# The algebraic and geometric classification of Zinbiel algebras 

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#### Abstract

This paper is devoted to the complete algebraic and geometric classification of complex 5-dimensional Zinbiel algebras. In particular, we proved that the variety of complex 5-dimensional Zinbiel algebras has dimension 24, it is defined by 16 irreducible components and it has 11 rigid algebras.


Keywords: nilpotent algebra, Zinbiel algebra, dual Leibniz algebra, algebraic classification, central extension, geometric classification, degeneration.

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## Introduction

The algebraic classification (up to isomorphism) of algebras of dimension $n$ from a certain variety defined by a certain family of polynomial identities is a classic problem in the theory of non-associative algebras. There are many results related to the algebraic classification of smalldimensional algebras in the varieties of Jordan, Lie, Leibniz, Zinbiel and many other algebras [16, 18, 36]. Geometric properties of a variety of algebras defined by a family of polynomial identities have been an object of study since 1970's (see, [4-7, 14, 15, 25, 26, 30, 31, 38, 46, 47, 49, 50]). Gabriel described the irreducible components of the variety of 4-dimensional unital associative algebras [26]. Cibils considered rigid associative algebras with 2 -step nilpotent radical [15]. Grunewald and O'Halloran computed the degenerations for the variety of 5 -dimensional nilpotent Lie algebras [30]. All irreducible components of 2-step nilpotent commutative and anticommutative algebras have been described in [33,47]. Chouhy proved that in the case of finite-dimensional associative algebras, the $N$-Koszul property is preserved under the degeneration relation [14]. Degenerations have also been used to study a level of complexity of an algebra [29, 49, 50]. The study of degenerations of algebras is very rich and closely related to deformation theory, in the sense of Gerstenhaber [28].

Loday introduced a class of symmetric operads generated by one bilinear operation subject to one relation making each left-normed product of three elements equal to a linear combination of rightnormed products: $\left(a_{1} a_{2}\right) a_{3}=\sum_{\sigma \in \mathbb{S}_{3}} x_{\sigma} a_{\sigma(1)}\left(a_{\sigma(2)} a_{\sigma(3)}\right)$; such an operad is called a parametrized onerelation operad. For a particular choice of parameters $\left\{x_{\sigma}\right\}$, this operad is said to be regular if each of its components is the regular representation of the symmetric group; equivalently, the corresponding free algebra on a vector space $V$ is, as a graded vector space, isomorphic to the tensor algebra of $V$. Bremner and Dotsenko classified, over an algebraically closed field of characteristic zero, all regular parametrized one-relation operads. In fact, they proved that each such operad is isomorphic to one of the following five operads: the left-nilpotent operad, the associative operad, the Leibniz operad, the Zinbiel operad, and the Poisson operad [10]. An algebra A is called a (left) Zinbiel algebra if it satisfies the identity $(x y) z=x(y z+z y)$. Zinbiel algebras were introduced by Loday in [41]. Under the Koszul duality, the operad of Zinbiel algebras is dual to the operad of Leibniz algebras. Zinbiel algebras are also known as pre-commutative algebras [40] and chronological algebras [39]. A Zinbiel algebra is equivalent to a commutative dendriform algebra [2]. It plays an important role in the definition of pre-Gerstenhaber algebras [3]. The variety of Zinbiel algebras is a proper subvariety in the variety of right commutative algebras. Each Zinbiel algebra with the commutator multiplication gives a Tortkara algebra [22], which has sprung up in unexpected areas of mathematics [19, 20]. Recently, the notion of matching Zinbiel algebras was introduced in [27]. Recently, Zinbiel algebras also appeared in a study of rack cohomology [17], number theory [13] and in a construction of a Cartesian differential category [34]. In recent years, there has been a strong interest in the study of Zinbiel algebras in the algebraic and the operad context [8, 11, 21,-24, 27, 35, 38, 42,-45].

Free Zinbiel algebras were shown to be precisely the shuffle product algebra [42]. Naurazbekova proved that, over a field of characteristic zero, free Zinbiel algebras are the free associativecommutative algebras (without unity) with respect to the symmetrization multiplication and their free generators are found; also she constructed examples of subalgebras of the two-generated free Zinbiel algebra that are free Zinbiel algebras of countable rank [44]. Nilpotent algebras play an important role in the class of Zinbiel algebras. So, Dzhumadildaev and Tulenbaev proved that each complex finite dimensional Zinbiel algebra is nilpotent [23]; Naurazbekova and Umirbaev proved that in characteristic zero any proper subvariety of the variety of Zinbiel algebras is nilpotent [45]. Finite-dimensional Zinbiel algebras with a "big" nilpotency index are classified in [1, 11]. Central extensions of three dimensional Zinbiel algebras were calculated in [35] and of filiform Zinbiel algebras in [12]. The full system of degenerations of complex four dimensional Zinbiel algebras is given in [38].

Our method for classifying nilpotent Zinbiel algebras is based on the calculation of central extensions of nilpotent algebras of smaller dimensions from the same variety. The algebraic study of central extensions of algebras has been an important topic for years [32, 37, 48]. First, Skjelbred and Sund used central extensions of Lie algebras to obtain a classification of nilpotent Lie algebras [48]. Note that the Skjelbred-Sund method of central extensions is an important tool in the classification of nilpotent algebras. Using the same method, small dimensional nilpotent (associative, terminal [36], Jordan, Lie [16, 18], anticommutative algebras, and some others) have been described. Our main results related to the algebraic classification of cited varieties are summarized below.

Theorem A. Up to isomorphism, there are infinitely many isomorphism classes of complex 5dimensional non-split non-2-step nilpotent Zinbiel algebras, described explicitly in section 1.3 in terms of 6 one-parameter families and 53 additional isomorphism classes.

The degenerations between the (finite-dimensional) algebras from a certain variety $\mathfrak{V}$ defined by a set of identities have been actively studied in the past decade. The description of all degenerations allows one to find the so-called rigid algebras and families of algebras, i.e. those whose orbit closures under the action of the general linear group form irreducible components of $\mathfrak{V}$ (with respect to the Zariski topology). We list here some works in which the rigid algebras of the varieties of all 4dimensional Leibniz algebras, all 4-dimensional nilpotent terminal algebras [36], all 4-dimensional nilpotent commutative algebras, all 6-dimensional nilpotent binary Lie algebras, all 6-dimensional nilpotent anticommutative algebras have been found. A full description of degenerations has been obtained for 2-dimensional algebras, for 4-dimensional Lie algebras, for 4-dimensional Zinbiel and 4-dimensional nilpotent Leibniz algebras in [38], for 6-dimensional nilpotent Lie algebras in [30, 46], for 8 -dimensional 2 -step nilpotent anticommutative algebras [4], and for $(n+1)$-dimensional $n$-Lie algebras. Our main results related to the geometric classification of cited varieties are summarized below.

Theorem B. The variety of complex 5-dimensional Zinbiel algebras has dimension 24 and it has 16 irreducible components (in particular, there are only 11 rigid algebras in this variety).
0.1. Symmetric Zinbiel algebras. Similar to the Leibniz case, obvious that there are left and right Zinbiel algebras. Hence, the notation of symmetric Zinbiel algebra should be introduced by the similar way with symmetric Leibniz algebras (about symmetric Leibniz algebras see [9] and references therein). An algebra $\mathbf{A}$ is called a symmetric Zinbiel algebra if it satisfies the identities

$$
(x y) z=x(y z+z y), x(y z)=(x y+y x) z .
$$

The operad of symmetric Zinbiel algebras is dual to the operad of symmetric Leibniz algebras. The variety of symmetric Zinbiel algebras is a proper subvariety in the variety of bicommutative algebras. The following Lemma can be obtained by some tedious calculations.

Lemma 1. Let $\mathcal{S}$ be a symmetric Zinbiel algebra. Then $\mathcal{S}$ is a 3-step nilpotent algebra and it satisfies the following two identities

$$
(x y) z=-y(z x) \text { and }(x y) z=-z(y x)
$$

Corollary 2. Each n-dimensional $(n<6)$ symmetric Zinbiel algebra is 2 -step nilpotent. There is a non-2-step nilpotent 6-dimensional symmetric Zinbiel algebra.

Proof. The first part of the present statement follows from Theorem A, because there are no symmetric algebras in the classification of Theorem A. The required 6-dimensional symmetric Zinbiel algebra is given below:

$$
\begin{array}{llll}
e_{1} e_{2}=e_{3} & e_{2} e_{1}=e_{4} & e_{2} e_{2}=e_{5} & e_{1} e_{5}=e_{6} \\
e_{2} e_{4}=-2 e_{6} & e_{4} e_{2}=-e_{6} & e_{2} e_{3}=e_{6} & e_{3} e_{2}=2 e_{6}
\end{array}
$$

## 1. THE ALGEBRAIC CLASSIFICATION OF ZINBIEL ALGEBRAS

1.1. Method of classification of nilpotent algebras. Throughout this paper, we use the notations and methods well written in [32, 37], which we have adapted for the Zinbiel case with some modifications. Further in this section we give some important definitions.

Let $(\mathbf{A}, \cdot)$ be a Zinbiel algebra over $\mathbb{C}$ and let $\mathbb{V}$ be a vector space over $\mathbb{C}$. Then the $\mathbb{C}$-linear space $Z^{2}(\mathbf{A}, \mathbb{V})$ is defined as the set of all bilinear maps $\theta: \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$, such that

$$
\theta(x y, z)=\theta(x, y z+z y) .
$$

These elements will be called cocycles. For a linear map $f$ from $\mathbf{A}$ to $\mathbb{V}$, if we define $\delta f: \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ by $\delta f(x, y)=f(x y)$, then $\delta f \in \mathrm{Z}^{2}(\mathbf{A}, \mathbb{V})$. We define $\mathrm{B}^{2}(\mathbf{A}, \mathbb{V})=$ $\{\theta=\delta f: f \in \operatorname{Hom}(\mathbf{A}, \mathbb{V})\}$. We define the second cohomology space $\mathrm{H}^{2}(\mathbf{A}, \mathbb{V})$ as the quotient space $\mathrm{Z}^{2}(\mathbf{A}, \mathbb{V}) / \mathrm{B}^{2}(\mathbf{A}, \mathbb{V})$.

Let $\operatorname{Aut}(\mathbf{A})$ be the automorphism group of $\mathbf{A}$ and let $\phi \in \operatorname{Aut}(\mathbf{A})$. For $\theta \in \mathrm{Z}^{2}(\mathbf{A}, \mathbb{V})$ define the action of the group $\operatorname{Aut}(\mathbf{A})$ on $\mathrm{Z}^{2}(\mathbf{A}, \mathbb{V})$ by $\phi \theta(x, y)=\theta(\phi(x), \phi(y))$. It is easy to verify that $B^{2}(\mathbf{A}, \mathbb{V})$ is invariant under the action of $\operatorname{Aut}(\mathbf{A})$. So, we have an induced action of $\operatorname{Aut}(\mathbf{A})$ on $H^{2}(\mathbf{A}, \mathbb{V})$.

Let $\mathbf{A}$ be a Zinbiel algebra of dimension $m$ over $\mathbb{C}$ and $\mathbb{V}$ be a $\mathbb{C}$-vector space of dimension $k$. For the bilinear map $\theta$, define on the linear space $\mathbf{A}_{\theta}=\mathbf{A} \oplus \mathbb{V}$ the bilinear product " $[-,-]_{\mathbf{A}_{\theta}}$ " by $\left[x+x^{\prime}, y+y^{\prime}\right]_{\mathbf{A}_{\theta}}=x y+\theta(x, y)$ for all $x, y \in \mathbf{A}, x^{\prime}, y^{\prime} \in \mathbb{V}$. The algebra $\mathbf{A}_{\theta}$ is called a $k$ dimensional central extension of $\mathbf{A}$ by $\mathbb{V}$. One can easily check that $\mathbf{A}_{\theta}$ is a Zinbiel algebra if and only if $\theta \in \mathrm{Z}^{2}(\mathbf{A}, \mathbb{V})$.

Call the set $\operatorname{Ann}(\theta)=\{x \in \mathbf{A}: \theta(x, \mathbf{A})+\theta(\mathbf{A}, x)=0\}$ the annihilator of $\theta$. We recall that the annihilator of an algebra $\mathbf{A}$ is defined as the ideal $\operatorname{Ann}(\mathbf{A})=\{x \in \mathbf{A}: x \mathbf{A}+\mathbf{A} x=0\}$. Observe that $\operatorname{Ann}\left(\mathbf{A}_{\theta}\right)=(\operatorname{Ann}(\theta) \cap \operatorname{Ann}(\mathbf{A})) \oplus \mathbb{V}$.

The following result shows that every algebra with a nonzero annihilator is a central extension of a smaller-dimensional algebra.

Lemma 3. Let $\mathbf{A}$ be an $n$-dimensional Zinbiel algebra such that $\operatorname{dim}(\operatorname{Ann}(\mathbf{A}))=m \neq 0$. Then there exists, up to isomorphism, an unique $(n-m)$-dimensional Zinbiel algebra $\mathbf{A}^{\prime}$ and a bilinear map $\theta \in Z^{2}\left(\mathbf{A}^{\prime}, \mathbb{V}\right)$ with $\operatorname{Ann}\left(\mathbf{A}^{\prime}\right) \cap \operatorname{Ann}(\theta)=0$, where $\mathbb{V}$ is a vector space of dimension $m$, such that $\mathbf{A} \cong \mathbf{A}_{\theta}^{\prime}$ and $\mathbf{A} / \operatorname{Ann}(\mathbf{A}) \cong \mathbf{A}^{\prime}$.

Proof. Let $\mathbf{A}^{\prime}$ be a linear complement of $\operatorname{Ann}(\mathbf{A})$ in $\mathbf{A}$. Define a linear map $P: \mathbf{A} \longrightarrow \mathbf{A}^{\prime}$ by $P(x+v)=x$ for $x \in \mathbf{A}^{\prime}$ and $v \in \operatorname{Ann}(\mathbf{A})$, and define a multiplication on $\mathbf{A}^{\prime}$ by $[x, y]_{\mathbf{A}^{\prime}}=P(x y)$ for $x, y \in \mathbf{A}^{\prime}$. For $x, y \in \mathbf{A}$, we have

$$
P(x y)=P((x-P(x)+P(x))(y-P(y)+P(y)))=P(P(x) P(y))=[P(x), P(y)]_{\mathbf{A}^{\prime}} .
$$

Since $P$ is a homomorphism, $P(\mathbf{A})=\mathbf{A}^{\prime}$ is a Zinbiel algebra and $\mathbf{A} / \operatorname{Ann}(\mathbf{A}) \cong \mathbf{A}^{\prime}$, which gives us the uniqueness. Now, define the map $\theta: \mathbf{A}^{\prime} \times \mathbf{A}^{\prime} \longrightarrow \operatorname{Ann}(\mathbf{A})$ by $\theta(x, y)=x y-[x, y]_{\mathbf{A}^{\prime}}$. Thus, $\mathbf{A}_{\theta}^{\prime}$ is $\mathbf{A}$ and therefore $\theta \in \mathrm{Z}^{2}\left(\mathbf{A}^{\prime}, \mathbb{V}\right)$ and $\operatorname{Ann}\left(\mathbf{A}^{\prime}\right) \cap \operatorname{Ann}(\theta)=0$.

Definition 4. Let $\mathbf{A}$ be an algebra and $I$ be a subspace of $\operatorname{Ann}(\mathbf{A})$. If $\mathbf{A}=\mathbf{A}_{0} \oplus I$ then $I$ is called an annihilator component of $\mathbf{A}$. A central extension of an algebra $\mathbf{A}$ without annihilator component is called a non-split central extension.

Our task is to find all central extensions of an algebra $\mathbf{A}$ by a space $\mathbb{V}$. In order to solve the isomorphism problem we need to study the action of $\operatorname{Aut}(\mathbf{A})$ on $\mathrm{H}^{2}(\mathbf{A}, \mathbb{V})$. To do that, let us fix a basis $\left\{e_{1}, \ldots, e_{s}\right\}$ of $\mathbb{V}$, and $\theta \in \mathbb{Z}^{2}(\mathbf{A}, \mathbb{V})$. Then $\theta$ can be uniquely written as $\theta(x, y)=\sum_{i=1}^{s} \theta_{i}(x, y) e_{i}$, where $\theta_{i} \in \mathrm{Z}^{2}(\mathbf{A}, \mathbb{C})$. Moreover, $\operatorname{Ann}(\theta)=\operatorname{Ann}\left(\theta_{1}\right) \cap \operatorname{Ann}\left(\theta_{2}\right) \cap \ldots \cap \operatorname{Ann}\left(\theta_{s}\right)$. Furthermore, $\theta \in \mathrm{B}^{2}(\mathbf{A}, \mathbb{V})$ if and only if all $\theta_{i} \in \mathrm{~B}^{2}(\mathbf{A}, \mathbb{C})$. It is not difficult to prove (see [32, Lemma 13]) that given a Zinbiel algebra $\mathbf{A}_{\theta}$, if we write as above $\theta(x, y)=\sum_{i=1}^{s} \theta_{i}(x, y) e_{i} \in \mathrm{Z}^{2}(\mathbf{A}, \mathbb{V})$ and $\operatorname{Ann}(\theta) \cap \operatorname{Ann}(\mathbf{A})=0$, then $\mathbf{A}_{\theta}$ has an annihilator component if and only if $\left[\theta_{1}\right],\left[\theta_{2}\right], \ldots,\left[\theta_{s}\right]$ are linearly dependent in $\mathrm{H}^{2}(\mathbf{A}, \mathbb{C})$.

Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{C}$. The Grassmannian $G_{k}(\mathbb{V})$ is the set of all $k$-dimensional linear subspaces of $\mathbb{V}$. Let $G_{s}\left(\mathrm{H}^{2}(\mathbf{A}, \mathbb{C})\right)$ be the Grassmannian of subspaces of dimension $s$ in $\mathrm{H}^{2}(\mathbf{A}, \mathbb{C})$. There is a natural action of $\operatorname{Aut}(\mathbf{A})$ on $G_{s}\left(\mathrm{H}^{2}(\mathbf{A}, \mathbb{C})\right)$. Let $\phi \in \operatorname{Aut}(\mathbf{A})$. For $W=\left\langle\left[\theta_{1}\right],\left[\theta_{2}\right], \ldots,\left[\theta_{s}\right]\right\rangle \in G_{s}\left(\mathrm{H}^{2}(\mathbf{A}, \mathbb{C})\right)$ define $\phi W=\left\langle\left[\phi \theta_{1}\right],\left[\phi \theta_{2}\right], \ldots,\left[\phi \theta_{s}\right]\right\rangle$. We denote the orbit of $W \in G_{s}\left(\mathrm{H}^{2}(\mathbf{A}, \mathbb{C})\right)$ under the action of $\operatorname{Aut}(\mathbf{A})$ by $\operatorname{Orb}(W)$. Given

$$
W_{1}=\left\langle\left[\theta_{1}\right],\left[\theta_{2}\right], \ldots,\left[\theta_{s}\right]\right\rangle, W_{2}=\left\langle\left[\vartheta_{1}\right],\left[\vartheta_{2}\right], \ldots,\left[\vartheta_{s}\right]\right\rangle \in G_{s}\left(\mathrm{H}^{2}(\mathbf{A}, \mathbb{C})\right)
$$

we easily have that if $W_{1}=W_{2}$, then $\bigcap_{i=1}^{s} \operatorname{Ann}\left(\theta_{i}\right) \cap \operatorname{Ann}(\mathbf{A})=\bigcap_{i=1}^{s} \operatorname{Ann}\left(\vartheta_{i}\right) \cap \operatorname{Ann}(\mathbf{A})$, and therefore we can introduce the set

$$
\mathbf{T}_{s}(\mathbf{A})=\left\{W=\left\langle\left[\theta_{1}\right],\left[\theta_{2}\right], \ldots,\left[\theta_{s}\right]\right\rangle \in G_{s}\left(\mathrm{H}^{2}(\mathbf{A}, \mathbb{C})\right): \bigcap_{i=1}^{s} \operatorname{Ann}\left(\theta_{i}\right) \cap \operatorname{Ann}(\mathbf{A})=0\right\}
$$

which is stable under the action of $\operatorname{Aut}(\mathbf{A})$.
Now, let $\mathbb{V}$ be an $s$-dimensional linear space and let us denote by $\mathbf{E}(\mathbf{A}, \mathbb{V})$ the set of all non-split s-dimensional central extensions of $\mathbf{A}$ by $\mathbb{V}$. By above, we can write

$$
\mathbf{E}(\mathbf{A}, \mathbb{V})=\left\{\mathbf{A}_{\theta}: \theta(x, y)=\sum_{i=1}^{s} \theta_{i}(x, y) e_{i} \text { and }\left\langle\left[\theta_{1}\right],\left[\theta_{2}\right], \ldots,\left[\theta_{s}\right]\right\rangle \in \mathbf{T}_{s}(\mathbf{A})\right\}
$$

We also have the following result, which can be proved as in [32, Lemma 17].

