The Algebraic and Ordering Construction of Lattice

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Abstract. Based on the order theory, a study of the differences between the preordering and the ordering and a discussion of the isotone map and homomorphism map are obtained. According to the differences of the constructions of the algebraic lattice and the ordering lattice, we prove the algebraic sublattice and the ordering sublattice are equivalent with the strengthen condition.

Introduction

George Grätzer gives a concise development of the basic concepts of lattice theory. Generally, our focus is on lattice structure theory, however, this paper intends to illustrate if the algebraic sublattice and the ordering sublattice are equivalent. We draw conclusions from "General Lattice Theory", "The Congruence of a Finite Lattice"[3] by George Grätzer:

Theorem 1

(1)Let the poset $L=\left(L,\leq\right)$ be a lattice. Set $a\wedge b=\inf\{a,b\}$, $a\vee b=\sup\{a,b\}$ .Then the algebra $L^a=(L,\wedge,\vee)$ is a lattice.

(2)Let the algebra $L=(L,\wedge,\vee)$ be a lattice. Set $a\leq b$ iff $a\wedge b=a$ .Then $L^P=(L,\leq)$ is a poset, and the poset is a lattice.

(3)Let the poset $L=(L,\leq)$ be a lattice. Then $(L^a)^P=L$.

(4)Let the algebra $L=\left(L,\wedge,\vee\right)$ be a lattice. Then $\left(L^P\right)^a=L$.

In other words, a lattice as an algebra and a lattice as a poset are "equivalent" concepts. We wonder if the concepts of a sublattice as an algebra and a sublattice as a poset are the same. We will discuss about it later.

Theorem 2 [1]Every homomorphism map is an isotone map.

Theorem 3 Let P be a finite order. Then $a\leq b$ if and only if $a=b$ or if there exists a finite sequence of elements x_1 , x_2 ,$\cdots$, x_n such that $a=x_1\prec x_2\prec\cdots\prec x_n=b$.

Preliminaries

Definition 1 [2] A preorder is a nonempty set Q with a binary relation $\leq$ that is reflexive and transitive.

Definition 2 A partially ordered set is a system consisting of a nonempty set P and a binary relation $\leq$ in P such that the following conditions are satisfied for all x , y , $z\in P$:

(1) Reflexivity $x\leq x$

(2) Antisymmetry $x\leq y$, $y\leq x\Rightarrow x=y$

(3) Transitivity $x\leq y$, $y\leq z\Rightarrow x\leq z$

The relation $\leq$ is a partial order in the set P , and P is said to be partially ordered by the relation $\leq$.

Definition 3 $(P,\ \leq)$ is a poset, S is a subset of P , $a\in P$:

(1)If $\forall\ s\in S$, $a\leq s(s\leq a)$,then a is a lower(upper) bound of S ;

(2)If $\forall\ s\in S$, $a\in S$, $a\leq s(s\leq a)$,then a is called the least(greatest) element of S ;

(3)A lower bound a of S is the greatest lower bounder of S, for any lower bound b of S,we have $a \geq b$. An upper bound a of S is the least upper bounder of S, for any upper bound b of S, we have $a \leq b$.

Definition 4 [5] A poset $L=(L,\leq)$ is a lattice, if $a,b \in L$, $\sup\{a,b\}$ and $\inf\{a,b\}$ exist.

Definition 5 If L is a lattice, then it is readily verified that the following conditions hold for all $a,b,c \in L$

(1) $a \wedge a = a$, $a \vee a = a$

(2) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$

(3) $(a \wedge b) \vee a = a$, $(a \vee b) \wedge a = a$

(4) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$

Definition 6 A lattice $L_1 = (L_1, \wedge, \vee)$ is called a algebraic sublattice of $L=(L,\wedge,\vee)$ if it satisfies the following two conditions:

(1) $L_1 \subseteq L$ is a nonvoid subset of L

(2) L_1 is closed under the same $\vee$ and $\wedge$

Definition 7 A lattice $L_1=(L_1,\leq)$ is called a partially sublattice of $L=(L,\leq)$, if it satifies the following two conditions:

(1) $L_1 \subseteq L$ is a nonvoid subset of L

(2) $(L_1,\leq)$ is a suborder of $(L,\leq)$

Definition 8 A homomorphism φ of the lattice L_0 into a lattice L_1 is a map of L_0 into L_1 satisfying both $\varphi(a \vee b)=\varphi(a) \vee \varphi(b)$ and $\varphi(a \wedge b)=\varphi(a) \wedge \varphi(b)$.

