THE ALGEBRAIC THEORY OF SURGERY
I. FOUNDATIONS

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Introduction

An algebraic theory of surgery on chain complexes with an abstract Poincaré duality should be a 'simple and satisfactory algebraic version of the whole setup' to quote § 17G of the book of Wall [**25**] on the surgery of compact manifolds. The theory of Mishchenko [**10**] describes the symmetric part of the surgery obstruction, and so determines it modulo 8-torsion. The theory presented here obtains the quadratic structure as well, capturing all of the surgery obstruction. Our theory of surgery is homotopy invariant in geometry and chain homotopy invariant in algebra.

An n-dimensional algebraic Poincaré complex over a ring A with an involution $^-: A \to A; a \mapsto \bar{a}$ is an A-module chain complex C with an n-dimensional Poincaré duality $H^*(C) = H_{n-*}(C)$.

We shall use n-dimensional algebraic Poincaré complexes to define two sequences of covariant functors

$$L^n \; \{L_n\}: (\text{rings with involution}) \to (\text{abelian groups}) \quad (n \in \mathbf{Z})$$

such that $L^0(A)$ {respectively $L_0(A)$} is the Witt group of non-singular symmetric {quadratic} forms over A. The quadratic L-groups $L_n(A)$ will turn out to be the surgery obstruction groups of Wall [**25**], with a 4-periodicity

$$L_n(A) = L_{n+4}(A) \quad (n \in \mathbf{Z}).$$

The higher symmetric L-groups $L^n(A)\,(n \geqslant 0)$ were introduced by Mishchenko [**10**], and are not 4-periodic in general (contrary to the claim made there). The lower symmetric L-groups are defined to be such that

$$L^n(A) = L_n(A) \quad (n \leqslant -3).$$

There are defined symmetrization maps

$$1+T: L_n(A) \to L^n(A) \quad (n \in \mathbf{Z}),$$

which are isomorphisms modulo 8-torsion for general A, and actually isomorphisms if 2 is invertible in A.

The symmetric {quadratic} L-groups $L^n(A)$ $\{L_n(A)\}$ $(n \in \mathbf{Z})$, are to the symmetric {quadratic} Witt group $L^0(A)$ $\{L_0(A)\}$ what the algebraic K-groups $K_n(A)$ $(n \in \mathbf{Z})$ are to the projective class group $K_0(A)$.

Part I of the paper covers only those algebraic aspects of the theory which are needed for the applications to topology considered in Part II (Ranicki [17]), namely the construction of the L-groups by means of an algebraic Poincaré cobordism relation, algebraic surgery, the identification of the quadratic L-groups $L_*(\mathbf{Z}[\pi])$ with the surgery obstruction groups $L_*(\pi)$, and the products

$$\left\{\begin{array}{l}\otimes: L^m(A) \otimes_{\mathbf{Z}} L^n(B) \to L^{m+n}(A \otimes_{\mathbf{Z}} B) \\ \otimes: L_m(A) \otimes_{\mathbf{Z}} L_n(B) \to L_{m+n}(A \otimes_{\mathbf{Z}} B)\end{array}\right. \quad (m, n \in \mathbf{Z}).$$

Part II uses algebraic Poincaré complexes to give a chain homotopy invariant account of geometric surgery theory, including formulae for the surgery obstructions of products and composites of normal maps of geometric Poincaré complexes. Later parts will be devoted to the following topics:

a change of rings exact sequence for a morphism $f: A \to B$ of rings with involution

$$\left\{\begin{array}{l}\ldots \longrightarrow L^{n+1}(f) \longrightarrow L^n(A) \xrightarrow{f} L^n(B) \longrightarrow L^n(f) \longrightarrow L^{n-1}(A) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow \ldots, \\ \ldots \longrightarrow L_{n+1}(f) \longrightarrow L_n(A) \xrightarrow{f} L_n(B) \longrightarrow L_n(f) \longrightarrow L_{n-1}(A) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow \ldots,\end{array}\right.$$

involving relative L-groups $L^*(f)$ $\{L_*(f)\}$;

a localization exact sequence, identifying the relative L-groups of a localization map $A \to S^{-1}A$ inverting a multiplicative subset S of A with the L-groups of algebraic Poincaré complexes over A which become contractible over $S^{-1}A$;

Mayer–Vietoris exact sequences for cartesian and localization-completion squares of rings with involution

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

of the type

$$\begin{cases} \ldots \to L^{n+1}(D) \to L^n(A) \to L^n(B) \oplus L^n(C) \to L^n(D) \to L^{n-1}(A) \to \ldots, \\ \ldots \to L_{n+1}(D) \to L_n(A) \to L_n(B) \oplus L_n(C) \to L_n(D) \to L_{n-1}(A) \to \ldots; \end{cases}$$

splitting theorems for the L-groups $L^*(D)$ $\{L_*(D)\}$ of a free product with amalgamation $D = B *_A C$, of the type

$$L_n\begin{pmatrix} A & \longrightarrow & B \\ \Big\downarrow & & \Big\downarrow \\ C & \longrightarrow & D \end{pmatrix} = \text{Cappell's } \mathrm{Unil}_n, \quad L_n(D) = L_n(A \to B \oplus C) \oplus \mathrm{Unil}_n,$$

and similarly for generalized Laurent extensions;

simplicial spectra $\mathbf{L}^*(A)$ $\{\mathbf{L}_*(A)\}$ such that

$$\begin{cases} \pi_n(\mathbf{L}^*(A)) = L^n(A) \\ \pi_n(\mathbf{L}_*(A)) = L_n(A) \end{cases} \quad (n \in \mathbf{Z});$$

the application of algebraic L-theory to the classification of topological bundle structures on spherical fibrations, and of topological manifold structures on geometric Poincaré complexes;

codimension-2 surgery (for example, knot theory) and the Cappell–Shaneson Γ-groups, involving the algebraic cobordism groups of quadratic complexes over A which become Poincaré complexes over B, for some morphism of rings with involution $A \to B$.

The reader is referred to Ranicki [**18, 19**] for a preliminary account of some of these topics, and to Ranicki [**20**] for an application of the theory to the surgery obstruction of a disjoint union.

A 'surgery' on an n-dimensional manifold M is the process of obtaining a new manifold M', by first cutting out from M an embedded $S^r \times D^{n-r}$ $(0 \leqslant r < n)$, and then glueing in $D^{r+1} \times S^{n-r-1}$

$$M' = \overline{M \backslash S^r \times D^{n-r}} \cup_{S^r \times S^{n-r-1}} D^{r+1} \times S^{n-r-1}.$$

If M is compact, oriented, smooth and closed then so is M', and the $(n+1)$-dimensional manifold

$$N = M \times [0,1] \cup_{S^r \times D^{n-r} \times 1} D^{r+1} \times D^{n-r}$$

is an oriented cobordism from M to M', that is

$$\partial N = M \cup -M'.$$

The surgery technique was initiated by Milnor [**9**] (where the idea of surgery is credited to Thom); it was proved there that every oriented cobordism is obtained by stringing together such elementary cobordisms.

Surgery has turned out to be instrumental in the classification of compact manifolds, particularly in dimensions greater than 4, starting with the classification of homotopy spheres due to Kervaire and Milnor. In the applications it is necessary to keep track of the behaviour of the stable normal bundle ν_M of M under the surgeries, in order to produce further embeddings $S^r \hookrightarrow M$ with trivial normal bundle (that is, with an extension $S^r \times D^{n-r} \hookrightarrow M$) on which to perform surgery. The situation was formalized by Browder [2], who introduced the concept of a 'normal map'. This is a degree 1 map from a manifold M to a geometric Poincaré complex X

$$f\colon M \to X$$

together with a covering map of stable bundles

$$b\colon \nu_M \to \nu_X.$$

Surgery obstruction theory has to determine whether a normal map $(f, b)\colon M \to X$ extends to a normal bordism

$$((g; f, f'), (c; b, b'))\colon (N; M, M') \to (X \times [0,1]; X \times 0, X \times 1)$$

such that $f'\colon M' \to X$ is a homotopy equivalence. For example, a geometric Poincaré complex X is homotopy equivalent to a smooth manifold if and only if there exists a stable vector bundle ν_X in the Spivak normal class for which the resulting normal map $(f, b)\colon (M, \nu_M) \to (X, \nu_X)$ obtained by the Browder–Novikov transversality construction is normal bordant to a homotopy equivalence. The surgery obstruction theory of Wall [25] associates to an n-dimensional normal map $(f, b)\colon M \to X$ $(n \geqslant 5)$ an element of an abelian group

$$\theta(f, b) \in L_n(\pi_1(X))$$

such that $\theta(f, b) = 0$ if and only if (f, b) is normal bordant to a homotopy equivalence. The surgery obstructions take their values in the algebraic L-groups $L_n(\pi)$ defined for $n \pmod 4$ by

$L_{2i}(\pi) =$ the Witt group of non-singular $(-)^i$quadratic forms over $\mathbf{Z}[\pi]$,

$L_{2i+1}(\pi) =$ the commutator quotient of a stable $(-)^i$unitary group over $\mathbf{Z}[\pi]$.

The construction of $\theta(f, b)$ makes use of the geometric intersection properties of the kernel $\mathbf{Z}[\pi_1(X)]$-modules

$$K_*(M) = \ker(\tilde{f}_*\colon H_*(\tilde{M}) \to H_*(\tilde{X}))$$

remaining after surgery below the middle dimension. The present paper views the L-groups $L_n(\pi)$ as being defined for $n \geqslant 0$ by the cobordism groups of n-dimensional $\mathbf{Z}[\pi]$-module chain complexes C with a quadratic

Poincaré duality $\psi\colon H^{n-*}(C) \cong H_*(C)$, and makes use of homotopy theory to express the surgery obstruction (without preliminary surgeries) as

$$\theta(f,b) = (C,\psi) \in L_n(\pi_1(X))$$

for some $\mathbf{Z}[\pi_1(X)]$-module chain complex C such that

$$H_*(C) = K_*(M).$$

An 'algebraic Poincaré complex' in the sense of Mishchenko [**10**] is a chain complex of finitely generated projective A-modules

$$C\colon C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \dots \xrightarrow{d} C_1 \xrightarrow{d} C_0$$

together with a collection of A-module morphisms

$$\varphi_s\colon C^{n-r+s} \to C_r \quad (s \geqslant 0)$$

such that

$$d\varphi_s + (-)^r\varphi_s d^* + (-)^{n+s-1}(\varphi_{s-1} + (-)^s T\varphi_{s-1}) = 0\colon C^{n-r+s-1} \to C_r \qquad (\varphi_{-1} = 0),$$

and such that the chain map

$$\varphi_0\colon C^{n-*} \to C$$

is a chain equivalence, inducing abstract Poincaré duality isomorphisms

$$\varphi_0\colon H^{n-r}(C) \to H_r(C).$$

Here, C^{n-*} is the chain complex of dual A-modules

$$C^r = C_r^* = \mathrm{Hom}_A(C_r, A)$$

with

$$(C^{n-*})_r = C^{n-r}, \quad d_{C^{n-*}} = (-)^r d^*\colon C^{n-r} \to C^{n-r+1},$$

and T is the duality involution

$$T\colon \mathrm{Hom}_A(C^p, C_q) \to \mathrm{Hom}_A(C^q, C_p);\ \varphi \mapsto (-)^{pq}\varphi^* \quad (C_p^{**} = C_p).$$

The definition was inspired by the symmetry properties of the chain equivalence

$$\varphi_0 = [M]\cap - : C(M)^{n-*} \to C(M) \quad ([M] = 1 \in H_n(M) = \mathbf{Z})$$

inducing the Poincaré duality isomorphisms

$$\varphi_0 = [M]\cap - : H^{n-r}(M) \to H_r(M)$$

of a compact oriented n-dimensional manifold M, with φ_1 a chain homotopy between φ_0 and $T\varphi_0$, φ_2 a higher chain homotopy between φ_1 and $T\varphi_1$, and so on. The 'algebraic Poincaré bordism' groups $\Omega_n(A)$ of Mishchenko [**10**] (which we denote by $L^n(A)$) are the abelian groups of equivalence classes of such n-dimensional **symmetric** Poincaré complexes over A (C,φ) (as we shall call them) under a cobordism relation given by abstract Poincaré–Lefschetz duality, with addition by

$$(C,\varphi)+(C',\varphi') = (C\oplus C',\ \varphi\oplus\varphi') \in L^n(A).$$

An n-dimensional geometric Poincaré complex X determines in a natural way an n-dimensional symmetric Poincaré complex over $\mathbf{Z}[\pi_1(X)]$,

$$\sigma^*(X) = (C(\tilde{X}),\varphi_{\tilde{X}}),$$

with $\tilde{X}$ the universal covering space of X. The 'higher signature' of Mishchenko [**10**] is the symmetric Poincaré cobordism class

$$\sigma^*(X) \in L^n(\mathbf{Z}[\pi_1(X)]).$$

The symmetric signature (as we shall call it) is a geometric Poincaré bordism invariant which is a $\pi_1(X)$-equivariant generalization of the signature.

