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# The algebraic dimension of compact complex threefolds with vanishing second Betti number

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## ABSTRACT

We study compact complex three-dimensional manifolds with vanishing second Betti number. In particular, we show that a compact complex manifold homeomorphic to the six-dimensional sphere does carry any non-constant meromorphic function.

## 1. Introduction

The paper [CDP98] studied compact complex threefolds  $X$  such that the second Betti number  $b_2(X) = 0$ . The main result is based on Lemma 1.5, which happens to be incorrect in general (but might still hold in the context of the paper). In any case, some of the statements and proofs need to be adapted to fill the possible gaps; this is done in the present corrigendum, with special regard to potential complex structures on the 6-sphere.

## 2. Statement of the results

We prove [CDP98, Theorem 2.1] in full generality in the case where  $X$  has a meromorphic non-holomorphic map  $X \dashrightarrow \mathbb{P}_1$ . In the remaining case,  $X$  has algebraic dimension 1 and the algebraic reduction  $f : X \rightarrow C$  is holomorphic. In this case we prove that  $c_3(X) \leq 0$ ; for simplicity, we will assume not only that  $b_2(X) = 0$  but slightly more strongly that  $H^2(X, \mathbb{Z}) = 0$  and, moreover, that  $H^1(X, \mathbb{Z}) = 0$ , hence  $C \simeq \mathbb{P}_1$ . This suffices to treat the main application of complex structures on  $S^6$ .

In summary, we shall prove the following result.

**THEOREM 2.1.** *Let  $X$  be a three-dimensional compact complex manifold with  $b_2(X) = 0$ . Assume that there exists a non-holomorphic meromorphic non-constant map  $g : X \dashrightarrow \mathbb{P}_1$ . Let  $B$  be a holomorphic vector bundle on  $X$ . Then:*

- (a)  $H^i(X, B \otimes \mathcal{M}) = 0$  for  $i \geq 0$  and  $\mathcal{M} \in \text{Pic}^\circ(X)$  generic;
- (b)  $\chi(X, B \otimes \mathcal{M}) = 0$  for all  $\mathcal{M} \in \text{Pic}^\circ(X)$ ;
- (c)  $c_3(X) = 0$ , that is, either  $b_1(X) = 0$  and  $b_3(X) = 2$ , or  $b_1(X) = 1$  and  $b_3(X) = 0$ .

Another proof has been given in [LRS18].

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Theorem 2.1 takes care of all threefolds  $X$  with  $1 \leq a(X) \leq 2$  and  $b_2(X) = 0$  except for those whose algebraic reduction  $f : X \rightarrow C$  is holomorphic onto a curve  $C$ . In this case the general fiber has Kodaira dimension  $\kappa(X_c) \leq 0$ .

By topological considerations and surface classification, the general fiber of  $f$  is either a torus, a primary Kodaira surface or a surface of type VII; in the latter case it is actually a primary Hopf surface or an Inoue surface, by Lemmas 4.2 and 4.3. The case where the general fiber is a Kodaira surface is ruled out in Corollary 5.3. Then we show the following theorem.

**THEOREM 2.2.** *Let  $X$  be a three-dimensional compact complex manifold with*

$$H^1(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = 0$$

*and algebraic dimension  $a(X) = 1$ . Assume that the algebraic reduction  $f : X \rightarrow C$  is holomorphic. Then:*

- (a)  $H^i(X, T_X \otimes \mathcal{M}) = 0$  for  $i \neq 1$ ;
- (b)  $\chi(X, T_X \otimes \mathcal{M}) \leq 0$ ;
- (c)  $c_3(X) \leq 0$ .

As a consequence we deduce the following corollary.

**COROLLARY 2.3.** *Let  $X$  be a three-dimensional compact complex manifold homeomorphic to  $S^6$ . Then  $a(X) = 0$ .*

*Proof.* Obviously,  $a(X) \neq 3$ , otherwise  $X$  is Moishezon and therefore  $b_2(X) \neq 0$ . If  $a(X) = 2$ , then there exists a meromorphic non-holomorphic map  $g : X \dashrightarrow \mathbb{P}_1$ . Then we conclude by Theorem 2.1 that  $c_3(X) = 0$ . By Hopf’s theorem,  $c_3(X) = \chi_{\text{top}}(S^6) = 2$ , a contradiction. If  $a(X) = 1$  and the algebraic reduction  $g : X \dashrightarrow C$  is not holomorphic, then  $C \simeq \mathbb{P}_1$ , and we conclude again by Theorem 2.1. If  $a(X) = 1$  and the algebraic reduction  $g : X \dashrightarrow C$  is holomorphic, then we apply Theorem 2.2 and obtain the same contradiction as before.  $\square$

We now comment on the strategy to prove Theorem 2.2. The arguments of [CDP98] show the following proposition.

**PROPOSITION 2.4.** *Let  $X$  be a three-dimensional compact complex manifold with  $b_2(X) = 0$  and algebraic dimension  $a(X) = 1$ . Assume that the algebraic reduction  $f : X \rightarrow C$  is holomorphic. If*

$$R^2 f_*(T_X \otimes \mathcal{L}) = 0 \tag{1}$$

*for some (or general)  $\mathcal{L} \in \text{Pic}(X)$ , then the assertions of Theorem 2.2 hold.*

Equation (1) is equivalent to the vanishing

$$H^2(X_c, T_X \otimes \mathcal{L}|_{X_c}) = 0$$

for all complex-analytic fibers  $X_c$  (with the natural fiber structure) of  $f$ , or equivalently

$$H^0(X_c, \Omega_X^1 \otimes \mathcal{L}^*|_{X_c}) = 0.$$

The key is to show that the restriction  $\mathcal{L}|_{X_c}$  of some or the general line bundle  $\mathcal{L}$  to any fiber is never torsion. Then we compute directly on  $X_c$ ; here the case where  $X_c$  is singular, in particular non-normal, needs special care.

For further information on the problem of complex structures on  $S^6$ , we refer to [Ete15] and to volume 57 of the journal *Differential Geometry and Its Applications*.

**3. Proof of Theorem 2.1**

Instead of simply pointing out the additions to be made in the proof of [CDP98, Theorem 2.1], we give full details for the benefit of the reader. Notice first that (2.1)(b) follows from (2.1)(a) since  $\chi(X, B \otimes \mathcal{M})$  does not depend on  $\mathcal{M}$ , and (2.1)(c) is a consequence of (2.1)(b) by applying Riemann–Roch to  $B = T_X$  and  $\mathcal{M} = \mathcal{O}_X$ . So it remains to prove (2.1)(a). By Serre duality, (2.1)(a) only needs to be shown for  $i = 0$  and  $i = 2$ . The case  $i = 0$  follows from [CDP98, Corollary 1.3], since  $X$  does carry effective non-zero divisors. Thus it remains to prove that

$$H^2(X, B \otimes \mathcal{M}) \tag{2}$$

for generic, equivalently one, line bundle  $\mathcal{M}$ . Let  $g : X \dashrightarrow \mathbb{P}_1$  be a non-constant non-holomorphic meromorphic map and  $\sigma : \hat{X} \rightarrow X$  be a resolution of indeterminacies of  $g$ . Let

$$f : \hat{X} \rightarrow C \simeq \mathbb{P}_1$$

be the fiber space given by the Stein factorization of the holomorphic map  $\sigma \circ g$ . Replacing  $g$  by the induced meromorphic map  $X \dashrightarrow C$ , we may assume from the beginning that  $g$  has connected fibers, hence no Stein factorization has to be taken. Note that the canonical map

$$H^2(X, B \otimes \mathcal{M}) \rightarrow H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M}))$$

is injective (by the Leray spectral sequence). Hence equation (2) follows from

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M})) = 0. \tag{3}$$

Fix an ample divisor  $A$  on  $C$ . Then  $f^*(\mathcal{O}_C(A))$  can be written as

$$f^*(\mathcal{O}_C(A)) = \sigma^*(\mathcal{L}) \otimes \mathcal{O}_{\hat{X}}(-E) \tag{4}$$

with a line bundle  $\mathcal{L}$  on  $X$  and a suitable effective divisor  $E$  which is supported on the exceptional set of  $\sigma$  and projects onto  $C$ . To verify equation (3) it suffices to show that