Definition 9 The map $\varphi: P_0 \rightarrow P_1$ is an isotone map of the poset P_0 into the poset P_1 iff $a \leq b$ in P_0 implies that $\varphi(a) \leq \varphi(b)$ in P_1.

Definition 10 A chain is an order with no incomparable elemets.

Definition 11 There exists a finite sequence of elements $x_1,\cdots,x_n \in Q$ such that $x_1 \prec \cdots \prec x_n \prec x_1$(n>1), then Q is called a circle.

Definition 12 The subset K of the lattice L is called convex iff $a,b \in K, c \in L$,and $a \leq c \leq b$ imply that $c \in K$.

Algebraic Sublattices and Partial Sublattices

As we know, a lattice as an algebra and a lattice as a poset are "equivalent" concepts. There are some differences between algebraic construction and partial construction [4] .

Theorem 1 L_1 is a partial sublattice of L,if $\forall a,b \in L_1$, $\inf\{a,b\}$, $\sup\{a,b\} \in L_1$ and $\inf\{a,b\}$, $\sup\{a,b\} \in L$ are equivalent, then partial sublattice L_1 and algebraic sublattice L_1 are equivalent.

Proof: Let L be a lattice, L_1 is an algebraic sublattice of L. L_1 is also a lattice. The conclusion of theorem1, L_1^P is a partial lattice. In view of the results, $a \leq b \Leftrightarrow a \wedge b=a$. Certainly $(a \wedge b) \wedge a=a \wedge b$ implies $a \wedge b \leq a$. Similarly $(a \wedge b) \wedge b=a \wedge b$ implies $a \wedge b \leq b$.It is very easy to infer $a \wedge b$ is a lower bound of a,b. Let d be a lower bounder of a,b, so $d \leq a$ and $d \leq b$. Certainly $d \wedge a=d$ and $d \wedge b=d$, thus we can infer $d \wedge (a \wedge b)=(d \wedge a) \wedge b=d \wedge b=d$ .So we get $d \leq a \wedge b$,then $a \wedge b$ is the greatest lower bound of a,b in L_1^P.In the definition of algebraic sublattice, $a \wedge b$ is also the greatest lower bound of a,b in L.

In view of the results, $a \leq b \Leftrightarrow a \wedge b=a \Leftrightarrow b=a \vee b$. Since $(a \vee b) \vee a=a \vee b$,we infer

$a \le a \vee b$.Obviously $(a \vee b) \vee b = a \vee b$ implies $b \le a \vee b$.we can also infer that $a \vee b$ is an upper bound of a, b. Let d' is an upper bound of a, b.Since $a \le d'$ and $b \le d'$, thus $a \vee d' = d'$ and $b \vee d' = d'$, thus $d' \vee (a \vee b) = (d' \vee a) \vee b = d' \vee b = d'$. Thus $a \vee b$ is the least upper bound of a, b in L_1^P and in L.

We conclude that L_1 is a partial sublattice of L, if $\forall a, b \in L_1$, $\inf\{a,b\}$, $\sup\{a,b\} \in L_1$ and $\inf\{a,b\}$, $\sup\{a,b\} \in L$ are equivalent, then partial sublattice L_1 and algebraic sublattice L_1 are equivalent.

Theorem 2 Every algebraic sublattive is the partial sublattice.

Proof: Let L be a lattice and L_1 be a partial sublattice of L.Assume L_1 does not satisfy the definition of partial sublattice. However, $\forall a, b \in L_1$, $a \wedge b$, $a \vee b \in L_1$, in the view of the preceding results, we get the conclusion that $a \wedge b = \inf\{a,b\}$, $a \vee b = \sup\{a,b\}$,contrary to the assumption. Consequently, every algebraic sublattice is the partial sublattice.

Theorem 3 Every convex partial sublattice is the partial sublattice.