Lemma 5. Let $\mathbf{A}_{\theta}, \mathbf{A}_{\vartheta} \in \mathbf{E}(\mathbf{A}, \mathbb{V})$. Suppose that $\theta(x, y)=\sum_{i=1}^{s} \theta_{i}(x, y) e_{i}$ and $\vartheta(x, y)=$ $\sum_{i=1}^{s} \vartheta_{i}(x, y) e_{i}$. Then the Zinbiel algebras $\mathbf{A}_{\theta}$ and $\mathbf{A}_{\vartheta}$ are isomorphic if and only if

$$
\operatorname{Orb}\left\langle\left[\theta_{1}\right],\left[\theta_{2}\right], \ldots,\left[\theta_{s}\right]\right\rangle=\operatorname{Orb}\left\langle\left[\vartheta_{1}\right],\left[\vartheta_{2}\right], \ldots,\left[\vartheta_{s}\right]\right\rangle .
$$

This shows that there exists a one-to-one correspondence between the set of $\operatorname{Aut}(\mathbf{A})$-orbits on $\mathbf{T}_{s}(\mathbf{A})$ and the set of isomorphism classes of $\mathbf{E}(\mathbf{A}, \mathbb{V})$. Consequently we have a procedure that allows us, given a Zinbiel algebra $\mathbf{A}^{\prime}$ of dimension $n-s$, to construct all non-split central extensions of $\mathbf{A}^{\prime}$. This procedure is:
(1) For a given Zinbiel algebra $\mathbf{A}^{\prime}$ of dimension $n-s$, determine $H^{2}\left(\mathbf{A}^{\prime}, \mathbb{C}\right), \operatorname{Ann}\left(\mathbf{A}^{\prime}\right)$ and $\operatorname{Aut}\left(\mathbf{A}^{\prime}\right)$.
(2) Determine the set of $\operatorname{Aut}\left(\mathbf{A}^{\prime}\right)$-orbits on $\mathbf{T}_{s}\left(\mathbf{A}^{\prime}\right)$.
(3) For each orbit, construct the Zinbiel algebra associated with a representative of it.
1.1.1. Notations. Let us introduce the following notations. Let $\mathbf{A}$ be a nilpotent algebra with a basis $\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$. Then by $\Delta_{i j}$ we will denote the bilinear form $\Delta_{i j}: \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{C}$ with $\Delta_{i j}\left(e_{l}, e_{m}\right)=$ $\delta_{i l} \delta_{j m}$. The set $\left\{\Delta_{i j}: 1 \leq i, j \leq n\right\}$ is a basis for the linear space of bilinear forms on $\mathbf{A}$, so every $\theta \in \mathrm{Z}^{2}(\mathbf{A}, \mathbb{V})$ can be uniquely written as $\theta=\sum_{1 \leq i, j \leq n} c_{i j} \Delta_{i j}$, where $c_{i j} \in \mathbb{C}$. Let us fix the following notations for our nilpotent algebras:
$\mathfrak{N}_{j} \quad$ - $\quad j$ th 4-dimensional 2-step nilpotent algebra.
$[\mathfrak{A}]_{j}^{i}-j$ th $i$-dimensional central extension of 3-dimensional Zinbiel algebra $\mathfrak{A}$ (see, [35]).
1.2. The algebraic classification of complex 4-dimensional Zinbiel algebras and their cohomology spaces. The present table collects all information about multiplication tables and cohomology spaces of 4 -dimensional Zinbiel algebras (for the classification of 4-dimensional Zinbiel algebras see, [38] and the Corrigendum of [38]), which will be used in our main classification.

| The list of 2-step nilpotent 4-dimensional Zinbiel algebras |
| :--- |
| $\mathfrak{N}_{01}: e_{1} e_{1}=e_{2}$ |
| $\mathrm{H}^{2}\left(\mathfrak{N}_{01}\right)=\left\langle\left[\Delta_{13}\right],\left[\Delta_{14}\right],\left[\Delta_{31}\right],\left[\Delta_{33}\right],\left[\Delta_{34}\right],\left[\Delta_{41}\right],\left[\Delta_{43}\right],\left[\Delta_{44}\right],\left[\Delta_{12}+2 \Delta_{21}\right]\right\rangle$ |
| $\mathfrak{N}_{02} \quad: e_{1} e_{1}=e_{3} \quad e_{2} e_{2}=e_{4}$ |
| $\mathrm{H}^{2}\left(\mathfrak{N}_{02}\right)=\left\langle\left[\Delta_{12}\right],\left[\Delta_{21}\right],\left[\Delta_{13}+2 \Delta_{31}\right],\left[\Delta_{24}+2 \Delta_{42}\right]\right\rangle$ |
| $\mathfrak{N}_{03} \quad: e_{1} e_{2}=e_{3} \quad e_{2} e_{1}=-e_{3}$ |
| $\mathrm{H}^{2}\left(\mathfrak{N}_{03}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{12}\right],\left[\Delta_{14}\right],\left[\Delta_{22}\right],\left[\Delta_{24}\right],\left[\Delta_{41}\right],\left[\Delta_{42}\right],\left[\Delta_{44}\right],\left[\Delta_{13}\right],\left[\Delta_{23}\right],\left[\Delta_{43}\right]\right\rangle$ |


| $\begin{aligned} & \hline \mathfrak{N}_{04}^{\alpha} \quad: e_{1} e_{1}=e_{3} \quad e_{1} e_{2}=e_{3} \quad e_{2} e_{2}=\alpha e_{3} \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{04}^{\alpha}\right)=\left\langle\left[\Delta_{12}\right],\left[\Delta_{14}\right],\left[\Delta_{21}\right],\left[\Delta_{22}\right],\left[\Delta_{24}\right],\left[\Delta_{41}\right],\left[\Delta_{42}\right],\left[\Delta_{44}\right]\right\rangle \end{aligned}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |
| $\begin{array}{lll} \hline \mathfrak{N}_{05} \quad: \quad e_{1} e_{1}=e_{3} & e_{1} e_{2}=e_{3} & e_{2} e_{1}=e_{3} \\ \mathrm{H}^{2}\left(\mathfrak{N}_{05}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{12}\right],\left[\Delta_{14}\right],\left[\Delta_{22}\right],\left[\Delta_{24}\right],\left[\Delta_{41}\right],\left[\Delta_{42}\right],\left[\Delta_{44}\right]\right\rangle \\ \hline \end{array}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{aligned} & \hline \mathfrak{N}_{06} \quad: \quad e_{1} e_{2}=e_{4} \quad e_{3} e_{1}=e_{4} \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{06}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{12}\right],\left[\Delta_{13}\right],\left[\Delta_{21}\right],\left[\Delta_{22}\right],\left[\Delta_{23}\right],\left[\Delta_{32}\right],\left[\Delta_{33}\right]\right\rangle \\ & \hline \end{aligned}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{aligned} & \hline \mathfrak{N}_{07} \quad: e_{1} e_{2}=e_{3} \quad e_{2} e_{1}=e_{4} \quad e_{2} e_{2}=-e_{3} \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{07}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{22}\right],\left[\Delta_{13}-\Delta_{14}-\Delta_{23}+\Delta_{24}-2 \Delta_{32}\right]\right\rangle \end{aligned}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{aligned} & \hline \mathfrak{N}_{08}^{\alpha} \quad: \quad e_{1} e_{1}=e_{3} \quad e_{1} e_{2}=e_{4} \quad e_{2} e_{1}=-\alpha e_{3} \quad e_{2} e_{2}=-e_{4} \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{08}^{\alpha \neq 0,1}\right)=\left\langle\left[\Delta_{12}\right],\left[\Delta_{21}\right]\right\rangle \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{08}^{0}\right)=\left\langle\left[\Delta_{12}\right],\left[\Delta_{21}\right],\left[\Delta_{13}+2 \Delta_{31}\right]\right\rangle \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{08}^{1}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{12}\right],\left[\Delta_{23}-\Delta_{13}-2 \Delta_{31}+\Delta_{32}+\Delta_{41}\right],\left[\Delta_{24}-\Delta_{14}-\Delta_{32}-\Delta_{41}+2 \Delta_{42}\right]\right\rangle \end{aligned}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{array}{lllll} \hline \mathfrak{N}_{09}^{\alpha} & : & e_{1} e_{1}=e_{4} & e_{1} e_{2}=\alpha e_{4} & e_{2} e_{1}=-\alpha e_{4} \end{array} e_{2} e_{2}=e_{4} \quad e_{3} e_{3}=e_{4},\left[\begin{array}{lll}  \\ \mathrm{H}^{2}\left(\mathfrak{N}_{09}^{\alpha}\right)=\left\langle\left[\Delta_{12}\right],\left[\Delta_{13}\right],\left[\Delta_{21}\right],\left[\Delta_{22}\right],\left[\Delta_{23}\right],\left[\Delta_{31}\right],\left[\Delta_{32}\right],\left[\Delta_{33}\right]\right\rangle \end{array}\right.$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{array}{llll} \hline \mathfrak{N}_{10} \quad: \quad e_{1} e_{2}=e_{4} & e_{1} e_{3}=e_{4} & e_{2} e_{1}=-e_{4} & e_{2} e_{2}=e_{4} \\ e_{3} e_{1}=e_{4} \\ \mathrm{H}^{2}\left(\mathfrak{N}_{10}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{12}\right],\left[\Delta_{13}\right],\left[\Delta_{21}\right],\left[\Delta_{23}\right],\left[\Delta_{31}\right],\left[\Delta_{32}\right],\left[\Delta_{33}\right]\right\rangle \end{array}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{aligned} & \hline \mathfrak{N}_{11} \quad: \quad e_{1} e_{1}=e_{4} \quad e_{1} e_{2}=e_{4} \quad e_{2} e_{1}=-e_{4} \quad e_{3} e_{3}=e_{4} \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{11}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{12}\right],\left[\Delta_{13}\right],\left[\Delta_{21}\right],\left[\Delta_{22}\right],\left[\Delta_{23}\right],\left[\Delta_{31}\right],\left[\Delta_{32}\right]\right\rangle \end{aligned}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{aligned} & \hline \mathfrak{N}_{12} \quad: \quad e_{1} e_{2}=e_{3} \quad e_{2} e_{1}=e_{4} \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{12}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{22}\right],\left[\Delta_{14}-\Delta_{13}\right],\left[\Delta_{24}-\Delta_{23}\right]\right\rangle \end{aligned}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{array}{llll} \hline \mathfrak{N}_{13} \quad: e_{1} e_{1}=e_{4} & e_{1} e_{2}=e_{3} & e_{2} e_{1}=-e_{3} & e_{2} e_{2}=2 e_{3}+e_{4} \\ \mathrm{H}^{2}\left(\mathfrak{N}_{13}\right)=\left\langle\left[\Delta_{21}\right],\left[\Delta_{22}\right]\right\rangle & & \\ \hline \end{array}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{aligned} & \hline \mathfrak{N}_{14}^{\alpha}: e_{1} e_{2}=e_{4} \quad e_{2} e_{1}=\alpha e_{4} \quad e_{2} e_{2}=e_{3} \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{14}^{\alpha \neq-1}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{21}\right],\left[\Delta_{23}+2 \Delta_{32}\right],\left[2 \alpha \Delta_{24}+(\alpha+1)\left(\Delta_{13}+2 \alpha \Delta_{31}+2 \Delta_{42}\right)\right]\right\rangle \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{14}^{-1}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{21}\right],\left[\Delta_{14}\right],\left[\Delta_{24}\right],\left[\Delta_{23}+2 \Delta_{32}\right]\right\rangle \\ & \hline \end{aligned}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\begin{aligned} & \hline \mathfrak{N}_{15} \quad: e_{1} e_{2}=e_{4} \quad e_{2} e_{1}=-e_{4} \quad e_{3} e_{3}=e_{4} \\ & \mathrm{H}^{2}\left(\mathfrak{N}_{15}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{13}\right],\left[\Delta_{21}\right],\left[\Delta_{22}\right],\left[\Delta_{23}\right],\left[\Delta_{31}\right],\left[\Delta_{32}\right],\left[\Delta_{33}\right]\right\rangle \\ & \hline \end{aligned}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| The list of 3-step nilpotent 4-dimensional Zinbiel algebras |  |  |  |  |  |  |
| $\begin{aligned} & \hline \mathcal{Z}_{1} \quad: e_{1} e_{1}=e_{2} \quad e_{1} e_{2}=e_{3} \quad e_{2} e_{1}=2 e_{3} \\ & \mathrm{H}^{2}\left(\mathfrak{Z}_{1}\right)=\left\langle\left[\Delta_{13}+3 \Delta_{22}+3 \Delta_{31}\right],\left[\Delta_{14}\right],\left[\Delta_{41}\right],\left[\Delta_{44}\right],\right\rangle \end{aligned}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $\left[\mathfrak{N}_{1}^{\mathrm{C}}\right]_{01}^{1}: e_{1} e_{1}=e_{2} \quad e_{1} e_{2}=e_{4} \quad e_{2} e_{1}=2 e_{4} \quad e_{3} e_{3}=e_{4}$ |  |  |  |  |  |  |



Remark 6. From the previous list, we will only consider $\mathfrak{N}_{01}, \mathfrak{N}_{02}, \mathfrak{N}_{03}, \mathfrak{N}_{07}, \mathfrak{N}_{08}^{1}, \mathfrak{N}_{12}, \mathfrak{N}_{14}^{\alpha}, \mathfrak{Z}_{1},\left[\mathfrak{Z}_{1}\right]_{1}^{1}$. All $n$-dimensional central extensions of the remaining algebras are split or have $(n+1)$-dimensional annihilator.

Remark 7. In what follows, when we consider an automorphism, the entries of the matrix that are choosen to be zero will be omitted in the description.
1.2.1. Central extensions of $\mathfrak{N}_{01}$. Let us use the following notations:

$$
\begin{array}{lll}
\nabla_{1}=\left[\Delta_{12}+2 \Delta_{21}\right], & \nabla_{2}=\left[\Delta_{13}\right], & \nabla_{3}=\left[\Delta_{14}\right], \\
\nabla_{6}=\left[\nabla_{34}=\left[\Delta_{31}\right],\right. & \nabla_{7}=\left[\Delta_{41}\right], & \nabla_{8}=\left[\Delta_{43}\right],
\end{array} \nabla_{9}=\left[\Delta_{33}\right], ~\left[\Delta_{44}\right] .
$$

Take $\theta=\sum_{i=1}^{9} \alpha_{i} \nabla_{i} \in \mathrm{H}^{2}\left(\mathfrak{N}_{01}\right)$. The automorphism group of $\mathfrak{N}_{01}$ is generated by invertible matrices of the form

$$
\phi=\left(\begin{array}{cccc}
x & 0 & 0 & 0 \\
y & x^{2} & z & w \\
q & 0 & s & t \\
r & 0 & u & v
\end{array}\right)
$$

Since

$$
\phi^{T}\left(\begin{array}{cccc}
0 & \alpha_{1} & \alpha_{2} & \alpha_{3} \\
2 \alpha_{1} & 0 & 0 & 0 \\
\alpha_{4} & 0 & \alpha_{5} & \alpha_{6} \\
\alpha_{7} & 0 & \alpha_{8} & \alpha_{9}
\end{array}\right) \phi=\left(\begin{array}{cccc}
\alpha^{*} & \alpha_{1}^{*} & \alpha_{2}^{*} & \alpha_{3}^{*} \\
2 \alpha_{1}^{*} & 0 & 0 & 0 \\
\alpha_{4}^{*} & 0 & \alpha_{5}^{*} & \alpha_{6}^{*} \\
\alpha_{7}^{*} & 0 & \alpha_{8}^{*} & \alpha_{9}^{*}
\end{array}\right),
$$

we have that the action of $\operatorname{Aut}\left(\mathfrak{N}_{01}\right)$ on the subspace $\left\langle\sum_{i=1}^{9} \alpha_{i} \nabla_{i}\right\rangle$ is given by $\left\langle\sum_{i=1}^{9} \alpha_{i}^{*} \nabla_{i}\right\rangle$, where

$$
\alpha_{1}^{*}=\alpha_{1} x^{3}
$$

$$
\begin{aligned}
\alpha_{2}^{*} & =\alpha_{1} x z+\alpha_{2} s x+\alpha_{3} u x+\alpha_{5} q s+\alpha_{6} q u+\alpha_{8} r s+\alpha_{9} r u \\
\alpha_{3}^{*} & =\alpha_{1} w x+\alpha_{2} t x+\alpha_{3} v x+\alpha_{5} q t+\alpha_{6} q v+\alpha_{8} r t+\alpha_{9} r v \\
\alpha_{4}^{*} & =2 \alpha_{1} x z+\alpha_{4} s x+\alpha_{5} q s+\alpha_{6} r s+\alpha_{7} u x+\alpha_{8} q u+\alpha_{9} r u \\
\alpha_{5}^{*} & =\alpha_{5} s^{2}+\left(\alpha_{6} s+\alpha_{8} s+\alpha_{9} u\right) u \\
\alpha_{6}^{*} & =\alpha_{5} s t+\alpha_{6} s v+\alpha_{8} t u+\alpha_{9} u v \\
\alpha_{7}^{*} & =2 \alpha_{1} w x+\alpha_{4} t x+\alpha_{5} q t+\alpha_{6} r t+\alpha_{7} v x+\alpha_{8} q v+\alpha_{9} r v \\
\alpha_{8}^{*} & =\alpha_{5} s t+\alpha_{6} t u+\alpha_{8} s v+\alpha_{9} u v \\
\alpha_{9}^{*} & =\alpha_{5} t^{2}+\left(\alpha_{6} t+\alpha_{8} t+\alpha_{9} v\right) v
\end{aligned}
$$