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M}) \otimes \mathcal{O}_{\hat{X}}(-tE)) = 0 \tag{5}$$

for some effective divisor  $E$  supported on the exceptional locus of  $\sigma$  and some  $t \geq 0$ . In fact, consider the exact sequence

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M}) \otimes \mathcal{O}_{\hat{X}}(-tE)) \rightarrow H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M})) \rightarrow H^2(tE, \sigma^*(B \otimes \mathcal{M}))$$

and note that

$$H^2(tE, \sigma^*(B \otimes \mathcal{M})) = 0.$$

This last vanishing is seen as follows. Let  $Z_t$  be the complex subspace of  $X$  defined by the ideal sheaf  $\sigma_*(\mathcal{O}_{\hat{X}}(-tE))$ . Then by

$$R^q(\sigma|_{tE})_*(\mathcal{O}_{tE}) = 0$$

for  $q = 1, 2$  and the Leray spectral sequence,

$$H^2(tE, \sigma^*(B \otimes \mathcal{M})) = H^2(Z_t, B \otimes \mathcal{M}).$$

Now the last group vanishes since  $\dim Z_t = 1$ .

By replacing  $\mathcal{M}$  by  $\mathcal{M} \otimes \mathcal{L}^{t+k}$  for some positive integer  $k$  and using (4), equation (5) reads

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^{t+k} \otimes \mathcal{O}_{\hat{X}}(-tE))) = H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k \otimes f^*(\mathcal{O}_C(tA))). \tag{6}$$

By the Leray spectral sequence applied to  $f$ , equation (6) comes down to verifying

$$R^2 f_*(\sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k)) = 0 \tag{7}$$

and

$$H^1(C, R^1 f_*(\sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k)) \otimes \mathcal{O}_C(tA)) = 0 \tag{8}$$

for suitable positive integers  $k$  and  $t$  and some line bundle  $\mathcal{M}$ .

To prove (7), let  $C^* \subset C$  be the finite set of points  $c \in C$  such that some component of the fiber  $X_c$  does not meet  $E$ . In particular, if  $X_c$  is irreducible, then  $c \in C \setminus C^*$ . Notice also that

$$R^2 f_*(\sigma^*(B \otimes \mathcal{L}^k))|_{\{c\}} \simeq H^2(\hat{X}_c, \sigma^*(B \otimes \mathcal{L}^k)),$$

by the standard base change theorem. Applying Serre duality, we obtain

$$\begin{aligned} H^2(\hat{X}_c, \sigma^*(B \otimes \mathcal{L}^k)) &\simeq H^0(\hat{X}_c, \sigma^*(B^* \otimes \mathcal{L}^{-k}) \otimes \omega_{\hat{X}_c}) \\ &\simeq H^0(\hat{X}_c, \sigma^*(B^*) \otimes \mathcal{O}_{\hat{X}_c}(-kE) \otimes \omega_{\hat{X}_c}). \end{aligned}$$

We claim first that there is a number  $k_0$  such that for  $k \geq k_0$ ,

$$\text{supp } R^2 f_*(\sigma^*(B \otimes \mathcal{L}^k)) \subset C^*. \tag{9}$$

This is equivalent to saying that

$$H^0(\hat{X}_c, \sigma^*(B^* \otimes \mathcal{M}^*) \otimes \mathcal{O}_{\hat{X}_c}(-kE) \otimes \omega_{\hat{X}_c}) = 0$$

for  $c \notin C^*$ . Fixing any point  $c_0 \in C^*$ , this number  $k_0 = k_0(c)$  clearly exists; in the case where  $X_c$  is reducible, we apply [CDP98, Proposition 1.1]. Hence the support of the direct image sheaf  $R^2 f_*(\sigma^*(B \otimes \mathcal{L}^{k_0}))$  is contained in a finite set  $C_{k_0}$  in  $C$ . Since  $\sigma^*(\mathcal{L})|_{X_c}$  is effective, it follows that  $C_k \subset C_{k_0}$ . Thus, enlarging  $k_0$  if necessary, (9) is verified.

Hence we only need to consider the fibers  $\hat{X}_c$  with  $c \in C^*$ . Let  $P$  be the set of line bundles  $\mathcal{M}$  of the form

$$\mathcal{M} = \mathcal{O}_X \left( \sum m_i S_i \right)$$

with  $m_i$  positive integers and  $S_i$  fiber components of  $f$  not meeting  $E$  (the  $S_i$  considered as surfaces in  $X$ ).

Since all line bundles  $\mathcal{M} \in P$  are trivial on  $X \setminus f^{-1}(C^*)$ , our previous considerations imply the existence of a number  $k_0$  such that for all  $k \geq k_0$  and all  $\mathcal{M} \in P$ ,

$$\text{supp } R^2 f_*(\sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k)) \subset C^*.$$

We will now construct a line bundle  $\mathcal{M} \in P$  such that

$$R^2 f_*(\sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k)) = 0$$

for a suitable number  $k$ . Fix a point  $c \in C^*$ . Let  $F_0 \subset X_c$  be the sub-divisor of  $\hat{X}_c$  consisting of all components meeting  $E$ ; let further  $F_1 \subset \hat{X}_c$  be the sub-divisor consisting of all components which meet  $F_1$  but not  $E$ . Continuing in this way, we obtain a decomposition

$$\hat{X}_c = \sum_{i=0}^r F_r$$

of sub-divisors  $F_j \subset X_c$  which pairwise do not have common components and which have the property that all components of  $F_j$  meet  $F_{j-1}$  but do not meet  $F_k$  for  $k < j - 1$ . Now choose  $m_r \gg 0$  such that

$$H^0(F_r, \mathcal{O}_{\hat{X}}(-m_r F_{r-1}) \otimes \sigma^*(B^*) \otimes \omega_{\hat{X}_c}|_{F_r}) = 0.$$

This is possible by our construction. Next choose  $m_{r-1} \geq m_r$  such that

$$H^0(F_{r-1}, \mathcal{O}_{\hat{X}}(-m_{r-1} F_{r-2}) \otimes \sigma^*(B^*) \otimes \omega_{\hat{X}_c}|_{F_{r-1}}) = 0.$$

Since  $\text{supp}(F_{r-2}) \cap \text{supp}(F_r) = \emptyset$ , we obtain

$$H^0(F_{r-1} + F_r, \mathcal{O}_{\hat{X}}(-m_{r-1}(F_{r-2} + F_{r-1})) \otimes \sigma^*(B^*) \otimes \omega_{\hat{X}_c}|_{F_{r-1}}) = 0.$$

