S is countably fixed if and only if it recognizes all clopen partitions of S of nonmeasurable cardinality.*

Proof. If U recognizes all clopen partitions of S of nonmeasurable cardinality, then, by Lemma 5, U is countably fixed. Conversely, suppose U is countably fixed and $\{A_i: i \in I\}$ is a clopen partition of S of nonmeasurable cardinality. Then U induces, via the quotient mapping, a countably fixed ultrafilter U' on the family $\{A_i: i \in I\}$. Since this family is of nonmeasurable cardinality, U' is fixed—that is, U' contains a singleton $\{A_j\}$. Hence A_j is in U , so U recognizes the partition $\{A_i: i \in I\}$.

Taking the dual versions of our compactness definitions, and using Lemma 6, we easily have the following lemmas:

LEMMA 7. *S is mildly compact if and only if every ultracofilter on S is fixed.*

LEMMA 8. *The following are equivalent:*

- (1) *S is mildly countably compact.*
- (2) *Every ultracofilter on S is countably fixed.*
- (3) *Every ultracofilter on S recognizes all clopen partitions of S of nonmeasurable cardinality.*

LEMMA 9. *The following are equivalent:*

- (1) *S is mildly Lindelof.*
- (2) *Every countably fixed ultracofilter on S is fixed.*
- (3) *Every ultracofilter on S which recognizes all clopen partitions of S of nonmeasurable cardinality is fixed.*

2. The density of a nonarchimedean Banach space. Let X be a nonarchimedean Banach space over F . We will assume that $|F| \subseteq \|X\| \subseteq \text{Cl}(|F|)$; i.e., (1) X has a unit vector, and (2) if F is discrete, then $\|X\| = |F|$.

A sphere $T(x, d) = \{y \in X: \|y - x\| \leq d\}$ in X will be called a *closphere*; a sphere $W(x, d) = \{y \in X: \|y - x\| < d\}$ will be called an *osphere*. $V(X)$ will denote the closphere and subring $T(0, 1)$. All clospheres and osppheres are clopen sets; every point of a closphere or ospphere is a center; and the clospheres, or osppheres, of any fixed radius form a partition of X [1]. For any $x \in X$, $a \in F$, and $d > 0$, we have:

$$a \cdot T(x, d) = T(ax, |a| \cdot d) \quad a \cdot W(x, d) = W(ax, |a| \cdot d).$$

THEOREM 6. *If X is locally compact, then every partition of a ciosphere in X into ciospheres of a fixed smaller radius is finite.*

Proof. If X is locally compact, it contains a compact sphere K . Since every other sphere in X is homeomorphic, by a translation and scalar multiplication, to a clopen subset of K , it follows that every sphere in X is compact. The theorem follows.

The remainder of this section is devoted to proving the companion theorem:

THEOREM 7. *If X is not locally compact, there exists an infinite cardinal $D(X)$ such that: if $d \in \text{Cl}(|F|)$, then every partition of a ciosphere of radius d in X into ciospheres of a fixed smaller radius is of cardinality $D(X)$.*

For $0 < d < 1$, let $r(X, d)$ denote the cardinality of the partition of $V(X)$ into ciospheres of radius d . Obviously, if $0 < c < d < 1$, then $r(X, c) \geq r(X, d)$.

LEMMA 10. *If $0 < c < 1$, $0 < d < 1$, and $d \in |F|$, then $r(X, cd) = r(X, c) \cdot r(X, d)$.*

Proof. Choose $a \in F$ with $|a| = d$. Since $V(X)$ contains $r(X, c)$ distinct ciospheres of radius c , it follows that $T(0, d) = a$. $V(X)$ contains $r(X, c)$ distinct ciospheres of radius cd . By translation, every ciosphere in X of radius d contains $r(X, c)$ distinct ciospheres of radius cd ; so $V(X)$, which contains $r(X, d)$ ciospheres of radius d , must contain $r(X, c) \cdot r(X, d)$ ciospheres of radius cd .