First note that we can not have $\alpha_{1}=0, \alpha_{2}=\alpha_{4}=\alpha_{5}=\alpha_{6}=\alpha_{8}=0$ or $\alpha_{3}=\alpha_{6}=\alpha_{7}=\alpha_{8}=$ $\alpha_{9}=0$. Besides, considering $\phi$ given by $x=s=v=1, z=-\frac{\alpha_{4}}{2 \alpha_{1}}$ and $w=-\frac{\alpha_{7}}{2 \alpha_{1}}$, we can suppose $\alpha_{4}=\alpha_{7}=0$. On the other hand, if we choose $\phi$ given by $x=u=t=1$ and $z=w$ we have the following change of position: $\alpha_{2} \leftrightarrow \alpha_{3}, \alpha_{5} \leftrightarrow \alpha_{9}$ and $\alpha_{6} \leftrightarrow \alpha_{8}$. Finally, we note that if $\alpha_{5} \neq 0$ then we can choose $\alpha_{8}=0$ (just take $\phi$ given by $x=s=v=1$ and $t=-\frac{\alpha_{8}}{\alpha_{5}}$ ) and applying the automorphism given by $x=s=v=1, z=\frac{\alpha_{6} \alpha_{3}}{\alpha_{9}}, w=\alpha_{3}$ and $r=-\frac{2 \alpha_{3}}{\alpha_{9}}$, we can suppose $\alpha_{3}=0$. In what follow, we will consider $\alpha_{1}=1$ and $\alpha_{4}=\alpha_{7}=0$. Then:
(1) Suppose $\alpha_{5}=\alpha_{9}=0$ and $\alpha_{6}=-\alpha_{8}$.
(a) If $\alpha_{6}=0$, note that $\alpha_{3} \neq 0$. Therefore, by choosing $x=s=1, u=-\frac{\alpha_{2}}{\alpha_{3}}$ and $v=\frac{1}{\alpha_{3}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{3}\right\rangle$, that we do not consider since the associated algebra would have 2 -dimensional annihilator.
(b) If $\alpha_{6} \neq 0$, by choosing $x=v=-u=1, z=\frac{\alpha_{3}}{3}, w=-\frac{\alpha_{2}+\alpha_{3} \alpha_{6}}{3 \alpha_{6}}, q=-\frac{2 \alpha_{3}}{3 \alpha_{6}}, t=\frac{1}{\alpha_{6}}$ and $r=\frac{2 \alpha_{2}}{3 \alpha_{6}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{6}-\nabla_{8}\right\rangle$.
(2) Suppose $\alpha_{5}=\alpha_{9}=0$ and $\alpha_{6} \neq-\alpha_{8}$.
(a) If $\alpha_{6}=2 \alpha_{8}$, we have $\alpha_{8} \neq 0$. Then, there are two cases to consider:
(i) If $\alpha_{2}=0$, by choosing $x=t=1, u=\frac{1}{\alpha_{8}}, z=\frac{\alpha_{3}}{3 \alpha_{8}}$ and $q=-\frac{2 \alpha_{3}}{3 \alpha_{8}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{6}+2 \nabla_{8}\right\rangle$.
(ii) If $\alpha_{2} \neq 0$, by choosing $x=1, t=\frac{1}{\alpha_{2}}, u=\frac{\alpha_{2}}{\alpha_{8}}, z=\frac{\alpha_{2} \alpha_{3}}{3 \alpha_{8}}$ and $r=-\frac{2 \alpha_{3}}{3 \alpha_{8}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{3}+\nabla_{6}+2 \nabla_{8}\right\rangle$.
(b) Suppose $\alpha_{6} \neq 2 \alpha_{8}$.
(i) Suppose $\alpha_{8}=2 \alpha_{6}$ and, thus, $\alpha_{6} \neq 0$.
(A) If $\alpha_{3}=0$, by choosing $x=v=1, s=\frac{1}{\alpha_{6}}, z=\frac{\alpha_{2}}{3 \alpha_{6}}$ and $r=-\frac{2 \alpha_{2}}{3 \alpha_{6}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{6}+2 \nabla_{8}\right\rangle$.
(B) If $\alpha_{3} \neq 0$, by choosing $x=1, s=\frac{\alpha_{3}}{\alpha_{6}}, v=\frac{1}{\alpha_{3}}, z=\frac{\alpha_{2} \alpha_{3}}{3 \alpha_{6}}$ and $r=-\frac{2 \alpha_{2}}{3 \alpha_{6}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{3}+\nabla_{6}+2 \nabla_{8}\right\rangle$.
(ii) If $\alpha_{8} \neq 2 \alpha_{6}$, since we can change position between $\alpha_{6}$ and $\alpha_{8}$, we can suppose $\alpha_{8} \neq 0$. Then, by choosing $x=u=1, t=\frac{1}{\alpha_{8}}, z=\frac{\alpha_{3} \alpha_{8}}{2 \alpha_{6}-\alpha_{8}} w=\frac{\alpha_{2} \alpha_{6}}{\left(2 \alpha_{8}-\alpha_{6}\right) \alpha_{8}}, q=$ $\frac{2 \alpha_{3}}{\alpha_{8}-2 \alpha_{6}}$ and $r=\frac{2 \alpha_{2}}{\alpha_{6}-2 \alpha_{8}}$, we obtain the family of representatives $\left\langle\nabla_{1}+\nabla_{6}+\alpha \nabla_{8}\right\rangle$, with $\alpha \neq-1, \frac{1}{2}, 2$. Note that for $\alpha \neq 0,\left\langle\nabla_{1}+\nabla_{6}+\alpha \nabla_{8}\right\rangle$ and $\left\langle\nabla_{1}+\nabla_{6}+\frac{1}{\alpha} \nabla_{8}\right\rangle$ are in the same orbit.
(3) Suppose $\alpha_{5}=0$ and $\alpha_{9} \neq 0$.
(a) Suppose $\alpha_{8}=-\alpha_{6}$.
(i) If $\alpha_{6} \neq 0$, by choosing $x=\sqrt[3]{\alpha_{6}^{2} \alpha_{9}}, z=-\frac{3 \alpha_{3} \alpha_{6}+4 \alpha_{2} \alpha_{9}}{9}, w=-\frac{\alpha_{2} \alpha_{9}}{3}$, $q=-\frac{2 \sqrt[3]{\alpha_{9}}\left(3 \alpha_{3} \alpha_{6}+\alpha_{2} \alpha_{9}\right)}{9 \sqrt[3]{\alpha_{6}^{4}}}, t=\alpha_{9}, r=\frac{2 \alpha_{2} \sqrt[3]{\alpha_{9}}}{3 \sqrt[3]{\alpha_{6}}}$ and $u=-\alpha_{6}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{5}+\nabla_{6}-\nabla_{8}\right\rangle$.
(ii) If $\alpha_{6}=0$, note that $\alpha_{2} \neq 0$. By choosing $x=1, s=\frac{1}{\alpha_{2}}, t=-\frac{\alpha_{3}}{\alpha_{2} \sqrt{\alpha_{9}}}$ and $v=\frac{1}{\sqrt{\alpha_{9}}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{2}+\nabla_{9}\right\rangle$.
(b) If $\alpha_{8} \neq-\alpha_{6}$, by choosing $x=s=t=1$ and $v=-\frac{\alpha_{6}+\alpha_{8}}{\alpha_{9}}$, we return to the case $\alpha_{5}=\alpha_{9}=0$.
(4) Suppose $\alpha_{5} \alpha_{9} \neq 0$. By choosing $x=t=u=1$ and $s=-\frac{\alpha_{6}+\alpha_{8}+\sqrt{\left(\alpha_{6}+\alpha_{8}\right)^{2}-4 \alpha_{5} \alpha_{9}}}{2 \alpha_{5}}$, we return to the case $\alpha_{5}=0$ and $\alpha_{9} \neq 0$
Summarizing, we have the following distinct orbits

$$
\left\langle\nabla_{1}+\nabla_{2}+\nabla_{9}\right\rangle,\left\langle\nabla_{1}+\nabla_{6}+\alpha \nabla_{8}\right\rangle^{O(\alpha)=O\left(\frac{1}{\alpha}\right)},\left\langle\nabla_{1}+\nabla_{5}+\nabla_{6}-\nabla_{8}\right\rangle,\left\langle\nabla_{1}+\nabla_{3}+\nabla_{6}+2 \nabla_{8}\right\rangle,
$$

which give the following new algebras:

$$
\begin{array}{llllllll}
\mathcal{Z}_{01} & : & e_{1} e_{1}=e_{2} & e_{1} e_{2}=e_{5} & e_{1} e_{3}=e_{5} & e_{2} e_{1}=2 e_{5} & e_{4} e_{4}=e_{5} & \\
\hline \mathcal{Z}_{02}^{\alpha} & : & e_{1} e_{1}=e_{2} & e_{1} e_{2}=e_{5} & e_{2} e_{1}=2 e_{5} & e_{3} e_{4}=e_{5} & e_{4} e_{3}=\alpha e_{5} & \\
\hline \mathcal{Z}_{03} & : & e_{1} e_{1}=e_{2} & e_{1} e_{2}=e_{5} & e_{2} e_{1}=2 e_{5} & e_{3} e_{3}=e_{5} & e_{3} e_{4}=e_{5} & e_{4} e_{3}=-e_{5} \\
\hline \mathcal{Z}_{04} & : & e_{1} e_{1}=e_{2} & e_{1} e_{2}=e_{5} & e_{1} e_{4}=e_{5} & e_{2} e_{1}=2 e_{5} & e_{3} e_{4}=e_{5} & e_{4} e_{3}=2 e_{5}
\end{array}
$$

1.2.2. Central extensions of $\mathfrak{N}_{02}$. Let us use the following notations:

$$
\nabla_{1}=\left[\Delta_{12}\right], \quad \nabla_{2}=\left[\Delta_{21}\right], \quad \nabla_{3}=\left[\Delta_{13}+2 \Delta_{31}\right], \quad \nabla_{4}=\left[\Delta_{24}+2 \Delta_{42}\right]
$$

Take $\theta=\sum_{i=1}^{4} \alpha_{i} \nabla_{i} \in \mathrm{H}^{2}\left(\mathfrak{N}_{02}\right)$. The automorphism group of $\mathfrak{N}_{02}$ is generated by invertible matrices of the form

$$
\phi_{1}=\left(\begin{array}{cccc}
x & 0 & 0 & 0 \\
0 & y & 0 & 0 \\
z & t & x^{2} & 0 \\
u & v & 0 & y^{2}
\end{array}\right) \text { and } \phi_{2}=\left(\begin{array}{cccc}
0 & x & 0 & 0 \\
y & 0 & 0 & 0 \\
z & t & 0 & x^{2} \\
u & v & y^{2} & 0
\end{array}\right)
$$

Since

$$
\phi_{1}^{T}\left(\begin{array}{cccc}
0 & \alpha_{1} & \alpha_{3} & 0 \\
\alpha_{2} & 0 & 0 & \alpha_{4} \\
2 \alpha_{3} & 0 & 0 & 0 \\
0 & 2 \alpha_{4} & 0 & 0
\end{array}\right) \phi_{1}=\left(\begin{array}{cccc}
\alpha^{*} & \alpha_{1}^{*} & \alpha_{3}^{*} & 0 \\
\alpha_{2}^{*} & \alpha^{* *} & 0 & \alpha_{4}^{*} \\
2 \alpha_{3}^{*} & 0 & 0 & 0 \\
0 & 2 \alpha_{4}^{*} & 0 & 0
\end{array}\right),
$$

we have that the action of $\operatorname{Aut}\left(\mathfrak{N}_{02}\right)$ on the subspace $\left\langle\sum_{i=1}^{4} \alpha_{i} \nabla_{i}\right\rangle$ is given by $\left\langle\sum_{i=1}^{4} \alpha_{i}^{*} \nabla_{i}\right\rangle$, where

$$
\alpha_{1}^{*}=\alpha_{1} x y+\alpha_{3} t x+2 \alpha_{4} u y, \quad \alpha_{2}^{*}=\alpha_{2} x y+2 \alpha_{3} t x+\alpha_{4} u y, \quad \alpha_{3}^{*}=\alpha_{3} x^{3}, \quad \alpha_{4}^{*}=\alpha_{4} y^{3}
$$

Since $\operatorname{Ann}\left(\mathfrak{N}_{02}\right)=\left\langle e_{3}, e_{4}\right\rangle$, we can not have $\alpha_{3} \alpha_{4}=0$. Thus, we can suppose $\alpha_{4}=1$. Then, by choosing $x=\frac{1}{\sqrt[3]{\alpha_{3}}}, y=1, t=\frac{\alpha_{1}-2 \alpha_{2}}{3 \alpha_{3}}$ and $u=\frac{\alpha_{2}-2 \alpha_{1}}{3 \sqrt[3]{\alpha_{3}}}$, we obtain the representative

$$
\left\langle\nabla_{3}+\nabla_{4}\right\rangle,
$$

which gives the following new algebra:

$$
z_{05}: e_{1} e_{1}=e_{3} \quad e_{1} e_{3}=e_{5} \quad e_{2} e_{2}=e_{4} \quad e_{2} e_{4}=e_{5} \quad e_{3} e_{1}=2 e_{5} \quad e_{4} e_{2}=2 e_{5}
$$

1.2.3. Central extensions of $\mathfrak{N}_{03}$. Let us use the following notations:

$$
\begin{array}{lll}
\nabla_{1}=\left[\Delta_{11}\right], & \nabla_{2}=\left[\Delta_{12}\right], & \nabla_{3}=\left[\Delta_{13}\right], \\
\nabla_{6}=\left[\nabla_{23}\right], & \nabla_{7}=\left[\Delta_{24}\right], & \Delta_{8}=\left[\Delta_{41}\right],
\end{array}, \quad \nabla_{5}=\left[\Delta_{22}\right],\left[\Delta_{42}\right], \quad \nabla_{10}=\left[\Delta_{43}\right], \quad \nabla_{11}=\left[\Delta_{44}\right] .
$$

Take $\theta=\sum_{i=1}^{11} \alpha_{i} \nabla_{i} \in \mathrm{H}^{2}\left(\mathfrak{N}_{03}\right)$. The automorphism group of $\mathfrak{N}_{03}$ is generated by invertible matrices of the form

$$
\phi=\left(\begin{array}{cccc}
x & w & 0 & 0 \\
z & y & 0 & 0 \\
q & r & x y-w z & s \\
t & u & 0 & v
\end{array}\right)
$$

Since

$$
\phi^{T}\left(\begin{array}{cccc}
\alpha_{1} & \alpha_{2} & \alpha_{3} & \alpha_{4} \\
0 & \alpha_{5} & \alpha_{6} & \alpha_{7} \\
0 & 0 & 0 & 0 \\
\alpha_{8} & \alpha_{9} & \alpha_{10} & \alpha_{11}
\end{array}\right) \phi=\left(\begin{array}{cccc}
\alpha_{1}^{*} & \alpha_{2}^{*}+\alpha^{*} & \alpha_{3}^{*} & \alpha_{4}^{*} \\
-\alpha^{*} & \alpha_{5}^{*} & \alpha_{6}^{*} & \alpha_{7}^{*} \\
0 & 0 & 0 & 0 \\
\alpha_{8}^{*} & \alpha_{9}^{*} & \alpha_{10}^{*} & \alpha_{11}^{*}
\end{array}\right)
$$

we have that the action of $\operatorname{Aut}\left(\mathfrak{N}_{03}\right)$ on the subspace $\left\langle\sum_{i=1}^{11} \alpha_{i} \nabla_{i}\right\rangle$ is given by $\left\langle\sum_{i=1}^{11} \alpha_{i}^{*} \nabla_{i}\right\rangle$, where

$$
\begin{aligned}
& \alpha_{1}^{*}= \alpha_{1} x^{2}+\alpha_{2} x z+\alpha_{3} x q+\alpha_{4} x t+\alpha_{5} z^{2}+\alpha_{6} z q+\alpha_{7} z t+\alpha_{8} x t+\alpha_{9} z t+\alpha_{10} q t+\alpha_{11} t^{2}, \\
& \alpha_{2}^{*}= 2 \alpha_{1} w x+\alpha_{2}(x y+w z)+\alpha_{3}(q w+r x)+\alpha_{4}(t w+u x)+2 \alpha_{5} z y+\alpha_{6}(r z+q y)+ \\
& \quad \alpha_{7}(t y+z u)+\alpha_{8}(t w+u x)+\alpha_{9}(t y+z u)+\alpha_{10}(r t+q u)+2 \alpha_{11} t u, \\
& \alpha_{3}^{*}=\left(\alpha_{3} x+\alpha_{6} z+\alpha_{10} t\right)(x y-w z), \\
& \alpha_{4}^{*}= \alpha_{3} s x+\alpha_{4} x v+\alpha_{6} z s+\alpha_{7} z v+\alpha_{10} s t+\alpha_{11} t v, \\
& \alpha_{5}^{*}=\left(\alpha_{3} r+\alpha_{4} u+\alpha_{8} u\right) w+\left(\alpha_{5} y+\alpha_{6} r+\alpha_{7} u+\alpha_{9} u\right) y+\alpha_{10} r u+\alpha_{11} u^{2}, \\
& \alpha_{6}^{*}=\left(\alpha_{3} w+\alpha_{6} y+\alpha_{10} u\right)(x y-w z), \\
& \alpha_{7}^{*}= \alpha_{3} s w+\alpha_{4} v w+\alpha_{6} s y+\alpha_{7} y v+\alpha_{10} s u+\alpha_{11} u v, \\
& \alpha_{8}^{*}=\left(\alpha_{8} x+\alpha_{9} z+\alpha_{10} q+\alpha_{11} t\right) v, \\
& \alpha_{9}^{*}=\left(\alpha_{8} w+\alpha_{9} y+\alpha_{10} r+\alpha_{11} u\right) v, \\
& \alpha_{10}^{*}= \alpha_{10}(x y-w z) v, \\
& \alpha_{11}^{*}=\left(\alpha_{10} s+\alpha_{11} v\right) v .
\end{aligned}
$$

First note that we can not have $\alpha_{3}=\alpha_{6}=\alpha_{10}=0$ or $\alpha_{4}=\alpha_{7}=\alpha_{8}=\alpha_{9}=\alpha_{10}=\alpha_{11}=0$. Besides, a standard computation shows that we can change the position of $\alpha_{3}$ and $\alpha_{6}$.