Given a degree 1 map of n-dimensional geometric Poincaré complexes

$$f\colon M \to X$$

there is defined a kernel n-dimensional symmetric Poincaré complex over $\mathbf{Z}[\pi_1(X)]$,

$$\sigma^*(f) = (C(f^!),\varphi_f),$$

such that up to a chain equivalence preserving the symmetric structure

$$\sigma^*(M) = \sigma^*(f)\oplus\sigma^*(X),$$

where $C(f^!)$ is the algebraic mapping cone of the Umkehr chain map

$$f^!\colon C(\tilde{X}) \xrightarrow{([X]\cap -)^{-1}} C(\tilde{X})^{n-*} \xrightarrow{\quad \tilde{f}^* \quad} C(\tilde{M})^{n-*} \xrightarrow{([M]\cap -)} C(\tilde{M}).$$

Here, $\tilde{M}$ is the covering space of M induced by f from the universal cover $\tilde{X}$ of X, and $\sigma^*(M)$ is constructed using $\tilde{M}$ rather than the universal cover of M. The $\mathbf{Z}[\pi_1(X)]$-module chain complex $C(f^!)$ has homology modules

$$H_*(C(f^!)) = K_*(M) = \ker(\tilde{f}_*\colon H_*(\tilde{M}) \to H_*(\tilde{X})).$$

A normal map of geometric Poincaré complexes

$$(f\colon M \to X,\ b\colon \nu_M \to \nu_X)$$

is a degree 1 map f together with a covering map b of Spivak normal fibrations. The 'quadratic construction' of § 1 of Part II of this paper refines the symmetric structure φ_f in the kernel $\sigma^*(f) = (C(f^!),\varphi_f)$ of a normal map (f,b) to a quadratic structure ψ_b depending only on the fibre homotopy class of b. The surgery obstruction of (f,b) is the equivalence class

$$\sigma_*(f,b) \in L_n(\mathbf{Z}[\pi_1(X)])$$

of the quadratic kernel

$$\sigma_*(f,b) = (C(f^!),\psi_b)$$

in the abstract cobordism group of such n-dimensional quadratic Poincaré complexes over $\mathbf{Z}[\pi_1(X)]$.

In the first version of the present theory of quadratic structures an n-dimensional quadratic Poincaré complex was defined to be a chain complex of finitely generated projective A-modules,

$$C\colon C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \dots \xrightarrow{d} C_1 \xrightarrow{d} C_0,$$

together with a collection of A-module morphisms,

$$\psi_s\colon C^{n-r-s} \to C_r \quad (s \geqslant 0),$$

such that

$$d\psi_s + (-)^r\psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T\psi_{s+1}) = 0\colon C^{n-r-s-1} \to C_r$$

and such that the chain map

$$(1+T)\psi_0\colon C^{n-*} \to C$$

is a chain equivalence, inducing abstract Poincaré duality isomorphisms

$$(1+T)\psi_0\colon H^{n-*}(C) \to H_*(C).$$

The cobordism groups $L_n(A)$ of n-dimensional quadratic Poincaré complexes over A (C,ψ) turned out to be 4-periodic,

$$L_n(A) = L_{n+4}(A) \quad (n \geqslant 0),$$

agreeing with the surgery obstruction groups of Wall [25] for a group ring $A = \mathbf{Z}[\pi]$,

$$L_n(\mathbf{Z}[\pi]) = L_n(\pi) \quad (n(\mathrm{mod}\ 4)).$$

There remained the problem of exhibiting such a quadratic structure ψ on the chain complex kernel $C(f^!)$ of a normal map $(f,b)\colon M \to X$, without the use of preliminary surgeries below the middle dimension. Graeme Segal pointed out that for the chain complex $C = C(X)$ of a topological space X and $A = \mathbf{Z}$ such collections $\psi = \{\psi_s \mid s \geqslant 0\}$ are cycles of homology classes in the 'quadratic construction'

$$\psi \in H_n((S^\infty \times X \times X)/\mathbf{Z}_2) = H_n(W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C(X) \otimes_{\mathbf{Z}} C(X)))$$

with the generator $T \in \mathbf{Z}_2$ acting by the antipodal map on S^∞ and by the transposition $T\colon (x_1,x_2) \mapsto (x_2,x_1)$ on $X \times X$, and

$$W = C(S^\infty)\colon \dots \longrightarrow \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1-T} \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1+T} \mathbf{Z}[\mathbf{Z}_2] \xrightarrow{1-T} \mathbf{Z}[\mathbf{Z}_2].$$

This led to the present formulation of the quadratic theory, in which a quadratic structure on an A-module chain complex C is defined to be a class $\psi \in Q_n(C)$ in the $\mathbf{Z}_2$-hyperhomology group

$$Q_n(C) = H_n(\mathbf{Z}_2, C \otimes_A C) = H_n(W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C \otimes_A C))$$

with $T \in \mathbf{Z}_2$ acting on $C \otimes_A C$ by

$$T: C_p \otimes_A C_q \to C_q \otimes_A C_p;\ x \otimes y \mapsto (-)^{pq} y \otimes x,$$

which corresponds to the duality involution T on $\mathrm{Hom}_A(C^*, C)$ under the natural identification $C \otimes_A C = \mathrm{Hom}_A(C^*, C)$ for finitely generated projective C. In turn, this led to the present formulation of the symmetric theory, in which the symmetric structure $\{\varphi_s \mid s \geqslant 0\}$ of Mishchenko [**10**] is considered as a cycle of a class $\varphi \in Q^n(C)$ in the $\mathbf{Z}_2$-hypercohomology group

$$Q^n(C) = H^n(\mathbf{Z}_2, C \otimes_A C) = H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C \otimes_A C)).$$

Transfer defines a map from quadratic structures to symmetric structures

$$(1+T): Q_n(C) \to Q^n(C);\ \psi \mapsto (1+T)\psi,$$

$$((1+T)\psi)_s = \begin{cases} (1+T)\psi_0 & \text{if } s = 0, \\ 0 & \text{if } s \geqslant 1. \end{cases}$$

The problem was reduced to finding a natural lifting of the $\mathbf{Z}_2$-hypercohomology class $\varphi_f \in Q^n(C(f^!))$ appearing in the symmetric kernel $\sigma^*(f) = (C(f^!), \varphi_f)$ of a normal map $(f: M \to X,\ b: \nu_M \to \nu_X)$ to a $\mathbf{Z}_2$-hyperhomology class $\psi_b \in Q_n(C(f^!))$ such that

$$(1+T)\psi_b = \varphi_f \in Q^n(C(f^!)).$$

Ib Madsen suggested using a more refined version of the construction of the Arf invariant used by Browder in [**2**, § III.4] which involved the S-dual of the induced map of Thom spaces $T(b): T(\nu_M) \to T(\nu_X)$, a stable map $F: \Sigma^\infty X_+ \to \Sigma^\infty M_+$ inducing the Umkehr $f^!: C(X) \to C(M)$ on the chain level. (Specifically, note that there is a natural map

$$\coprod_{k \geqslant 0} E\Sigma_k \times_{\Sigma_k} (\prod_k M) \to \Omega^\infty \Sigma^\infty M_+ \quad (M_+ = M \cup \{\mathrm{pt.}\})$$

which is a group completion in homology, with the symmetric group Σ_k on k letters acting by permutation on the k-fold cartesian product $\prod_k M$, and $E\Sigma_k$ a contractible space with a free Σ_k-action. The image of the fundamental class $[X] \in H_n(X)$ under the composite

$$\begin{aligned} \psi: H_n(X) = \tilde{H}_n(X_+) &\xrightarrow{(\text{adjoint } F)_*} \tilde{H}_n(\Omega^\infty \Sigma^\infty M_+) \\ &= \Big(\bigoplus_{k=1}^{\infty} H_n(E\Sigma_k \times_{\Sigma_k} (\prod_k M))\Big) \otimes_{\mathbf{Z}[\mathbf{N}]} \mathbf{Z}[\mathbf{Z}] \\ &\xrightarrow{\text{projection}} H_n(E\Sigma_2 \times_{\Sigma_2} (M \times M)) = Q_n(C(M)) \\ &\xrightarrow{e_\%} Q_n(C(f^!)) \end{aligned}$$

(where e = inclusion: $C(M) \to C(f^!)$, $\mathbf{N} = \{1, 2, 3, \ldots\}$) is a $\mathbf{Z}_2$-hyperhomology class $\psi_b = \psi[X] \in Q_n(C(f^!))$ such that $(1+T)\psi_b = \varphi_f \in Q^n(C(f^!))$, and which agrees with the class ψ_b constructed in § 4 of Part II by a direct chain level operation.) This led to the observation that an abstract $\mathbf{Z}_2$-hypercohomology class $\varphi \in Q^n(C)$ lies in $\mathrm{im}((1+T): Q_n(C) \to Q^n(C))$ if and only if

$$S^p\varphi = 0 \in Q^{n+p}(S^pC) \quad (p \text{ large}),$$

where SC is the suspension of the chain complex C ($SC_r = C_{r-1}$) and

$$S: Q^n(C) \to Q^{n+1}(SC); \; \varphi \mapsto S\varphi,$$

$$(S\varphi)_s = \begin{cases} \varphi_{s-1} & \text{if } s \geqslant 1, \\ 0 & \text{if } s = 0 \end{cases}$$

(Proposition 1.3). If there exists a $\pi_1(X)$-equivariant map

$$F: \Sigma^p \tilde{X}_+ \to \Sigma^p \tilde{M}_+ \quad (p \geqslant 0)$$

inducing the Umkehr $f^!: C(\tilde{X}) \to C(\tilde{M})$ on the chain level then the $\mathbf{Z}_2$-hypercohomology class $\varphi_f \in Q^n(C(f^!))$ is of this type, and the stable $\pi_1(X)$-equivariant homotopy class of F determines a $\mathbf{Z}_2$-hyperhomology class $\psi_F \in Q_n(C(f^!))$ such that

$$(1+T)\psi_F = \varphi_f \in Q^n(C(f^!))$$

(Proposition II.2.3). A normal map of n-dimensional geometric Poincaré complexes $(f: M \to X, b: \nu_M \to \nu_X)$ gives rise to a $\pi_1(X)$-equivariant geometric Umkehr map $F: \Sigma^\infty \tilde{X}_+ \to \Sigma^\infty \tilde{M}_+$ via an equivariant S-duality theory (Proposition II.4.2), and $\psi_b = \psi_F \in Q_n(C(f^!))$ defines the kernel n-dimensional quadratic Poincaré complex over $\mathbf{Z}[\pi_1(X)]$,

$$\sigma_*(f, b) = (C(f^!), \psi_b).$$

The quadratic Poincaré cobordism class

$$\sigma_*(f, b) \in L_n(\mathbf{Z}[\pi_1(X)])$$

is the surgery obstruction of Wall [**25**] (Proposition II.7.1).

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The sections of Part I are as follows:

§ 1. Algebraic Poincaré complexes;
§ 2. Forms and formations;
§ 3. Algebraic Poincaré cobordism;
§ 4. Algebraic surgery;
§ 5. Witt groups;
§ 6. Lower L-theory;
§ 7. Dedekind rings;
§ 8. Products;
§ 9. Change of K-theory;
§ 10. Laurent extensions.