COROLLARY 11. *If $0 < d < 1$ and $d \in |F|$, then $r(X, d^n) = r(X, d)^n$ for any positive integer n .*

LEMMA 12. *If X is not locally compact, and $0 < d < 1$, then $r(X, d)$ is infinite.*

Proof. Shilkret [14] has shown that if X is not locally compact, then it is not discrete or has infinite residue space. If X is not discrete, then for some $d > 0$ there is a family $\{x_n: n = 1, 2, 3, \dots\}$ in X such that $d < \|x_n\| < 1$ for all n and $\|x_n\| \neq \|x_m\|$ for $n \neq m$. Hence the ciospheres $\{T(x_n, d): n = 1, 2, 3, \dots\}$ are distinct in $V(X)$, so $r(X, d)$ is infinite.

If X has infinite residue space, then the members of the residue

space form a partition of $V(X)$ into ospheres of radius 1. Since this partition is infinite, the equivalent or finer partition into closspheres of radius d must also be infinite. Hence $r(X, d)$ is infinite.

LEMMA 13. *If X is not locally compact, then the cardinal numbers $\{r(X, d): 0 < d < 1\}$ are all equal.*

Proof. (1) Suppose X and F are not discrete, and $0 < c < d < 1$. Choose $e \in |F|$ and a positive integer n such that $d < e < 1$ and $e^n < c$. Applying Corollary 11 and Lemma 12, we have:

$$r(X, d) \geq r(X, e) = r(X, e)^n = r(X, e^n) \geq r(X, c) \geq r(X, d) .$$

Hence $r(X, c) = r(X, d)$.

(2) Suppose X and F are discrete, with $\|X\| = |F|$, and $0 < d < 1$. Let $e < 1$ be a generator of the multiplicative group $|F - \{0\}|$, and let n be the integer satisfying $e^n \leq d < e^{n+1}$. Then every clossphere of radius d is a clossphere of radius e^n ; hence, by Corollary 11 and Lemma 12,

$$r(X, d) = r(X, e^n) = r(X, e)^n = r(X, e) .$$

For X not locally compact, we now define $D(X)$, the *density* of X , to be $r(X, d)$ for any d between 0 and 1. For X locally compact, it will be convenient to define $D(X)$ to be 1.

Proof of Theorem 7. Suppose X is not locally compact, $T(x, d)$ is a clossphere in X with $d \in \text{Cl}(|F|)$, and P is a partition of $T(x, d)$ into closspheres of radius $c < d$.

(1) If $d \in |F|$, then $T(x, d)$ is homeomorphic, by a translation and scalar multiplication, to $V(X)$; and this homeomorphism carries P , in a one-to-one fashion, onto the partition of $V(X)$ into closspheres of radius c/d . Hence, $\text{card}(P) = r(X, c/d) = D(X)$.

(2) If $d \in \text{Cl}(|F|)$, then there exist $e, f \in |F|$ such that $c < e < d < f$. Obviously, $T(x, e) \subseteq T(x, d) \subseteq T(x, f)$; and by part (1), $T(x, e)$ and $T(x, f)$ both contain precisely $D(X)$ closspheres of radius c . It follows that $T(x, d)$ contains $D(X)$ closspheres of radius c , so $\text{card}(P) = D(X)$.

3. Maximal ideals of $C^*(S, F)$. Let X be a commutative non-archimedean Banach algebra over F , with identity e of norm 1.

Each quotient field $F_M = X/M$, for M a maximal ideal of X , is both a field extending F (identifying each $b \in F$ with $b \cdot e + M \in F_M$), and a normed algebra over F whose quotient norm extends the valuation on F [11].

Each $x \in X$ gives rise to a function x^* on \mathfrak{M} , the family of maximal ideals of X , given by $x^*(M) = x + M \in F_M (M \in \mathfrak{M})$. The family $X^* = \{x^*: x \in X\}$, with the supremum norm, is a normed algebra over F .

After Shilkret [15], we let $\mathfrak{M}' = \{M \in \mathfrak{M}: F_M = F\}$, and $X_0 = \{x \in X: x^*(M) \in F (M \in \mathfrak{M})\}$. On \mathfrak{M} , the *Gelfand topology* is the weak- X_0^* topology; on \mathfrak{M}' , and all subsets of \mathfrak{M}' , the *strong Gelfand topology* is the weak- X^* topology, and the *weak Gelfand topology* is the weak- X_0^* topology.

THEOREM 8. (1) *All Gelfand topologies are ultraregular.* (2) *The strong Gelfand topology on \mathfrak{M}' is ultrahausdorff.*

Proof. We apply Lemma 2, with $A = \mathfrak{M}'$ or \mathfrak{M} , $B = F$, and $G = X_0^*$ or X^* . Part (1) follows immediately; part (2) follows from the observation that X^* separates points of \mathfrak{M}' .