Since $\alpha_{10}^{*}=0$ if, and only if, $\alpha_{10}=0$, we have:
(1) Suppose $\alpha_{10}=\alpha_{11}=0$. Note that in this case we can not have $\alpha_{3}=\alpha_{6}=0$.
(a) Suppose $\alpha_{3} \alpha_{6} \neq 0$ and $\alpha_{9}=0$. Write $H=\alpha_{3}^{2} \alpha_{5}-\alpha_{2} \alpha_{3} \alpha_{6}+\alpha_{1} \alpha_{6}^{2}$.
(i) If $\alpha_{8}=0$ and $\alpha_{4} \alpha_{6} \neq \alpha_{3} \alpha_{7}$, by choosing $x=1, w=-\frac{\alpha_{6}}{\alpha_{3}^{2}}, y=\frac{1}{\alpha_{3}}, q=-\frac{\alpha_{1}}{\alpha_{3}}$, $r=\frac{2 \alpha_{1} \alpha_{3} \alpha_{6} \alpha_{7}-\alpha_{1} \alpha_{4} \alpha_{6}^{2}+\alpha_{3}^{2}\left(\alpha_{4} \alpha_{5}-\alpha_{2} \alpha_{7}\right)}{\alpha_{3}^{3}\left(\alpha_{3} \alpha_{7}-\alpha_{4} \alpha_{6}\right)}, s=\frac{\alpha_{3} \alpha_{4}}{\alpha_{4} \alpha_{6}-\alpha_{3} \alpha_{7}}, u=\frac{H}{\alpha_{3}^{2}\left(\alpha_{3} \alpha_{7}-\alpha_{4} \alpha_{6}\right)}$ and $v=$ $\frac{\alpha_{3}^{2}}{\alpha_{3} \alpha_{7}-\alpha_{4} \alpha_{6}}$, we obtain the representative $\left\langle\nabla_{3}+\nabla_{7}\right\rangle$.
(ii) Suppose $\alpha_{8}=0$ and $\alpha_{4} \alpha_{6}=\alpha_{3} \alpha_{7}$.
(A) If $H=0$, by choosing $x=v=1, w=-\frac{\alpha_{6}}{\alpha_{3}^{2}}, y=\frac{1}{\alpha_{3}}, q=-\frac{\alpha_{1}}{\alpha_{3}}, r=$ $\frac{2 \alpha_{1} \alpha_{6}-\alpha_{2} \alpha_{3}}{\alpha_{3}^{3}}$ and $s=-\frac{\alpha_{7}}{\alpha_{6}}$, we obtain the representative $\left\langle\nabla_{3}\right\rangle$, that we do not consider since it does not satisfy the above condition.
(B) If $H \neq 0$, by choosing $x=\frac{\sqrt[4]{H}}{\alpha_{3}}, w=-\frac{\alpha_{6}}{\sqrt{H}}, y=\frac{\alpha_{3}}{\sqrt{H}}, q=-\frac{\alpha_{1} \sqrt[4]{H}}{\alpha_{3}^{2}}, r=$ $\frac{2 \alpha_{1} \alpha_{6}-\alpha_{2} \alpha_{3}}{\alpha_{3} \sqrt{H}}$, $s=-\frac{\alpha_{7}}{\alpha_{6}}$ and $v=1$, we obtain the representative $\left\langle\nabla_{3}+\nabla_{5}\right\rangle$, that we do not consider since it does not satisfy the above condition.
(iii) Suppose $\alpha_{8} \neq 0$ and $\alpha_{4} \alpha_{6}+\alpha_{6} \alpha_{8} \neq \alpha_{3} \alpha_{7}$. By choosing $x=\frac{1}{\alpha_{6}}, z=-\frac{\alpha_{3}}{\alpha_{6}^{2}}$, $y=1, q=\frac{\alpha_{1} \alpha_{6}^{2} \alpha_{7}-\alpha_{3}^{2} \alpha_{5} \alpha_{7}+2 \alpha_{3} \alpha_{5} \alpha_{6}\left(\alpha_{4}+\alpha_{8}\right)-\alpha_{2} \alpha_{6}^{2}\left(\alpha_{4}+\alpha_{8}\right)}{\alpha_{6}^{3}\left(\alpha_{4} \alpha_{6}-\alpha_{3} \alpha_{7}+\alpha_{6} \alpha_{8}\right)}, r=-\frac{\alpha_{5}}{\alpha_{6}}, s=-\frac{\alpha_{7}}{\alpha_{8}}, t=$ $\frac{\alpha_{2} \alpha_{3} \alpha_{6}-\alpha_{3}^{2} \alpha_{5}-\alpha_{1} \alpha_{6}^{2}}{\alpha_{6}^{2}\left(\alpha_{4} \alpha_{6}-\alpha_{3} \alpha_{7}+\alpha_{6} \alpha_{8}\right)}$ and $v=\frac{\alpha_{6}}{\alpha_{8}}$, we obtain the family of representatives $\left\langle\alpha \nabla_{4}+\right.$ $\left.\nabla_{6}+\nabla_{8}\right\rangle$.
(iv) Suppose $\alpha_{8} \neq 0$ and $\alpha_{4} \alpha_{6}+\alpha_{6} \alpha_{8}=\alpha_{3} \alpha_{7}$.
(A) If $H \neq 0$, by choosing $x=\frac{\alpha_{6}}{\sqrt{H}}, z=-\frac{\alpha_{3}}{\sqrt{H}}, y=\frac{\sqrt[4]{H}}{\alpha_{6}}, q=\frac{2 \alpha_{3} \alpha_{5}-\alpha_{2} \alpha_{6}}{\alpha_{6} \sqrt{H}}$, $r=-\frac{\alpha_{5} \sqrt[4]{H}}{\alpha_{6}^{2}}, s=-\frac{\alpha_{7} \sqrt{H}}{\alpha_{6}^{2} \alpha_{8}}$ and $v=\frac{\sqrt{H}}{\alpha_{6} \alpha_{8}}$, we obtain the representative $\left\langle\nabla_{1}-\right.$ $\left.\nabla_{4}+\nabla_{6}+\nabla_{8}\right\rangle$.
(B) If $H=0$, by choosing $x=1, z=-\frac{\alpha_{3}}{\alpha_{6}}, y=\frac{1}{\sqrt{\alpha_{6}}}, q=\frac{2 \alpha_{3} \alpha_{5}-\alpha_{2} \alpha_{6}}{\alpha_{6}^{2}}, r=-\frac{\alpha_{5}}{\sqrt{\alpha_{6}^{3}}}$, $s=-\frac{\alpha_{7}}{\alpha_{6} \alpha_{8}}$ and $v=\frac{1}{\alpha_{8}}$, we obtain the representative $\left\langle-\nabla_{4}+\nabla_{6}+\nabla_{8}\right\rangle$.
(b) Suppose $\alpha_{6}=\alpha_{9}=0$ and $\alpha_{3} \neq 0$.
(i) If $\alpha_{7} \alpha_{8} \neq 0$, by choosing $x=\alpha_{7}, y=\alpha_{8}, q=-\frac{\alpha_{1} \alpha_{7}}{\alpha_{3}}, r=\frac{\alpha_{8}\left(\alpha_{4} \alpha_{5}-\alpha_{2} \alpha_{7}+\alpha_{5} \alpha_{8}\right)}{\alpha_{3} \alpha_{7}}, s=$ $-\alpha_{4} \alpha_{7}, q=-\frac{\alpha_{5} \alpha_{8}}{\alpha_{7}}$ and $v=\alpha_{3} \alpha_{7}$, we obtain the representative $\left\langle\nabla_{3}+\nabla_{7}+\nabla_{8}\right\rangle$.
(ii) If $\alpha_{5} \alpha_{8} \neq 0$ and $\alpha_{7}=0$, by choosing $x=\alpha_{5}, y=\alpha_{3} \alpha_{5}, q=-\frac{\alpha_{1} \alpha_{5}}{\alpha_{3}}, r=-\alpha_{2} \alpha_{5}$, $s=-\frac{\alpha_{3} \alpha_{4} \alpha_{5}^{2}}{\alpha_{8}}$ and $v=\frac{\alpha_{3}^{2} \alpha_{5}^{2}}{\alpha_{8}}$, we obtain the representative $\left\langle\nabla_{3}+\nabla_{5}+\nabla_{8}\right\rangle$.
(iii) If $\alpha_{8} \neq 0$ and $\alpha_{5}=\alpha_{7}=0$, by choosing $x=1, y=\alpha_{8}, q=-\frac{\alpha_{1}}{\alpha_{3}}, r=-\frac{\alpha_{2} \alpha_{8}}{\alpha_{3}}$, $s=-\alpha_{4}$ and $v=\alpha_{3}$, we obtain the representative $\left\langle\nabla_{3}+\nabla_{8}\right\rangle$.
(iv) If $\alpha_{8}=0$ and $\alpha_{7} \neq 0$, by choosing $x=\alpha_{7}, y=1, q=-\frac{\alpha_{1} \alpha_{7}}{\alpha_{3}}, r=\frac{\alpha_{4} \alpha_{5}-\alpha_{2} \alpha_{7}}{\alpha_{3} \alpha_{7}}$, $s=-\alpha_{4} \alpha_{7}, u=-\frac{\alpha_{5}}{\alpha_{7}}$ and $v=\alpha_{3} \alpha_{7}$, we obtain the representative $\left\langle\nabla_{3}+\nabla_{7}\right\rangle$.
(v) If $\alpha_{7} \alpha_{8}=0$, by choosing $v=1$ and $s=-\frac{\alpha_{4}}{\alpha_{3}}$, we obtain the representative $\left\langle\alpha_{1} \nabla_{1}+\alpha_{2} \nabla_{2}+\alpha_{3} \nabla_{3}+\alpha_{5} \nabla_{5}\right\rangle$, that we do not consider since it does not satisfy the above condition.
(c) If $\alpha_{3} \alpha_{9} \neq 0$ or $\alpha_{6} \alpha_{9} \neq 0$, by choosing $w=z=v=1$ and $y=-\frac{\alpha_{8}}{\alpha_{9}}$, we return to one of the above cases.
(d) If $\alpha_{3}=\alpha_{9}=0$ and $\alpha_{6} \neq 0$, by choosing $x=y=z=v=1$, we return to the case when $\alpha_{3} \alpha_{6} \neq 0$ and $\alpha_{9}=0$.
(2) Suppose $\alpha_{10}=0$ and $\alpha_{11} \neq 0$. Again, in this case we can not have $\alpha_{3}=\alpha_{6}=0$.
(a) Suppose $\alpha_{6} \neq 0$ and $\alpha_{3}=0$.
(i) If $\alpha_{8} \neq \alpha_{4}$, by choosing $x=\frac{\sqrt{\alpha_{11}}}{\alpha_{4}-\alpha_{8}}, y=\frac{\sqrt{\alpha_{4}-\alpha_{8}}}{\sqrt[4]{\alpha_{11} \alpha_{6}^{2}}}, q=\frac{\alpha_{7} \alpha_{8}-\alpha_{11} \alpha_{2}+\alpha_{4} \alpha_{9}}{\alpha_{6} \sqrt{\alpha_{11}\left(\alpha_{4}-\alpha_{8}\right)}}, r=$ $\frac{\sqrt{\alpha_{4}-\alpha_{8}}\left(\alpha_{7} \alpha_{9}-\alpha_{11} \alpha_{5}\right)}{\sqrt[4]{\alpha_{11}^{5} \alpha_{6}^{6}}}, s=\frac{\alpha_{9}-\alpha_{7}}{\alpha_{6} \sqrt{\alpha_{11}}}, t=\frac{\alpha_{8}}{\sqrt{\alpha_{11}}\left(\alpha_{8}-\alpha_{4}\right)}, u=\frac{\alpha_{9} \sqrt{\alpha_{4}-\alpha_{8}}}{\sqrt[4]{\alpha_{6}^{2} \alpha_{11}^{5}}}$ and $v=\frac{1}{\sqrt{\alpha_{11}}}$, we obtain the family of representatives $\left\langle\alpha \nabla_{1}+\nabla_{4}+\nabla_{6}+\nabla_{11}\right\rangle$.
(ii) If $\alpha_{8}=\alpha_{4}$ and $\alpha_{1} \alpha_{11} \neq \alpha_{4}^{2}$, by choosing $x=\frac{\sqrt{\alpha_{11}}}{\sqrt{\alpha_{1} \alpha_{11}-\alpha_{4}^{2}}}, y=\frac{\sqrt[4]{\alpha_{1} \alpha_{11}-\alpha_{4}^{2}}}{\sqrt[4]{\alpha_{11} \alpha_{6}^{2}}}, q=$ $\frac{\alpha_{4}\left(\alpha_{7}+\alpha_{9}\right)-\alpha_{11} \alpha_{2}}{\alpha_{6} \sqrt{\alpha_{11}\left(\alpha_{1} \alpha_{11}-\alpha_{4}^{2}\right)}}, r=\frac{\left(\alpha_{7} \alpha_{9}-\alpha_{11} \alpha_{5}\right) \sqrt[4]{\alpha_{1} \alpha_{11}-\alpha_{4}^{2}}}{\sqrt[4]{\alpha_{11}^{5} \alpha_{6}^{6}}}, s=\frac{\alpha_{9}-\alpha_{7}}{\alpha_{6} \sqrt{\alpha_{11}}}, t=-\frac{\alpha_{4}}{\sqrt{\alpha_{11}\left(\alpha_{1} \alpha_{11}-\alpha_{4}^{2}\right)}}$, $u=-\frac{\alpha 9 \sqrt[4]{\alpha_{1} \alpha_{11}-\alpha_{4}^{2}}}{\sqrt[4]{\alpha_{11}^{5} \alpha_{6}^{2}}}$ and $v=\frac{1}{\sqrt{\alpha_{11}}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{6}+\nabla_{11}\right\rangle$.
(iii) If $\alpha_{8}=\alpha_{4}$ and $\alpha_{1} \alpha_{11}=\alpha_{4}^{2}$, by choosing $x=\frac{1}{\alpha_{6}}, y=1, q=\frac{\alpha_{4}\left(\alpha_{7}+\alpha_{9}\right)-\alpha_{11} \alpha_{2}}{\alpha_{6}^{2} \alpha_{11}}$, $r=\frac{\alpha_{7} \alpha_{9}-\alpha_{11} \alpha_{5}}{\alpha_{11} \alpha_{6}}, s=\frac{\alpha_{9}-\alpha_{7}}{\alpha_{6} \sqrt{\alpha_{11}}}, t=-\frac{\alpha_{4}}{\alpha_{11} \alpha_{6}}, u=-\frac{\alpha_{9}}{\alpha_{11}}$ and $v=\frac{1}{\sqrt{\alpha_{11}}}$, we obtain the representative $\left\langle\nabla_{6}+\nabla_{11}\right\rangle$.
(b) If $\alpha_{3} \alpha_{6} \neq 0$, by choosing $x=-\frac{\alpha_{6}}{\alpha_{3}}$ and $z=y=v=1$, we return to the case $\alpha_{6} \neq 0$ and $\alpha_{3}=0$.
(3) Suppose $\alpha_{10} \neq 0$. Note that, by choosing $x=y=v=1, q=\frac{\alpha_{11} \alpha_{3}-\alpha_{10} \alpha_{8}}{\alpha_{10}^{2}}, r=\frac{\alpha_{11} \alpha_{6}-\alpha_{10} \alpha_{9}}{\alpha_{10}^{2}}$, $s=-\frac{\alpha_{11}}{\alpha_{10}}, t=-\frac{\alpha_{3}}{\alpha_{10}}$ and $u=-\frac{\alpha_{6}}{\alpha_{10}}$, we can suppose $\alpha_{3}=\alpha_{6}=\alpha_{8}=\alpha_{9}=\alpha_{11}=0$.
(a) Suppose $\alpha_{7} \neq 0$. By choosing $x=y=v=1$ and $z=-\frac{\alpha_{4}}{\alpha_{7}}$ we can suppose $\alpha_{4}=0$. Therefore, we have the following cases:
(i) If $\alpha_{1}\left(\alpha_{2}^{2}-4 \alpha_{1} \alpha_{5}\right) \neq 0$, by choosing $x=\frac{\alpha_{7}}{\alpha_{10}}, w=-\frac{\alpha_{2} \alpha_{7}}{\alpha_{10} \sqrt{4 \alpha_{1} \alpha_{5}-\alpha_{2}^{2}}}, y=$ $\frac{2 \alpha_{1} \alpha_{7}}{\alpha_{10} \sqrt{4 \alpha_{1} \alpha_{5}-\alpha_{2}^{2}}}$ and $v=\frac{\sqrt{4 \alpha_{1} \alpha_{5}-\alpha_{2}^{2}}}{2 \alpha_{10}}$, we have the representative $\left\langle\nabla_{1}+\nabla_{5}+\nabla_{7}+\right.$ $\left.\nabla_{10}\right\rangle$.
(ii) If $\alpha_{1} \neq 0$ and $\alpha_{2}^{2}=4 \alpha_{1} \alpha_{5}$, by choosing $x=\frac{\alpha_{7}}{\alpha_{10}}, w=-\frac{\alpha_{2}}{\alpha_{1}}, y=1$ and $v=\frac{\alpha_{1} \alpha_{7}}{\alpha_{10}^{2}}$, we have the representative $\left\langle\nabla_{1}+\nabla_{7}+\nabla_{10}\right\rangle$.
(iii) If $\alpha_{1}=0$ and $\alpha_{2} \neq 0$, by choosing $x=\frac{\alpha_{7}}{\alpha_{10}}, w=-\frac{\alpha_{5}}{\alpha_{2}}, y=1$ and $v=\frac{\alpha_{2}}{\alpha_{10}}$, we have the representative $\left\langle\nabla_{2}+\nabla_{7}+\nabla_{10}\right\rangle$.
(iv) If $\alpha_{1}=\alpha_{2}=0$, we obtain the representatives $\left\langle\nabla_{5}+\nabla_{7}+\nabla_{10}\right\rangle$ and $\left\langle\nabla_{7}+\nabla_{10}\right\rangle$.
(b) Suppose $\alpha_{7}=0$ and $\alpha_{4} \neq 0$. By choosing $w=z=v=1$, we return to the case $\alpha_{7} \neq 0$.
(c) Suppose $\alpha_{7}=\alpha_{4}=0$. If $\alpha_{5} \neq 0$, then by choosing $w=z=v=1$ and $y=$ $\frac{\sqrt{\alpha_{2}^{2}-4 \alpha_{1} \alpha_{5}}-\alpha_{2}}{2 \alpha_{5}}$, we can suppose $\alpha_{5}=0$.
(i) If $\alpha_{2} \neq 0$, by choosing $x=y=1, z=-\frac{\alpha_{1}}{\alpha_{2}}$ and $v=\frac{\alpha_{2}}{\alpha_{10}}$, we obtain the representative $\left\langle\nabla_{2}+\nabla_{10}\right\rangle$.
(ii) If $\alpha_{2}=0$, we have the representatives $\left\langle\nabla_{1}+\nabla_{10}\right\rangle$ and $\left\langle\nabla_{10}\right\rangle$.

Summarizing, we have the following distinct orbits

$$
\begin{gathered}
\left\langle\nabla_{3}+\nabla_{7}\right\rangle,\left\langle\nabla_{3}+\nabla_{8}\right\rangle,\left\langle\nabla_{3}+\nabla_{5}+\nabla_{8}\right\rangle,\left\langle\nabla_{3}+\nabla_{7}+\nabla_{8}\right\rangle,\left\langle\alpha \nabla_{4}+\nabla_{6}+\nabla_{8}\right\rangle, \\
\left\langle\nabla_{1}-\nabla_{4}+\nabla_{6}+\nabla_{8}\right\rangle,\left\langle\nabla_{6}+\nabla_{11}\right\rangle,\left\langle\nabla_{1}+\nabla_{6}+\nabla_{11}\right\rangle,\left\langle\alpha \nabla_{1}+\nabla_{4}+\nabla_{6}+\nabla_{11}\right\rangle,\left\langle\nabla_{10}\right\rangle, \\
\left\langle\nabla_{1}+\nabla_{10}\right\rangle,\left\langle\nabla_{2}+\nabla_{10}\right\rangle,\left\langle\nabla_{7}+\nabla_{10}\right\rangle,\left\langle\nabla_{1}+\nabla_{7}+\nabla_{10}\right\rangle,\left\langle\nabla_{2}+\nabla_{7}+\nabla_{10}\right\rangle,\left\langle\nabla_{5}+\nabla_{7}+\nabla_{10}\right\rangle,
\end{gathered}
$$

$$
\left\langle\nabla_{1}+\nabla_{5}+\nabla_{7}+\nabla_{10}\right\rangle
$$

which give the following new algebras:

| $Z_{06}$ | $:$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $Z_{07}$ | $:$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{4} e_{1}=e_{5}$ |  |  |
| $Z_{08}$ | $:$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{5}$ | $e_{4} e_{1}=e_{5}$ |  |
| $Z_{09}$ | $:$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{1}=e_{5}$ |  |
| $Z_{10}^{\alpha}$ | $:$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{4}=\alpha e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{1}=e_{5}$ |  |
| $Z_{11}$ | $:$ | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{4}=-e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{1}=e_{5}$ |
| $Z_{12}$ | $:$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{4}=e_{5}$ |  |  |
| $Z_{13}$ | $:$ | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{4}=e_{5}$ |  |
| $Z_{14}^{\alpha}$ | $:$ | $e_{1} e_{1}=\alpha e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{4}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{4}=e_{5}$ |
| $Z_{15}$ | $:$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{4} e_{3}=e_{5}$ |  |  |  |
| $Z_{16}$ | $:$ | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{4} e_{3}=e_{5}$ |  |  |
| $Z_{17}$ | $:$ | $e_{1} e_{2}=e_{3}+e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{4} e_{3}=e_{5}$ |  |  |  |
| $Z_{18}$ | $:$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{3}=e_{5}$ |  |  |
| $Z_{19}$ | $:$ | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{3}=e_{5}$ |  |
| $Z_{20}$ | $:$ | $e_{1} e_{2}=e_{3}+e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{3}=e_{5}$ |  | $e_{4} e_{3}=e_{5}$ |
| $Z_{21}$ | $:$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{5}$ | $e_{2} e_{4}=e_{5}$ | $e_{0}$ |  |
| $Z_{22}$ | $:$ | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{5}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{3}=e_{5}$ |

1.2.4. Central extensions of $\mathfrak{N}_{07}$. Let us use the following notations:

$$
\nabla_{1}=\left[\Delta_{11}\right], \quad \nabla_{2}=\left[\Delta_{22}\right], \quad \nabla_{3}=\left[\Delta_{13}-\Delta_{14}-\Delta_{23}+\Delta_{24}-2 \Delta_{32}\right]
$$

Take $\theta=\sum_{i=1}^{3} \alpha_{i} \nabla_{i} \in \mathrm{H}^{2}\left(\mathfrak{N}_{07}\right)$. The automorphism group of $\mathfrak{N}_{07}$ is generated by invertible matrices of the form

$$
\phi=\left(\begin{array}{cccc}
x & 0 & 0 & 0 \\
0 & x & 0 & 0 \\
r & s & x^{2} & 0 \\
t & u & 0 & x^{2}
\end{array}\right)
$$

Since

$$
\phi^{T}\left(\begin{array}{cccc}
\alpha_{1} & 0 & \alpha_{3} & -\alpha_{3} \\
0 & \alpha_{2} & -\alpha_{3} & \alpha_{3} \\
0 & -2 \alpha_{3} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \phi=\left(\begin{array}{cccc}
\alpha_{1}^{*} & -\alpha^{*} & \alpha_{3}^{*} & -\alpha_{3}^{*} \\
\alpha^{* *} & \alpha^{*}+\alpha_{2}^{*} & -\alpha_{3}^{*} & \alpha_{3}^{*} \\
0 & -2 \alpha_{3}^{*} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

we have that the action of $\operatorname{Aut}\left(\mathfrak{N}_{07}\right)$ on the subspace $\left\langle\sum_{i=1}^{3} \alpha_{i} \nabla_{i}\right\rangle$ is given by $\left\langle\sum_{i=1}^{3} \alpha_{i}^{*} \nabla_{i}\right\rangle$, where

$$
\alpha_{1}^{*}=\alpha_{3}(r-t) x+\alpha_{1} x^{2}, \quad \alpha_{2}^{*}=\alpha_{2} x^{2}-2 \alpha_{3}(r+s) x, \quad \alpha_{3}^{*}=\alpha_{3} x^{3} .
$$

Since $\alpha_{3} \neq 0$, we suppose that $\alpha_{3}=1$. Then, by choosing $x=1, r=\frac{\alpha_{2}}{2}$ and $t=\frac{2 \alpha_{1}+\alpha_{2}}{2}$, we obtain the representative $\left\langle\nabla_{3}\right\rangle$, which gives the following new algebra:

$$
\begin{aligned}
& Z_{23}: e_{1} e_{2}=e_{3} \quad e_{1} e_{3}=e_{5} \quad e_{1} e_{4}=-e_{5} \quad e_{2} e_{1}=e_{4} \\
& e_{2} e_{2}=-e_{3} \quad e_{2} e_{3}=-e_{5} \quad e_{2} e_{4}=e_{5} \quad e_{3} e_{2}=-2 e_{5}
\end{aligned}
$$

1.2.5. Central extensions of $\mathrm{H}^{2}\left(\mathfrak{N}_{08}^{1}\right)$. Let us use the following notations:

$$
\left.\begin{array}{rlrl}
\nabla_{1} & =\left[\Delta_{11}\right], & \nabla_{3}=\left[\Delta_{23}-\Delta_{13}-2 \Delta_{31}+\Delta_{32}+\Delta_{41}\right], \\
\nabla_{2} & =\left[\Delta_{12}\right], & \nabla_{4}=\left[\Delta_{24}-\Delta_{14}-\Delta_{32}-\Delta_{41}+2 \Delta_{42}\right.
\end{array}\right] .
$$

Take $\theta=\sum_{i=1}^{4} \alpha_{i} \nabla_{i} \in \mathrm{H}^{2}\left(\mathfrak{N}_{08}^{1}\right)$. The automorphism group of $\mathfrak{N}_{08}^{1}$ is generated by invertible matrices of the form

$$
\phi=\left(\begin{array}{cccc}
a & x & 0 & 0 \\
a+x-y & y & 0 & 0 \\
z & t & a(y-x) & x(y-x) \\
u & v & (a+x-y)(y-x) & y(y-x)
\end{array}\right) .
$$