1. Algebraic Poincaré complexes

Given a ring with involution A, an element $\varepsilon \in A$ such that $\bar{\varepsilon} = \varepsilon^{-1} \in A$, and an A-module chain complex C, we define the $\mathbf{Z}_2$-hypercohomology {respectively $\mathbf{Z}_2$-hyperhomology, Tate $\mathbf{Z}_2$-hypercohomology} group $Q^n(C, \varepsilon)$ {respectively $Q_n(C, \varepsilon)$, $\hat{Q}^n(C, \varepsilon)$} of n-dimensional ε-symmetric {respectively ε-quadratic, ε-hyperquadratic} structures on C, depending only on the chain homotopy type of C. The Q-groups are related to each other by an exact sequence

$$\ldots \longrightarrow \hat{Q}^{n+1}(C, \varepsilon) \xrightarrow{H} Q_n(C, \varepsilon) \xrightarrow{1+T_\varepsilon} Q^n(C, \varepsilon) \xrightarrow{J} \hat{Q}^n(C, \varepsilon) \xrightarrow{H} Q_{n-1}(C, \varepsilon) \longrightarrow \ldots,$$

and there is defined a forgetful map

$$Q^n(C, \varepsilon) \to H_n(C \otimes_A C);\ \varphi \mapsto \varphi_0.$$

An n-dimensional ε-symmetric {ε-quadratic} Poincaré complex over A is an n-dimensional finitely generated projective A-module chain complex C together with a class $\varphi \in Q^n(C, \varepsilon)$ {$\psi \in Q_n(C, \varepsilon)$} such that slant product with $\varphi_0 \otimes H_n(C \otimes_A C)$ {$(1+T_\varepsilon)\psi_0 \in H_n(C \otimes_A C)$} defines Poincaré duality isomorphisms

$$\varphi_0\colon H^{n-*}(C) \to H_*(C) \quad \{(1+T_\varepsilon)\psi_0\colon H^{n-*}(C) \to H_*(C)\}.$$

In § 2 below we shall express the theory of n-dimensional ε-symmetric {ε-quadratic} complexes over A $(C, \varphi \in Q^n(C, \varepsilon))$ {$(C, \psi \in Q_n(C, \varepsilon))$} for $n = 0$ (respectively 1) in terms of ε-symmetric {ε-quadratic} forms (respectively formations) over A. In § 2 {§ 4, § 9} of Part II we shall show that an n-dimensional geometric Poincaré complex X {a normal map of geometric Poincaré complexes $(f\colon M \to X, b\colon \nu_M \to \nu_X)$, a stable spherical fibration $p\colon X \to BG$ over an n-dimensional CW complex X} determines in a natural way an n-dimensional 1-symmetric Poincaré {1-quadratic Poincaré, 1-hyperquadratic} complex over $\mathbf{Z}[\pi_1(X)]$ $\sigma^*(X)$ $\{\sigma_*(f,b), \hat{\sigma}^*(p)\}$ such that

$$(1+T)\sigma_*(f,b) \oplus \sigma^*(X) = \sigma^*(M),$$

$$J\sigma^*(X) = \hat{\sigma}^*(\nu_X).$$

A *ring with involution* A is an associative ring with 1, together with a function

$$\bar{\ }: A \to A;\ a \mapsto \bar{a}$$

such that

$$\begin{aligned} (\overline{a+b}) &= \bar{a}+\bar{b}, \\ (\overline{ab}) &= (\bar{b})(\bar{a}), \\ \bar{\bar{a}} &= a, \\ \bar{1} &= 1 \qquad (a,b \in A). \end{aligned}$$

Given a left A-module M let M^t denote the right A-module defined by the additive group of M, with A acting by

$$M^t \times A \to M^t;\ (x,a) \mapsto \bar{a}x.$$

We shall be mainly concerned with left A-modules, so that 'an A-module' is to be taken to mean 'a left A-module' unless a right A-action is specified.

The *dual* of an A-module M is the A-module

$$M^* = \mathrm{Hom}_A(M,A),$$

with A acting by

$$A \times M^* \to M^*;\ (a,f) \mapsto (x \mapsto f(x).\bar{a}).$$

The *dual* of an A-module morphism $f \in \mathrm{Hom}_A(M,N)$ is the A-module morphism

$$f^*: N^* \to M^*;\ g \mapsto (x \mapsto g(f(x))).$$

The dual of a finitely generated (f.g.) projective A-module M is a f.g. projective A-module M^*, and there is defined a natural A-module isomorphism

$$M \to M^{**};\ x \mapsto (f \mapsto \overline{f(x)})$$

which we shall use as an identification.

The *homology* {*cohomology*} A-modules $H_*(C)$ $\{H^*(C)\}$ of an A-module chain complex C,

$$C: \dots \longrightarrow C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \longrightarrow \dots \quad (r \in \mathbf{Z},\ d^2 = 0),$$

are defined by

$$\begin{cases} H_r(C) = \ker(d: C_r \to C_{r-1})/\mathrm{im}(d: C_{r+1} \to C_r) \\ H^r(C) = \ker(d^*: C^r \to C^{r+1})/\mathrm{im}(d^*: C^{r-1} \to C^r) \end{cases} \quad (r \in \mathbf{Z},\ C^r = C_r^*).$$

A *chain map* {*homotopy*} of A-module chain complexes {maps}

$$\begin{cases} f\colon C \to D \\ g\colon f \simeq f'\colon C \to D \end{cases}$$

is a collection of A-module morphisms $\{f \in \mathrm{Hom}_A(C_r, D_r) \mid r \in \mathbf{Z}\}$ $\{\{g \in \mathrm{Hom}_A(C_r, D_{r+1}) \mid r \in \mathbf{Z}\}\}$ such that

$$\begin{cases} d_D f = f d_C\colon C_r \to D_{r-1} \\ f' - f = d_D g + g d_C\colon C_r \to D_r \end{cases} \quad (r \in \mathbf{Z}).$$

A *chain equivalence* is a chain map which admits a chain homotopy inverse.

An A-module chain complex C is *n-dimensional* if it is chain equivalent to a f.g. projective A-module chain complex of the type

$$C\colon \ldots \longrightarrow 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \ldots$$

for some integer $n \geqslant 0$. We recall that a chain map $f\colon C \to D$ of finite-dimensional A-module chain complexes is a chain equivalence if and only if it induces A-module isomorphisms $f_*\colon H_*(C) \to H_*(D)$ in homology (or, equivalently, if it induces A-module isomorphisms $f^*\colon H^*(D) \to H^*(C)$ in cohomology). A finite f.g. projective A-module chain complex C is n-dimensional if and only if $H_r(C) = 0$ for $r < 0$ and $H^r(C) = 0$ for $r > n$. We shall be mainly concerned with finite-dimensional chain complexes.

Given A-module chain complexes C, D, let $C^t \otimes_A D$, $\mathrm{Hom}_A(C, D)$ be the abelian group chain complexes defined by

$$(C^t \otimes_A D)_n = \sum_{p+q=n} C^t_p \otimes_A D_q, \quad d_{C^t \otimes_A D}(x \otimes y) = x \otimes d_D(y) + (-)^q d_C(x) \otimes y,$$

$$\mathrm{Hom}_A(C, D)_n = \sum_{q-p=n} \mathrm{Hom}_A(C_p, D_q), \quad d_{\mathrm{Hom}_A(C,D)}(f) = d_D f + (-)^q f d_C.$$

The *slant* chain map

$$\backslash : C^t \otimes_A D \to \mathrm{Hom}_A(C^{-*}, D);\ x \otimes y \mapsto (f \mapsto \overline{f(x)}.y)$$

is a chain equivalence (respectively isomorphism) if C is finite-dimensional (respectively f.g. projective), where C^{-*} is the A-module chain complex defined by

$$(C^{-*})_r = C^{-r}, \quad d_{C^{-*}} = (d_C)^*.$$

Let $\varepsilon \in A$ be a central unit such that

$$\bar{\varepsilon} = \varepsilon^{-1} \in A,$$

for example, $\varepsilon = \pm 1$. Given an A-module chain complex C let the generator

$T \in \mathbf{Z}_2$ act on $C^t \otimes_A C$ by the *ε-transposition* involution

$$T_\varepsilon : C^t_p \otimes_A C_q \to C^t_q \otimes_A C_p \,;\; x \otimes y \mapsto (-)^{pq} y \otimes \varepsilon x,$$

and define the *Q-groups*

$$\begin{cases} Q^n_{[i,j]}(C,\varepsilon) = H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[i,j], C^t \otimes_A C)) \\ Q^{[i,j]}_n(C,\varepsilon) = H_n(W[i,j] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C^t \otimes_A C)) \end{cases} \quad (-\infty \leqslant i \leqslant j \leqslant \infty, n \in \mathbf{Z})$$

with $W[i,j]$ the $\mathbf{Z}[\mathbf{Z}_2]$-module chain complex given by

$$W[i,j]_r = \begin{cases} \mathbf{Z}[\mathbf{Z}_2] & \text{if } i \leqslant r \leqslant j, \\ 0 & \text{otherwise}, \end{cases}$$

$$d_{W[i,j]} = 1 + (-)^r T : W[i,j]_r \to W[i,j]_{r-1} \quad (i < r \leqslant j).$$

An element $\varphi \in Q^n_{[i,j]}(C,\varepsilon)$ $\{\psi \in Q^{[i,j]}_n(C,\varepsilon)\}$ is represented by a collection of chains

$$\begin{cases} \varphi = \{\varphi_s \in (C^t \otimes_A C)_{n+s} \mid i \leqslant s \leqslant j\} \\ \psi = \{\psi_s \in (C^t \otimes_A C)_{n-s} \mid i \leqslant s \leqslant j\} \end{cases}$$

such that

$$\begin{cases} d_{C^t \otimes_A C}(\varphi_s) + (-)^{n+s-1}(\varphi_{s-1} + (-)^s T_\varepsilon \varphi_{s-1}) = 0 \in (C^t \otimes_A C)_{n+s-1} \\ \qquad\qquad (i \leqslant s \leqslant j,\ \varphi_{i-1} = 0), \\ d_{C^t \otimes_A C}(\psi_s) + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T_\varepsilon \psi_{s+1}) = 0 \in (C^t \otimes_A C)_{n-s-1} \\ \qquad\qquad (i \leqslant s \leqslant j,\ \psi_{j+1} = 0). \end{cases}$$

The notation is somewhat redundant, allowing identifications

$$Q^n_{[i,j]}(C,\varepsilon) = Q^{n+k}_{[i-k,j-k]}(C,(-)^k\varepsilon) = Q^{[k-j,k-i]}_{n+k}(C,(-)^{k+1}\varepsilon) \quad (k \in \mathbf{Z}).$$

For $i = j$ we have

$$Q^n_{[i,i]}(C,\varepsilon) = H_{n+i}(C^t \otimes_A C), \quad Q^{[i,i]}_n(C,\varepsilon) = H_{n-i}(C^t \otimes_A C).$$

The isomorphism type of the Q-groups $Q^n_{[i,j]}(C,\varepsilon)$ $\{Q^{[i,j]}_n(C,\varepsilon)\}$ depends only on the chain homotopy type of C, despite the quadratic nature of the construction.

PROPOSITION 1.1. (i) *An A-module chain map $f: C \to D$ $\{$homotopy $g: f \simeq f' : C \to D\}$ induces a $\mathbf{Z}$-module chain map $\{$homotopy$\}$*

$$\begin{cases} f^{\%} \\ g^{\%} : f^{\%} \simeq f'^{\%} \end{cases} : \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[i,j], C^t \otimes_A C) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[i,j], D^t \otimes_A D),$$

$$\begin{cases} f_{\%} \\ g_{\%} : f_{\%} \simeq f'_{\%} \end{cases} : W[i,j] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C^t \otimes_A C) \to W[i,j] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (D^t \otimes_A D).$$

(ii) *If C is an m-dimensional A-module chain complex then*

$$\begin{cases} Q^n_{[i,j]}(C,\varepsilon) = 0 \\ Q_n^{[i,j]}(C,\varepsilon) = 0 \end{cases} \text{if} \begin{cases} n+i > 2m \text{ or } n+j < 0, \\ n-j > 2m \text{ or } n-i < 0. \end{cases}$$

(iii) *For $-\infty \leqslant i \leqslant j \leqslant k \leqslant \infty$ there is defined a long exact sequence of Q-groups*

$$\begin{cases} \dots \to Q^n_{[j+1,k]}(C,\varepsilon) \to Q^n_{[i,k]}(C,\varepsilon) \to Q^n_{[i,j]}(C,\varepsilon) \to Q^{n-1}_{[j+1,k]}(C,\varepsilon) \to \dots \\ \dots \to Q_n^{[i,j]}(C,\varepsilon) \to Q_n^{[i,k]}(C,\varepsilon) \to Q_n^{[j+1,k]}(C,\varepsilon) \to Q_{n-1}^{[i,j]}(C,\varepsilon) \to \dots \end{cases} \quad (n \in \mathbf{Z}).$$

Proof. (i) Given $\varphi \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[i,j], C^t \otimes_A C)_n$ set

$$\begin{cases} f^{\%}(\varphi)_s = (f^t \otimes_A f)\varphi_s \in (D^t \otimes_A D)_{n+s}, \\ g^{\%}(\varphi)_s = (f^t \otimes_A g + (-)^q g^t \otimes_A f')\varphi_s + (-)^{q+s-1}(g^t \otimes_A g)T_s\varphi_{s-1} \\ \qquad \in (D^t \otimes_A D)_{n+s+1} = \sum\limits_{q=-\infty}^{\infty} D^t_{n-q+s+1} \otimes_A D_q \quad (i \leqslant s \leqslant j,\ \varphi_{i-1} = 0). \end{cases}$$

The other case is similar.