Continuing in this way, we obtain a line bundle

$$\mathcal{M}'(c) = \mathcal{O}_X\left(\sum_{i=2}^r m_i F_i\right)$$

(bearing in mind that the divisors  $F_i, i \geq 1$ , do not meet the exceptional locus of  $\sigma$ ), such that

$$H^0\left(\sum_{i=2}^r F_i, \sigma^*(B^* \otimes \mathcal{M}'(c)) \otimes \omega_{\hat{X}_c} \mid \left(\sum_{i=2}^r F_i\right)\right) = 0.$$

Since the component  $F_0$  meets  $E$ , it needs special treatment. We observe that

$$\mathcal{O}_{\hat{X}}(F_0) \simeq \sigma^*(\mathcal{O}_X(\sigma(F_0))) \otimes \mathcal{O}_{\hat{X}}(-E')$$

with some effective  $\sigma$ -exceptional divisor  $E'$ . Hence, choosing  $m_1 \gg 0$  and setting

$$\mathcal{M}(c) = \mathcal{M}'(c) \otimes \mathcal{O}_X(m_1 \sigma(F_0)),$$

then

$$H^0\left(\sum_{i=1}^r F_i, \sigma^*(B^* \otimes \mathcal{M}(c)^*) \otimes \omega_{\hat{X}_c} \mid \left(\sum_{i=1}^r F_i\right)\right) = 0.$$

Finally, enlarging  $k$ , we get

$$H^0(\hat{X}_c, \sigma^*(B^* \otimes \mathcal{M}(c)^*) \otimes \mathcal{O}_{\hat{X}}(-kE) \otimes \omega_{\hat{X}_c}|_{X_c}) = 0.$$

Setting

$$\mathcal{M} = \bigotimes_{c \in C^*} \mathcal{M}(c),$$

this settles (7).

As to (8), we fix  $k$  as in (7) and then apply Serre's vanishing theorem to the ample divisor  $A$  to obtain  $t$ .

#### 4. General structure of the fibers

From now on, for the rest of the paper, we let  $X$  be a compact complex manifold of dimension 3 with

$$H^1(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = 0$$

and holomorphic algebraic reduction  $f : X \rightarrow C$  to the curve  $C \simeq \mathbb{P}_1$ . We will freely use the theory of compact complex surfaces, in particular of non-Kähler surfaces, and refer to [BHPV04] as general reference. Deviating from [BHPV04], we call bi-elliptic surfaces hyperelliptic.

An application of Riemann–Roch gives the following lemma.

LEMMA 4.1.  $\chi(D, \mathcal{O}_D) = 0$  for all effective divisors  $D$  on  $X$ .

*Proof.* By Riemann–Roch,

$$\chi(X, \mathcal{O}_X(-D)) = \chi(X, \mathcal{O}_X),$$

hence

$$\chi(D, \mathcal{O}_D) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-D)) = 0. \quad \square$$

LEMMA 4.2. Let  $s$  be the number of singular fibers and  $r$  be the numbers of irreducible components of the singular fibers. Then

$$r = s - 1 + b_1(X_c),$$

where  $X_c$  is a smooth fiber. Moreover,  $H_1(X_c, \mathbb{Z})$  is torsion-free for all smooth fibers  $X_c$ .

*Proof.* The first assertion is [CDP98, Lemma 3.2]. For the second, fix a smooth fiber  $X_c$ , let  $A$  be a subset of the union of all singular fibers of  $f$ , and set  $X' = X \setminus A$ . As seen in the proof of [CDP98, Lemma 3.2],

$$H_1(X', \mathbb{Z}) \simeq H_1(C \setminus f(A), \mathbb{Z}) \oplus H_1(X_c, \mathbb{Z}),$$

hence it suffices to show that  $H_1(X', \mathbb{Z})$  is torsion-free. To do this, we consider the cohomology sequence for pairs,

$$0 = H^4(X, \mathbb{Z}) \rightarrow H^4(A, \mathbb{Z}) \rightarrow H^5(X, A, \mathbb{Z}) \rightarrow H^5(X, \mathbb{Z}) \rightarrow 0.$$

Notice first that  $H^4(A, \mathbb{Z})$  is torsion-free. Further,  $H^5(X, \mathbb{Z})$  is torsion-free by the universal coefficient theorem, since  $H_4(X, \mathbb{Z})$  is torsion-free: by Poincaré duality,

$$H_4(X, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) = 0.$$

Actually,  $H^5(X, \mathbb{Z}) = 0$ . Consequently,

$$H^5(X, A, \mathbb{Z}) \simeq H_1(X', \mathbb{Z})$$

is torsion-free. □

LEMMA 4.3. Let  $X_c$  be a smooth fiber of  $f$ . Then  $X_c$  is either a primary Hopf surface, an Inoue surface, a torus, a hyperelliptic surface with torsion-free first homology group or a primary Kodaira surface with torsion-free first homology group.

*Proof.* Note first that  $K_{X_c}$  is topologically trivial, since  $K_X$  is topologically trivial, due to  $b_2(X) = 0$ . Then we use the tangent sequence

$$0 \rightarrow T_{X_c} \rightarrow T_X|_{X_c} \rightarrow N_{X_c/X} \simeq \mathcal{O}_{X_c} \rightarrow 0$$

and observe that  $c_2(X) = 0$ , since  $b_4(X) = 0$ . Thus  $c_2(X_c) = 0$ . Since the (sufficiently) general fiber of an algebraic reduction has non-positive Kodaira dimension [Uen75, Theorem 12.1], so does every smooth fiber (see, for example, [BHPV04, VI.8.1]). Then we conclude by surface classification and using the torsion-freeness of  $H_1(X_c, \mathbb{Z})$  (Lemma 4.2). Note here that  $H_1(X_c, \mathbb{Z})$  for a secondary Kodaira surface  $X_c$  has torsion, since  $c_1(X_c)$  is torsion in  $H^2(X_c, \mathbb{Z})$  (then apply the universal coefficient theorem) and that a secondary Hopf surface has torsion in  $H_1(X_c, \mathbb{Z})$  by definition.  $\square$

**COROLLARY 4.4.** *All fibers of  $f$  are irreducible unless the general fiber of  $f$  is a torus, a hyperelliptic surface with torsion-free first homology or a primary Kodaira surface with torsion-free first homology.*

We fix some notation for the rest of the paper.

*Notation 4.5.* Let  $S \subset X$  be an irreducible reduced surface. In particular,  $S$  is Gorenstein. We denote by  $\omega_S$  the dualizing sheaf, which is a line bundle. Let

$$\eta : \tilde{S} \rightarrow S$$

be the normalization of  $S$ ; denote by  $N \subset S$  the non-normal locus, equipped with the complex structure given by the conductor ideal. Let  $\tilde{N} \subset \tilde{S}$  be the complex-analytic preimage. Let

$$\pi : \hat{S} \rightarrow \tilde{S}$$

be a minimal desingularization and

$$\sigma : \hat{S} \rightarrow S_0$$

be a minimal model. For the class of  $\omega_S$  we write  $K_S$ , and analogously for  $\omega_{\tilde{S}}$ , etc.