Since

$$
\phi^{T}\left(\begin{array}{cccc}
\alpha_{1} & \alpha_{2} & -\alpha_{3} & -\alpha_{4} \\
0 & 0 & \alpha_{3} & \alpha_{4} \\
-2 \alpha_{3} & \alpha_{3}-\alpha_{4} & 0 & 0 \\
\alpha_{3}-\alpha_{4} & 2 \alpha_{4} & 0 & 0
\end{array}\right) \phi=\left(\begin{array}{cccc}
\alpha^{*}+\alpha_{1}^{*} & \alpha^{* *}+\alpha_{2}^{*} & -\alpha_{3}^{*} & -\alpha_{4}^{*} \\
-\alpha^{*} & -\alpha^{* *} & \alpha_{3}^{*} & \alpha_{4}^{*} \\
-2 \alpha_{3}^{*} & \alpha_{3}^{*}-\alpha_{4}^{*} & 0 & 0 \\
\alpha_{3}^{*}-\alpha_{4}^{*} & 2 \alpha_{4}^{*} & 0 & 0
\end{array}\right)
$$

we have that the action of $\operatorname{Aut}\left(\mathfrak{N}_{08}^{1}\right)$ on the subspace $\left\langle\sum_{i=1}^{4} \alpha_{i} \nabla_{i}\right\rangle$ is given by $\left\langle\sum_{i=1}^{4} \alpha_{i}^{*} \nabla_{i}\right\rangle$, where

$$
\alpha_{1}^{*}=\left(\alpha_{1}+\alpha_{2}\right) a^{2}+\left(\alpha_{1} x+2 \alpha_{2} x-\alpha_{2} y+\alpha_{3}(u+v-t-z)+\alpha_{4}(u+v-t-z)\right) a+
$$

$$
\begin{aligned}
& \quad\left(\alpha_{2} x+\alpha_{3}(t+z)-\alpha_{4}(t-2 u-2 v+z)\right)(x-y), \\
& \alpha_{2}^{*}= \alpha_{1}\left(x^{2}+a x\right)+\alpha_{2}(x y+a y)+\alpha_{4}(2 u y+2 v y-y z-u x-v x-t y)+ \\
& \alpha_{3}^{*}=\left(\left(\alpha_{3}+\alpha_{4}\right) a+\alpha_{4}(x-y)\right)(x-y)^{2}, \\
& \alpha_{4}^{*}=\left(\alpha_{3} x+\alpha_{4} y\right)(x-y)^{2} .
\end{aligned}
$$

Note that we can not have $\left(\alpha_{3}, \alpha_{4}\right)=(0,0)$ and by the action of $\operatorname{Aut}\left(\mathfrak{N}_{08}^{1}\right)$, we need to consider only the following two cases:
(1) If $\alpha_{4}=0$ and $\alpha_{3} \neq 0$, by choosing $x=1, t=-\frac{\alpha_{2}}{\alpha_{3}}$ and $u=-\frac{\alpha_{1}+2 \alpha_{2}}{\alpha_{3}}$, we obtain the representative $\left\langle\nabla_{4}\right\rangle$.
(2) Suppose $\alpha_{3} \alpha_{4} \neq 0$.
(a) If $\alpha_{3} \neq-\alpha_{4}$, by choosing $a=-\frac{\alpha_{4}}{\sqrt[3]{\alpha_{3}^{4}}}, x=\frac{3 \alpha_{3}+\alpha_{4}}{2 \sqrt[3]{\alpha_{3}^{4}}}, y=\frac{2}{\sqrt[3]{\alpha_{3}}}, z=\frac{2 \alpha_{1} \alpha_{4}-\alpha_{2}\left(\alpha_{3}-\alpha_{4}\right)}{\sqrt[3]{\alpha_{3}\left(\alpha_{3}+\alpha_{4}\right)^{2}}}$, $t=\frac{2\left(2 \alpha_{1} \alpha_{4}-\alpha_{2}\left(\alpha_{3}-\alpha_{4}\right)\right.}{\sqrt[3]{\alpha_{3}\left(\alpha_{3}+\alpha_{4}\right)^{2}}}, u=\frac{\left(2 \alpha_{2} \alpha_{3}+\alpha_{1}\left(\alpha_{3}-\alpha_{4}\right)\right) \alpha_{4}}{2 \sqrt[3]{\alpha_{3}^{4}}\left(\alpha_{3}+\alpha_{4}\right)^{2}}$ and $v=\frac{\left(\alpha_{1}\left(\alpha_{3}-\alpha_{4}\right)-2 \alpha_{2} \alpha_{3}\right)\left(3 \alpha_{3}+\alpha_{4}\right)}{2 \sqrt[3]{\alpha_{3}^{4}}\left(\alpha_{3}+\alpha_{4}\right)^{2}}$, we obtain the representative $\left\langle\nabla_{4}\right\rangle$.
(b) Suppose $\alpha_{3}=-\alpha_{4}$.
(i) If $\alpha_{1} \neq-\alpha_{2}$, by choosing $x=\frac{\alpha_{1}+\alpha_{2}}{\alpha_{4}}$ and $v=\frac{\alpha_{1}\left(\alpha_{1}+\alpha_{2}\right)}{2 \alpha_{4}^{2}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{3}-\nabla_{4}\right\rangle$.
(ii) If $\alpha_{1}=-\alpha_{2}$, by choosing $x=-1$ and $v=\frac{\alpha_{2}}{2 \alpha_{4}}$, we obtain the representative $\left\langle-\nabla_{3}+\nabla_{4}\right\rangle$.
Summarizing, we have the distinct orbits

$$
\left\langle\nabla_{4}\right\rangle,\left\langle\nabla_{1}+\nabla_{3}-\nabla_{4}\right\rangle,\left\langle\nabla_{3}-\nabla_{4}\right\rangle
$$

which give the following new algebras:

| $\mathcal{Z}_{24}$ | $:$ | $e_{1} e_{1}=e_{3}$ |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $e_{2} e_{4}=e_{5}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{4}=-e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=-e_{4}$ |  |
| $Z_{25}$ | $:$ | $e_{1} e_{1}=e_{3}+e_{5}$ | $e_{1} e_{2}=e_{4}$ | $e_{4} e_{1}=-e_{5}$ | $e_{4} e_{2}=2 e_{5}$ |  |
|  |  | $e_{2} e_{3}=e_{5}$ | $e_{2} e_{4}=-e_{5}$ | $e_{3} e_{1}=-2 e_{5}$ | $e_{3} e_{4}=e_{5}=2 e_{5}$ | $e_{2} e_{1}=-e_{3} e_{1}=2 e_{5}$ |$e_{2} e_{2}=-e_{4} e_{2}=-2 e_{5}$.

1.2.6. Central extensions of $\mathfrak{N}_{12}$. Let us use the following notations:

$$
\nabla_{1}=\left[\Delta_{11}\right], \quad \nabla_{2}=\left[\Delta_{22}\right], \quad \nabla_{3}=\left[\Delta_{14}-\Delta_{13}\right], \quad \nabla_{4}=\left[\Delta_{24}-\Delta_{23}\right] .
$$

Take $\theta=\sum_{i=1}^{4} \alpha_{i} \nabla_{i} \in \mathrm{H}^{2}\left(\mathfrak{N}_{12}\right)$. The automorphism group of $\mathfrak{N}_{12}$ is generated by invertible matrices of the form

$$
\phi_{1}=\left(\begin{array}{cccc}
0 & x & 0 & 0 \\
y & 0 & 0 & 0 \\
z & t & 0 & x y \\
u & v & x y & 0
\end{array}\right) \text { and } \phi_{2}=\left(\begin{array}{cccc}
x & 0 & 0 & 0 \\
0 & y & 0 & 0 \\
z & t & x y & 0 \\
u & v & 0 & x y
\end{array}\right)
$$

Since

$$
\phi_{1}^{T}\left(\begin{array}{cccc}
\alpha_{1} & 0 & -\alpha_{3} & \alpha_{3} \\
0 & \alpha_{2} & -\alpha_{4} & \alpha_{4} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \phi_{1}=\left(\begin{array}{cccc}
\alpha_{1}^{*} & \alpha^{*} & -\alpha_{3}^{*} & \alpha_{3}^{*} \\
\beta^{*} & \alpha_{2}^{*} & -\alpha_{4}^{*} & \alpha_{4}^{*} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right),
$$

we have that the action of $\operatorname{Aut}\left(\mathfrak{N}_{12}\right)$ on the subspace $\left\langle\sum_{i=1}^{4} \alpha_{i} \nabla_{i}\right\rangle$ is given by $\left\langle\sum_{i=1}^{4} \alpha_{i}^{*} \nabla_{i}\right\rangle$, where

$$
\alpha_{1}^{*}=\left(\alpha_{2} y+\alpha_{4} u-\alpha_{4} z\right) y, \quad \alpha_{2}^{*}=\left(\alpha_{1} x-\alpha_{3} t+\alpha_{3} v\right) x, \quad \alpha_{3}^{*}=-\alpha_{4} x y^{2}, \quad \alpha_{4}^{*}=-\alpha_{3} x^{2} y
$$

Since $\mathrm{H}^{2}\left(\mathfrak{N}_{12}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{22}\right],\left[\Delta_{14}-\Delta_{13}\right],\left[\Delta_{24}-\Delta_{23}\right]\right\rangle$ and we are interested only in new algebras, we must have $\left(\alpha_{3}, \alpha_{4}\right) \neq(0,0)$. Then
(1) if $\alpha_{4} \alpha_{3} \neq 0$, then by choosing $x=\alpha_{4}, y=\alpha_{3}, z=\frac{\alpha_{2} \alpha_{3}}{\alpha_{4}}$ and $t=\frac{\alpha_{1} \alpha_{4}}{\alpha_{3}}$, we have the representative $\left\langle\nabla_{3}+\nabla_{4}\right\rangle$.
(2) if $\alpha_{4} \alpha_{1} \neq 0$ and $\alpha_{3}=0$, then by choosing $x=-\alpha_{1}, y=\frac{\alpha_{1}}{\sqrt{\alpha_{4}}}$ and $z=\frac{\alpha_{2} y}{\alpha_{4}}$, we have the representative $\left\langle\nabla_{2}+\nabla_{3}\right\rangle$.
(3) if $\alpha_{4} \neq 0$ and $\alpha_{1}=\alpha_{3}=0$, then by choosing $x=y=1$ and $z=\frac{\alpha_{2}}{\alpha_{4}}$, we have the representative $\left\langle\nabla_{3}\right\rangle$.
(4) if $\alpha_{3} \neq 0$ and $\alpha_{4}=0$, by applying $\phi_{2}$ we obtain the case $\alpha_{4}^{*}=0$ which has been considered above.
Summarizing, we have the following distinct orbits

$$
\left\langle\nabla_{3}+\nabla_{4}\right\rangle,\left\langle\nabla_{2}+\nabla_{3}\right\rangle,\left\langle\nabla_{3}\right\rangle,
$$

which give the following new algebras:

$$
\begin{array}{lllllll}
\mathcal{Z}_{27} & : & e_{1} e_{2}=e_{3} & e_{1} e_{3}=-e_{5} & e_{1} e_{4}=e_{5} & e_{2} e_{1}=e_{4} & e_{2} e_{3}=-e_{5} \quad e_{2} e_{4}=e_{5} \\
\hline \mathcal{Z}_{28} & : & e_{1} e_{2}=e_{3} & e_{1} e_{3}=-e_{5} & e_{1} e_{4}=e_{5} & e_{2} e_{1}=e_{4} & e_{2} e_{2}=e_{5} \\
\hline \mathcal{Z}_{29} & : & e_{1} e_{2}=e_{3} & e_{1} e_{3}=-e_{5} & e_{1} e_{4}=e_{5} & e_{2} e_{1}=e_{4}
\end{array}
$$

1.2.7. Central extensions of $\mathfrak{N}_{14}^{\alpha \neq-1}$. Let us use the following notations:

$$
\nabla_{1}=\left[\Delta_{11}\right], \quad \nabla_{2}=\left[\Delta_{21}\right], \quad \nabla_{3}=\left[\Delta_{23}+2 \Delta_{32}\right], \quad \nabla_{4}=\left[2 \alpha \Delta_{24}+(\alpha+1)\left(\Delta_{13}+2 \alpha \Delta_{31}+2 \Delta_{42}\right)\right] .
$$

Take $\theta=\sum_{i=1}^{4} \alpha_{i} \nabla_{i} \in \mathrm{H}^{2}\left(\mathfrak{N}_{14}^{\alpha \neq-1}\right)$. The automorphism group of $\mathfrak{N}_{14}^{\alpha \neq-1}$ consists of invertible matrices of the form

$$
\phi=\left(\begin{array}{cccc}
x & y & 0 & 0 \\
0 & z & 0 & 0 \\
t & u & z^{2} & 0 \\
v & w & y z(\alpha+1) & x z
\end{array}\right)
$$

Since

$$
\phi^{T}\left(\begin{array}{cccc}
\alpha_{1} & 0 & (\alpha+1) \alpha_{4} & 0 \\
\alpha_{2} & 0 & \alpha_{3} & 2 \alpha \alpha_{4} \\
2 \alpha(\alpha+1) \alpha_{4} & 2 \alpha_{3} & 0 & 0 \\
0 & 2(\alpha+1) \alpha_{4} & 0 & 0
\end{array}\right) \phi=\left(\begin{array}{cccc}
\alpha_{1}^{*} & \beta^{*} & (\alpha+1) \alpha_{4}^{*} & 0 \\
\alpha_{2}^{*}+\alpha \beta^{*} & \gamma^{*} & \alpha_{3}^{*} & 2 \alpha \alpha_{4}^{*} \\
2 \alpha(\alpha+1) \alpha_{4}^{*} & 2 \alpha_{3}^{*} & 0 & 0 \\
0 & 2(\alpha+1) \alpha_{4}^{*} & 0 & 0
\end{array}\right),
$$

we have that the action of $\operatorname{Aut}\left(\mathfrak{N}_{14}^{\alpha \neq-1}\right)$ on the subspace $\left\langle\sum_{i=1}^{4} \alpha_{i} \nabla_{i}\right\rangle$ is given by $\left\langle\sum_{i=1}^{4} \alpha_{i}^{*} \nabla_{i}\right\rangle$, where

$$
\begin{aligned}
\alpha_{1}^{*}= & \left(\alpha_{1} x+\left(2 \alpha^{2}+3 \alpha+1\right) \alpha_{4} t\right) x, \\
\alpha_{2}^{*}= & (\alpha+1)\left(1-2 \alpha^{2}\right) \alpha_{4} t y+\alpha(\alpha+1) \alpha_{4} u x+ \\
& (1-\alpha) \alpha_{1} x y-2 \alpha^{2} \alpha_{4} v z+\alpha_{2} x z+(1-2 \alpha) \alpha_{3} t z, \\
\alpha_{3}^{*}= & \left(\alpha_{3} z+\left(2 \alpha^{2}+3 \alpha+1\right) \alpha_{4} y\right) z^{2}, \\
\alpha_{4}^{*}= & \alpha_{4} x z^{2} .
\end{aligned}
$$

Since $H^{2}\left(\mathfrak{N}_{14}^{\alpha \neq-1}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{21}\right],\left[\Delta_{23}+2 \Delta_{32}\right],\left[2 \alpha \Delta_{24}+(\alpha+1)\left(\Delta_{13}+2 \alpha \Delta_{31}+2 \Delta_{42}\right)\right]\right\rangle$ and we are interested only in new algebras, we must have $\alpha_{4} \neq 0$. Then
(1) if $\alpha \neq 0,-\frac{1}{2}$, then by choosing $x=z=-\alpha_{4}\left(2 \alpha^{2}+3 \alpha+1\right), y=\alpha_{3}, t=\alpha_{1}$ and $v=$ $-\frac{\alpha_{2} \alpha_{4}\left(4 \alpha^{3}+8 \alpha^{2}+5 \alpha+1\right)+\alpha_{1} \alpha_{3}\left(4 \alpha^{2}-\alpha-1\right)}{2 \alpha^{2}(2 \alpha+1) \alpha_{4}}$, we have the representative $\left\langle\nabla_{4}\right\rangle$.
(2) if $\alpha=-\frac{1}{2}$,
(a) Suppose $\alpha_{3}=0$.
(i) If $\alpha_{1}=0$, then by choosing $x=z=1$ and $u=\frac{4 \alpha_{2}}{\alpha_{4}}$, we obtain the representative $\left\langle\nabla_{4}\right\rangle$.
(ii) If $\alpha_{1} \neq 0$, then by choosing $x=\alpha_{1} \alpha_{4}, z=\alpha_{1}$ and $u=\frac{4 \alpha_{1} \alpha_{2}}{\alpha_{4}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{4}\right\rangle$.
(b) Suppose $\alpha_{3} \neq 0$.
(i) If $\alpha_{1}=0$, then by choosing $x=\alpha_{3}, z=\alpha_{4}$ and $u=4 \alpha_{2}$, we obtain the representative $\left\langle\nabla_{3}+\nabla_{4}\right\rangle$.
(ii) If $\alpha_{1} \neq 0$, then by choosing $x=\frac{\alpha_{1} \alpha_{3}^{2}}{\alpha_{4}^{3}}, z=\frac{\alpha_{1} \alpha_{3}}{\alpha_{4}^{2}}$ and $u=\frac{4 \alpha_{1} \alpha_{2} \alpha_{3}}{\alpha_{4}^{3}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{3}+\nabla_{4}\right\rangle$.
(3) if $\alpha=0$ and $\alpha_{2} \alpha_{4}-\alpha_{1} \alpha_{3} \neq 0$, then by choosing $x=\alpha_{4}, y=\frac{\alpha_{3}\left(\alpha_{1} \alpha_{3}-\alpha_{2} \alpha_{4}\right)}{\alpha_{4}^{3}}, z=\frac{\alpha_{2} \alpha_{4}-\alpha_{1} \alpha_{3}}{\alpha_{4}^{2}}$ and $t=-\alpha_{1}$, we have the representative $\left\langle\nabla_{2}+\nabla_{4}\right\rangle$.
(4) if $\alpha=0$ and $\alpha_{2} \alpha_{4}-\alpha_{1} \alpha_{3}=0$, then by choosing $x=z=\alpha_{4}, y=-\alpha_{3}$ and $t=-\alpha_{1}$, we have the representative $\left\langle\nabla_{4}\right\rangle$.
Summarizing, we have

$$
\left\langle\nabla_{4}\right\rangle_{\alpha \neq-1},\left\langle\nabla_{2}+\nabla_{4}\right\rangle_{\alpha=0},\left\langle\nabla_{1}+\nabla_{4}\right\rangle_{\alpha=-\frac{1}{2}},\left\langle\nabla_{3}+\nabla_{4}\right\rangle_{\alpha=-\frac{1}{2}},\left\langle\nabla_{1}+\nabla_{3}+\nabla_{4}\right\rangle_{\alpha=-\frac{1}{2}}
$$

which give the following new algebras:

| $\mathcal{Z}_{30}^{\alpha \neq-1}$ | $:$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=(\alpha+1) e_{5}$ | $e_{2} e_{1}=\alpha e_{4}$ | $e_{2} e_{2}=e_{3}$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  |  | $e_{2} e_{4}=2 \alpha e_{5}$ | $e_{3} e_{1}=2 \alpha(\alpha+1) e_{5}$ | $e_{4} e_{2}=2(\alpha+1) e_{5}$ |  |  |
| $\mathcal{Z}_{31}$ | $:$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=e_{5}$ | $e_{2} e_{2}=e_{3}$ | $e_{4} e_{2}=2 e_{5}$ |
| $Z_{32}$ | $:$ | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=\frac{1}{2} e_{5}$ | $e_{2} e_{1}=-\frac{1}{2} e_{4}$ |  |
|  |  | $e_{2} e_{2}=e_{3}$ | $e_{2} e_{4}=-e_{5}$ | $e_{3} e_{1}=-\frac{1}{2} e_{5}$ | $e_{4} e_{2}=e_{5}$ |  |
| $Z_{33}$ | $:$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=\frac{1}{2} e_{5}$ | $e_{2} e_{1}=-\frac{1}{2} e_{4}$ | $e_{2} e_{2}=e_{3}$ | $e_{2} e_{3}=e_{5}$ |
|  |  | $e_{2} e_{4}=-e_{5}$ | $e_{3} e_{1}=-\frac{1}{2} e_{5}$ | $e_{3} e_{2}=2 e_{5}$ | $e_{4} e_{2}=e_{5}$ |  |
| $\mathcal{Z}_{34}$ | $:$ | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=\frac{1}{2} e_{5}$ | $e_{2} e_{1}=-\frac{1}{2} e_{4}$ | $e_{2} e_{2}=e_{3}$ |
|  |  | $e_{2} e_{3}=e_{5}$ | $e_{2} e_{4}=-e_{5}$ | $e_{3} e_{1}=-\frac{1}{2} e_{5}$ | $e_{3} e_{2}=2 e_{5}$ | $e_{4} e_{2}=e_{5}$ |