(ii) By the chain homotopy invariance (verified in (i)) it may be assumed that $C_r = 0$ for $r < 0$ or $r > m$.

(iii) Given intervals $[i,j], [i',j']$ such that $i \leqslant i' \leqslant j \leqslant j'$ define a $\mathbf{Z}[\mathbf{Z}_2]$-module chain map $W[i,j] \to W[i',j']$ by

$$1\colon W[i,j]_r = \mathbf{Z}[\mathbf{Z}_2] \to W[i',j']_r = \mathbf{Z}[\mathbf{Z}_2] \quad (i' \leqslant r \leqslant j).$$

There are induced contravariantly {covariantly} abelian group morphisms

$$\begin{cases} Q^n_{[i',j']}(C,\varepsilon) \to Q^n_{[i,j]}(C,\varepsilon) \\ Q_n^{[i,j]}(C,\varepsilon) \to Q_n^{[i',j']}(C,\varepsilon) \end{cases} \quad (n \in \mathbf{Z}).$$

For $-\infty \leqslant i \leqslant j < k \leqslant \infty$ there is defined a split short exact sequence of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$0 \to W[i,j] \to W[i,k] \to W[j+1,k] \to 0.$$

Apply $\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(-, C^t \otimes_A C)$ $\{- \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C^t \otimes_A C)\}$ to this sequence, and consider the associated long exact sequence in homology.

(The chain homotopy invariance of the Q-groups is fundamental to the methods of this paper—it is further clarified by the following discussion. Define the $\mathbf{Z}_2$-*isovariant* category, with objects $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes and morphisms $f\colon C \to D$ $\mathbf{Z}_2$-hypercohomology classes,

$$f \in H^0(\mathbf{Z}_2; \mathrm{Hom}_{\mathbf{Z}}(C,D)) = H_0(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \mathrm{Hom}_{\mathbf{Z}}(C,D))) \qquad (W = W[0,\infty]),$$

with $T \in \mathbf{Z}_2$ acting on $\mathrm{Hom}_{\mathbf{Z}}(C,D)$ by

$$T\colon \mathrm{Hom}_{\mathbf{Z}}(C,D) \to \mathrm{Hom}_{\mathbf{Z}}(C,D);\ g \mapsto T_D g T_C.$$

A $\mathbf{Z}_2$-isovariant morphism $f\colon C \to D$ is thus an equivalence class of collections $\{f_s \in \mathrm{Hom}_{\mathbf{Z}}(C_r, D_{r+s}) \mid r \in \mathbf{Z},\ s \geqslant 0\}$ such that

$$d_D f_s + (-)^{s-1} f_s d_C + (-)^{s-1}(f_{s-1} + (-)^s T_D f_{s-1} T_C) = 0\colon C_r \to D_{r+s-1} \qquad (f_{-1} = 0),$$

corresponding to a $\mathbf{Z}$-module chain map $f_0\colon C \to D$ together with a $\mathbf{Z}$-module chain homotopy $f_1\colon f_0 \simeq T_D f_0 T_C\colon C \to D$ and higher chain homotopies $f_2, f_3, \ldots$. The diagonal $\mathbf{Z}[\mathbf{Z}_2]$-module chain map

$$\Delta\colon W[i,j] \to W \otimes_{\mathbf{Z}} W[i,j];\ 1_s \mapsto \sum_{r=0}^{s-i} 1_r \otimes (T_{s-r})^r \quad (i \leqslant s \leqslant j)$$

can be used to define products

$$H^0(\mathbf{Z}_2; \mathrm{Hom}_{\mathbf{Z}}(C,D)) \otimes_{\mathbf{Z}} Q^n_{[i,j]}(C,\varepsilon) \to Q^n_{[i,j]}(D,\varepsilon);\ f \otimes \varphi \mapsto (f \otimes \varphi)\Delta,$$

so that a $\mathbf{Z}_2$-isovariant morphism $f\colon C \to D$ induces abelian group morphisms

$$f^{\%}\colon Q^n_{[i,j]}(C,\varepsilon) \to Q^n_{[i,j]}(D,\varepsilon)$$

in a natural way. An A-module chain map $f\colon C \to D$ of A-module chain complexes induces a $\mathbf{Z}[\mathbf{Z}_2]$-module chain map $f^t \otimes_A f\colon C^t \otimes_A C \to D^t \otimes_A D$ of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes, but an A-module chain homotopy $g\colon f \simeq f'\colon C \to D$ does not in general induce a $\mathbf{Z}[\mathbf{Z}_2]$-module chain homotopy $f^t \otimes_A f \simeq f'^t \otimes_A f'\colon C^t \otimes_A C \to D^t \otimes_A D$. However, $f^t \otimes_A f$ and $f'^t \otimes_A f'$ represent the same $\mathbf{Z}_2$-isovariant morphism $C^t \otimes_A C \to D^t \otimes_A D$, so that

$$f^{\%} = f'^{\%}\colon Q^n_{[i,j]}(C,\varepsilon) \to Q^n_{[i,j]}(D,\varepsilon)$$

(as was explicitly verified in the proof of Proposition 1.1(i)). Note that for the singular chain complex $C(X)$ of a topological space X the Eilenberg–Zilber theorem gives a natural $\mathbf{Z}_2$-isovariant equivalence

$$C(X \times X) \to C(X) \otimes_{\mathbf{Z}} C(X),$$

as used in the construction of the Steenrod squares in singular cohomology (cf. § 1 of Part II).)

Given an A-module chain complex C define the *$\mathbf{Z}_2$-hypercohomology* {*$\mathbf{Z}_2$-hyperhomology*, *Tate $\mathbf{Z}_2$-hypercohomology*} groups of the $\mathbf{Z}_2$-action on $C^t \otimes_A C$ by the ε-transposition involution T_ε

$$\begin{cases} Q^n(C,\varepsilon) = Q^n_{[0,\infty]}(C,\varepsilon) = H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C^t \otimes_A C)) \\ Q_n(C,\varepsilon) = Q^{[0,\infty]}_n(C,\varepsilon) = H_n(W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C^t \otimes_A C)) \qquad (n \in \mathbf{Z}), \\ \hat{Q}^n(C,\varepsilon) = Q^n_{[-\infty,\infty]}(C,\varepsilon) = H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(\hat{W}, C^t \otimes_A C)) \end{cases}$$

with $W = W[0,\infty]$ a free $\mathbf{Z}[\mathbf{Z}_2]$-resolution of $\mathbf{Z}$ and $\hat{W} = W[-\infty,\infty]$ a complete resolution for $\mathbf{Z}_2$ (cf. Cartan and Eilenberg [4, Chapters XII and XVII]). If C is n-dimensional

$$\begin{cases} Q^n(C,\varepsilon) = Q^n_{[0,n+1]}(C,\varepsilon) \\ Q_n(C,\varepsilon) = Q_n^{[0,n+1]}(C,\varepsilon) \\ \hat{Q}^n(C,\varepsilon) = Q^n_{[-n-1,n+1]}(C,\varepsilon) \end{cases}$$

by Proposition 1.1.

PROPOSITION 1.2. *Given an A-module chain complex C there is defined a long exact sequence of abelian groups*

$$\ldots \longrightarrow \hat{Q}^{n+1}(C,\varepsilon) \xrightarrow{H} Q_n(C,\varepsilon) \xrightarrow{1+T_\varepsilon} Q^n(C,\varepsilon) \xrightarrow{J} \hat{Q}^n(C,\varepsilon) \xrightarrow{H} Q_{n-1}(C,\varepsilon) \longrightarrow \ldots \quad (n \in \mathbf{Z})$$

with

$$H\colon \hat{Q}^{n+1}(C,\varepsilon) \to Q_n(C,\varepsilon);\ \theta \mapsto \{(H\theta)_s = \theta_{-s-1} \mid s \geqslant 0\},$$

$$1+T_\varepsilon\colon Q_n(C,\varepsilon) \to Q^n(C,\varepsilon);\ \psi \mapsto \left\{((1+T_\varepsilon)\psi)_s = \begin{cases} (1+T_\varepsilon)\psi_0 & \textit{if } s = 0 \\ 0 & \textit{if } s \geqslant 1 \end{cases}\right\},$$

$$J\colon Q^n(C,\varepsilon) \to \hat{Q}^n(C,\varepsilon);\ \varphi \mapsto \left\{(J\varphi)_s = \begin{cases} \varphi_s & \textit{if } s \geqslant 0 \\ 0 & \textit{if } s \leqslant -1 \end{cases}\right\}.$$

Proof. This is just the special case of the exact sequence of Proposition 1.1(iii) for Q_*, with $i = -\infty$, $j = -1$, $k = \infty$.

(The exact sequence of Proposition 1.2 is related to the EHP sequence in homotopy theory, cf. Proposition 5.1 of Part II.)

An *n-dimensional ε-symmetric $\{\varepsilon$-quadratic, ε-hyperquadratic$\}$ complex over A* (C,φ) $\{(C,\psi),\ (C,\theta)\}$ is an n-dimensional A-module chain complex C together with an element $\varphi \in Q^n(C,\varepsilon)$ $\{\psi \in Q_n(C,\varepsilon),\ \theta \in \hat{Q}^n(C,\varepsilon)\}$. The *$\varepsilon$-hyperquadratization $\{\varepsilon$-symmetrization, ε-quadratization$\}$* of an n-dimensional ε-symmetric $\{\varepsilon$-quadratic, ε-hyperquadratic$\}$ complex over A (C,φ) $\{(C,\psi),\ (C,\theta)$ such that $H^n(C) = 0\}$ is the n- $\{n$-, $(n-1)$-$\}$ dimensional ε-hyperquadratic $\{\varepsilon$-symmetric, ε-quadratic$\}$ complex over A

$$\begin{cases} J(C,\varphi) = (C, J\varphi \in \hat{Q}^n(C,\varepsilon)), \\ (1+T_\varepsilon)(C,\psi) = (C, (1+T_\varepsilon)\psi \in Q^n(C,\varepsilon)), \\ H(C,\theta) = (C, H\theta \in Q_{n-1}(C,\varepsilon)). \end{cases}$$

If there exists a central element $a \in A$ such that $a + \bar{a} = 1$ (for example, $a = \frac{1}{2}$) the ε-symmetrization map $(1+T_\varepsilon)\colon Q_n(C,\varepsilon) \to Q^n(C,\varepsilon)$

is an isomorphism (the inverse being given by $Q^n(C,\varepsilon) \to Q_n(C,\varepsilon)$; $\varphi \mapsto \psi = \{\psi_s = (1\otimes a)(1+T_\varepsilon)(1\otimes a)\varphi_0$ if $s = 0$, $\psi_s = 0$ if $s \geqslant 1\}$) and $\hat{Q}^n(C,\varepsilon) = 0$ so that there is no difference between n-dimensional ε-symmetric $\{\varepsilon$-hyperquadratic$\}$ complexes over A and n-dimensional ε-quadratic $\{$chain$\}$ complexes over A. The groups $\hat{Q}^n(C,\varepsilon)$ are of exponent 2 (for any A).

An n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex over A $(C, \varphi \in Q^n(C,\varepsilon))$ $\{(C, \psi \in Q_n(C,\varepsilon))\}$ is a *Poincaré* complex if the evaluation of the slant products

$$\backslash : H^r(C) \otimes_{\mathbf{Z}} H_n(C^t \otimes_A C) \to H_{n-r}(C);\, f \otimes (x \otimes y) \mapsto \overline{f(x)}.y$$

on $\varphi_0 \in H_n(C^t \otimes_A C)$ $\{(1+T_\varepsilon)\psi_0 \in H_n(C^t \otimes_A C)\}$ defines A-module isomorphisms

$$\begin{cases} \varphi_0 \colon H^r(C) \to H_{n-r}(C) \\ (1+T_\varepsilon)\psi_0 \colon H^r(C) \to H_{n-r}(C) \end{cases} \qquad (0 \leqslant r \leqslant n).$$

The ε-symmetrization of an n-dimensional ε-quadratic Poincaré complex (C, ψ) is evidently an n-dimensional ε-symmetric Poincaré complex $(1+T_\varepsilon)(C,\psi)$.