For the remainder of this paper, X will be the algebra $C^*(S, F)$. $T(b, d)$ and $W(b, d)$ will denote closspheres and ospheres in F . \mathfrak{M}'' will denote the family $\{M_s: s \in S\}$ of fixed maximal ideals of X , where $M_s = \{x \in X: x(s) = 0\}$. We note that $x^*(M_s) = x(s) \in F$ for all $x \in X$ and $M_s \in \mathfrak{M}''$; hence $\mathfrak{M}'' \subseteq \mathfrak{M}' \subseteq \mathfrak{M}$.

THEOREM 9. *For $s, t \in S$: $M_s = M_t$ if and only if $Q(s) = Q(t)$.*

Proof. If $M_s \neq M_t$, then $x(s) = 0 \neq x(t)$ for some $x \in X$. Let C be a clopen neighborhood of 0 in F excluding $x(t)$; then $x^{-1}(C)$ is a clopen set in S containing s but not t . Hence $Q(s) \neq Q(t)$.

Conversely, if $Q(s) \neq Q(t)$, there is a clopen set K in S containing s but not t . Since the characteristic function of K belongs to M_t but not M_s , we have $M_s \neq M_t$.

COROLLARY 14. *Each member of X is constant on quasicomponents of S .*

COROLLARY 15. *There is a natural one-to-one correspondence between $QK(S)$ and \mathfrak{M}'' , given by $Q(s) \rightarrow M_s (s \in S)$.*

COROLLARY 16. *The natural mapping of S onto \mathfrak{M}'' is a bijection if and only if S is ultrahausdorff.*

LEMMA 17. X_0 contains all characteristic functions in X .

Proof. If $x \in X$ is a characteristic function, then $x^2 - x = 0$. Hence, for all $M \in \mathfrak{M}$, $x^*(M)^2 - x^*(M) = 0$; so $x^*(M) = 0$ or 1. It follows that $x \in X_0$.

THEOREM 10. *The weak and strong Gelfand topologies on \mathfrak{M}'' are identical. This topology is ultrahausdorff and ultraregular.*

Proof. Let $(x^*)^{-1}(P)$ be a strong-Gelfand subbasic set in \mathfrak{M}'' , where P is an osphere in F , $x \in X$, and x^* is regarded as a function on \mathfrak{M}'' . Then

$(x^*)^{-1}(P) = \{M_s: x^*(M_s) \in P\} = \{M_s: x(s) \in P\} = \{M_s: s \in x^{-1}(P)\}$. Since $x^{-1}(P)$ is clopen in S , we have $x^{-1}(P) = y^{-1}(Q)$, where y is the characteristic function of $x^{-1}(P)$, Q is a clopen set in F containing 1 but not 0, and y^* is regarded as a function on \mathfrak{M}'' . It follows that $(x^*)^{-1}(P) = (y^*)^{-1}(Q)$, a weak-Gelfand open set. Hence the two Gelfand topologies on \mathfrak{M}'' are identical; by Theorem 8, this topology is ultrahausdorff and ultraregular.

We can now speak of *the Gelfand topology on \mathfrak{M}''* without ambiguity.

We now show that $C^*(S, F)$ is congruent to the algebra of bounded, continuous, F -valued functions on an ultrahausdorff, ultraregular space.

THEOREM 11. *$C^*(S, F)$ is congruent to $C^*(QK(S), F)$.*

Proof. Let $Q': C^*(QK(S), F) \rightarrow C^*(S, F)$ be given by $Q'(x') = x' \circ Q$. Obviously, Q' is a ring homomorphism and an isometry. To show that Q' is onto $C^*(S, F)$, consider any $x \in C^*(S, F)$. Let $x' \in C^*(QK(S), F)$ be given by $x'(Q(s)) = x(s)$ ($s \in S$). By Corollary 14, x' is well-defined; since x is continuous on S , x' is continuous on $QK(S)$; and obviously $Q'(x') = x$.

Thus, in general, we can only hope to recover the structure of $QK(S)$ from that of X . Only where S is homeomorphic to $QK(S)$ —i.e., where S is ultraregular and T_1 —can we hope to recover S itself.