**LEMMA 4.6.** *By Notation 4.5, we have*

$$\omega_{\tilde{S}} \simeq \eta^*(\omega_S) \otimes \mathcal{I}_{\tilde{N}}$$

and

$$\omega_{\hat{S}} \simeq \pi^*\eta^*(\omega_S) \otimes \pi^*(\mathcal{I}_{\tilde{N}}) \otimes \mathcal{O}_{\hat{S}}(E) = \pi^*\eta^*(\omega_S) \otimes \mathcal{O}_{\hat{S}}(-\hat{N}) \otimes \mathcal{O}_{\hat{S}}(-\hat{E}),$$

with an effective divisor  $E$  supported on the exceptional locus of  $\pi$ , and  $\hat{N}$  the strict transform of  $\tilde{N}$  in  $\hat{S}$ . Moreover, there are exact sequences

$$0 \rightarrow \mathcal{O}_S \rightarrow \eta_*(\mathcal{O}_{\tilde{S}}) \rightarrow \omega_S^{-1} \otimes \omega_N \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_N \rightarrow \eta_*(\mathcal{O}_{\tilde{N}}) \rightarrow \omega_S^{-1} \otimes \omega_N \rightarrow 0.$$

*Proof.* [Mor82, ch. 3, § 8].  $\square$

As an immediate consequence, we note the following proposition.



PROPOSITION 4.7. *Let  $S$  be any irreducible reduced compact Gorenstein surface with  $\omega_S \equiv 0$  and  $\chi(S, \mathcal{O}_S) = 0$ . Then:*

- (a)  $\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \chi(N, \omega_S^{-1} \otimes \omega_N) = -\chi(N, \mathcal{O}_N)$ ;
- (b)  $\chi(\tilde{N}, \mathcal{O}_{\tilde{N}}) = 0$ .

*Proof.* The first equation in (a) follows from Lemma 4.6. As to the second equation in (a), observe by Serre duality that

$$\chi(N, \omega_S^{-1} \otimes \omega_N) = -\chi(N, \omega_{S|N}) = -\chi(N, \mathcal{O}_N),$$

since  $\omega_S \equiv 0$ . For the same reasons

$$\chi(\tilde{N}, \mathcal{O}_{\tilde{N}}) = \chi(N, \mathcal{O}_N) + \chi(N, \omega_S^{-1} \otimes \omega_N) = 0. \quad \square$$

PROPOSITION 4.8. *Let  $X_c$  be a smooth fiber of  $f$ . Then*

$$H^0(X_c, T_{X|X_c}) \neq 0.$$

*Proof.* Consider the exact sequence

$$0 \rightarrow H^0(X_c, T_{X_c}) \rightarrow H^0(X_c, T_{X|X_c}) \xrightarrow{\kappa} H^0(X_c, N_{X_c/X}) \simeq \mathbb{C}.$$

If  $H^0(X_c, T_{X_c}) \neq 0$ , the assertion is clear. So it remains to treat the case where  $X_c$  has no vector fields. Then by Lemma 4.3 and [Ino74],  $X_c$  is an Inoue surface of type  $S_M$  or  $S_N^-$ , in which cases  $H^1(X_c, T_{X_c}) = 0$ . Thus  $X_c$  is rigid and  $\kappa$  is surjective, so that  $H^0(X_c, T_{X|X_c}) \neq 0$  also in these cases.  $\square$

COROLLARY 4.9. *Let  $X_c = \lambda S$  be a fiber with  $S$  an irreducible singular surface and  $\lambda \geq 1$ . Then there exists a finite étale cover  $S' \rightarrow S$  such that  $H^0(S', T_{S'}) \neq 0$ .*

*Proof.* We consider the tangent sequence

$$0 \rightarrow T_S \rightarrow T_{X|S} \xrightarrow{\kappa} N_{S/X}.$$

If  $\lambda = 1$ , then  $N_{S/X} \simeq \mathcal{O}_S$ . However,  $\kappa$  is not surjective, since  $S$  is singular. Hence

$$H^0(S, T_S) = H^0(S, T_{X|S}).$$

By semicontinuity and Proposition 4.8,  $H^0(S, T_{X|S}) \neq 0$  and we conclude.

If  $\lambda \geq 2$ , arguing in the same way, we obtain a torsion line bundle  $\mathcal{L}$  on  $X$  such that

$$H^0(S, T_S \otimes \mathcal{L}|_S) \neq 0.$$

Then we pass to a finite étale cover  $\tilde{S} \rightarrow S$  to trivialize  $\mathcal{L}|_S$ .  $\square$

Remark 4.10. A vector field  $v \in H^0(S, T_S)$  canonically induces a vector field  $v_0 \in H^0(S_0, T_{S_0})$ . For brevity, we say that  $v_0$  comes from  $S$ .

PROPOSITION 4.11. *Let  $X_c = \lambda S$  with  $S$  singular and irreducible. Then  $S$  is non-normal.*

*Proof.* Suppose  $S$  normal and let  $\pi : \hat{S} \rightarrow S$  be a minimal desingularization and  $\sigma : \hat{S} \rightarrow S_0$  a minimal model.

(a) Suppose that  $S$  has only rational singularities, hence  $S$  has only rational double points. Then  $K_{\hat{S}} \equiv 0$ , in particular  $\hat{S}$  is a minimal surface containing  $(-2)$ -curves. By surface classification,  $\hat{S}$  is either a K3 surface, an Enriques surface, of type VII or non-Kähler of Kodaira dimension  $\kappa(\hat{S}) = 1$ . The first two cases are impossible since

$$\chi(\hat{S}, \mathcal{O}_{\hat{S}}) = \chi(S, \mathcal{O}_S) = 0.$$

If  $\hat{S}$  is of type VII, then it must be a Hopf surface or an Inoue surface, since  $K_{\hat{S}} \equiv 0$ , but these surfaces do not contain  $(-2)$ -curves. If  $\kappa(\hat{S}) = 1$ , then, since  $\hat{S}$  has a vector field, it does not any rational curve; see, for example, [GH90, Satz 1].

(b) Suppose now that  $S$  has at least one irrational singularity. Then  $\chi(\hat{S}, \mathcal{O}_{\hat{S}}) < \chi(S, \mathcal{O}_S) = 0$ , hence

$$h^1(S_0, \mathcal{O}_{S_0}) = h^1(\hat{S}, \mathcal{O}_{\hat{S}}) \geq 2.$$

Suppose first that  $S_0$  is not Kähler. By classification,  $S_0$  has to be a primary Kodaira surface or  $\kappa(S_0) = 1$ . By Corollary 4.9,  $H^0(S_0, T_{S_0}) \neq 0$  (up to finite étale cover). Choose a non-zero vector field  $v_0$  coming from  $S$ ; then  $v_0$  does not have zeros by classification [GH90, Satz 1]; note that if  $\kappa(S_0) = 1$ ,  $S_0$  is an elliptic bundle over a curve of genus at least 2. Hence we must have  $\hat{S} = S_0$ . But then  $\hat{S}$  does not contain contractible curves, so that  $S$  is smooth, a contradiction.

Thus  $S_0$  is Kähler. Since  $K_{\hat{S}} = \pi^*(K_S) - E$  with  $E$  a non-zero effective divisor,  $\kappa(S_0) = -\infty$  and  $S_0$  is a ruled surface over a curve  $B$  of genus  $g = g(B) \geq 2$ . Since  $S$  has an irrational singularity,  $\pi$  must contract an irrational curve whose normalization necessarily has genus at least  $g$ . Thus

$$h^0(S, R^1\pi_*(\mathcal{O}_{\hat{S}})) \geq g$$

and therefore

$$1 - g = \chi(\hat{S}, \mathcal{O}_{\hat{S}}) \leq \chi(S, \mathcal{O}_S) - g = -g,$$

which is absurd. □

### 5. Kodaira surfaces, hyperelliptic surfaces and tori

In this section we consider the case where the general fiber of  $f$  is a Kodaira surface, a hyperelliptic surface or a torus. We rule out the case of Kodaira and hyperelliptic fibers and show in the torus case that for general line bundles  $\mathcal{L}$  on  $X$ , the restriction  $\mathcal{L}|_{X_c}$  to any fiber is never torsion.