A *map* (respectively *homotopy equivalence*) of n-dimensional ε-symmetric $\{\varepsilon$-quadratic, ε-hyperquadratic$\}$ complexes over A

$$\begin{cases} f \colon (C,\varphi) \to (C',\varphi') \\ f \colon (C,\psi) \to (C',\psi') \\ f \colon (C,\theta) \to (C',\theta') \end{cases}$$

is an A-module chain map (respectively chain equivalence)

$$f \colon C \to C'$$

such that

$$\begin{cases} f^{\%}(\varphi) = \varphi' \in Q^n(C',\varepsilon), \\ f_{\%}(\psi) = \psi' \in Q_n(C',\varepsilon), \\ \hat{f}^{\%}(\theta) = \theta' \in \hat{Q}^n(C',\varepsilon). \end{cases}$$

Homotopy equivalence is an equivalence relation.

In §2 below we shall identify the homotopy equivalence classes of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes over A for $n = 0$ (respectively 1) with the (stable) isomorphism classes of ε-symmetric $\{\varepsilon$-quadratic$\}$ forms (respectively formations) over A.

For $\varepsilon = 1 \in A$ we shall contract the terminology by writing

$$\begin{cases} Q^n(C,1) = Q^n(C), \\ Q_n(C,1) = Q_n(C), \\ \hat{Q}^n(C,1) = \hat{Q}^n(C), \end{cases}$$

calling 1-symmetric {1-quadratic, 1-hyperquadratic} complexes symmetric {quadratic, hyperquadratic}.

Our notion of a symmetric Poincaré complex is a chain homotopy invariant version of an 'algebraic Poincaré complex' due to Mishchenko [**10**].

For f.g. projective A-module chain complexes C the slant map isomorphism $C^t \otimes_A C \to \mathrm{Hom}_A(C^*, C)$ allows us to represent elements $\varphi \in Q^n(C,\varepsilon)$ $\{\psi \in Q_n(C,\varepsilon),\ \ \theta \in \hat{Q}^n(C,\varepsilon)\}$ by collections of A-module morphisms

$$\begin{cases} \{\varphi_s \in \mathrm{Hom}_A(C^{n-r+s}, C_r) \mid r \in \mathbf{Z},\ s \geqslant 0\} \\ \{\psi_s \in \mathrm{Hom}_A(C^{n-r-s}, C_r) \mid r \in \mathbf{Z},\ s \geqslant 0\} \\ \{\theta_s \in \mathrm{Hom}_A(C^{n-r+s}, C_r) \mid r \in \mathbf{Z},\ s \in \mathbf{Z}\} \end{cases}$$

such that

$$\begin{cases} d_C\varphi_s + (-)^r\varphi_s d_C^* + (-)^{n+s-1}(\varphi_{s-1} + (-)^s T_\varepsilon \varphi_{s-1}) = 0 \colon C^{n-r+s-1} \to C_r \\ \hfill (s \geqslant 0,\ \varphi_{-1} = 0), \\ d_C\psi_s + (-)^r\psi_s d_C^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T_\varepsilon \psi_{s+1}) = 0 \colon C^{n-r-s-1} \to C_r \\ \hfill (s \geqslant 0), \\ d_C\theta_s + (-)^r\theta_s d_C^* + (-)^{n+s-1}(\theta_{s-1} + (-)^s T_\varepsilon \theta_{s-1}) = 0 \colon C^{n-r+s-1} \to C_r \\ \hfill (s \in \mathbf{Z}), \end{cases}$$

where T_ε is the *ε-duality* involution

$$T_\varepsilon \colon \mathrm{Hom}_A(C^p, C_q) \to \mathrm{Hom}_A(C^q, C_p);\ \theta \mapsto (-)^{pq}\varepsilon.\theta^*.$$

An n-dimensional ε-symmetric {ε-quadratic} complex $(C, \varphi \in Q^n(C,\varepsilon))$ $\{(C, \psi \in Q_n(C,\varepsilon))\}$ with C f.g. projective is a Poincaré complex if and only if the chain map

$$\begin{cases} \varphi_0 \colon C^{n-*} \to C \\ (1+T_\varepsilon)\psi_0 \colon C^{n-*} \to C \end{cases}$$

is a chain equivalence, with

$$(C^{n-*})_r = C^{n-r}, \quad d_{C^{n-*}} = (-)^r d_C^* \colon C^{n-r} \to C^{n-r+1}.$$

The *suspension* of an A-module chain complex C is the A-module chain complex SC obtained by dimension shift -1

$$(SC)_r = C_{r-1}, \quad d_{SC} = d_C.$$

We shall denote the inverse operation by ΩC $((\Omega C)_r = C_{r+1}, d_{\Omega C} = d_C)$, and can identify

$$\begin{cases} H_r(SC) = H_{r-1}(C), & H_r(\Omega C) = H_{r+1}(C), \\ H^r(SC) = H^{r-1}(C), & H^r(\Omega C) = H^{r+1}(C), \end{cases} \quad S\Omega C = \Omega SC = C,$$

$$\begin{cases} Q^n_{[i,j]}(C,\varepsilon) = Q^{n+1}_{[i+1,j+1]}(SC,\varepsilon) = Q^{n+2}_{[i,j]}(SC,-\varepsilon), \\ Q^{[i,j]}_n(C,\varepsilon) = Q^{[i-1,j-1]}_{n+1}(SC,\varepsilon) = Q^{[i,j]}_{n+2}(SC,-\varepsilon). \end{cases}$$

In particular,

$$\hat{Q}^n(C,\varepsilon) = \hat{Q}^{n+1}(SC,\varepsilon).$$

The *skew-suspension* of an n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ (Poincaré) complex over A $(C, \varphi \in Q^n(C,\varepsilon))$ $\{(C,\psi \in Q_n(C,\varepsilon))\}$ is the $(n+2)$-dimensional $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic$\}$ (Poincaré) complex over A

$$\begin{cases} \bar{S}(C,\varphi) = (SC, \bar{S}\varphi \in Q^{n+2}(SC,-\varepsilon)), \\ \bar{S}(C,\psi) = (SC, \bar{S}\psi \in Q_{n+2}(SC,-\varepsilon)), \end{cases}$$

with $\bar{S}\colon Q^n(C,\varepsilon) \to Q^{n+2}(SC,-\varepsilon)$ $\{\bar{S}\colon Q_n(C,\varepsilon) \to Q_{n+2}(SC,-\varepsilon)\}$ the abelian group isomorphism induced by the isomorphism of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$\bar{S}\colon C^t \otimes_A C \to \Omega^2(SC^t \otimes_A SC);\ x \otimes y \mapsto (-)^p x \otimes y \quad (x \in C_p,\ y \in C_q).$$

Here $T \in \mathbf{Z}_2$ acts by T_ε on $C^t \otimes_A C$ and by $T_{-\varepsilon}$ on $SC^t \otimes_A SC$. An $(n+2)$-dimensional $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic$\}$ complex over A $(D, \varphi \in Q^{n+2}(D,-\varepsilon))$ $\{(D,\psi \in Q_{n+2}(D,-\varepsilon))\}$ is the skew-suspension of an n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex over A

$$(\Omega D, \bar{S}^{-1}(\varphi) \in Q^n(\Omega D,\varepsilon)) \quad \{(\Omega D, \bar{S}^{-1}(\psi) \in Q_n(\Omega D,\varepsilon))\}$$

if and only if ΩD is an n-dimensional A-module chain complex, that is if

$$H_0(D) = 0, \quad H^{n+2}(D) = 0.$$

Thus the study of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes (C,φ) $\{(C,\psi)\}$ such that $H_r(C) = 0$, $H^r(C) = 0$ for $r < i$ reduces to the study of $(n-2i)$-dimensional $(-)^i\varepsilon$-symmetric $\{(-)^i\varepsilon$-quadratic$\}$ Poincaré complexes $(\Omega^i C, \bar{S}^{-i}(\varphi))$ $\{(\Omega^i C, \bar{S}^{-i}(\psi))\}$ for $n \geqslant 2i$.

Given an A-module chain complex C define the *suspension* chain map

$$\begin{cases} S\colon \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C^t \otimes_A C) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[-1,\infty], C^t \otimes_A C) \\ \qquad = \Omega\,\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, SC^t \otimes_A SC)\,;\ \varphi \mapsto \{(S\varphi)_s = \varphi_{s-1} \mid s \geqslant 0\}\ \ (\varphi_{-1} = 0) \\ S\colon W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C^t \otimes_A C) \to W[1,\infty] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C^t \otimes_A C) \\ \qquad = \Omega(W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (SC^t \otimes_A SC))\,;\ \psi \mapsto \{(S\psi)_s = \psi_{s+1} \mid s \geqslant 0\} \end{cases}$$

using the natural chain map

$$W[-1,\infty] \to W[0,\infty] = W \quad \{W = W[0,\infty] \to W[1,\infty]\},$$

so that there are induced suspension maps in $\mathbf{Z}_2$-hypercohomology $\{\mathbf{Z}_2$-hyperhomology$\}$,

$$\begin{cases} S\colon Q^n(C,\varepsilon) \to Q^{n+1}(SC,\varepsilon), \\ S\colon Q_n(C,\varepsilon) \to Q_{n+1}(SC,\varepsilon). \end{cases}$$

Now $W[-\infty,\infty] = \varprojlim_p W[-p,\infty]$ so that the Tate $\mathbf{Z}_2$-hypercohomology groups are the direct limits

$$\hat{Q}^n(C,\varepsilon) = \varinjlim_p Q^{n+p}(S^pC,\varepsilon)$$

of the directed systems of suspension maps

$$Q^n(C,\varepsilon) \xrightarrow{\ S\ } Q^{n+1}(SC,\varepsilon) \xrightarrow{\ S\ } Q^{n+2}(S^2C,\varepsilon) \xrightarrow{\ S\ } \cdots$$

with $J\colon Q^n(C,\varepsilon) \to \hat{Q}^n(C,\varepsilon)$ the natural map. Thus the relation $\ker J = \mathrm{im}(1+T_\varepsilon)$ in the exact sequence of Proposition 1.2 can be interpreted as saying that an n-dimensional ε-symmetric complex $(C, \varphi \in Q^n(C,\varepsilon))$ is the ε-symmetrization $(1+T_\varepsilon)(C,\psi)$ of an n-dimensional ε-quadratic complex $(C, \psi \in Q_n(C,\varepsilon))$ if and only if $S^p\varphi = 0 \in Q^{n+p}(S^pC,\varepsilon)$ for some $p \geqslant 0$. This is the mechanism by which we shall obtain quadratic structures in the topological context, in the 'quadratic construction' of § 1 of Part II.