Using the facts that the topology on $QK(S)$ is the weak- $C^*(QK(S), F)$ topology, and the topology on \mathfrak{M}'' is the weak- $C^*(S, F)^*$ topology, we easily have:

THEOREM 12. *The natural bijection of $QK(S)$ onto \mathfrak{M}'' is a homeomorphism.*

COROLLARY 18. *The natural mapping of S onto \mathfrak{M}'' is continuous; it is a homeomorphism if and only if S is ultraregular and T_1 .*

We now establish a one-to-one correspondence between the closed ideals of X and the cofilters on S .

For each $x \in X$, a *smallset* of x will be a set

$$Sm(x, d) = \{s \in S: |x(s)| < d\} = x^{-1}(W(0, d)) \text{ for some } d > 0.$$

Obviously, all smallsets are clopen. The *zero-set* $Z(x)$ will be the set

$x^{-1}(0)$. For any clopen set L in S , $C(L)$ will denote the characteristic function of L .

For any proper ideal I of X , let $G'(I)$ denote the family of all smallsets of members of I , and let $G''(I)$ denote the family of zero-sets of characteristic functions of I .

LEMMA 19. $G'(I) = G''(I)$ for any ideal I in X .

Proof. Let $N = Sm(x, d) \in G'(I)$. Let $y \in X$ be given by $y(s) = 0(s \in N)$, $y(s) = x(s)^{-1}(s \notin N)$. Then $xy = C(S - N)$ is in I , so $N = Z(xy)$ is in $G''(I)$. Thus, $G'(I) \subseteq G''(I)$. The reverse inclusion is trivial.

We can now show that $G'(I)$ generates a cofilter $G(I)$ on S . First, if $L, M \in G'(I)$, then $C(S - L), C(S - M) \in I$; hence $C(S - L \cap M) = C(S - L) + C(S - M) - C(S - L) \cdot C(S - M) \in I$; so $L \cap M \in G'(I)$. Second, $\phi = Z(e) \notin G'(I)$.

For any cofilter G on S , let $I(G)$ denote the family of members of X all of whose smallsets are in G .

LEMMA 20. $I(G)$ is a closed ideal in X , for any cofilter G on S .

Proof. If $x, y \in I(G)$, then for any $d > 0$, $Sm(x + y, d) \in G$, since $Sm(x + y, d) \supseteq Sm(x, d) \cap Sm(y, d) \in G$; hence $x + y \in I(G)$. If $x \in I(G)$ and $y \in X$, then for any $d > 0$, $Sm(xy, d) \in G$, since $Sm(xy, d) \supseteq Sm(x, d \|y\|^{-1}) \in G$; hence $xy \in I(G)$, so $I(G)$ is an ideal. If $z \in Cl(I(G))$, then for any $d > 0$, $\|z - x\| < d$ for some $x \in I(G)$, so $Sm(z, d) = Sm(x, d) \in G$; hence $z \in I(G)$, so $I(G)$ is closed.

LEMMA 21. (1) $I(G(I)) = Cl(I)$ for any ideal I in X . (2) $G(I(G)) = G$ for any cofilter G on S .

Proof. (1) Let $x \in I(G(I))$. For any $d > 0$, $Sm(x, d) \in G(I)$; hence $Sm(x, d) = z(C(L))$ for some $C(L) \in I$; so $x \cdot C(L) \in I$ and $\|x - x \cdot C(L)\| < d$. Thus $x \in Cl(I)$, so $I(G(I)) \subseteq Cl(I)$. The reverse inclusion follows from the fact that $I(G(I))$ is a closed ideal containing I .

(2) Let $L \in G(I(G))$. Then $L \supseteq M$ for some $M \in G'(I(G))$; hence $C(S - M) \in I(G)$, so $M = Sm(C(S - M), 1/2) \in G$; so $L \in G$. Thus $G(I(G)) \subseteq G$. The reverse inclusion is trivial.

The following theorem is now obvious.

THEOREM 13. (1) There is a one-to-one correspondence between the proper closed ideals of X and the cofilters on S , given by $I \rightarrow G(I)$, with inverse $G \rightarrow I(G)$.

(2) This correspondence carries the maximal ideals of X onto

the ultracofilters on S , and the fixed maximal ideals onto the fixed ultracofilters.

THEOREM 14. \mathfrak{M}'' is Gelfand-dense in \mathfrak{M} , and weak-Gelfand dense in \mathfrak{M}' .