**PROPOSITION 5.1.** *Assume that the general fiber of  $f$  is a Kodaira surface or a torus. Then  $R^j f_*(\mathcal{O}_X)$  is locally free for all  $j$ ; in fact,  $h^j(X_c, \mathcal{O}_{X_c})$  is independent on  $c \in C$ .*

*Proof.* It suffices to show that  $h^2(X_c, \mathcal{O}_{X_c})$  is independent of  $c$ . Indeed, since  $h^0(X_c, \mathcal{O}_{X_c}) = 1$  for all  $c$  and since  $\chi(X_c, \mathcal{O}_{X_c})$  is constant,  $h^1(X_c, \mathcal{O}_{X_c})$  does not depend on  $c$  as well, and the assertions follow by Grauert’s theorem. By Serre duality,

$$H^2(X_c, \mathcal{O}_{X_c}) = H^0(X_c, \omega_{X_c}) = H^0(X_c, \omega_X|_{X_c}).$$

Setting  $\mathcal{L} = f_*(\omega_X)$ , a locally free sheaf of rank one, we obtain

$$\omega_X = f^*(\mathcal{L}) \otimes \mathcal{O}_X \left( \sum_i (m_i - 1) F_i \right),$$

where  $F_i$  are the non-reduced fiber components. In particular,  $\omega_{X|X_c} = \mathcal{O}_{X_c}$  for all reduced fibers  $X_c$  and therefore

$$h^0(X_c, \omega_{X_c}) = 1$$

for all those  $c$ . So let  $X_c$  be a non-reduced fiber and set  $Y = \text{red}(X_c)$ . We consider the complex subspace

$$Z = \sum (m_i - 1)F_i$$

of  $X_c$  and have an induced exact sequence

$$0 \rightarrow \mathcal{I}_{Z/X_c} \otimes \omega_{X|X_c} \rightarrow \omega_{X|X_c} \rightarrow \omega_{X|Z} \rightarrow 0.$$

Applying  $f_*$  and observing that

$$f_*(\omega_{X|X_c}) = \mathcal{O}_{\{c\}}$$

shows that the restriction map

$$H^0(X_c, \omega_{X|X_c}) \rightarrow H^0(Z, \omega_{X|Z})$$

vanishes. Since

$$\dim H^0(X_c, \omega_{X|X_c}) = \dim H^0(X_c, \mathcal{O}_{X_c}) = 1,$$

we conclude  $h^0(X_c, \omega_{X_c}) = 1$ . □

**COROLLARY 5.2.** *Assume that the general smooth fiber of  $F$  is a Kodaira surface, a hyperelliptic surface or a torus. Then the restriction map*

$$r_F : H^1(X, \mathcal{O}_X) \rightarrow H^1(F, \mathcal{O}_F)$$

*is surjective.*

*Proof.* Suppose first that  $F$  is a Kodaira surface or a torus. Then the assertion is [CDP98, Theorem 3.1]; the proof works since we now know that  $R^1f_*(\mathcal{O}_X)$  is locally free. If  $F$  is hyperelliptic, then  $H^1(F, \mathcal{O}_F)$  is one-dimensional, hence it suffices to show that  $r_F \neq 0$ . Let  $\mu : H^1(X, \mathcal{O}_X) \rightarrow \text{Pic}(X)$  be the canonical isomorphism and write  $\mu(\alpha) = \omega_X$ . Then  $r_F(\alpha) \neq 0$ . In fact, otherwise  $\omega_F = \omega_X|_F = \mathcal{O}_F$ , noticing also that  $c_1(\omega_F) = c_1(\omega_X|_F) = 0$  since  $H^2(X, \mathbb{Z}) = 0$ . But  $\omega_F \neq \mathcal{O}_F$ , a contradiction. □

As a consequence, we obtain the following corollary.

**COROLLARY 5.3.** *The general fiber of  $f$  cannot be a Kodaira or hyperelliptic surface.*

*Proof.* This is [CDP98, Proposition 3.6]. In the proof of Proposition 3.6, Theorem 3.1 is used which is now established by Corollary 5.2. Notice that in step 2 of the proof of Proposition 3.6 in [CDP98], the local freeness of  $R^j f_*(\tilde{\mathcal{L}})$  is used only generically. □

*Remark 5.4.* The same arguments also rule out Hopf surfaces of algebraic dimension 1.

From now, for the remainder of this section, we assume that the general fiber of  $f$  is a torus.

**PROPOSITION 5.5.** *There is an isomorphism  $R^1f_*(\mathcal{O}_X) \simeq \mathcal{O}_C(b_1) \oplus \mathcal{O}_C(b_2)$  with  $b_j \geq 0$ .*

*Proof.* By Proposition 5.1, the sheaf  $R^1 f_*(\mathcal{O}_X)$  is locally free of rank two. Write

$$R^1 f_*(\mathcal{O}_X) = \mathcal{O}_C(b_1) \oplus \mathcal{O}_C(b_2).$$

We observe that  $R^1 f_*(\mathcal{O}_X)$  is generically spanned by Corollary 5.2, since

$$H^1(X, \mathcal{O}_X) = H^0(C, R^1 f_*(\mathcal{O}_X)).$$

Hence  $b_j \geq 0$ . □

PROPOSITION 5.6. *For general  $\mathcal{L} \in \text{Pic}(X)$ , the restriction  $\mathcal{L}|_{X_c}$  is non-torsion for all  $c \in C$ .*

*Proof.* By Proposition 5.5,  $h^1(X, \mathcal{O}_X) \geq 2$  and the restriction

$$H^1(X, \mathcal{O}_X) \rightarrow H^1(X_c, \mathcal{O}_{X_c})$$

is surjective for all  $c$ . Consequently, the kernel of the restriction

$$\text{Pic}(X) = H^1(X, \mathcal{O}_X) \rightarrow \text{Pic}^\circ(X_c) = H^1(X_c, \mathcal{O}_{X_c})/H^1(X_c, \mathbb{Z})$$

is discrete for all  $c$  plus a linear subspace of codimension 2. Since  $\dim C = 1$ , it follows that for  $\mathcal{L} \in \text{Pic}(X)$  general, the restriction  $\mathcal{L}|_{X_c}$  is never trivial and thus also non-torsion. □

### 6. Hopf and Inoue surfaces

In this section we assume that the general fiber of  $f$  is a Hopf or Inoue surface and show that for general line bundles  $\mathcal{L}$  on  $X$ , the restriction  $\mathcal{L}|_{X_c}$  is never torsion.