1.2.8. Central extensions of $\mathfrak{N}_{14}^{-1}$. Let us use the following notations:

$$
\nabla_{1}=\left[\Delta_{11}\right], \quad \nabla_{2}=\left[\Delta_{14}\right], \quad \nabla_{3}=\left[\Delta_{21}\right], \quad \nabla_{4}=\left[\Delta_{24}\right], \quad \nabla_{5}=\left[\Delta_{23}+2 \Delta_{32}\right] .
$$

Take $\theta=\sum_{i=1}^{5} \alpha_{i} \nabla_{i} \in \mathrm{H}^{2}\left(\mathfrak{N}_{14}^{-1}\right)$. The automorphism group of $\mathfrak{N}_{14}^{-1}$ consists of invertible matrices of the form

$$
\phi=\left(\begin{array}{cccc}
x & y & 0 & 0 \\
0 & z & 0 & 0 \\
t & u & z^{2} & 0 \\
v & w & 0 & x z
\end{array}\right) .
$$

Since

$$
\phi^{T}\left(\begin{array}{cccc}
\alpha_{1} & 0 & 0 & \alpha_{2} \\
\alpha_{3} & 0 & \alpha_{5} & \alpha_{4} \\
0 & 2 \alpha_{5} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \phi=\left(\begin{array}{cccc}
\alpha_{1}^{*} & \beta^{*} & 0 & \alpha_{2}^{*} \\
\alpha_{3}^{*}-\beta^{*} & \gamma^{*} & \alpha_{5}^{*} & \alpha_{4}^{*} \\
0 & 2 \alpha_{5}^{*} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right),
$$

we have that the action of $\operatorname{Aut}\left(\mathfrak{N}_{14}^{-1}\right)$ on the subspace $\left\langle\sum_{i=1}^{5} \alpha_{i} \nabla_{i}\right\rangle$ is given by $\left\langle\sum_{i=1}^{5} \alpha_{i}^{*} \nabla_{i}\right\rangle$, where

$$
\begin{aligned}
& \alpha_{1}^{*}=\left(\alpha_{1} x+\alpha_{2} v\right) x, \\
& \alpha_{2}^{*}=\alpha_{2} x^{2} z, \\
& \alpha_{3}^{*}=2 \alpha_{1} x y+\alpha_{2} v y+\alpha_{2} w x+\alpha_{3} x z+\alpha_{4} v z+3 \alpha_{5} t z, \\
& \alpha_{4}^{*}=\left(\alpha_{2} y+\alpha_{4} z\right) x z, \\
& \alpha_{5}^{*}=\alpha_{5} z^{3} .
\end{aligned}
$$

Since $\mathrm{H}^{2}\left(\mathfrak{N}_{14}^{-1}\right)=\left\langle\left[\Delta_{11}\right],\left[\Delta_{21}\right],\left[\Delta_{14}\right],\left[\Delta_{24}\right],\left[\Delta_{23}+2 \Delta_{32}\right]\right\rangle$ and we are interested only in new algebras, we must have $\alpha_{5} \neq 0$ and $\left(\alpha_{2}, \alpha_{4}\right) \neq(0,0)$. Then:
(1) If $\alpha_{2} \neq 0$, then by choosing $x=\sqrt{\alpha_{2} \alpha_{5}}, y=-\alpha_{4}, z=\alpha_{2}, v=-\frac{\alpha_{1} \sqrt{\alpha_{5}}}{\sqrt{\alpha_{2}}}$ and $w=\frac{2 \alpha_{1} \alpha_{4}-\alpha_{2} \alpha_{3}}{\alpha_{2}}$, we have the representative $\left\langle\nabla_{2}+\nabla_{5}\right\rangle$.
(2) If $\alpha_{2}=0$ and $\alpha_{4} \alpha_{1} \neq 0$, then by choosing $x=\frac{\alpha_{1} \alpha_{5}^{2}}{\alpha_{4}^{3}}, z=\frac{\alpha_{1} \alpha_{5}}{\alpha_{4}^{2}}$ and $t=-\frac{\alpha_{1} \alpha_{3} \alpha_{5}}{3 \alpha_{4}^{2}}$, we have the representative $\left\langle\nabla_{1}+\nabla_{4}+\nabla_{5}\right\rangle$.
(3) If $\alpha_{4} \neq 0$ and $\alpha_{2}=\alpha_{1}=0$, then by choosing $x=\alpha_{5}, z=\alpha_{4}$ and $t=-\frac{\alpha_{3}}{3}$, we have the representative $\left\langle\nabla_{4}+\nabla_{5}\right\rangle$.
Summarizing, we have the following distinct orbits:

$$
\left\langle\nabla_{2}+\nabla_{5}\right\rangle,\left\langle\nabla_{1}+\nabla_{4}+\nabla_{5}\right\rangle,\left\langle\nabla_{4}+\nabla_{5}\right\rangle,
$$

which give the following new algebras:

$$
\begin{array}{llllllll}
\mathcal{Z}_{35} & : & e_{1} e_{2}=e_{4} & e_{1} e_{4}=e_{5} & e_{2} e_{1}=-e_{4} & e_{2} e_{2}=e_{3} & e_{2} e_{3}=e_{5} & e_{3} e_{2}=2 e_{5} \\
\hline \mathcal{Z}_{36} & : & e_{1} e_{1}=e_{5} & e_{1} e_{2}=e_{4} & e_{2} e_{1}=-e_{4} & e_{2} e_{2}=e_{3} & e_{2} e_{3}=e_{5} & e_{2} e_{4}=e_{5}
\end{array} e_{3} e_{2}=2 e_{5}, ~\left(\mathcal{Z}_{37}: e_{1} e_{2}=e_{4} \quad e_{2} e_{1}=-e_{4} \quad e_{2} e_{2}=e_{3} \quad e_{2} e_{3}=e_{5} \quad e_{2} e_{4}=e_{5} \quad e_{3} e_{2}=2 e_{5}\right)
$$

1.2.9. Central extensions of $\mathfrak{Z}_{1}$. Let us use the following notations:

$$
\nabla_{1}=\left[\Delta_{13}+3 \Delta_{22}+3 \Delta_{31}\right], \quad \nabla_{2}=\left[\Delta_{14}\right], \quad \nabla_{3}=\left[\Delta_{41}\right], \quad \nabla_{4}=\left[\Delta_{44}\right] .
$$

Take $\theta=\sum_{i=1}^{4} \alpha_{i} \nabla_{i} \in \mathrm{H}^{2}\left(\mathfrak{Z}_{1}\right)$. The automorphism group of $\mathfrak{Z}_{1}$ is generated by invertible matrices of the form

$$
\phi=\left(\begin{array}{cccc}
x & 0 & 0 & 0 \\
y & x^{2} & 0 & 0 \\
z & 3 x y & x^{3} & t \\
u & 0 & 0 & v
\end{array}\right) .
$$

Since

$$
\phi^{T}\left(\begin{array}{cccc}
0 & 0 & \alpha_{1} & \alpha_{2} \\
0 & 3 \alpha_{1} & 0 & 0 \\
3 \alpha_{1} & 0 & 0 & 0 \\
\alpha_{3} & 0 & 0 & \alpha_{4}
\end{array}\right) \phi=\left(\begin{array}{cccc}
\alpha^{*} & \beta^{*} & \alpha_{1}^{*} & \alpha_{2}^{*} \\
2 \beta^{*} & 3 \alpha_{1}^{*} & 0 & 0 \\
3 \alpha_{1}^{*} & 0 & 0 & 0 \\
\alpha_{3}^{*} & 0 & 0 & \alpha_{4}^{*}
\end{array}\right),
$$

we have that the action of $\operatorname{Aut}\left(\mathfrak{Z}_{1}\right)$ on the subspace $\left\langle\sum_{i=1}^{4} \alpha_{i} \nabla_{i}\right\rangle$ is given by $\left\langle\sum_{i=1}^{4} \alpha_{i}^{*} \nabla_{i}\right\rangle$, where

$$
\begin{array}{ll}
\alpha_{1}^{*}=\alpha_{1} x^{4}, & \alpha_{2}^{*}=\alpha_{1} x t+\left(\alpha_{2} x+\alpha_{4} u\right) v, \\
\alpha_{3}^{*}=\left(3 \alpha_{1} t+\alpha_{3} v\right) x+\alpha_{4} v u, & \alpha_{4}^{*}=\alpha_{4} v^{2} .
\end{array}
$$

Since $\mathrm{H}^{2}\left(\mathfrak{Z}_{1}\right)=\left\langle\left[\Delta_{13}+3 \Delta_{22}+3 \Delta_{31}\right],\left[\Delta_{14}\right],\left[\Delta_{41}\right],\left[\Delta_{44}\right]\right\rangle$ and we are interested only in new algebras, we must have $\alpha_{1} \neq 0$ and $\left(\alpha_{2}, \alpha_{3}, \alpha_{4}\right) \neq(0,0,0)$. Then
(1) If $\alpha_{4} \neq 0$, then by taking $x=1, t=\frac{\alpha_{2}-\alpha_{3}}{2 \sqrt{\alpha_{1} \alpha_{4}}}, u=\frac{\alpha_{3}-3 \alpha_{2}}{2 \alpha_{4}}$ and $v=\sqrt{\frac{\alpha_{1}}{\alpha_{4}}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{4}\right\rangle$.
(2) If $\alpha_{4}=0$ and $\alpha_{3} \neq 3 \alpha_{2}$, then by taking $x=1, t=\frac{\alpha_{3}}{\alpha_{3}-3 \alpha_{2}}$ and $v=\frac{3 \alpha_{1}}{3 \alpha_{2}-\alpha_{3}}$, we obtain the representative $\left\langle\nabla_{1}+\nabla_{2}\right\rangle$.
(3) If $\alpha_{4}=0$ and $\alpha_{3}=3 \alpha_{2}$, then by taking $x=v=1$ and $t=-\frac{\alpha_{2}}{\alpha_{1}}$, then we obtain the representative $\left\langle\nabla_{1}\right\rangle$, which does not satisfy condition $\left(\alpha_{2}, \alpha_{3}, \alpha_{4}\right) \neq 0$. So we will not consider it.
Summarizing, we have the following distinct orbits:

$$
\left\langle\nabla_{1}+\nabla_{4}\right\rangle,\left\langle\nabla_{1}+\nabla_{2}\right\rangle
$$

which give the following new algebras:

$$
\begin{array}{lllllll}
\mathcal{Z}_{38} & : & e_{1} e_{1}=e_{2} & e_{1} e_{2}=e_{3} & e_{1} e_{3}=e_{5} & e_{2} e_{1}=2 e_{3} & e_{2} e_{2}=3 e_{5}
\end{array} e_{3} e_{1}=3 e_{5} \quad e_{4} e_{4}=e_{5}, ~\left(\mathcal{Z}_{39}: e_{1} e_{1}=e_{2} \quad e_{1} e_{2}=e_{3} \quad e_{1} e_{3}=e_{5} \quad e_{1} e_{4}=e_{5} \quad e_{2} e_{1}=2 e_{3} \quad e_{2} e_{2}=3 e_{5} \quad e_{3} e_{1}=3 e_{5}\right.
$$

1.3. Classification theorem for 5-dimensional Zinbiel algebras. Thanks to [23] each complex finite-dimensional Zinbiel algebra is nilpotent. Hence, the algebraic classification of complex 5dimensional Zinbiel algebras consists of two parts:
(1) 5-dimensional algebras with identity $x y z=0$ (also known as 2 -step nilpotent algebras) is the intersection of all varieties of algebras defined by a family of polynomial identities of degree three or more; for example, it is in intersection of associative, Zinbiel, Leibniz, etc, algebras. All these algebras can be obtained as central extensions of zero-product algebras. The geometric classification of 2 -step nilpotent algebras is given in [33]. It is the reason why we are not interested in it.
(2) 5-dimensional Zinbiel (non-2-step nilpotent) algebras, which are central extensions of Zinbiel algebras with nonzero product of a smaller dimension. These algebras are classified by several steps:
(a) split complex 5-dimensional Zinbiel algebras are classified in [23, 35];
(b) non-split complex 5-dimensional Zinbiel algebras with 2-dimensional annihilator are classified in [35];
(c) non-split complex 5-dimensional Zinbiel algebras with 1-dimensional annihilator are classified in Theorem A (see below).

Theorem A. Let Z be a non-split complex 5-dimensional Zinbiel (non-2-step nilpotent) algebra. Then Z is isomorphic to one algebra from the following list:

$$
\begin{array}{llllll}
z_{01} & : e_{1} e_{1}=e_{2} & e_{1} e_{2}=e_{5} & e_{1} e_{3}=e_{5} & e_{2} e_{1}=2 e_{5} & e_{4} e_{4}=e_{5} \\
Z_{02}^{0} & : e_{1} e_{1}=e_{2} & e_{1} e_{2}=e_{5} & e_{2} e_{1}=2 e_{5} & e_{3} e_{4}=e_{5} & e_{4} e_{3}=\alpha e_{5} \\
z_{03} & : e_{1} e_{1}=e_{2} & e_{1} e_{2}=e_{5} & e_{2} e_{1}=2 e_{5} & & \\
& & e_{3} e_{3}=e_{5} & e_{3} e_{4}=e_{5} & e_{4} e_{3}=-e_{5} & \\
z_{04} & : e_{1} e_{1}=e_{2} & e_{1} e_{2}=e_{5} & e_{1} e_{4}=e_{5} & & \\
& e_{2} e_{1}=2 e_{5} & e_{3} e_{4}=e_{5} & e_{4} e_{3}=2 e_{5} & &
\end{array}
$$

| $z_{05}$ | : $e_{1} e_{1}=e_{3}$ |  | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{2}=e_{4}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $e_{2} e_{4}=e_{5}$ | $e_{3} e_{1}=2 e_{5}$ | $e_{4} e_{2}=2 e_{5}$ |  |  |
| $z_{06}$ | : | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ |  |
| $z_{07}$ | : | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{4} e_{1}=e_{5}$ |  |
| $z_{08}$ |  | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{5}$ | $e_{4} e_{1}=e_{5}$ |
| $z_{09}$ | : | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{1}=e_{5}$ |
| $z_{10}^{\alpha}$ | : | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{4}=\alpha e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{1}=e_{5}$ |
| $z_{11}$ | : | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{4}=-e_{5}$ |  |  |
|  |  | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{1}=e_{5}$ |  |  |
| $z_{12}$ | : | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{4}=e_{5}$ |  |
| $z_{13}$ |  | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{4}=e_{5}$ |
| $z_{14}^{\alpha}$ |  | $e_{1} e_{1}=\alpha e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{4}=e_{5}$ |  |  |
|  |  | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{4} e_{4}=e_{5}$ |  |  |
| $z_{15}$ | : | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{4} e_{3}=e_{5}$ |  |  |
| $z_{16}$ | : | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{4} e_{3}=e_{5}$ |  |
| $z_{17}$ |  | $e_{1} e_{2}=e_{3}+e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{4} e_{3}=e_{5}$ |  |  |
| $z_{18}$ |  | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{3}=e_{5}$ |  |
| $z_{19}$ |  | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{3}=e_{5}$ |
| $z_{20}$ |  | $e_{1} e_{2}=e_{3}+e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{3}=e_{5}$ |  |
| $z_{21}$ |  | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{5}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{3}=e_{5}$ |
| $z_{22}$ | : | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{3}$ | $e_{2} e_{1}=-e_{3}$ |  |  |
|  |  | $e_{2} e_{2}=e_{5}$ | $e_{2} e_{4}=e_{5}$ | $e_{4} e_{3}=e_{5}$ |  |  |
| $z_{23}$ | : | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{1} e_{4}=-e_{5}$ | $e_{2} e_{1}=e_{4}$ |  |
|  |  | $e_{2} e_{2}=-e_{3}$ | $e_{2} e_{3}=-e_{5}$ | $e_{2} e_{4}=e_{5}$ | $e_{3} e_{2}=-2 e_{5}$ |  |
| $z_{24}$ | : | $e_{1} e_{1}=e_{3}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{4}=-e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=-e_{4}$ |
|  |  | $e_{2} e_{4}=e_{5}$ | $e_{3} e_{2}=-e_{5}$ | $e_{4} e_{1}=-e_{5}$ | $e_{4} e_{2}=2 e_{5}$ |  |
| $z_{25}$ |  | $e_{1} e_{1}=e_{3}+e_{5}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=-e_{5}$ | $e_{1} e_{4}=e_{5}$ |  |
|  |  | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=-e_{4}$ | $e_{2} e_{3}=e_{5}$ | $e_{2} e_{4}=-e_{5}$ |  |
|  |  | $e_{3} e_{1}=-2 e_{5}$ | $e_{3} e_{2}=2 e_{5}$ | $e_{4} e_{1}=2 e_{5}$ | $e_{4} e_{2}=-2 e_{5}$ |  |
| $z_{26}$ |  | $e_{1} e_{1}=e_{3}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=-e_{5}$ | $e_{1} e_{4}=e_{5}$ |  |
|  |  | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=-e_{4}$ | $e_{2} e_{3}=e_{5}$ | $e_{2} e_{4}=-e_{5}$ |  |
|  |  | $e_{3} e_{1}=-2 e_{5}$ | $e_{3} e_{2}=2 e_{5}$ | $e_{4} e_{1}=2 e_{5}$ | $e_{4} e_{2}=-2 e_{5}$ |  |
| $z_{27}$ | : | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=-e_{5}$ | $e_{1} e_{4}=e_{5}$ |  |  |
|  |  | $e_{2} e_{1}=e_{4}$ | $e_{2} e_{3}=-e_{5}$ | $e_{2} e_{4}=e_{5}$ |  |  |
| $z_{28}$ |  | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=-e_{5}$ | $e_{1} e_{4}=e_{5}$ | $e_{2} e_{1}=e_{4}$ | $e_{2} e_{2}=e_{5}$ |
| $z_{29}$ |  | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=-e_{5}$ | $e_{1} e_{4}=e_{5}$ | $e_{2} e_{1}=e_{4}$ |  |
| $z_{30}^{\alpha \neq-1}$ |  | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=(\alpha+$ | $e_{2} e_{1}=\alpha e_{4}$ | $e_{2} e_{2}=e_{3}$ |  |
|  |  | $e_{2} e_{4}=2 \alpha e_{5}$ | $e_{3} e_{1}=2 \alpha(\alpha$ |  | $e_{4} e_{2}=2(\alpha+$ |  |
| $z_{31}$ |  | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=e_{5}$ | $e_{2} e_{2}=e_{3}$ | $e_{4} e_{2}=2 e_{5}$ |
| $z_{32}$ |  | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=\frac{1}{2} e_{5}$ | $e_{2} e_{1}=-\frac{1}{2} e_{4}$ |  |