Proof. Let $N \in \mathfrak{M}$, and let $V = \{M \in \mathfrak{M} : |x_i^*(M)| < d(1 \leq i \leq n)\}$ be a typical Gelfand-basic neighborhood of N , where $d > 0$ and $x_i \in X_0 \cap N(1 \leq i \leq n)$. Then the smallsets $\{Sm(x_i, d) : 1 \leq i \leq n\}$ all belong to the ultracofilter $G(N)$; so the intersection of these smallsets is nonempty and contains a point s . Hence $|x_i^*(M_s)| = |x_i(s)| < d(1 \leq i \leq n)$, so $M_s \in V$. Thus N is in the Gelfand-closure of \mathfrak{M}'' , and \mathfrak{M}'' is Gelfand-dense in \mathfrak{M} . The proof that \mathfrak{M}'' is weak-Gelfand dense in \mathfrak{M}' is similar.

COROLLARY 22. The natural bijection of $QK(S)$ onto \mathfrak{M}'' carries $QK(S)$ onto a dense subspace of both \mathfrak{M} and \mathfrak{M}' .

From Lemma 7 and Theorem 13, we now have the key result:

THEOREM 15. $\mathfrak{M}'' = \mathfrak{M}$ if and only if S is mildly compact.

The following corollary will be of use later.

COROLLARY 23. Suppose T is a mildly compact topological space, I is an ideal in $C^*(T, F)$, and the members of I do not all vanish at any point of T . Then $I = C^*(T, F)$.

Collecting results, we now have the following theorem on the natural injection $B: Q(S) \rightarrow \mathfrak{M}$ and the mapping $B \circ Q: S \rightarrow \mathfrak{M}$.

THEOREM 16. (1) B is a homeomorphism if and only if S is mildly compact.

(2) $B \circ Q$ is a bijection if and only if S is mildly compact and ultrahausdorff.

(3) $B \circ Q$ is a homeomorphism if and only if S is compact and ultrahausdorff.

We note that, by Theorem 13, $\mathfrak{M} = \{I(U) : U \text{ an ultracofilter on } S\}$.

THEOREM 17. For any $x \in X$, $\|x\| = \|x^*(I(U))\|$ for some $I(U)$ in \mathfrak{M} ; i.e., each member of X realizes its norm on some maximal ideal.

Proof. Let $x \in X$. For $d > 0$, let $K(d) = \{s \in S : |x(s)| > d\}$. The

family $\{K(d): 0 < d < \|x\|\}$ generates a cofilter on S , and is therefore contained in an ultracofilter U .

For $y \in I(U)$ and $0 < d < \|x\|$, the sets $Sm(y, d)$ and $K(d)$ are both in U ; hence some $s \in S$ belongs to $Sm(y, d) \cap K(d)$; so $|y(s)| < d$, $|x(s)| > d$, and $\|x - y\| > d$. Therefore, $\|x - y\| \geq \|x\|$ for all $y \in I(U)$, so $\|x^*(I(U))\| = \inf \{\|x - y\|: y \in I(U)\} \geq \|x\|$. The reverse inequality is trivial.

THEOREM 18. *The following are equivalent:*

(1) *For any $x \in X$, $\|x\| = |x(s)|$ for some $s \in S$; i.e., each member of X realizes its norm at some point of S .*

(2) *F is discrete, or S is mildly countably compact.*

Proof. Suppose (2) is false. Then there is a clopen partition $\{A_i: i = 1, 2, 3, \dots\}$ of S , and a bounded sequence $\{b_i: i = 1, 2, 3, \dots\}$ in F such that $\{|b_i|: i = 1, 2, 3, \dots\}$ is strictly increasing. Let $x \in X$ be given by $x(s) = b_i(x \in A_i, i \geq 1)$. Then for any $s \in S$, $|x(s)| < \sup\{|b_i|: i = 1, 2, 3, \dots\} = \|x\|$. The proof of the converse is similar.

For $x \in X$, and U an ultracofilter on S , $x(U)$ will denote the filter on F generated by the family $\{x(T): T \in U\}$.