The exact sequence of Proposition 1.2 admits a generalization:

PROPOSITION 1.3. *Given an A-module chain complex C there is defined in a natural way a chain equivalence of $\mathbf{Z}$-module chain complexes*

$$C(S^p) \to S(W[0,p-1] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C^t \otimes_A C))$$

for each $p \geqslant 0$, with $C(S^p)$ the algebraic mapping cone of the p-fold suspension chain map

$$S^p\colon \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C^t \otimes_A C) \to \Omega^p\,\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, S^pC^t \otimes_A S^pC),$$

and there is also defined a commutative braid of exact sequences of **Z**-*modules*

$$\begin{tikzcd}[column sep=small]
{} \arrow[dr] \arrow[rr, bend left] & & Q_n^{[0,p-1]}(C,\varepsilon) \arrow[dr] \arrow[rr, bend left] & & Q^n(C,\varepsilon) \arrow[dr, "S^p"] \arrow[rr, bend left, "J"] & & \hat{Q}^n(C,\varepsilon) \arrow[dr, "H"] \arrow[rr, bend left] & & {} \\
& Q^{n+p+1}(S^pC,\varepsilon) \arrow[ur] \arrow[dr] & & Q_n(C,\varepsilon) \arrow[ur, "1+T_\varepsilon"] \arrow[dr, "S^p"] & & Q^{n+p}(S^pC,\varepsilon) \arrow[ur] \arrow[dr] & & Q_{n-1}(C,\varepsilon) \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] \arrow[rr, bend right] & & \hat{Q}^{n+1}(C,\varepsilon) \arrow[ur, "H"] \arrow[rr, bend right] & & Q_{n+p}(S^pC,\varepsilon) \arrow[ur, "1+T_\varepsilon"'] \arrow[rr, bend right] & & Q_{n-1}^{[0,p-1]}(C,\varepsilon) \arrow[ur] \arrow[rr, bend right] & & {}
\end{tikzcd}$$

If C is n-dimensional then for $p \geqslant n+1$

$$Q_{n+p}(S^pC,\varepsilon) = 0, \quad Q_n^{[0,p]}(C,\varepsilon) = Q_n(C,\varepsilon), \quad Q^{n+p}(S^pC,\varepsilon) = \hat{Q}^n(C,\varepsilon),$$

and the braid collapses to the exact sequence

$$\ldots \longrightarrow \hat{Q}^{n+1}(C,\varepsilon) \xrightarrow{H} Q_n(C,\varepsilon) \xrightarrow{1+T_\varepsilon} Q^n(C,\varepsilon) \xrightarrow{J} \hat{Q}^n(C,\varepsilon)$$
$$\xrightarrow{H} Q_{n-1}(C,\varepsilon) \longrightarrow \ldots .$$

Proof. Applying $\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(-, C^t \otimes_A C)$ to the chain equivalence of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$SW[-p,-1] \to C(W[-p,\infty] \to W[0,\infty])$$

arising from the split short exact sequence

$$0 \to W[-p,-1] \to W[-p,\infty] \to W[0,\infty] \to 0$$

we have a chain equivalence of **Z**-module chain complexes

$$C(S^p) = C(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[0,\infty], C^t \otimes_A C) \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[-p,\infty], C^t \otimes_A C))$$
$$\to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(SW[-p,-1], C^t \otimes_A C) = S(W[0,p-1] \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (C^t \otimes_A C)).$$

To obtain the braid apply $-\otimes_{\mathbf{Z}[\mathbf{Z}_2]}(C^t \otimes_A C)$ to the commutative diagram of split short exact sequences of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$\begin{tikzcd}[column sep=small]
& & 0 \arrow[dr] \arrow[rr, bend left] & & W[0,p-1] \arrow[dr] \arrow[rr, bend left] & & 0 \arrow[dr] & & \\
& 0 \arrow[ur] \arrow[dr] & & W[-\infty,p-1] \arrow[ur] \arrow[dr] & & W[0,\infty] \arrow[ur] \arrow[dr] & & 0 \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, bend right] & & W[-\infty,-1] \arrow[ur] \arrow[rr, bend right] & & W[-\infty,\infty] \arrow[ur] \arrow[rr, bend right] & & W[p,\infty] \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

and consider the associated long exact sequences of homology groups (which are all special cases of those of Proposition 1.1(iii)).

Given A-module chain complexes C, C' there are defined *direct sum* operations

$$\begin{cases} \oplus : Q^n_{[i,j]}(C,\varepsilon) \oplus Q^n_{[i,j]}(C',\varepsilon) \to Q^n_{[i,j]}(C \oplus C',\varepsilon)\,;\ (\varphi,\varphi') \mapsto \varphi \oplus \varphi', \\ \oplus : Q_n^{[i,j]}(C,\varepsilon) \oplus Q_n^{[i,j]}(C',\varepsilon) \to Q_n^{[i,j]}(C \oplus C',\varepsilon)\,;\ (\psi,\psi') \mapsto \psi \oplus \psi'. \end{cases}$$

The direct sum of n-dimensional ε-symmetric {ε-quadratic} (Poincaré) complexes over A $(C, \varphi \in Q^n(C,\varepsilon)), (C', \varphi' \in Q^n(C',\varepsilon))$ $\{(C, \psi \in Q_n(C,\varepsilon)), (C', \psi' \in Q_n(C',\varepsilon))\}$ is an n-dimensional ε-symmetric {ε-quadratic} (Poincaré) complex over A

$$\begin{cases} (C,\varphi) \oplus (C',\varphi') = (C \oplus C', \varphi \oplus \varphi' \in Q^n(C \oplus C',\varepsilon)), \\ (C,\psi) \oplus (C',\psi') = (C \oplus C', \psi \oplus \psi' \in Q_n(C \oplus C',\varepsilon)). \end{cases}$$

The $\mathbf{Z}_2$-hypercohomology {$\mathbf{Z}_2$-hyperhomology, Tate $\mathbf{Z}_2$-hypercohomology} groups behave as follows under the direct sum operation.

PROPOSITION 1.4. (i) *Given A-module chain complexes C, D there are natural direct sum decompositions of abelian groups*

$$\begin{cases} Q^n(C \oplus D,\varepsilon) = Q^n(C,\varepsilon) \oplus Q^n(D,\varepsilon) \oplus H_n(C^t \otimes_A D), \\ Q_n(C \oplus D,\varepsilon) = Q_n(C,\varepsilon) \oplus Q_n(D,\varepsilon) \oplus H_n(C^t \otimes_A D), \\ \hat{Q}^n(C \oplus D,\varepsilon) = \hat{Q}^n(C,\varepsilon) \oplus \hat{Q}^n(D,\varepsilon). \end{cases}$$

The ε-symmetrization $(1+T_\varepsilon) : Q_n(C \oplus D,\varepsilon) \to Q^n(C \oplus D,\varepsilon)$ is an isomorphism on $H_n(C^t \otimes_A D)$.

(ii) *Given A-module chain maps $f, g : C \to D$ there are defined factorizations*

$$\begin{cases} (f+g)^\% - f^\% - g^\% : Q^n(C,\varepsilon) \longrightarrow H_n(C^t \otimes_A C) \xrightarrow{f^t \otimes_A g} H_n(D^t \otimes_A D) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow Q^n(D,\varepsilon) \\ (f+g)_\% - f_\% - g_\% : Q_n(C,\varepsilon) \longrightarrow H_n(C^t \otimes_A C) \xrightarrow{f^t \otimes_A g} H_n(D^t \otimes_A D) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow Q_n(D,\varepsilon) \\ (f+g)^{\wedge\%} - \hat{f}^\% - \hat{g}^\% = 0 : \hat{Q}^n(C,\varepsilon) \longrightarrow \hat{Q}^n(D,\varepsilon) \end{cases}$$

with

$$Q^n(C,\varepsilon) \to H_n(C^t \otimes_A C)\,;\ \varphi \mapsto \varphi_0,$$

$$H_n(D^t \otimes_A D) \to Q^n(D,\varepsilon)\,;\ \theta \mapsto \left\{ \varphi_s = \begin{cases} (1+T_\varepsilon)\theta & \text{if } s = 0 \\ 0 & \text{if } s \geqslant 1 \end{cases} \right\},$$

$$Q_n(C,\varepsilon) \to H_n(C^t \otimes_A C)\,;\ \psi \mapsto (1+T_\varepsilon)\psi_0,$$

$$H_n(D^t \otimes_A D) \to Q_n(D,\varepsilon)\,;\ \theta \mapsto \left\{ \psi_s = \begin{cases} \theta & \text{if } s = 0 \\ 0 & \text{if } s \geqslant 1 \end{cases} \right\}.$$

Proof. (i) Applying $H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[i,j], -))$ to the direct sum decomposition of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$(C\oplus D)^t\otimes_A (C\oplus D) = (C^t\otimes_A C)\oplus (D^t\otimes_A D)\oplus ((C^t\otimes_A D)\oplus (D^t\otimes_A C))$$

we have a direct sum decomposition of abelian groups

$$\begin{aligned} Q^n_{[i,j]}(C\oplus D,\varepsilon) &= Q^n_{[i,j]}(C,\varepsilon)\oplus Q^n_{[i,j]}(D,\varepsilon) \\ &\quad \oplus H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[i,j],(C^t\otimes_A D)\oplus(D^t\otimes_A C))) \end{aligned}$$

with $T\in\mathbf{Z}_2$ acting on $(C^t\otimes_A D)\oplus(D^t\otimes_A C)$ by

$$T_\varepsilon\colon C^t_p\otimes_A D_q\oplus D^t_r\otimes_A C_s \to C^t_s\otimes_A D_r\oplus D^t_q\otimes_A C_p;$$
$$(u\otimes v, x\otimes y)\mapsto ((-)^{rs}y\otimes\varepsilon x, (-)^{pq}v\otimes\varepsilon u).$$

By direct computation

$$\begin{cases} H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[0,\infty],(C^t\otimes_A D)\oplus(D^t\otimes_A C))) = H_n(C^t\otimes_A D), \\ H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[-\infty,-1],(C^t\otimes_A D)\oplus(D^t\otimes_A C))) = H_n(C^t\otimes_A D), \\ H_n(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[-\infty,\infty],(C^t\otimes_A D)\oplus(D^t\otimes_A C))) = 0. \end{cases}$$

(ii) Substitute the decomposition

$$Q^n(C\oplus C,\varepsilon) = Q^n(C,\varepsilon)\oplus Q^n(C,\varepsilon)\oplus H_n(C^t\otimes_A C)$$

in

$$(f+g)^\%\colon Q^n(C,\varepsilon)\xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}^\%} Q^n(C\oplus C,\varepsilon)\xrightarrow{(f\ g)^\%} Q^n(D,\varepsilon)$$

and similarly for Q_n, $\hat{Q}^n$.

An A-module chain complex C is *strictly n-dimensional* if each C_r is a f.g. projective A-module, and $C_r = 0$ for $r<0$ and $r>n$,

$$C\colon \ldots \longrightarrow 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \ldots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \ldots .$$

The chain equivalence classes of n-dimensional A-module chain complexes are in a natural one–one correspondence with the stable isomorphism classes of strictly n-dimensional A-module chain complexes, the stability being with respect to the chain contractible complexes. We shall now obtain an analogous result for algebraic Poincaré complexes.

An n-dimensional ε-symmetric {ε-quadratic, ε-hyperquadratic} complex over A (C,φ) {(C,ψ), (C,θ)} is *strictly n-dimensional* (respectively *contractible*) if the underlying A-module chain complex C is strictly n-dimensional (respectively chain contractible). Note that the Q-groups of a chain contractible complex C are 0, by Proposition 1.1 (i).

A *stable isomorphism* of strictly n-dimensional ε-symmetric $\{\varepsilon$-quadratic, ε-hyperquadratic$\}$ complexes over A

$$\begin{cases} [f]\colon (C,\varphi) \to (C',\varphi') \\ [f]\colon (C,\psi) \to (C',\psi') \\ [f]\colon (C,\theta) \to (C',\theta') \end{cases}$$

is an isomorphism

$$\begin{cases} f\colon (C,\varphi)\oplus(D,0) \to (C',\varphi')\oplus(D',0) \\ f\colon (C,\psi)\oplus(D,0) \to (C',\psi')\oplus(D',0) \\ f\colon (C,\theta)\oplus(D,0) \to (C',\theta')\oplus(D',0) \end{cases}$$

for some chain contractible strictly n-dimensional A-module chain complexes D, D'.

PROPOSITION 1.5. *The homotopy equivalence classes of n-dimensional ε-symmetric $\{\varepsilon$-quadratic, ε-hyperquadratic$\}$ complexes over A are in a natural one–one correspondence with the stable isomorphism classes of strictly n-dimensional ε-symmetric $\{\varepsilon$-quadratic, ε-hyperquadratic$\}$ complexes over A.*

Proof. We need only consider the ε-symmetric case, the ε-quadratic and ε-hyperquadratic cases being entirely similar.