PROPOSITION 6.1. *Assume that the general fiber of  $f$  is a Hopf or Inoue surface. Let  $\mathcal{L} \in \text{Pic}(X)$  be general. Then  $\mathcal{L}|_{X_c}$  is non-torsion for all  $c \in C$ , and the restriction map  $\text{Pic}(X) \rightarrow \text{Pic}(X_c)$  is surjective for any smooth fiber  $X_c$ .*

*Proof.* The exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^* \rightarrow 1$$

and our assumptions give

$$H^1(X, \mathbb{C}^*) = 0.$$

Moreover,  $H^2(X, \mathbb{C}^*)$  is torsion. Consider the canonical morphism

$$\lambda : H^0(C, R^1 f_*(\mathbb{C}^*)) \rightarrow H^2(C, \mathbb{C}^*) \simeq \mathbb{C}^*.$$

Then by the Leray spectral sequence,  $\lambda$  is injective and the cokernel is torsion. Hence

$$H^0(C, R^1 f_*(\mathbb{C}^*)) \simeq \mathbb{C}^*.$$

Choose

$$1 \neq u \in H^0(C, R^1 f_*(\mathbb{C}^*))$$

non-torsion. This section defines an inclusion

$$\iota : \mathbb{C}^* \rightarrow R^1 f_*(\mathbb{C}^*).$$

Let  $C_0$  be the smooth locus of  $f$  in  $C$ . We claim that

$$R^1 f_*(\mathbb{C}^*)|_{C_0} = R^1 f_{0*}(\mathbb{C}^*) \simeq \mathbb{C}^*. \tag{10}$$

Suppose first that claim (10) holds. Then we conclude as follows. Certainly,  $\iota$  is an isomorphism over  $C_0$ . Thus we obtain a sequence

$$0 \rightarrow \mathbb{C}^* \rightarrow R^1 f_*(\mathbb{C}^*) \rightarrow Q \rightarrow 0,$$

where  $Q$  is supported on the finite set  $C \setminus C_0$ . Since  $H^0(C, R^1 f_*(\mathbb{C}^*)) \simeq \mathbb{C}^*$  and since  $H^1(C, \mathbb{C}^*) = 0$ , it follows that  $H^0(C, Q) = 0$ , hence  $Q = 0$ . Thus  $\iota$  is an isomorphism everywhere and consequently  $u$  never takes value 1, nor does, by our choice of  $u$ , any multiple  $u^m$ . Hence  $u$  defines a line bundle  $\mathcal{L}$  such that  $\mathcal{L}|_{X_c}$  is non-torsion for all  $c \in C$ .

It remains to prove claim (10). As before, set  $\Delta = C \setminus C_0$ ,  $A = f^{-1}(\Delta)$  and  $X_0 = X \setminus A$ . Then, as in the proof of Lemma 4.2,

$$H^4(A, \mathbb{C}^*) = H^1(X_0, \mathbb{C}^*) = H^1(C_0, \mathbb{C}^*) \oplus H^0(C_0, R^1 f_{0*}(\mathbb{C}^*)).$$

Since  $H^4(A, \mathbb{C}^*) \simeq (\mathbb{C}^*)^s$ , it follows that

$$H^0(C_0, R^1 f_{0*}(\mathbb{C}^*)) \simeq \mathbb{C}^*.$$

Since  $R^1 f_{0*}(\mathbb{C}^*)$  is locally constant of rank one, the claim follows. □

As a consequence we obtain the following corollary.

**COROLLARY 6.2.**

- (a)  $R^1 f_*(\mathcal{O}_X) \simeq \mathcal{O}_C$ .
- (b)  $R^2 f_*(\mathcal{O}_X) = 0$ .
- (c) For general  $\mathcal{L} \in \text{Pic}(X)$  and all  $c \in C$ , we have

$$H^0(X_c, \mathcal{L}|_{X_c}) = H^0(X_c, \mathcal{L}^*|_{X_c}) = 0.$$

*Proof.* (a) Since  $R^1 f_*(\mathcal{O}_X)$  has rank one, we may write

$$R^1 f_*(\mathcal{O}_X) \simeq \mathcal{O}_C(a) \oplus \text{torsion}.$$

By Proposition 6.1,  $a = 0$ . So it remains to show that  $R^1 f_*(\mathcal{O}_X)$  is torsion-free. If not, there exists a line bundle  $\mathcal{M}$ , such that  $\mathcal{M}|_{X_{c_0}}$  is non-torsion for some  $c_0$  but  $\mathcal{M}|_{X_c} \simeq \mathcal{O}_{X_c}$  for  $c \neq c_0$ . Write  $S = \text{red}(X_{c_0})$ . Then

$$\mathcal{M} = f^* f_*(\mathcal{M}) \otimes \mathcal{O}_X(D),$$

with an effective divisor  $D$  supported on  $S$ . Since  $S$  is irreducible,  $D = mS$  and therefore  $\mathcal{O}_X(D)|_{X_c}$  is a torsion line bundle, which is a contradiction.

(b) By (a),  $h^1(X, \mathcal{O}_X) = 1$ . Since the general fiber of  $f$  has negative Kodaira dimension, we have

$$H^3(X, \mathcal{O}_X) = H^0(X, \omega_X) = 0.$$

Thus we conclude from  $\chi(X, \mathcal{O}_X) = 0$  that

$$H^2(X, \mathcal{O}_X) = 0.$$

Hence, by the Leray spectral sequence,  $R^2f_*(\mathcal{O}_X)$  must be torsion-free, therefore

$$R^2f_*(\mathcal{O}_X) = 0.$$

(c) As a consequence of (b),  $R^2f_*(\mathcal{L}) = 0$  for general  $\mathcal{L}$ , hence

$$H^2(X_c, \mathcal{L}_{X_c}) = 0$$

for all  $c$ . Thus

$$H^0(X_c, \mathcal{L}_{X_c}) = 0$$

for general  $\mathcal{L}$  and all  $c$  as well. In summary, we may say that

$$H^0(X_c, \mathcal{L}|_{X_c}) = H^0(X_c, \mathcal{L}^*_{|X_c})$$

for general  $\mathcal{L}$  and all  $c$ .

Now  $\omega_X^m$  defines a section  $t_m \in H^0(C, R^1f_*(\mathcal{O}_X^*))$ . Notice that for  $c \in C_0$ , the smooth locus of  $f$ , the bundle  $\omega_{X|X_c}^m = \omega_{|X_c}^m$  is never trivial and thus  $t_m$  does not take value 1 on  $C_0$ . Since  $R^1f_*(\mathcal{O}_X^*) \simeq \mathcal{O}_C^*$ , the section never takes value 1, hence our claim follows.  $\square$

We will need the following basic statement on Hopf and Inoue surfaces.

PROPOSITION 6.3. *Let  $S$  be a primary Hopf surface. Assume that*

$$H^0(S, \Omega_S^1 \otimes \mathcal{L}) \neq 0$$

*for some line bundle  $\mathcal{L}$  on  $S$ . Then*

$$H^0(S, \mathcal{L}) \neq 0.$$

*Proof.* Choose a vector field  $v$  on  $S$  and let  $C$  be the zero locus of  $v$  which is purely one-dimensional. We obtain an exact sequence

$$0 \rightarrow \mathcal{O}_S(C) \rightarrow T_S \rightarrow \mathcal{O}_S(-C) \otimes \omega_S^{-1} \rightarrow 0.$$

Dualizing,

$$0 \rightarrow \mathcal{O}_S(C) \otimes \omega_S \rightarrow \Omega_S^1 \otimes \mathcal{O}_S(-C) \rightarrow 0.$$

Hence

$$H^0(S, \mathcal{O}_S(C) \otimes \omega_S \otimes \mathcal{L}) \neq 0$$

or

$$H^0(S, \mathcal{O}_S(-C) \otimes \mathcal{L}) \neq 0.$$

In the latter case the claim is clear. In the first case we observe that

$$H^0(S, \omega_S^{-1} \otimes \mathcal{O}_S(-C)) \neq 0.$$

Indeed, there exists another vector field  $v'$ , and  $v \wedge v'$  is a section of  $\omega_S^{-1}$  vanishing on  $C$ .  $\square$

PROPOSITION 6.4. *Let  $S$  be an Inoue surface. Then there is a unique line bundle  $\mathcal{L}$  such that*

$$H^0(S, \Omega_S^1 \otimes \mathcal{L}) \neq 0.$$

*Moreover, one of the following statements holds:*

- (a) either  $H^0(S, T_S) \neq 0$  and  $\mathcal{L} = \omega_S^{-1}$ ;
- (b) or  $H^0(S, T_S) = 0$  and  $\mathcal{L}^{\otimes 2} \simeq \omega_S^{-1}$ .