LEMMA 24. *If $x \in X$, U is an ultracofilter on S , and $b \in F$, then $x^*(I(U)) = b$ if and only if $x(U)$ converges to b .*

Proof. Suppose $x^*(I(U)) = b$. Then $(x - b \cdot e)^*(I(U)) = 0$, so $x - b \cdot e \in I(U)$. Hence, for any $d > 0$,

$$Sm(x - b \cdot e, d) = \{s \in S: |x(s) - b| < d\} = x^{-1}(W(b, d))$$

is in U , so $W(b, d)$ is in $x(U)$. Thus, every osphere containing b is in $x(U)$, so $x(U)$ converges to b . The proof of the converse is similar.

COROLLARY 25. *A maximal ideal $I(U)$ is in \mathfrak{M}' if and only if $x(U)$ converges for all $x \in X$.*

THEOREM 19. *$\mathfrak{M}' = \mathfrak{M}$ (i.e., F is the only quotient field of X) if and only if F is locally compact or S is mildly countably compact.*

Proof. (1) Let $I(U) \in \mathfrak{M}$. For $x \in X$, and $d > 0$, let $\{T(b_i, d): i \in I\}$ be a partition of the closphere $T(0, \|x\|)$ into clospheres of radius d . Then the nonempty members of the family $\{x^{-1}(T(b_i, d)): i \in I\}$ form a clopen partition of S . If F is locally compact (in which case I is finite) or S is mildly countably compact, this partition must be finite;

hence one set $x^{-1}(T(b_j, d))$ is in U . Thus $T(b_j, d)$ is in $x(U)$, so $x(U)$ contains closspheres of arbitrarily small radius. Since F is complete, $x(U)$ converges. By Corollary 25, $I(U) \in \mathfrak{W}'$; so $\mathfrak{W}' = \mathfrak{W}$.

(2) Suppose F is not locally compact and S is not mildly countably compact. Then for some $d > 0$, there is a bounded sequence $\{b_i: i = 1, 2, 3, \dots\}$ in F such that $|b_i - b_j| > d$ for $i \neq j$; and there is a countable clopen partition $\{A_i: i = 1, 2, 3, \dots\}$ of S . The family $\{S - A_i: i = 1, 2, 3, \dots\}$ generates a cofilter on S , and is therefore contained in an ultracofilter U . Let $x \in X$ be given by $x(s) = b_i (s \in A_i, i \geq 1)$. Then $x(U)$ contains no clossphere of radius d , since it contains the complement of every such clossphere. Hence $x(U)$ does not converge in F , so $I(U)$ is not in \mathfrak{W}' . Thus $\mathfrak{W}' \neq \mathfrak{W}$.

LEMMA 26. *A maximal ideal $I(U)$ belongs to \mathfrak{W}' if and only if U recognizes all clopen partitions of S of cardinality less than or equal to $D(F)$.*

Proof. We may assume that F is not locally compact, for otherwise the lemma is trivial. Suppose that U recognizes all clopen partitions of S of cardinality less than or equal to $D(F)$, and let $x \in X$ and $0 < d < \|x\|$. Let $\{T(b_i, d): i \in I\}$ be a partition of $T(0, \|x\|)$; then $\text{card}(I) = D(F)$. The nonempty members of the family $\{x^{-1}(T(b_i, d)): i \in I\}$ form a clopen partition of S of cardinality less than or equal to $D(F)$; hence U contains a set $x^{-1}(T(b_j, d))$. Thus, $T(b_j, d)$ is in $x(U)$; $x(U)$ contains arbitrarily small spheres; $x(U)$ converges; and $I(U)$ is in \mathfrak{W}' . The proof of the converse is similar to part (2) of the proof of Theorem 19.

Recalling Lemma 6, we have:

COROLLARY 27. *If $D(F)$ is an infinite, nonmeasurable cardinal, then $I(U)$ is in \mathfrak{W}' if and only if U is countably fixed.*

Using Theorems 13, 15, and 19, Lemmas 9 and 26, and Corollary 27, we have the following theorems:

THEOREM 20. *$\mathfrak{W}'' = \mathfrak{W}'$ if and only if every cofilter on S which recognizes all clopen partitions of S of cardinality less than or equal to $D(F)$ is fixed.*

THEOREM 21. (1) *For F locally compact: $\mathfrak{W}'' = \mathfrak{W}'$ if and only if S is mildly compact.*

(2) *For F not locally compact: If S is mildly Lindelof, then $\mathfrak{W}'' = \mathfrak{W}'$.*

(3) *For F not locally compact and $D(F)$ nonmeasurable: $\mathfrak{W}'' =$*

\mathfrak{M}' if and only if S is mildly Lindelof.