Given a stable isomorphism of strictly n-dimensional ε-symmetric complexes

$$[f]\colon (C,\varphi) \to (C',\varphi')$$

there is defined a homotopy equivalence, namely the composite

$$g\colon (C,\varphi) \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} (C,\varphi)\oplus(D,0) \xrightarrow{f} (C',\varphi')\oplus(D',0) \xrightarrow{(1\ 0)} (C',\varphi').$$

Given an n-dimensional A-module chain complex C there exist a strictly n-dimensional A-module chain complex C' and a chain equivalence

$$g\colon C \to C'$$

(by definition). It follows that for any n-dimensional ε-symmetric complex over A $(C, \varphi \in Q^n(C,\varepsilon))$ there exist a strictly n-dimensional ε-symmetric complex over A (C',φ') and a homotopy equivalence

$$g\colon (C,\varphi) \to (C',\varphi'),$$

defining

$$\varphi' = g^{\%}(\varphi) \in Q^n(C',\varepsilon).$$

Given a homotopy equivalence of strictly n-dimensional ε-symmetric complexes

$$g\colon (C,\varphi) \to (C',\varphi')$$

there is defined a stable isomorphism

$$[f]\colon (C,\varphi) \to (C',\varphi'),$$

as follows. The algebraic mapping cone $C(g)$, as defined by

$$d_{C(g)} = \begin{pmatrix} d' & (-)^{r-1}g \\ 0 & d \end{pmatrix} : C(g)_r = C'_r \oplus C_{r-1} \to C(g)_{r-1} = C'_{r-1} \oplus C_{r-2},$$

is chain contractible. Choose a chain contraction

$$\Gamma = \begin{pmatrix} h' & k \\ g' & h \end{pmatrix} : C(g)_r = C'_r \oplus C_{r-1} \to C(g)_{r+1} = C'_{r+1} \oplus C_r,$$

such that

$$d_{C(g)}\Gamma + \Gamma d_{C(g)} = 1 : C(g)_r \to C(g)_r,$$

and define chain contractible strictly n-dimensional A-module chain complexes D, D' by

$$d_D = \begin{cases} \begin{bmatrix} d' & (-)^{n-1}g \\ 0 & d \end{bmatrix} : D_n = \operatorname{coker}\left(\begin{pmatrix} (-)^n g \\ d \end{pmatrix} : C_n \to C'_n \oplus C_{n-1}\right) \\ \qquad\qquad\qquad\qquad \to D_{n-1} = C'_{n-1} \oplus C_{n-2}, \\ \begin{pmatrix} d' & (-)^{r-1}g \\ 0 & d \end{pmatrix} : D_r = C'_r \oplus C_{r-1} \to D_{r-1} = C'_{r-1} \oplus C_{r-2} \\ \qquad\qquad\qquad\qquad (0 \leqslant r \leqslant n-1), \end{cases}$$

$$d_{D'} = \begin{cases} \begin{pmatrix} (-)^n \\ d \end{pmatrix} : D'_n = C_{n-1} \to D'_{n-1} = C_{n-1} \oplus C_{n-2}, \\ \begin{pmatrix} d & (-)^r \\ 0 & d \end{pmatrix} : D'_r = C_r \oplus C_{r-1} \to D'_{r-1} = C_{r-1} \oplus C_{r-2} \quad (0 \leqslant r \leqslant n-1). \end{cases}$$

The isomorphism of A-module chain complexes

$$f : C \oplus D \to C' \oplus D',$$

given by

$$f = \begin{cases} \begin{pmatrix} g & \begin{bmatrix} 1+(-)^{n-1}gg' & (-)^{n-1}gh \\ -dg' & 1-dh \end{bmatrix} \\ 0 & \end{pmatrix} : \\ \quad C_n \oplus D_n = C_n \oplus \operatorname{coker}\left(\begin{pmatrix} (-)^n g \\ d \end{pmatrix} : C_n \to C'_n \oplus C_{n-1}\right) \\ \qquad\qquad\qquad\qquad \to C'_n \oplus D'_n = C'_n \oplus C_{n-1}, \\ \begin{pmatrix} g & 1+(-)^{r-1}gg' & (-)^{r-1}gh \\ 1 & (-)^{r-1}g' & (-)^{r-1}h \\ 0 & 0 & 1 \end{pmatrix} : \\ \quad C_r \oplus D_r = C_r \oplus C'_r \oplus C_{r-1} \to C'_r \oplus D'_r = C'_r \oplus C_r \oplus C_{r-1} \\ \qquad\qquad\qquad\qquad (0 \leqslant r \leqslant n-1), \end{cases}$$

defines a stable isomorphism of strictly n-dimensional ε-symmetric complexes over A

$$[f]\colon (C,\varphi) \to (C',\varphi').$$

The sequence of homology A-modules $H_*(C)$ is the fundamental chain homotopy invariant of an A-module chain complex C. We shall now associate to an algebraic Poincaré complex over A (C,φ) a sequence of functions

$$v(\varphi)\colon H^*(C) \to (\text{subquotient group of } A),$$

which is a fundamental chain homotopy invariant of (C,φ). The functions will be called 'Wu classes', because they are closely related to the Wu classes of algebraic topology—the connection between the geometry and the algebra will be made precise in § 9 of Part II. The Wu class $v(\varphi)(x)$ is the obstruction to performing an algebraic surgery on (C,φ) to kill a cohomology class $x \in H^*(C)$, in the terminology to be developed in § 4 below.

Let $T \in \mathbf{Z}_2$ act on A by

$$T_\varepsilon\colon A \to A\,;\ a \mapsto \varepsilon\bar{a},$$

and define the $\mathbf{Z}_2$-cohomology $\{\mathbf{Z}_2$-homology, Tate $\mathbf{Z}_2$-cohomology$\}$ groups

$$\begin{cases} H^r(\mathbf{Z}_2;\, A,\varepsilon) = \begin{cases} \ker(1-T_\varepsilon\colon A \to A) & \text{if } r = 0, \\ \hat{H}^r(\mathbf{Z}_2;\, A,\varepsilon) & \text{if } r \geqslant 1, \\ 0 & \text{if } r < 0, \end{cases} \\ H_r(\mathbf{Z}_2;\, A,\varepsilon) = \begin{cases} \operatorname{coker}(1-T_\varepsilon\colon A \to A) & \text{if } r = 0, \\ \hat{H}^{r+1}(\mathbf{Z}_2;\, A,\varepsilon) & \text{if } r \geqslant 1, \\ 0 & \text{if } r < 0, \end{cases} \\ \hat{H}^r(\mathbf{Z}_2;\, A,\varepsilon) = \ker(1-(-)^rT_\varepsilon\colon A \to A)/\operatorname{im}(1+(-)^rT_\varepsilon\colon A \to A) \quad (r \in \mathbf{Z}). \end{cases}$$

The function

$$A \times \hat{H}^r(\mathbf{Z}_2;\, A,\varepsilon) \to \hat{H}^r(\mathbf{Z}_2;\, A,\varepsilon)\,;\ (a,x) \mapsto ax\bar{a}$$

defines an A-module structure on $\hat{H}^r(\mathbf{Z}_2;\, A,\varepsilon)$ (which is of exponent 2, and vanishes if $\frac{1}{2} \in A$). The functions

$$\begin{cases} A \times H^0(\mathbf{Z}_2;\, A,\varepsilon) \to H^0(\mathbf{Z}_2;\, A,\varepsilon)\,;\ (a,x) \mapsto ax\bar{a} \\ A \times H_0(\mathbf{Z}_2;\, A,\varepsilon) \to H_0(\mathbf{Z}_2;\, A,\varepsilon)\,;\ (a,x) \mapsto ax\bar{a} \end{cases}$$

are not linear in A, and so do not define A-module structures. Nevertheless, we shall write $\operatorname{Hom}_A(M, H^0(\mathbf{Z}_2;\, A,\varepsilon))$ $\{\operatorname{Hom}_A(M, H_0(\mathbf{Z}_2;\, A,\varepsilon))\}$ for the abelian group of functions $f\colon M \to H^0(\mathbf{Z}_2;\, A,\varepsilon)$ $\{f\colon M \to H_0(\mathbf{Z}_2;\, A,\varepsilon)\}$

defined on an A-module M such that

$$f(ax) = af(x)\bar{a} \in \begin{cases} H^0(\mathbf{Z}_2; A, \varepsilon) \\ H_0(\mathbf{Z}_2; A, \varepsilon) \end{cases} \quad (a \in A,\ x \in M),$$

calling such functions 'A-module morphisms'. For $\varepsilon = 1 \in A$ we write

$$\begin{cases} H^r(\mathbf{Z}_2; A, 1) = H^r(\mathbf{Z}_2; A), \\ H_r(\mathbf{Z}_2; A, 1) = H_r(\mathbf{Z}_2; A), \\ \hat{H}^r(\mathbf{Z}_2; A, 1) = \hat{H}^r(\mathbf{Z}_2; A). \end{cases}$$

The cohomology classes $f \in H^m(C)$ of an A-module chain complex C may be regarded as the chain homotopy classes of A-module chain maps

$$f\colon C \to S^m A,$$

where $S^m A$ is the A-module chain complex defined by

$$(S^m A)_r = \begin{cases} A & \text{if } r = m, \\ 0 & \text{otherwise.} \end{cases}$$

The induced abelian group morphisms

$$\begin{cases} f^{\%}\colon Q^n_{[i,j]}(C, \varepsilon) \to Q^n_{[i,j]}(S^m A, \varepsilon);\ \varphi \mapsto (f^t \otimes_A f)\varphi_{2m-n} & (\varphi_{2m-n} \in C^t_m \otimes_A C_m) \\ f_{\%}\colon Q^{[i,j]}_n(C, \varepsilon) \to Q^{[i,j]}_n(S^m A, \varepsilon);\ \psi \mapsto (f^t \otimes_A f)\psi_{n-2m} & (\psi_{n-2m} \in C^t_m \otimes_A C_m) \end{cases}$$

depend only on $f \in H^m(C)$, on account of the chain homotopy invariance of Proposition 1.1 (i). Using the $\mathbf{Z}[\mathbf{Z}_2]$-module isomorphism

$$A \to A^t \otimes_A A;\ a \mapsto 1 \otimes a$$

as an identification we have that these morphisms take values in

$$\begin{cases} Q^n_{[i,j]}(S^m A, \varepsilon) = \begin{cases} H^{2m-n-i}(\mathbf{Z}_2; A, (-)^{m-i}\varepsilon) & \text{if } 2m-n < j \neq i, \\ H_0(\mathbf{Z}_2; A, (-)^{n-m+1}\varepsilon) & \text{if } 2m-n = j \neq i, \\ A & \text{if } 2m-n = j = i, \\ 0 & \text{otherwise,} \end{cases} \\ Q^{[i,j]}_n(S^m A, \varepsilon) = \begin{cases} H_{n-2m-i}(\mathbf{Z}_2; A, (-)^{m-i}\varepsilon) & \text{if } n-2m < j \neq i, \\ H^0(\mathbf{Z}_2; A, (-)^{n-m+1}\varepsilon) & \text{if } n-2m = j \neq i, \\ A & \text{if } n-2m = j = i, \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

Define the *rth ε-symmetric* {*ε-quadratic, ε-hyperquadratic*} *Wu class* of an element $\varphi \in Q^n(C, \varepsilon)$ $\{\psi \in Q_n(C, \varepsilon),\ \theta \in \hat{Q}^n(C, \varepsilon)\}$ for some A-module chain complex C to be the A-module morphism

$$\begin{cases} v_r = v_r(\varphi)\colon H^{n-r}(C) \to Q^n(S^{n-r}A,\varepsilon) = H^{n-2r}(\mathbf{Z}_2;\, A, (-)^{n-r}\varepsilon); \\ \hfill f \mapsto (f^t \otimes_A f)\varphi_{n-2r}, \\ v^r = v^r(\psi)\colon H^{n-r}(C) \to Q_n(S^{n-r}A,\varepsilon) = H_{2r-n}(\mathbf{Z}_2;\, A, (-)^{n-r}\varepsilon); \\ \hfill f \mapsto (f^t \otimes_A f)\psi_{2r-n}, \\ \hat{v}_r = \hat{v}_r(\theta)\colon H^{n-r}(C) \to \hat{Q}^n(S^{n-r}A,\varepsilon) = \hat{H}^r(\mathbf{Z}_2;\, A,\varepsilon);\, f \mapsto (f^t \otimes_A f)\theta_{n-2r} \\ \hfill (f\colon C_{n-r} \to A,\ \varphi_{n-2r},\ \psi_{2r-n},\ \theta_{n-2r} \in C^t_{n-r} \otimes_A C_{n-r}). \end{cases}$$