*Proof.* The existence of  $\mathcal{L}$  is classical [Ino74].

If  $S$  has a non-zero vector field  $v$ , necessarily without zeroes, then  $v$  induces an exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow T_S \rightarrow \omega_S^* \rightarrow 0,$$

and the claim is immediate, since  $S$  has no curves and since  $H^0(S, \Omega_S^1) = 0$ . If  $H^0(S, T_S) = 0$ , consider the exact sequence

$$0 \rightarrow \mathcal{L}^* \rightarrow \Omega_S^1 \rightarrow \mathcal{L} \otimes \omega_S \rightarrow 0.$$

Since the sequence does not split,

$$H^1(S, \omega_S^{-1} \otimes \mathcal{L}^{\otimes -2}) \neq 0.$$

Hence either  $\omega_S^{-1} \simeq \mathcal{O}_{\otimes 2}$  or  $\omega_S \simeq \mathcal{L}$ , [Ino74, Lemma 1]. However, the second case cannot happen, since

$$H^0(S, \Omega_S^1 \otimes \omega_S) \simeq H^2(S, T_S) \neq 0$$

[Ino74, Proposition 2]. □

### 7. Proof of Theorem 2.2

As already said in the introduction, it suffices to prove Proposition 2.4. Thus we need to show that

$$H^2(X_c, T_{X|X_c} \otimes \mathcal{L}|_{X_c}) = 0$$

for all  $c \in C$ . By Serre duality, this comes down to showing that

$$H^0(X_c, \Omega_{X|X_c}^1 \otimes \mathcal{L}|_{X_c}) = 0$$

for some  $\mathcal{L} \in \text{Pic}(X)$  and for all  $c \in C$ .

We first consider irreducible fibers. Let

$$S = \text{red}X_c.$$

Using the (co)tangent sheaf sequence

$$0 \rightarrow N_S^* \rightarrow \Omega_{X|S}^1 \rightarrow \Omega_S^1 \rightarrow 0,$$

it is immediate, provided  $\mathcal{L}|_S$  is non-torsion, that it suffices to show the statement

$$H^0(S, \tilde{\Omega}_S^1 \otimes \mathcal{L}|_S) = 0, \tag{11}$$

where

$$\tilde{\Omega}_S^1 = \Omega_S^1 / \text{torsion}.$$

We first treat smooth fibers  $S = X_c$ .

**PROPOSITION 7.1.** *Equation (11) holds for smooth fibers  $S$  (independent on the structure of the general fiber), that is, for  $\mathcal{L} \in \text{Pic}(X)$  general,*

$$H^0(S, \Omega_S^1 \otimes \mathcal{L}|_S) = 0$$

*simultaneously for all smooth fibers  $S$ .*

*Proof.* (a) First, if  $S$  is a torus, then

$$H^0(S, T_S \otimes \mathcal{A}) = H^0(S, \Omega_S^1 \otimes \mathcal{A}) = 0$$

for all non-trivial  $\mathcal{A}$ , hence we may take any  $\mathcal{L}$  such that  $\mathcal{L}|_{X_c}$  is never trivial (Proposition 5.6).

(b) If  $S$  is a Hopf surface, then by Corollary 6.2,  $H^0(S, \mathcal{L}|_S) = 0$  for general  $\mathcal{L}$ , hence we conclude by Propositions 6.1 and 6.3.

(c) If  $S$  is an Inoue surface with a vector field, then for  $\mathcal{L}$  general,  $\mathcal{L}^* \otimes \omega_X$  is also general, hence

$$H^0(S, \mathcal{L}^*_{|S} \otimes (\omega_X)_{|S}) = H^0(S, \mathcal{L}^* \otimes \omega_S) = 0,$$

hence we conclude by Propositions 6.1 and 6.4.

(d) Finally, assume that  $S$  is an Inoue surface without vector field. Then we argue as in (c), observing that  $(\mathcal{L}^*)^{\otimes 2} \otimes \omega_X$  is general for general  $\mathcal{L}$ . □

*Remark 7.2.* Since the conormal bundle of a multiple fiber is torsion, the arguments also apply to fibers  $X_c = \lambda S$  with  $\lambda \geq 2$  and  $S$  smooth.

PROPOSITION 7.3. Equation (11) holds for singular reduced fibers.

*Proof.* Recall Notation 4.5. By Lemma 4.6 and Proposition 4.11,  $\kappa(S_0) = -\infty$ , the surface  $S$  is non-normal and

$$H^2(\tilde{S}, \mathcal{O}_{\tilde{S}}) = H^0(\tilde{S}, \omega_{\tilde{S}}) = 0.$$

Arguing by contradiction, there exists a one-dimensional family  $\mathcal{L}_t$  of line bundles on  $X$  such that

$$H^0(S, \tilde{\Omega}_S^1 \otimes \mathcal{L}_t|_S) \neq 0.$$

Passing to a desingularization and then to a minimal model  $S_0$  as in Notation 4.5, there are numerically trivial line bundles  $\mathcal{M}_t$  on  $S_0$  such that

$$\mathcal{M}_t = \sigma_* \pi^* \eta^* (\mathcal{L}_t)^{**}$$

with a one-dimensional family of sections in

$$H^0(S_0, \Omega_{S_0}^1 \otimes \mathcal{M}_t).$$

Thus

$$H^0(S_0, \Omega_{S_0}^1 \otimes \mathcal{M}_t) \neq 0. \tag{12}$$

Observe that all line bundles  $\mathcal{M}_t$  might be trivial.

*Step 1.* Suppose first that  $S_0$  is Kähler. Then by (12),  $S_0$  must be ruled over a curve of genus at least 2.

CLAIM.  $\tilde{S}$  has rational singularities only.

*Proof of the claim.* Assume to the contrary that  $\tilde{S}$  has an irrational singularity. We claim that  $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$ . In fact,  $\pi$  must contract a curve  $B_0$  projecting onto  $B$ . Thus  $h^1(B_0, \mathcal{O}_{B_0}) \geq g$  and therefore  $h^0(\tilde{S}, R^1 \pi_*(\mathcal{O}_{\tilde{S}})) \geq g$ . Since  $H^2(\tilde{S}, \omega_{\tilde{S}}) = 0$ , the Leray spectral sequence yields  $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$  (and  $h^0(\tilde{S}, R^1 \pi_*(\mathcal{O}_{\tilde{S}})) = g$ ).



Thus all line bundles  $\eta^*(\mathcal{L}_t)$  are trivial and we obtain a one-dimensional family  $\tilde{\omega}_t$  of holomorphic 1-forms on  $\tilde{S}$ . Moreover there exists a one-dimensional family  $\omega_t$  of 1-forms on  $B$  such that

$$\sigma^*p^*(\omega_t) = \pi^*(\tilde{\omega}_t),$$

where  $p : S_0 \rightarrow B$  is the ruling. Since  $p(\sigma(B_0)) = B$ , we have  $\iota_{B_0}^* \sigma^*p^*(\omega_t) \neq 0$ . On the other hand, since  $\pi$  contracts  $B_0$ , it follows that  $\iota_{B_0}^*(\pi^*(\omega_t)) = 0$ , a contradiction. This proves the claim and thus  $\tilde{S}$  has rational singularities, only.