(4) For S mildly countably compact: $\mathfrak{M}'' = \mathfrak{M}'$ if and only if S is mildly Lindelof.

The question of whether $\mathfrak{M}'' = \mathfrak{M}'$ always implies that S is mildly Lindelof remains open.

COROLLARY 28. *Suppose F is not locally compact. Then:*

(1) *If S is T_1 , ultraregular, and mildly Lindelof, then $B \circ Q$ is a homeomorphism of S onto \mathfrak{M}' .*

(2) *If $D(F)$ is nonmeasurable, and $B \circ Q$ carries S homeomorphically onto \mathfrak{M}' , then S is T_1 , ultraregular, and mildly Lindelof.*

4. **Stone-Weierstrass properties.** We will say that X has the *strong Stone-Weierstrass property* if every closed subalgebra which separates quasicomponents of S is either X itself or a fixed maximal ideal; and that X has the *weak Stone-Weierstrass property* if the only closed subalgebra which separates quasicomponents and contains the constants is X itself. This section is devoted to proving the following two theorems:

THEOREM 22. *X has the strong Stone-Weierstrass property if and only if S is mildly compact.*

THEOREM 23. *If X has the weak Stone-Weierstrass property, then S is mildly countably compact.*

We begin with a lemma of Kaplansky [7].

LEMMA 29. *If D is a compact set in F , and $0 \neq a \in F$, then there is a polynomial $p(t)$ over F , without constant term, such that $p(a) = 1$ and $|p(b)| \leq 1 (b \in D)$.*

Proof. We may assume that $a \in D$, for otherwise we can replace D with $D \cup \{a\}$. Let $d = |a|^2/|D|$, where $|D| = \sup\{|b| : b \in D\}$. Then $d \leq |a|$.

The set $D - T(0, |a|)$ is closed in D , hence compact; hence it has a finite partition $\{T(b_i, d) \cap D : 1 \leq i \leq n\}$ into clospheres of radius d . We may assume that $|b_1| \leq |b_2| \leq \dots \leq |b_n|$. We set $k(i) = 2^{i-1} (1 \leq i \leq n)$ and

$$p(t) = 1 - (1 - t/a) \cdot \prod_{i=1}^n (1 - t/b_i)^{k(i)}.$$

By straightforward computation, the lemma follows.

Proof of Theorem 22. (1) If S is not mildly compact, then, by Theorem 15, X has a nonfixed maximal ideal $I(U)$. If Q and R are distinct quasicomponents of S , then some clopen set $A \subseteq S$ contains Q but not R ; either A or $S - A$ is in U ; and hence a characteristic function in $I(U)$ separates Q and R . Thus, $I(U)$ is a closed subalgebra of X which separates quasicomponents. Since $I(U)$ is neither X itself nor a fixed maximal ideal, X does not have the strong Stone-Weierstrass property.

(2) Suppose S is mildly compact, and Y is a closed subalgebra of X which separates quasicomponents. We will prove that if Y is contained in some fixed maximal ideal M_s , then $Y = M_s$; a similar proof shows that if Y is not contained in any fixed ideal, then $Y = X$.

We therefore assume that $Y \subseteq M_s$ for some $s \in S$. First, we contend that if $u, v \in S$ and $Q(s) \neq Q(u) \neq Q(v)$, then some $x_v \in Y$ satisfies: $x_v(u) = 1, x_v(v) = 0, \|x_v\| = 1$.

To prove this, we note that some $y_1 \in Y$ separates u and v ; and some $y_2 \in Y$ does not vanish at u . Let $y = y_1 y_2 - y_1(v) \cdot y_2$; then $y \in Y, y(u) \neq 0$, and $y(v) = 0$. Let $a = y(u)$ and $D = y(S)$; then D is mildly compact, hence compact. Let $p(t)$ be the resulting polynomial of Lemma 29, and let $x_v = p(y)$.

Second, we contend that if V is a clopen set in S containing s , u is a point of S outside V , and $d > 0$, then some $x \in Y$ satisfies: $x(u) = 1, |x(v)| \leq d (v \in V)$, and $\|x\| = 1$.