| $z_{33}$ | $\begin{aligned} & e_{2} e_{2}=e_{3} \\ & e_{1} e_{2}=e_{4} \end{aligned}$ | $\begin{aligned} & e_{2} e_{4}=-e_{5} \\ & e_{1} e_{3}=\frac{1}{2} e_{5} \end{aligned}$ | $e_{3} e_{1}=-\frac{1}{2} e_{5}$ | $e_{4} e_{2}=e_{5}$ | $e_{2} e_{3}=e_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $e_{2} e_{4}=-e_{5}$ | $e_{3} e_{1}=-\frac{1}{2} e_{5}$ | $e_{3} e_{2}=2 e_{5}$ | $e_{4} e_{2}=e_{5}$ |  |
| $z_{34}$ | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=\frac{1}{2} e_{5}$ | $e_{2} e_{1}=-\frac{1}{2} e_{4}$ | $e_{2} e_{2}=e_{3}$ |
|  | $e_{2} e_{3}=e_{5}$ | $e_{2} e_{4}=-e_{5}$ | $e_{3} e_{1}=-\frac{1}{2} e_{5}$ | $e_{3} e_{2}=2 e_{5}$ | $e_{4} e_{2}=e_{5}$ |
| $z_{35}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{4}=e_{5}$ | $e_{2} e_{1}=-e_{4}$ |  |  |
|  | $e_{2} e_{2}=e_{3}$ | $e_{2} e_{3}=e_{5}$ | $e_{3} e_{2}=2 e_{5}$ |  |  |
| $z_{36}$ | $e_{1} e_{1}=e_{5}$ | $e_{1} e_{2}=e_{4}$ | $e_{2} e_{1}=-e_{4}$ | $e_{2} e_{2}=e_{3}$ |  |
|  | $e_{2} e_{3}=e_{5}$ | $e_{2} e_{4}=e_{5}$ | $e_{3} e_{2}=2 e_{5}$ |  |  |
| $z_{37}$ | $e_{1} e_{2}=e_{4}$ | $e_{2} e_{1}=-e_{4}$ | $e_{2} e_{2}=e_{3}$ |  |  |
|  | $e_{2} e_{3}=e_{5}$ | $e_{2} e_{4}=e_{5}$ | $e_{3} e_{2}=2 e_{5}$ |  |  |
| $z_{38}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=2 e_{3}$ |  |
|  | $e_{2} e_{2}=3 e_{5}$ | $e_{3} e_{1}=3 e_{5}$ | $e_{4} e_{4}=e_{5}$ |  |  |
| $z_{39}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{1} e_{4}=e_{5}$ |  |
|  | $e_{2} e_{1}=2 e_{3}$ | $e_{2} e_{2}=3 e_{5}$ | $e_{3} e_{1}=3 e_{5}$ |  |  |
| $z_{40}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=\frac{1}{2} e_{3}$ | $e_{1} e_{3}=2 e_{4}$ | $e_{1} e_{4}=e_{5}$ | $e_{2} e_{1}=e_{3}$ |
|  | $e_{2} e_{2}=3 e_{4}$ | $e_{2} e_{3}=8 e_{5}$ | $e_{3} e_{1}=6 e_{4}$ | $e_{3} e_{2}=12 e_{5}$ | $e_{4} e_{1}=4 e_{5}$ |
| $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{01}^{2}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=2 e_{4}$ |  |
| $\left[\mathfrak{N}_{1}^{\text {C }}\right]_{02}^{2, \alpha}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=\alpha e_{5}$ | $e_{2} e_{1}=2 e_{4}$ | $e_{3} e_{1}=e_{5}$ |
| $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{03}^{2}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=2 e_{4}$ | $e_{3} e_{3}=e_{5}$ |
| $\left[\mathfrak{N}_{1}^{C}\right]_{04}^{2}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{4}$ | $e_{2} e_{1}=2 e_{4}$ | $e_{3} e_{3}=e_{5}$ |  |
| $\left[\mathfrak{N}_{1}^{\mathrm{C}}\right]_{05}^{2}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=e_{4}$ | $e_{2} e_{1}=2 e_{4}$ | $e_{3} e_{3}=e_{5}$ |
| $\left[\mathfrak{N}_{1}^{C}\right]_{06}^{2}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=e_{4}+e_{5}$ | $e_{2} e_{1}=2 e_{4}$ | $e_{3} e_{3}=e_{5}$ |
| $\left[\mathfrak{N}_{1}^{C}\right]_{07}^{2}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=2 e_{4}$ | $e_{3} e_{1}=e_{4}+2 e_{5}$ |
| $\left[\mathfrak{N}_{1}^{\mathrm{C}}\right]_{08}^{2}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=2 e_{4}$ | $e_{3} e_{3}=e_{4}$ |
| $\left[\mathfrak{N}_{1}^{\mathrm{C}}\right]_{09}^{2, \alpha}$ | $e_{1} e_{1}=e_{2}$ | $e_{1} e_{2}=e_{4}$ | $e_{1} e_{3}=\alpha e_{5}$ |  |  |
|  | $e_{2} e_{1}=2 e_{4}$ | $e_{3} e_{1}=e_{5}$ | $e_{3} e_{3}=e_{4}$ |  |  |
| $\left[\mathfrak{N}_{1}\right]_{01}^{2}$ | $e_{1} e_{1}=e_{4}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ |  |
| $\left[\mathfrak{N} \mathfrak{N}_{1}\right]_{02}^{2}$ | $e_{1} e_{2}=e_{3}+e_{4}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ |  |  |
| $\left[\mathfrak{N} \mathfrak{N}_{1}\right]_{03}^{2}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{4}$ |  |
| $[\mathfrak{N}]_{04}^{2}$ | $e_{1} e_{1}=e_{4}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{4}$ |
| $\left[\mathfrak{N} \mathfrak{N}_{1}\right]_{05}^{2}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{4}$ |  |
| $\left[\mathfrak{N} \mathfrak{N}_{1}\right]_{06}^{2}$ | $e_{1} e_{1}=e_{4}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{4}$ |
| $\left.\mathfrak{N}_{1}\right]_{07}^{2}$ | $e_{1} e_{2}=e_{3}+e_{4}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{3}=e_{4}$ |  |
| $\left[\mathfrak{N}_{1}\right]_{08}^{2}$ | $e_{1} e_{1}=e_{4}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ |  |  |
|  | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{4}$ | $e_{2} e_{3}=e_{4}$ |  |  |
| $\left[\mathfrak{N}_{1}\right]_{09}^{2}$ | $e_{1} e_{1}=e_{4}$ | $e_{1} e_{2}=e_{3}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{5}$ |
| $[\mathfrak{N}]_{10}^{2}$ | $e_{1} e_{2}=e_{3}+e_{4}$ | $e_{1} e_{3}=e_{5}$ | $e_{2} e_{1}=-e_{3}$ | $e_{2} e_{2}=e_{5}$ |  |

All of these algebras are pairwise non-isomorphic, except for the following: $\mathcal{Z}_{02}^{\alpha} \cong \mathcal{Z}_{02}^{\alpha-1}$.

## 2. The geometric classification of Zinbiel algebras

2.1. Definitions and notation. Given an $n$-dimensional vector space $\mathbb{V}$, the set $\operatorname{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V}) \cong$ $\mathbb{V}^{*} \otimes \mathbb{V}^{*} \otimes \mathbb{V}$ is a vector space of dimension $n^{3}$. This space has the structure of the affine variety $\mathbb{C}^{n^{3}}$. Indeed, let us fix a basis $e_{1}, \ldots, e_{n}$ of $\mathbb{V}$. Then any $\mu \in \operatorname{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is determined by $n^{3}$ structure constants $c_{i j}^{k} \in \mathbb{C}$ such that $\mu\left(e_{i} \otimes e_{j}\right)=\sum_{k=1}^{n} c_{i j}^{k} e_{k}$. A subset of $\operatorname{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is Zariski-closed if it can be defined by a set of polynomial equations in the variables $c_{i j}^{k}(1 \leq i, j, k \leq n)$.

Let $T$ be a set of polynomial identities. The set of algebra structures on $\mathbb{V}$ satisfying polynomial identities from $T$ forms a Zariski-closed subset of the variety $\operatorname{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$. We denote this subset by $\mathbb{L}(T)$. The general linear group $\mathrm{GL}(\mathbb{V})$ acts on $\mathbb{L}(T)$ by conjugations:

$$
(g * \mu)(x \otimes y)=g \mu\left(g^{-1} x \otimes g^{-1} y\right)
$$

for $x, y \in \mathbb{V}, \mu \in \mathbb{L}(T) \subset \operatorname{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ and $g \in \operatorname{GL}(\mathbb{V})$. Thus, $\mathbb{L}(T)$ is decomposed into $\mathrm{GL}(\mathbb{V})$ orbits that correspond to the isomorphism classes of algebras. Let $O(\mu)$ denote the orbit of $\mu \in \mathbb{L}(T)$ under the action of $\mathrm{GL}(\mathbb{V})$ and $\overline{O(\mu)}$ denote the Zariski closure of $O(\mu)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $n$-dimensional algebras satisfying the identities from $T$, and let $\mu, \lambda \in \mathbb{L}(T)$ represent $\mathcal{A}$ and $\mathcal{B}$, respectively. We say that $\mathcal{A}$ degenerates to $\mathcal{B}$ and write $\mathcal{A} \rightarrow \mathcal{B}$ if $\lambda \in \overline{O(\mu)}$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of $\mu$ and $\lambda$. If $\mathcal{A} \neq \mathcal{B}$, then the assertion $\mathcal{A} \rightarrow \mathcal{B}$ is called a proper degeneration. We write $\mathcal{A} \nrightarrow \mathcal{B}$ if $\lambda \notin \overline{O(\mu)}$.

Let $\mathcal{A}$ be represented by $\mu \in \mathbb{L}(T)$. Then $\mathcal{A}$ is rigid in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an irreducible component. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra $\mathcal{A}$ is rigid in $\mathbb{L}(T)$ if and only if $\overline{O(\mu)}$ is an irreducible component of $\mathbb{L}(T)$.

Given the spaces $U$ and $W$, we write simply $U>W$ instead of $\operatorname{dim} U>\operatorname{dim} W$.
2.2. Method of the description of degenerations of algebras. In the present work we use the methods applied to Lie algebras in [30,31,46]. First of all, if $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, then $\mathfrak{D e r}(\mathcal{A})<\mathfrak{D e r}(\mathcal{B})$, where $\mathfrak{D e r}(\mathcal{A})$ is the Lie algebra of derivations of $\mathcal{A}$. We compute the dimensions of algebras of derivations and check the assertion $\mathcal{A} \rightarrow \mathcal{B}$ only for $\mathcal{A}$ and $\mathcal{B}$ such that $\mathfrak{D e r}(\mathcal{A})<\mathfrak{D e r}(\mathcal{B})$.

To prove degenerations, we construct families of matrices parametrized by $t$. Namely, let $\mathcal{A}$ and $\mathcal{B}$ be two algebras represented by the structures $\mu$ and $\lambda$ from $\mathbb{L}(T)$ respectively. Let $e_{1}, \ldots, e_{n}$ be a basis of $\mathbb{V}$ and $c_{i j}^{k}(1 \leq i, j, k \leq n)$ be the structure constants of $\lambda$ in this basis. If there exist $a_{i}^{j}(t) \in \mathbb{C}$ $\left(1 \leq i, j \leq n, t \in \mathbb{C}^{*}\right)$ such that $E_{i}^{t}=\sum_{j=1}^{n} a_{i}^{j}(t) e_{j}(1 \leq i \leq n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^{*}$, and the structure constants of $\mu$ in the basis $E_{1}^{t}, \ldots, E_{n}^{t}$ are such rational functions $c_{i j}^{k}(t) \in \mathbb{C}[t]$ that $c_{i j}^{k}(0)=c_{i j}^{k}$, then $\mathcal{A} \rightarrow \mathcal{B}$. In this case $E_{1}^{t}, \ldots, E_{n}^{t}$ is called a parametrized basis for $\mathcal{A} \rightarrow \mathcal{B}$. To
simplify our equations, we will use the notation $A_{i}=\left\langle e_{i}, \ldots, e_{n}\right\rangle, i=1, \ldots, n$ and write simply $A_{p} A_{q} \subset A_{r}$ instead of $c_{i j}^{k}=0(i \geq p, j \geq q, k<r)$.

Since the variety of 5 -dimensional Zinbiel algebras contains infinitely many non-isomorphic algebras, we have to do some additional work. Let $\mathcal{A}(*):=\{\mathcal{A}(\alpha)\}_{\alpha \in I}$ be a series of algebras, and let $\mathcal{B}$ be another algebra. Suppose that for $\alpha \in I, \mathcal{A}(\alpha)$ is represented by the structure $\mu(\alpha) \in \mathbb{L}(T)$ and $B \in \mathbb{L}(T)$ is represented by the structure $\lambda$. Then we say that $\mathcal{A}(*) \rightarrow \mathcal{B}$ if $\lambda \in \overline{\{O(\mu(\alpha))\}_{\alpha \in I}}$, and $\mathcal{A}(*) \nrightarrow \mathcal{B}$ if $\lambda \notin \overline{\{O(\mu(\alpha))\}_{\alpha \in I}}$.

Let $\mathcal{A}(*), \mathcal{B}, \mu(\alpha)(\alpha \in I)$ and $\lambda$ be as above. To prove $\mathcal{A}(*) \rightarrow \mathcal{B}$ it is enough to construct a family of pairs $(f(t), g(t))$ parametrized by $t \in \mathbb{C}^{*}$, where $f(t) \in I$ and $g(t) \in \mathrm{GL}(\mathbb{V})$. Namely, let $e_{1}, \ldots, e_{n}$ be a basis of $\mathbb{V}$ and $c_{i j}^{k}(1 \leq i, j, k \leq n)$ be the structure constants of $\lambda$ in this basis. If we construct $a_{i}^{j}: \mathbb{C}^{*} \rightarrow \mathbb{C}(1 \leq i, j \leq n)$ and $f: \mathbb{C}^{*} \rightarrow I$ such that $E_{i}^{t}=\sum_{j=1}^{n} a_{i}^{j}(t) e_{j}(1 \leq i \leq n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^{*}$, and the structure constants of $\mu_{f(t)}$ in the basis $E_{1}^{t}, \ldots, E_{n}^{t}$ are such rational functions $c_{i j}^{k}(t) \in \mathbb{C}[t]$ that $c_{i j}^{k}(0)=c_{i j}^{k}$, then $\mathcal{A}(*) \rightarrow \mathcal{B}$. In this case $E_{1}^{t}, \ldots, E_{n}^{t}$ and $f(t)$ are called a parametrized basis and a parametrized index for $\mathcal{A}(*) \rightarrow \mathcal{B}$, respectively.

We now explain how to prove $\mathcal{A}(*) \nrightarrow \mathcal{B}$. Note that if $\mathfrak{D e r} \mathcal{A}(\alpha)>\mathfrak{D e r} \mathcal{B}$ for all $\alpha \in I$ then $\mathcal{A}(*) \nrightarrow \mathcal{B}$. One can also use the following Lemma, whose proof is the same as the proof of Lemma 1.5 from [30].

Lemma 8. Let $\mathfrak{B}$ be a Borel subgroup of $\mathrm{GL}(\mathbb{V})$ and $\mathcal{R} \subset \mathbb{L}(T)$ be a $\mathfrak{B}$-stable closed subset. If $\mathcal{A}(*) \rightarrow \mathcal{B}$ and for any $\alpha \in I$ the algebra $\mathcal{A}(\alpha)$ can be represented by a structure $\mu(\alpha) \in \mathcal{R}$, then there is $\lambda \in \mathcal{R}$ representing $\mathcal{B}$.
2.3. The geometric classification of 5 -dimensional Zinbiel algebras. The main result of the present section is the following theorem.

Theorem B. The variety of complex 5-dimensional Zinbiel algebras has dimension 24 and it has 16 irreducible components defined by

$$
\begin{aligned}
& \mathcal{C}_{1}=\overline{\left\{\mathcal{O}\left(\mathfrak{V}_{4+1}\right)\right\}}, \quad \mathfrak{C}_{2}=\overline{\left\{\mathcal{O}\left(\mathfrak{V}_{3+2}\right)\right\}}, \quad \mathfrak{C}_{3}=\overline{\mathcal{O}\left(\left[\mathfrak{N}_{1}\right]_{08}^{2}\right)}, \quad \mathcal{C}_{4}=\overline{\mathcal{O}\left(\left[\mathfrak{N}_{1}^{\mathrm{C}}\right]_{06}^{2}\right)}, \quad \mathcal{C}_{5}=\overline{\left\{\mathcal{O}\left(\mathcal{Z}_{02}^{\alpha}\right)\right\}},
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{C}_{12}=\overline{\left\{\mathcal{O}\left(Z_{30}^{\alpha}\right)\right\}}, \quad \mathcal{C}_{13}=\overline{\mathcal{O}\left(\mathcal{Z}_{34}\right)}, \quad \mathfrak{C}_{14}=\overline{\mathcal{O}\left(\mathcal{Z}_{35}\right)}, \quad \mathfrak{C}_{15}=\overline{\mathcal{O}\left(Z_{38}\right)}, \quad \mathfrak{C}_{16}=\overline{\mathcal{O}\left(\mathcal{Z}_{40}\right)} .
\end{aligned}
$$

In particular, the variety of complex 5 -dimensional Zinbiel algebras has 11 rigid algebras

$$
\left[\mathfrak{N}_{1}\right]_{08}^{2},\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{06}^{2}, z_{05}, z_{22}, z_{23}, z_{24}, z_{27}, z_{34}, z_{35}, z_{38} \text { and } z_{40} .
$$

Proof. Thanks to [33], the variety of 5-dimensional 2-step nilpotent algebras has only three irreducible components defined by

$$
\begin{array}{llllll}
\mathfrak{V}_{4+1} & : & e_{1} e_{2}=e_{5} & e_{2} e_{1}=\lambda e_{5} & e_{3} e_{4}=e_{5} & e_{4} e_{3}=\mu e_{5} \\
\mathfrak{V}_{3+2} & : & e_{1} e_{1}=e_{4} & e_{1} e_{2}=\mu_{1} e_{5} & e_{1} e_{3}=\mu_{2} e_{5} & e_{2} e_{1}=\mu_{3} e_{5}
\end{array} e_{2} e_{2}=\mu_{4} e_{5}
$$

Thanks to [33, 38], all 5-dimensional split Zinbiel algebras are in the orbit closure of the families $\mathfrak{V}_{4+1}$ and $\mathfrak{V}_{3+2}$, and the algebras $\left[\mathfrak{Z}_{1}\right]_{1}^{1},\left[\mathfrak{N}_{1}\right]_{01}^{1},\left[\mathfrak{N}_{1}^{\mathrm{C}}\right]_{01}^{1}$.

Let us give some useful degenerations for our proof.

| $\begin{aligned} & z_{04} \quad \rightarrow \mathcal{Z}_{01} \\ & E_{2}^{t}=e_{2}+\frac{2\left(3 t^{2}-1\right)}{9 t^{2}} e_{5} \\ & E_{4}^{t}=-\frac{1}{9 t} e_{2}+\frac{t}{3 t^{2}-1} e_{3}+\frac{3 t^{2}-1}{3 t} e_{4} \end{aligned}$ | $\begin{aligned} & E_{1}^{t}=e_{1}+\frac{2\left(3 t^{2}-1\right)}{9 t^{2}} e_{4} \\ & E_{3}^{t}=-\frac{1}{3} e_{2}+\frac{3 t^{2}}{3 t^{2}-1} e_{3} \\ & E_{5}^{t}=e_{5} \end{aligned}$ |
| :---: | :---: |
| $\mathcal{Z}_{02}^{t-1} \rightarrow Z_{03}$ | $E_{1}^{t}=e_{1}$ |
| $E_{2}^{t}=e_{2}$ | $E_{3}^{t}=e_{3}+\frac{1}{t} e_{4}$ |
| $E_{4}^{t}=e_{4}$ | $E_{5}^{t}=e_{5}$ |
| $\mathcal{Z}_{02}^{t+2} \rightarrow \mathcal{Z}_{04}$ | $E_{1}^{t}=e_{1}-\frac{2}{t} e_{3}$ |
| $E_{2}^{t}=e_{2}$ | $E_{3}^{t}=e_{3}$ |
| $E_{4}^{t}=\frac{t+2}{t} e_{2}+e_{4}$ | $E_{5}^{t}=e_{5}$ |
| $z_{22} \quad \rightarrow z_{07}$ | $E_{1}^{t}=e_{1}+\frac{\sqrt{4 t^{2}-5}}{2\left(t^{2}-1\right)} e_{3}+\frac{2}{\sqrt{4 t^{2}-5}} e_{4}$ |
| $E_{2}^{t}=\frac{t}{2\left(1-t^{2}\right)} e_{1}+\frac{t \sqrt{4 t^{2}-5}}{2\left(t^{2}-1\right)} e_{2}$ | $E_{3}^{t}=\frac{t \sqrt{4 t^{2}-5}}{2\left(t^{2}-1\right)} e_{3}-\frac{t}{2\left(t^{2}-1\right)} e_{5}$ |
| $E_{4}^{t}=\frac{2\left(1-t^{2}\right)}{\sqrt{4 t^{2}-5}} e_{4}$ | $E_{5}^{t}=\frac{2\left(t t^{2}-1\right)}{t^{2}-1} e_{5} \quad 2\left(t^{2}-1\right){ }^{\text {a }}$ |
| $Z_{22} \quad \rightarrow \quad z_{08}$ | $E_{1}^{t}=e_{1}+\frac{\sqrt{t\left(4 t^{2}-t-4\right)}}{2 t\left(t^{2}-1\right)} e_{3}+2 \sqrt{\frac{t}{4 t^{2}-t-4}} e_{4}$ |
| $E_{2}^{t}=\frac{t}{2\left(1-t^{2}\right)} e_{1}+\frac{\sqrt{t\left(4 t^{2}-t-4\right)}}{2\left(t^{2}-1\right)}$ | $E_{3}^{t}=\frac{\sqrt{t\left(4 t^{2}-t-4\right)}}{2\left(t^{2}-1\right)} e_{3}+\frac{t}{2\left(1-t^{2}\right)} e_{5}$ |
|  |  |
| $E_{4}^{t}=2 t \sqrt{\frac{t}{4 t^{2}-t-4}} e_{4}$ | $E_{5}^{t}=\frac{t}{t^{2}-1} e_{5}$ |
| $z_{10}^{-\frac{1}{t}} \quad \rightarrow Z_{09}$ | $E_{1}^{t}=e_{1}+e_{2}$ |
| $E_{2}^{t}=t e_{2}$ | $E_{3}^{t}=t e_{3}$ |
| $E_{4}^{t}=e_{3}+t e_{4}$ | $E_{5}^{t}=t e_{5}$ |
| $z^{\frac{t^{2}-\alpha}{(\alpha-1)^{2}}} \rightarrow z^{\alpha}$ | $E_{1}^{t}=\frac{(\alpha-1)^{2}}{} e_{1}+\frac{\alpha-1}{t} e_{4}$ |
| $z_{14} \quad \rightarrow z_{10}^{\alpha}$ | $\begin{aligned} & E_{1}^{t}=\frac{\alpha-1)}{t} e_{1}+\frac{\alpha-1}{t} e_{4} \\ & E^{t}-(\alpha-1)^{2} \end{aligned}$ |
| $E_{2}^{t}=e_{2}$ | $E_{3}^{t}=\frac{(\alpha-1)^{2}}{(\alpha-1)^{2}} e_{3}$ |
| $E_{4}^{t}=(\alpha-1) e_{4}$ | $E_{5}^{t}=\frac{(\alpha-1)^{2}}{t} e_{5}$ |
| $z_{14}^{\frac{t+1}{4}} \rightarrow z_{11}$ | $E_{1}^{t}=\frac{4}{t} e_{1}-\frac{2}{t} e_{4}$ |
| $E_{2}^{t}=e_{2}$ | $E_{3}^{t}=\frac{4}{t} e_{3}$ |
| $E_{4}^{t}=-2 e_{4}$ | $E_{5}^{t}=\frac{4}{t} e_{5}$ |
| $z_{22} \quad \rightarrow \quad z_{17}$ | $E_{1}^{t}=\frac{1}{t} e_{1}$ |
| $E_{2}^{t}=\frac{1}{2 t^{2}} e_{1}+\frac{\sqrt{4 t^{2}-1}}{2 t^{2}} e_{2}$ | $E_{3}^{t}=\frac{\sqrt{4 t^{2}-1}}{2 t^{3}} e_{3}-\frac{1}{2 t^{3}} e_{5}$ |
| $E_{4}^{t}=-2 \sqrt{4 t^{2}-1} e_{4}$ | $E_{5}^{t}=\frac{1}{t^{3}} e_{5}$ |
| $\mathcal{Z}_{22} \rightarrow \rightarrow \mathcal{Z}_{20}$ | $E_{1}^{t}=e_{1}$ |
| $E_{2}^{t}=\frac{1}{2 t} e_{1}+\frac{\sqrt{4 t^{2}-1}}{2 t} e_{2}$ | $E_{3}^{t}=\frac{\sqrt{4 t^{2}-1}}{2 t} e_{3}-\frac{1}{2 t} e_{5}$ |
| $E_{4}^{t}=\frac{2}{\sqrt{4 t^{2}-1}} e_{4}$ | $E_{5}^{t}=\frac{1}{t} e_{5}$ |
| $z_{24} \rightarrow z_{25}$ | $E_{1}^{t}=-(t+1) e_{2}-\frac{(t+1)(t+2)}{t} e_{3}-\frac{(t+1)}{t} e_{4}$ |