In this case the morphism  $p_0 : S_0 \rightarrow B$  induced a morphism  $\tilde{p} : \tilde{S} \rightarrow B$ . In the language of divisors and using the notation of (4.5) and (4.6), we have

$$-K_{\hat{S}} \equiv \hat{N} + \hat{E},$$

where  $\hat{N}$  is the strict transform of  $\tilde{N}$  in  $\hat{S}$ . Set

$$N_0 = \sigma_*(\hat{N}).$$

We are now using the theory of ruled surfaces as in [Har77, § V.2], also adopting the notation from [Har77]. In particular, we have the invariant  $e$  and a section  $C_0$  with minimal self-intersection  $C_0^2 = -e$ . Moreover,

$$-K_{S_0} \equiv 2C_0 + (e + 2 - 2g)F,$$

where  $F$  is a fiber of  $p_0$  and  $g = g(B)$  the genus of  $B$ . Since  $\tilde{S}$  has rational singularities,  $\pi$  cannot contract any curve projecting onto  $B$ . Hence we must have

$$N \equiv 2C_0 + aF$$

with  $a \leq e + 2 - 2g$ . Taking into account the numerical description of irreducible curves in  $S_0$ , as given in [Har77, § V.2], it follows immediately that  $e > 0$  and that

$$N = 2C_0 + R$$

with an effective divisor  $R \sim aF$  (note that the curve  $C_0$  is the unique contractible curve in  $S_0$ ). Consequently,  $\tilde{N}$  has a unique component, say  $\tilde{N}_1$ , projecting onto  $B$ , and this component has multiplicity 2. The map  $\tilde{p} : \tilde{S} \rightarrow B$  induces a holomorphic map  $p : S \rightarrow B'$  and a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\eta} & S \\ \tilde{p} \downarrow & & \downarrow p \\ B & \xrightarrow{\tau} & B' \end{array}$$

The general fiber  $S_b$  of  $p$  is a reduced Gorenstein curve with

$$\omega_{S_b} \equiv 0$$

whose normalization of  $S_b$  is a disjoint union of smooth rational curves. Thus, if  $S_b$  is irreducible, then it is a rational curve with one node or cusp, and if  $S_b$  is reducible, it is a cycle of smooth rational curves. In the case where  $S_b$  has a node or is a cycle, the normalization map  $\eta$  is generically  $2 : 1$  along  $\tilde{N}_1$ . In these cases, however,  $\tilde{N}_1$  would be reduced (see [KW88]), a contradiction. In the remaining case,  $\eta$  has degree 1 along  $\tilde{N}_1$ , hence  $\tau$  also has degree 1.

Unless  $\tau$  is an isomorphism and  $g(B) = 2$ , we have  $h^1(B', \mathcal{O}_{B'}) \geq 3$ , hence  $h^1(S, \mathcal{O}_S) \geq 3$ . Since  $\chi(S, \mathcal{O}_S) = 0$ , we conclude

$$h^0(S, \omega_S) = h^2(S, \mathcal{O}_S) \geq 2.$$

Since  $S$  is Moishezon and  $\omega_S \equiv 0$ , this is impossible. Alternatively, apply Proposition 5.5 or Corollary 6.2, respectively.

Hence  $\tau$  is biholomorphic, that is,  $p$  maps to the smooth curve  $B$  of genus 2 and  $h^1(S, \mathcal{O}_S) = h^1(B, \mathcal{O}_B) = 2$ . Moreover,  $h^0(S, \omega_S) = 1$ , and therefore  $\omega_S \simeq \mathcal{O}_S$ . The map  $p$  being flat,  $R^1p_*(\mathcal{O}_S)$  is locally free of rank one, and by relative duality,

$$R^1p_*(\mathcal{O}_S) \simeq p_*(\omega_{S/B})^* = \omega_B,$$

hence

$$H^0(B, R^1p_*(\mathcal{O}_S)) \neq 0.$$

But then  $h^1(S, \mathcal{O}_S) > h^1(B, \mathcal{O}_B)$ , a contradiction. This shows that  $g(B) \geq 2$  is impossible and concludes the proof in the Kähler case.

*Step 2.* We are thus reduced to the case where  $S_0$  is not Kähler.

If  $S_0$  is of type VII, then  $H^0(S_0, \Omega_{S_0}^1) = 0$ , hence  $H^0(S_0, \Omega_{S_0}^1 \otimes \mathcal{M}) = 0$  for  $\mathcal{M}$  general, contradicting (12).

The same argument applies to a secondary Kodaira surface  $S_0$ . If  $S_0$  is a primary Kodaira surface, then the cotangent sequence reads

$$0 \rightarrow \mathcal{O}_{S_0} \rightarrow \Omega_{S_0}^1 \rightarrow \mathcal{O}_{S_0} \rightarrow 0,$$

which immediately gives a contradiction by tensorizing with  $\mathcal{M}_t$ .

It remains to exclude the case  $\kappa(S_0) = 1$ . Since  $H^0(S_0, T_{S_0}) \neq 0$  ((4.9) and (4.10)), the Iitaka fibration  $h_0 : S_0 \rightarrow B$  is an elliptic bundle over a curve of genus  $g(B) \geq 2$  [GH90, Satz 1] and, as already noticed, the induced vector field  $v_0$  has no zeros. Hence  $\tilde{S} = \hat{S} = S_0$ . Since  $\omega_{\tilde{S}} = \mathcal{I}_{\tilde{N}} \otimes \eta^*(\omega_S)$ , we have

$$h^2(S, \mathcal{O}_S) \geq h^2(\tilde{S}, \mathcal{O}_{\tilde{S}}),$$

hence  $h^2(S, \mathcal{O}_S) \geq 2$ , contradicting Proposition 5.5 or Corollary 6.2, respectively. □

*Remark 7.4.* If the fiber  $X_c = \lambda S$  with  $S$  an irreducible reduced singular surface and  $\lambda \geq 2$ , we argue in the same way, passing to a finite étale cover.

Finally, we have to treat reducible fibers.

**PROPOSITION 7.5.** *Equation (11) holds for reducible fibers.*

*Proof.* Let

$$F = \sum a_i S_i$$

be a reducible fiber. Arguing by contradiction, there is a one-dimensional family  $\mathcal{L}_t$  of line bundles on  $X$  such that

$$H^0(F, \Omega_X^1|_F \otimes (\mathcal{L}_t)|_F) \neq 0.$$

Hence there exists a number  $i_0$  such that

$$H^0(S_{i_0}, \tilde{\Omega}_X^1|_{S_{i_0}} \otimes (\mathcal{L}_t)|_{S_{i_0}}) \neq 0$$

for all  $t$ , and therefore

$$H^0(S_{i_0}, \tilde{\Omega}_{S_{i_0}}^1 \otimes (\mathcal{L}_t)_{|S_{i_0}}) \neq 0.$$

Now we argue as in Proposition 7.3 to obtain a contradiction. One might also use the line bundle  $\mathcal{O}_X(-kS_{i_1})$  for  $k \gg 0$ , where the surface  $S_{i_1}$  meets in  $S_{i_0}$  in a curve.  $\square$

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