To prove this, we note that V is mildly compact. For each $v \in V$, some $x_v \in Y$ satisfies: $x_v(u) = 1, x_v(v) = 0$, and $\|x_v\| = 1$. For $v \in V$, let $W_v = \{w \in V: |x_v(w)| < d\}$; then the family $\{W_v: v \in V\}$ is a clopen cover of V and has a finite subcover $\{W_{v(1)}, \dots, W_{v(n)}\}$. Let $x = x_{v(1)} \cdot x_{v(2)} \cdot \dots \cdot x_{v(n)}$.

Third, we contend that if W is a clopen set in S not containing s , then the characteristic function $C(W)$ is in Y .

To prove this, we note that if $0 < d < 1$ and $u \in W$, then some $x_u \in Y$ satisfies: $x_u(u) = 1, |x_u(v)| < d (v \in S - W)$, and $\|x_u\| = 1$. Each set $W_u = \{w \in W: |x_u(w) - 1| < d\}$ is clopen in W ; hence the family $\{W_u: u \in W\}$ is a clopen cover of W ; and since W is mildly compact, a finite subfamily $\{W_{u(1)}, \dots, W_{u(n)}\}$ covers W . Let

$$x = e - (e - x_{u(1)}) \cdot \dots \cdot (e - x_{u(n)});$$

then $x \in Y$ and $\|x - C(W)\| \leq d$. Since Y is closed, $C(W) \in Y$.

Finally, we show that $Y = M_s$. Let $x \in M_s$; for any $d > 0$, the preimages, under x , of the closspheres of radius d in F form a clopen partition of S . Since S is mildly compact, this partition is finite: $S = \bigcup \{W_i: 1 \leq i \leq n\}$, where each $W_i = x^{-1}(T(a_i, d)) = \{u \in S: |x(u) - a_i| \leq d\}$ for some $a_i \in F$. We may assume that $a_1 = 0$; i.e., that $s \in W_1$. Let $y = a_1 C(W_1) \cdot \dots \cdot a_n C(W_n)$; then $y \in Y$ and $\|x - y\| \leq d$. Since Y

is closed, $x \in Y$.

Proof of Theorem 23. Suppose S is not mildly countably compact. Then S has an infinite clopen partition $\{T_i: i \in I\}$. Let Y be the closed ideal in X generated by the characteristic functions of these sets; and let Z be the subalgebra $Y + F \cdot e$ of X . Then Z is a closed subalgebra of X which separates quasicomponents and contains the constants. However, every member of Z must take values arbitrarily close to some constant on all but a finite number of the sets $\{T_i: i \in I\}$; so Z is not equal to X itself. Hence X does not have the weak Stone-Weierstrass property.

We note that the question of whether the weak Stone-Weierstrass property is equivalent to mild compactness, or to mild countable compactness, or to some intermediate property, remains unresolved.

REFERENCES

1. N. Bourbaki, *General Topology* (2 vol.), Addison-Wesley, Reading, Mass., 1966.
2. D. G. Cantor, *On the Stone-Weierstrass approximation theorem for valued fields*, Pacific J. Math., **21** (1967), 473-478.
3. P. R. Chernoff, R. A. Rasala, and W. C. Waterhouse, *The Stone-Weierstrass theorem for valuable fields*, Pacific J. Math., **27** (1968), 233-240.
4. R. L. Ellis, *A nonarchimedean analog of the Tietze-Urysohn extension theorem*, Indag. Math., **70** (1967), 332-333.
5. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, New York, 1960.
6. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961.
7. I. Kaplansky, *The Weierstrass theorem in fields with valuation*, Proc. Amer. Math. Soc., **1** (1950), 356-357.
8. J. L. Kelley, *General Topology*, Van Nostrand, New York, 1955.
9. K. Kuratowski, *Topology* (2 vol.), Academic Press, New York, 1966-1968.
10. A. F. Monna, *Analyse Non-archimédienne*, Springer-Verlag, Berlin, 1970.
11. L. Narici, *On nonarchimedean Banach algebras*, Arch. Math., **19** (1968), 428-435.
12. L. Narici, E. Beckenstein, and G. Bachman, *Functional Analysis and Valuation Theory*, Dekker, New York, 1971.
13. M. van der Put, *Algebres de fonctions continues p -adiques*, Indag. Math., **30** (1968), 401-420.
14. N. Shilkret, *Non-archimedean Banach algebras*, Doctoral dissertation, Polytechnic Institute of Brooklyn, 1968.
15. ———, *Nonarchimedean Gelfand theory*, Pacific J. Math., **32** (1970), 541-550.

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