Proof: Let K be the convex partial sublattice of L, $a \le b \in K$ for all $c \in [a,b]$ satisfies $c \in K$. Assume that K does not satisfy the definition of algebraic sublattice. There exists $a, b \in K$ but $a \wedge b$ or $a \vee b \notin K$. Thus $a \le b \in K$, $c_1, c_2 \in [a,b]$, and $a \le c_1 \le c_2 \le b$. In the view of the preceding results, $a \le b \Leftrightarrow a \wedge b = a$ and $a \vee b = b$, so we infer that $c_1 \wedge c_2 = c_1$ and $c_1 \vee c_2 = c_2 \in K$,contrary to the assumption . Therefore every convex partial sublattice is the partial sublattice.

In fact, we can have the more general conclusion:

Theorem 4 Let L_1 be a parital sublattice of $L = (L, \wedge, \vee)$,then L_1 is an algebraic sublattice iff for all $a, b \in L_1$ satisfy $a \wedge b$ and $a \vee b \in L_1$ （$\wedge$, $\vee$ in L）.

Proof: $\Rightarrow$ obviously.

$\Leftarrow$ L_1 is a partial sublattice, according to the definition, for all $a, b \in L_1$, then $\inf\{\mathrm{a,b}\}$ and $\sup\{\mathrm{a,b}\} \in L_1$. Let $d = \inf\{a,b\} \in L_1$, according to the theorem, $d' = \inf\{a,b\} = a \wedge b \in L$.Since $d' \in L_1$, thus $d \ge d'$. If $d > d'$, contrary to the fact $d = \inf\{a,b\} \in L_1$, thus $d = d'$.

Similarly, for all $a, b \in L_1$, let $d = \sup\{a,b\} \in L_1$ and $d' = \sup\{a,b\} = a \vee b \in L$. Since $d' \in L_1$, so $d \le d'$. If $d < d'$, contrary to the fact $d = \sup\{a,b\} \in L_1$, therefore $d = d'$.

Ordering and its Properties

A preorder is a nonempty set Q with a binary relation $\le$ that is reflexive and transitive. We can add some conditions on the preorder in order that the preorder and the poset are equivalent.

Theorem 1 The finite preorder is a poset iff there does not exist circle.

Proof: Assume that there is a circle $a \prec x_0 \prec \cdots \prec x_n \prec a$ .We get $a \le x_0$, according to the transitivity , $x_0 \le a$. Combining the antisymmetry, $a \le b$, $b \le a \Rightarrow a = b$, then $a = x_0$, contrary to $a \prec x_0$.

For all a, b, $a \le b$ and $b \le a$, we have two situations:

(1) $a = b$

(2) $a \ne b$, then $a \prec x_1 \prec \cdots \prec b$ and $b \prec x_1' \prec \cdots \prec a$.

If $a \ne b$, we get $a \prec x_1 \prec \cdots \prec b \prec x_1' \prec \cdots \prec a$,contrary to the fact that there does not exist circle. Therefore, for all a, b if $a \le b$ and $b \le a$, we have $a = b$.

Theorem 2 Every isotone map of the finite chain is a homomorphism.

Proof: Let C be a finite chain. There exists a finite sequence of elements $x_1, x_2, \cdots, x_n \in C$ such that $x_1 \prec x_2 \prec \cdots \prec x_n$ .Since φ is an isotone map of C into P ,thus $\varphi(x_1) \prec \varphi(x_2) \prec \cdots \prec \varphi(x_n)$. For all $x_1, x_2 \in C$ if $x_1 \prec x_2$, we infer that $x_1 \wedge x_2 = x_1$ and $x_1 \vee x_2 = x_2$. Similarly, for all $\varphi(x_1), \varphi(x_2) \in P$, we get $\varphi(x_1) \wedge \varphi(x_2) = \varphi(x_1)$ and $\varphi(x_1) \vee \varphi(x_2) = \varphi(x_2)$.

For all $a, b \in C$, there are only two situations, $a \prec b$ or $b \prec a$. If $a \prec b$, then $\varphi(a \wedge b) = \varphi(a) = \varphi(a) \wedge \varphi(b)$ and $\varphi(a \vee b) = \varphi(a) = \varphi(a) \vee \varphi(b)$.

On the other hand, if $b \prec a$, then $\varphi(b \wedge a) = \varphi(b) = \varphi(b) \wedge \varphi(a)$ and $\varphi(b \vee a) = \varphi(b) = \varphi(b) \vee \varphi(a)$. Therefore every isotone map of the finite chain is a homomorphism.

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