| $\left[\mathfrak{N}_{1}\right]_{10}^{2} \rightarrow\left[\mathfrak{N}_{1}\right]_{09}^{2}$ | $E_{1}^{t}=e_{1}+\frac{1}{t} e_{2}$ |
| :--- | :--- |
| $E_{2}^{t}=e_{2}-\frac{1}{t} e_{3}$ | $E_{3}^{t}=e_{3}-\frac{1}{t} e_{5}$ |
| $E_{4}^{t}=\frac{1}{t} e_{4}+\frac{1}{t^{2}} e_{5}$ | $E_{5}^{t}=e_{5}$ |
| $\left[\mathfrak{N}_{1}\right]_{08}^{2} \rightarrow\left[\mathfrak{N}_{1}\right]_{10}^{2}$ | $E_{1}^{t}=t e_{2}-t e_{3}$ |
| $E_{2}^{t}=-t^{2} e_{1}$ | $E_{3}^{t}=t^{3} e_{3}-t^{3} e_{5}$ |
| $E_{4}^{t}=t^{3} e_{5}$ | $E_{5}^{t}=t^{4} e_{4}$ |
| $\left[\mathfrak{N}_{1}\right]_{08}^{2} \rightarrow\left[\mathfrak{N}_{1}\right]_{01}^{1}$ | $E_{1}^{t}=t e_{1}$ |
| $E_{2}^{t}=\sqrt{t} e_{2}$ | $E_{3}^{t}=\sqrt{t^{3}} e_{3}$ |
| $E_{4}^{t}=t e_{4}$ | $E_{5}^{t}=-e_{4}+\sqrt{t^{3}} e_{5}$ |
| $\left[\mathfrak{N}_{1}^{C}\right]_{06}^{2} \rightarrow\left[\mathfrak{N}_{1}^{C}\right]_{01}^{1}$ | $E_{1}^{t}=\frac{1}{t-1} e_{1}$ |
| $E_{2}^{t}=\frac{1}{(t-1)^{2}} e_{2}$ | $E_{3}^{t}=\frac{1}{t-1} e_{3}$ |
| $E_{4}^{t}=\frac{1}{(t-1)^{3}} e_{4}$ | $E_{5}^{t}=-\frac{1}{t(t-1)^{3}} e_{4}+\frac{1}{t(t-1)^{2}} e_{5}$ |
| $Z_{40} \rightarrow \mathfrak{V}_{2+3}$ | $E_{1}^{t}=e_{1}-\frac{1}{t} e_{2}+\frac{3(2+9 \lambda)}{8 t^{2}} e_{3}-\frac{6(2+9 \lambda)}{5 t^{3}} e_{4}$ |
| $E_{2}^{t}=-\frac{3}{t} e_{2}$ | $E_{3}^{t}=-\frac{3}{2 t} e_{3}+\frac{9}{t^{2}} e_{4}-\frac{27(2+9 \lambda}{2 t^{3}} e_{5}$ |
| $E_{4}^{t}=-\frac{3}{t} e_{3}+\frac{9}{t^{2}} e_{4}-\frac{9(2+9 \lambda)}{t^{3}} e_{5}$ | $E_{5}^{t}=\frac{27}{t^{2}} e_{4}$ |

For the rest of degenerations, in case of $E_{1}^{t}, \ldots, E_{n}^{t}$ is a parametric basis for $\mathbf{A} \rightarrow \mathbf{B}$, it will be denote as $\mathbf{A} \xrightarrow{\left(E_{1}^{t}, \ldots, E_{n}^{t}\right)} \mathbf{B}$.

| $Z_{09}$ | $\xrightarrow{\left(t e_{1}, e_{2}, t e_{3}, t^{2} e_{4}, t^{2} e_{5}\right)}$ | $z_{06}$ | $z_{13}$ | $\xrightarrow{\left(t e_{1}, e_{2}, t e_{3}, t^{\frac{1}{2}} e_{4}, t e_{5}\right)}$ | $z_{12}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $z_{14}^{\frac{1}{t^{2}}}$ | $\xrightarrow{\left(t^{2} e_{1}, e_{2}, t^{2} e_{3}, t e_{4}, t^{2} e_{5}\right)}$ | $z_{13}$ | $Z_{16}$ | $\xrightarrow{\left(t e_{1}, e_{2}, t e_{3}, e_{4}, t e_{5}\right)}$ | $z_{15}$ |
| $z_{22}$ | $\xrightarrow{\left(t^{-1} e_{1}, t^{-\frac{1}{2}} e_{2}, t^{-\frac{3}{2}} e_{3}, t^{-\frac{1}{2}} e_{4}, t^{-2} e_{5}\right)}$ | $z_{16}$ | $Z_{22}$ | $\xrightarrow{\left(e_{1}, e_{2}, e_{3}, t^{-1} e_{4}, t^{-1} e_{5}\right)}$ | $z_{18}$ |
| $Z_{22}$ | $\xrightarrow{\left(e_{1}, t e_{2}, t e_{3}, t^{-1} e_{4}, e_{5}\right)}$ | $Z_{19}$ | $Z_{22}$ | $\xrightarrow{\left(e_{1}, t^{-\frac{1}{2}} e_{2}, t^{-\frac{1}{2}} e_{3}, t^{-\frac{1}{2}} e_{4}, t^{-1} e_{5}\right)}$ | $z_{21}$ |
| $Z_{25}$ | $\xrightarrow{\left(t^{-1} e_{1}, t^{-1} e_{2}, t^{-2} e_{3}, t^{-2} e_{4}, t^{-3} e_{5}\right)}$ | $z_{26}$ | $Z_{28}$ | $\xrightarrow{\left(e_{1}, t e_{2}, t e_{3}, t e_{4}, t e_{5}\right)}$ | $z_{29}$ |
| $z_{34}$ | $\xrightarrow{\left(t^{-2} e_{1}, t^{-1} e_{2}, t^{-2} e_{3}, t^{-3} e_{4}, t^{-4} e_{5}\right)}$ | $z_{32}$ | $Z_{34}$ | $\xrightarrow{\left(t^{-1} e_{1}, t^{-1} e_{2}, t^{-2} e_{3}, t^{-2} e_{4}, t^{-3} e_{5}\right)}$ | $z_{33}$ |
| $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{02}^{2, \frac{1}{t}}$ | $\xrightarrow{\left(t e_{1}, t^{2} e_{2}, e_{3}, t^{3} e_{4}, e_{5}\right)}$ | $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{01}^{2}$ | $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{06}^{2}$ | $\xrightarrow{\left(t^{-1} e_{1}, t^{-2} e_{2}, t^{-1} e_{3}, t^{-3} e_{4}, t^{-2} e_{5}\right)}$ | $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{03}^{2}$ |
| $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{05}^{2}$ | $\xrightarrow{\left(e_{1}, e_{2}, t e_{3}, e_{4}, t^{2} e_{5}\right)}$ | $\left[\mathfrak{N}_{1}^{\mathrm{C}}\right]_{04}^{2}$ | $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{06}^{2}$ | $\xrightarrow{\left(t^{-1} e_{1}, t^{-2} e_{2}, t^{-2} e_{3}, t^{-3} e_{4}, t^{-4} e_{5}\right)}$ | $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{05}^{2}$ |
| $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{09}^{2, \frac{1}{t}}$ | $\xrightarrow{\left(t^{\frac{2}{3}} e_{1}, t^{\frac{4}{3}} e_{2}, t e_{3}, t^{2} e_{4}, t^{\frac{2}{3}} e_{5}\right)}$ | $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{08}^{2}$ | $\left[\mathfrak{N}_{1}\right]_{07}^{2}$ | $\xrightarrow{\left(e_{1}, t e_{2}, t e_{3}, t e_{4}, t e_{5}\right)}$ | $\left[\mathfrak{N}_{1}\right]_{02}^{2}$ |
| $\left[\mathfrak{N}_{1}\right]_{04}^{2}$ | $\xrightarrow{\left(t^{\frac{1}{2}} e_{1}, e_{2}, t^{\frac{1}{2}} e_{3}, e_{4}, t e_{5}\right)}$ | $\left[\mathfrak{N}_{1}\right]_{03}^{2}$ | $\left[\mathfrak{N}_{1}\right]_{08}^{2}$ | $\xrightarrow{\left(t e_{1}, t e_{2}, t^{2} e_{3}, t^{2} e_{4}, t^{3} e_{5}\right)}$ | $\left[\mathfrak{N}_{1}\right]_{04}^{2}$ |
| $\left[\mathfrak{N}_{1}\right]_{06}^{2}$ | $\xrightarrow{\left(t e_{1}, e_{2}, t e_{3}, t e_{4}, t^{2} e_{5}\right)}$ | $\left[\mathfrak{N}_{1}\right]_{05}^{2}$ | $Z_{40}$ | $\xrightarrow{\left(e_{1}, e_{2}, e_{3}, e_{4}, t^{-1} e_{5}\right)}$ | $\left[\mathfrak{Z}_{1}\right]_{1}^{1}$ |

The following dimensions of orbit closures are important for us:

$$
\begin{array}{llll}
\operatorname{dim} \mathcal{O}\left(\mathfrak{V}_{3+2}\right)=24, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{22}\right)=22, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{14}^{\alpha}\right)=21, & \operatorname{dim} \mathcal{O}\left(\left[\mathfrak{N}_{1}\right]_{08}^{2}\right)=21, \\
\operatorname{dim} \mathcal{O}\left(\mathfrak{V}_{4+1}\right)=20, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{02}^{\alpha}\right)=20, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{30}^{\alpha}\right)=20, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{05}\right)=20, \\
\operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{23}\right)=20, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{24}\right)=20, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{27}\right)=20, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{35}\right)=20, \\
\operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{38}\right)=20, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{40}\right)=20, & \operatorname{dim} \mathcal{O}\left(\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{06}^{\alpha}\right)=20, & \operatorname{dim} \mathcal{O}\left(\mathcal{Z}_{34}\right)=19 .
\end{array}
$$

Let us, also, give the list of dimensions of the square of principal algebras:

| $\operatorname{dim} A^{2}=2$ | $\mathfrak{Z}_{14}^{\alpha}, \mathfrak{Z}_{02}^{\alpha}, \mathfrak{Z}_{22}, \mathfrak{V}_{3+2}$ |
| :--- | :--- |
| $\operatorname{dim} A^{2}=3$ | $\left[\mathfrak{N}_{1}\right]_{08}^{2},\left[\mathfrak{N}_{1}^{C}\right]_{06}^{2}, \mathfrak{Z}_{05}, \mathfrak{Z}_{23}, \mathfrak{Z}_{24}, \mathfrak{Z}_{27}, \mathfrak{Z}_{30}^{\alpha}, \mathfrak{Z}_{34}, \mathfrak{Z}_{35}, \mathfrak{Z}_{38}$ |
| $\operatorname{dim} A^{2}=4$ | $\mathfrak{Z}_{40}$ |

Hence, and taking into account the annihilator dimension of this algebras (see item (2) in the beginning of Section (1.3), $\left[\mathfrak{N}_{1}\right]_{08}^{2}, z_{05}, z_{22}, z_{23}, z_{24}, z_{27}, z_{30}^{\alpha}, z_{35}, z_{38}, z_{40}$ and $\mathfrak{V}_{3+2}$ give 11 irreducible components. Below we have listed all necessary reasons for non-degenerations, which imply that $\left[\mathfrak{N}_{1}^{\mathbb{C}}\right]_{06}^{2}, \mathfrak{Z}_{02}^{\alpha}, \mathfrak{Z}_{14}^{\alpha}, \mathfrak{Z}_{34}$ and $\mathfrak{V}_{4+1}$ give another 5 irreducible components.

| Non-degenerations reasons |  |
| :---: | :---: |
| $\left[\mathfrak{N}_{1}\right]_{08}^{2} \quad \rightarrow \quad\left[\mathfrak{N}_{1}^{C}\right]_{06}^{2}$ | $\mathcal{R}=\left\{\begin{array}{l} A_{1}^{2} \subseteq A_{3}, A_{1} A_{4}+A_{4} A_{1}=0, \\ c_{11}^{3}=0, c_{22}^{3}=0, c_{12}^{3}=-c_{21}^{3} \end{array}\right\}$ |
| $z_{05} \quad \rightarrow \quad z_{34}$ | $\mathcal{R}=\left\{\begin{array}{l}A_{1}^{2} \subseteq A_{3}, A_{1} A_{2}+A_{2} A_{1} \subseteq A_{4}, A_{4} A_{3}+A_{3} A_{4}=0, \\ c_{12}^{4}=c_{21}^{4}, c_{12}^{3}=c_{21}^{3}=c_{22}^{3}=0,2 c_{24}^{5}=c_{42}^{5}, 2 c_{14}^{5}=c_{41}^{5}\end{array}\right\}$ |
| $\begin{aligned} & \hline \mathcal{Z}_{14}^{\alpha} \nrightarrow \begin{array}{l} z_{02}^{\alpha}, \\ \mathfrak{V}_{4+1} \end{array} . \end{aligned}$ | $\mathcal{R}=\left\{\begin{array}{l}A_{1}^{2} \subseteq A_{4}, A_{4} A_{1}=0 \\ \text { new base for } Z_{14}^{\alpha}: f_{1}=e_{1}, f_{2}=e_{2}, f_{3}=e_{4}, f_{4}=e_{3}, f_{5}=e_{5}\end{array}\right\}$ |
| $$ | $\mathcal{R}=\left\{\begin{array}{l} A_{1}^{2} \subseteq A_{4}, A_{1} A_{3}+A_{3} A_{1}+A_{2}^{2} \subseteq A_{5}, A_{4} A_{1}+A_{1} A_{5}=0, \\ \text { new base for } z_{22}: f_{1}=e_{1}, f_{2}=e_{2}, f_{3}=e_{4}, f_{4}=e_{3}, f_{5}=e_{5} \end{array}\right\}$ |
| $z_{23} \quad \nrightarrow \quad z_{34}$ | $\mathcal{R}=\left\{A_{1}^{2} \subseteq A_{3}, A_{1} A_{3}+A_{3} A_{1} \subseteq A_{5}, A_{1} A_{5}+A_{4} A_{1}=0\right\}$ |
| $z_{24} \nrightarrow z_{34}$ | $\mathcal{R}=\left\{\begin{array}{l} A_{1}^{2} \subseteq A_{3}, A_{1} A_{2} \subseteq A_{4}, \\ A_{1} A_{3}+A_{3} A_{1} \subseteq A_{5}, A_{1} A_{5}+A_{5} A_{1}=0, \\ c_{22}^{4} c_{11}^{3}=c_{11}^{4} c_{21}^{3}, c_{23}^{5} c_{11}^{3}=c_{13}^{5} c_{c 1}^{3}, 2 c_{22}^{4} c_{14}^{5}=c_{12}^{4} c_{42}^{5}, \\ c_{21}^{4} 1_{14}^{5}+c_{12}^{4} c_{14}^{5}+c_{13}^{5} c_{21}^{3}=c_{12}^{4} c_{41}^{5}, \\ c_{11}^{4} c_{14}^{5} c_{21}^{3}+c_{11}^{3}\left(c_{12}^{4} c_{14}^{5}+c_{13}^{5} c_{21}^{3}\right)=c_{11}^{3} c_{12}^{4} c_{41}^{5} \end{array}\right\}$ |
| $z_{27} \nrightarrow z_{34}$ | $\mathcal{R}=\left\{A_{1}^{2} \subseteq A_{3}, A_{3} A_{1}=0\right\}$ |
| $z_{30}^{\alpha} \rightarrow \rightarrow \quad z_{34}$ | $\mathcal{R}=\left\{\begin{array}{l} A_{1}^{2} \subseteq A_{3}, A_{1} A_{3}+A_{3} A_{1} \subseteq A_{5}, A_{1} A_{5}+A_{5} A_{1}=0, \\ c_{12}^{3}=c_{21}^{3}, c_{23}^{5} c_{42}^{5}=c_{24}^{5}{ }^{5} 5, \\ c_{31}^{5} c_{24}^{5}\left(c_{12}^{4}\left(c_{14}^{5}\right)^{2}+3 c_{11}^{3} c_{14}^{5} c_{23}^{5}-2 c_{11}^{3} c_{13}^{5} c_{24}^{5}\right)= \\ 2 c_{14}^{5}\left(c_{12}^{4} c_{14}^{5}+2 c_{11}^{3} c_{23}^{5}\right)\left(c_{14}^{5} c_{23}^{5}-c_{13}^{5} c_{24}^{5}\right) \end{array}\right\}$ |
| $z_{35} \quad \rightarrow \quad z_{34}$ | $\mathcal{R}=\left\{\begin{array}{l} A_{1}^{2} \subseteq A_{3}, A_{4} A_{1}+A_{3}^{2}+A_{1} A_{5}=0, \\ c_{11}^{3}\left(c_{12}^{4}+c_{21}^{4}\right)=2 c_{11}^{4} c_{12}^{3}, c_{22}^{3}\left(c_{12}^{4}+c_{21}^{4}\right)=2 c_{21}^{3} c_{22}^{4} \end{array}\right\}$ |


| $z_{38}$ | $\nrightarrow$ | $z_{34}$ | $\mathcal{R}=\left\{\begin{array}{l}A_{1}^{2} \subseteq A_{3}, A_{1} A_{2}+A_{2} A_{1} \subseteq A_{4}, \\ A_{2}^{2}+A_{1} A_{4}+A_{4} A_{1} \subseteq A_{5}, A_{1} A_{5}+A_{5} A_{1}=0, \\ c_{14}^{5} c_{11}^{4}+c_{11}^{3} c_{31}^{5}=2 c_{11}^{3} c_{13}^{5}, 2 c_{12}^{1}=c_{21}^{4} \\ \text { new base for } z_{38}: f_{1}=e_{1}, f_{2}=e_{4}, f_{3}=e_{2}, f_{4}=e_{3}, f_{5}=e_{5}\end{array}\right\}$ |
| :--- | :--- | :--- | :--- |
| $z_{40} \nrightarrow \quad z_{34}$ | $\mathcal{R}=\left\{\begin{array}{l}A_{1}^{2} \subseteq A_{2}, A_{1} A_{3}+A_{3} A_{1}+A_{2}^{2} \subseteq A_{4}, \\ A_{2} A_{3}+A_{3} A_{2} \subseteq A_{5}, A_{5} A_{1}+A_{1} A_{5}=0, \\ 4 c_{14}^{5}=c_{41}^{5}, 3 c_{23}^{5}=2 c_{32}^{5}, 3 c_{13}^{4}=c_{31}^{4}, c_{21}^{3}=2 c_{12}^{3}\end{array}\right\}$ |  |  |

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