Note that $v_r = 0$ for $2r > n$ $\{v^r = 0$ for $2r < n\}$, and that the Wu classes satisfy the addition formulae (cf. Proposition 1.4(ii))

$$\begin{cases} v_r(\varphi)(f+g) - v_r(\varphi)(f) - v_r(\varphi)(g) \\ \qquad = \begin{cases} (1 + T_{(-)^r\varepsilon})(f^t \otimes_A g)\varphi_0 \in H^0(\mathbf{Z}_2;\, A, (-)^r\varepsilon) & \text{if } n = 2r, \\ 0 \in \hat{H}^0(\mathbf{Z}_2;\, A, (-)^r\varepsilon) & \text{if } n > 2r, \end{cases} \\ v^r(\psi)(f+g) - v^r(\psi)(f) - v^r(\psi)(g) \\ \qquad = \begin{cases} (f^t \otimes_A g)(1 + T_\varepsilon)\psi_0 \in H_0(\mathbf{Z}_2;\, A, (-)^r\varepsilon) & \text{if } n = 2r, \\ 0 \in \hat{H}^0(\mathbf{Z}_2;\, A, (-)^{r+1}\varepsilon) & \text{if } n < 2r, \end{cases} \\ \hat{v}_r(\theta)(f+g) - \hat{v}_r(\theta)(f) - \hat{v}_r(\theta)(g) = 0 \in \hat{H}^0(\mathbf{Z}_2;\, A, (-)^r\varepsilon) \quad (r \in \mathbf{Z}). \end{cases}$$

The Wu classes are compatible with all the maps appearing in the braid of Proposition 1.3. In particular, the Wu classes commute with the suspension maps:

$$\begin{array}{ccc} Q^n(C,\varepsilon) & \xrightarrow{v_r} & \operatorname{Hom}_A(H^{n-r}(C), H^{n-2r}(\mathbf{Z}_2;\, A, (-)^{n-r}\varepsilon)) \\ {\scriptstyle S}\big\downarrow & & \big\downarrow{\scriptstyle S\ (=\text{ id. if } n > 2r)} \\ Q^{n+1}(SC,\varepsilon) & \xrightarrow{v_r} & \operatorname{Hom}_A(H^{n-r+1}(SC), H^{n-2r+1}(\mathbf{Z}_2;\, A, (-)^{n-r+1}\varepsilon)) \end{array}$$

$$\begin{array}{ccc} Q_n(C,\varepsilon) & \xrightarrow{v^r} & \operatorname{Hom}_A(H^{n-r}(C), H_{2r-n}(\mathbf{Z}_2;\, A, (-)^{n-r}\varepsilon)) \\ {\scriptstyle S}\big\downarrow & & \big\downarrow{\scriptstyle S\ (=\text{ id. if } n < 2r)} \\ Q_{n+1}(SC,\varepsilon) & \xrightarrow{v^r} & \operatorname{Hom}_A(H^{n-r+1}(SC,)\, H_{2r-n+1}(\mathbf{Z}_2;\, A, (-)^{n-r+1}\varepsilon)) \end{array}$$

and also with the skew-suspension isomorphisms:

$$\begin{tikzcd}
Q^n(C,\varepsilon) \arrow[r, "v_r"] \arrow[d, "\bar{S}"'] & \mathrm{Hom}_A(H^{n-r}(C), H^{n-2r}(\mathbf{Z}_2; A, (-)^{n-r}\varepsilon)) \arrow[d, "\mathrm{id.}"] \\
Q^{n+2}(SC,-\varepsilon) \arrow[r, "v_{r+1}"] & \mathrm{Hom}_A(H^{n-r+1}(SC), H^{n-2r}(\mathbf{Z}_2; A, (-)^{n-r}\varepsilon))
\end{tikzcd}$$

$$\begin{tikzcd}
Q_n(C,\varepsilon) \arrow[r, "v^r"] \arrow[d, "\bar{S}"'] & \mathrm{Hom}_A(H^{n-r}(C), H_{2r-n}(\mathbf{Z}_2; A, (-)^{n-r}\varepsilon)) \arrow[d, "\mathrm{id.}"] \\
Q_{n+2}(SC,-\varepsilon) \arrow[r, "v^{r+1}"] & \mathrm{Hom}_A(H^{n-r+1}(SC), H_{2r-n}(\mathbf{Z}_2; A, (-)^{n-r}\varepsilon))
\end{tikzcd}$$

The composite

$$Q_n(C,\varepsilon) \xrightarrow{1+T_\varepsilon} Q^n(C,\varepsilon) \xrightarrow{v_r} \mathrm{Hom}_A(H^{n-r}(C), H^{n-2r}(\mathbf{Z}_2; A, (-)^{n-r}\varepsilon))$$

is 0 for $n \neq 2r$. For $n = 2r$ there is defined a commutative diagram

$$\begin{tikzcd}
Q_{2r}(C,\varepsilon) \arrow[r, "1+T_\varepsilon"] \arrow[d, "v^r"'] & Q^{2r}(C,\varepsilon) \arrow[d, "v_r"] \\
\mathrm{Hom}_A(H^r(C), H_0(\mathbf{Z}_2; A, (-)^r\varepsilon)) \arrow[r, "1+T_{(-)^r\varepsilon}"] & \mathrm{Hom}_A(H^r(C), H^0(\mathbf{Z}_2; A, (-)^r\varepsilon))
\end{tikzcd}$$

There is also defined a commutative diagram

$$\begin{tikzcd}
\hat{Q}^{n+1}(C,\varepsilon) \arrow[r, "\hat{v}_{r+1}"] \arrow[d, "H"'] & \mathrm{Hom}_A(H^{n-r}(C), \hat{H}^{r+1}(\mathbf{Z}_2; A, \varepsilon)) \arrow[d, "H\ (=\mathrm{id.\ if}\ 2r>n)"] \\
Q_n(C,\varepsilon) \arrow[r, "v^r"] & \mathrm{Hom}_A(H^{n-r}(C), H_{2r-n}(\mathbf{Z}_2; A, (-)^{n-r}\varepsilon))
\end{tikzcd}$$

In § 2 below, and elsewhere, we shall need the following notions.

The *reduced 0th Wu class* of an n-dimensional ε-symmetric complex over A $(C, \varphi \in Q^n(C,\varepsilon))$ is the composite

$$\hat{v}_0(\varphi) : H^n(C) \xrightarrow{v_0(\varphi)} H^n(\mathbf{Z}_2; A, (-)^n\varepsilon) \xrightarrow{J} \hat{H}^n(\mathbf{Z}_2; A, (-)^n\varepsilon) = \hat{H}^0(\mathbf{Z}_2; A, \varepsilon).$$

(For $n > 0$, $\hat{v}_0(\varphi) = v_0(\varphi)$.)

An n-dimensional ε-symmetric complex over A $(C, \varphi \in Q^n(C,\varepsilon))$ is *even* if

$$\hat{v}_0(\varphi) = 0\colon H^n(C) \to \hat{H}^0(\mathbf{Z}_2; A, \varepsilon).$$

For example, skew-suspensions of ε-symmetric complexes and ε-symmetrizations of ε-quadratic complexes are even.

Finally, we investigate the behaviour of our constructions under a change of rings.

Given a morphism of rings with 1

$$f\colon A \to B$$

(such that $f(1_A) = 1_B$) regard B as a (B, A)-bimodule by

$$B \times B \times A \to B;\ (b, x, a) \mapsto b.x.f(a),$$

so that an A-module M induces a B-module $B \otimes_A M$, with $B \otimes_A A = B$. If f is a morphism of rings with involution, with

$$\overline{f(a)} = f(\bar{a}) \in B \quad (a \in A),$$

then for any f.g. projective A-module M there is defined a natural isomorphism of f.g. projective B-modules

$$B \otimes_A M^* \to (B \otimes_A M)^*;\ b \otimes g \mapsto (c \otimes m \mapsto c.fg(m).\bar{b}).$$

For any A-module chain complex C there are defined natural $\mathbf{Z}$-module chain maps

$$C \to B \otimes_A C;\ x \mapsto 1 \otimes x,$$

$$C^* \to (B \otimes_A C)^*;\ g \mapsto (b \otimes x \mapsto b.fg(x)),$$

inducing the *change of rings* maps in homology {cohomology}

$$\begin{cases} f\colon H_*(C) \to H_*(B \otimes_A C), \\ f\colon H^*(C) \to H^*(B \otimes_A C). \end{cases}$$

If $\varepsilon \in A$ is a central unit such that $\bar{\varepsilon} = \varepsilon^{-1} \in A$ and such that $\eta = f(\varepsilon) \in B$ is central (necessarily such that $\bar{\eta} = \eta^{-1} \in B$) then there are also induced change of rings maps

$$\begin{cases} f\colon Q^n_{[i,j]}(C, \varepsilon) \to Q^n_{[i,j]}(B \otimes_A C, \eta), \\ f\colon Q_n^{[i,j]}(C, \varepsilon) \to Q_n^{[i,j]}(B \otimes_A C, \eta). \end{cases}$$

An n-dimensional ε-symmetric {ε-quadratic} (Poincaré) complex over A (C, φ) $\{(C, \psi)\}$ induces an n-dimensional η-symmetric {η-quadratic} (Poincaré) complex over B,

$$B \otimes_A (C, \varphi) = (B \otimes_A C, 1 \otimes \varphi) \quad \{B \otimes_A (C, \psi) = (B \otimes_A C, 1 \otimes \psi)\}.$$

The Wu classes remain invariant under change of rings, in the sense that the following diagram commutes

$$\begin{array}{ccc} H^{n-r}(C) & \xrightarrow{\ f\ } & H^{n-r}(B\otimes_A C) \\ \Big\downarrow{\scriptstyle v_r(\varphi)} & & \Big\downarrow{\scriptstyle v_r(1\otimes\varphi)} \\ H^{n-2r}(\mathbf{Z}_2;\, A, (-)^{n-r}\varepsilon) & \xrightarrow{\ f\ } & H^{n-2r}(\mathbf{Z}_2;\, B, (-)^{n-r}\eta) \end{array} \qquad (\varphi\in Q^n(C,\varepsilon))$$

and similarly for Q_n, $\hat{Q}^n$.

2. Forms and formations

We shall now identify the homotopy theory of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes over A for $n = 0$ (respectively $n = 1$) with the isomorphism (respectively stable isomorphism) theory of ε-symmetric $\{\varepsilon$-quadratic$\}$ forms (respectively formations) over A.

Given a f.g. projective A-module M define the *ε-duality* involution

$$T_\varepsilon\colon \mathrm{Hom}_A(M,M^*) \to \mathrm{Hom}_A(M,M^*);\ \varphi \mapsto (\varepsilon\varphi^*\colon x\mapsto (y\mapsto \varepsilon.\overline{\varphi(y)(x)})),$$

and define the abelian groups

$$Q^\varepsilon(M) = \ker(1-T_\varepsilon\colon \mathrm{Hom}_A(M,M^*)\to \mathrm{Hom}_A(M,M^*)),$$
$$Q\langle v_0\rangle^\varepsilon(M) = \mathrm{im}(1+T_\varepsilon\colon \mathrm{Hom}_A(M,M^*)\to \mathrm{Hom}_A(M,M^*)) \subseteq Q^\varepsilon(M),$$
$$Q_\varepsilon(M) = \mathrm{coker}(1-T_\varepsilon\colon \mathrm{Hom}_A(M,M^*)\to \mathrm{Hom}_A(M,M^*)).$$

An *ε-symmetric $\{\varepsilon$-quadratic$\}$ form over A* (M,φ) $\{(M,\psi)\}$ is a f.g. projective A-module M together with an element $\varphi\in Q^\varepsilon(M)$ $\{\psi\in Q_\varepsilon(M)\}$, and it is *non-singular* if $\varphi\in \mathrm{Hom}_A(M,M^*)$ $\{(1+T_\varepsilon)\psi\in \mathrm{Hom}_A(M,M^*)\}$ is an isomorphism. A *morphism* (respectively *isomorphism*) of ε-symmetric $\{\varepsilon$-quadratic$\}$ forms over A

$$\begin{cases} f\colon (M,\varphi)\to(M',\varphi') \\ f\colon (M,\psi)\to(M',\psi') \end{cases}$$

is an A-module morphism (respectively isomorphism) $f\in\mathrm{Hom}_A(M,M')$ such that

$$\begin{cases} f^*\varphi' f = \varphi \in Q^\varepsilon(M), \\ f^*\psi' f = \psi\in Q_\varepsilon(M). \end{cases}$$

The *Wu class* of an ε-symmetric $\{\varepsilon$-quadratic$\}$ form over A (M,φ) $\{(M,\psi)\}$ is the quadratic function

$$\begin{cases} v_0(\varphi)\colon M\to H^0(\mathbf{Z}_2;\, A,\varepsilon);\ x\mapsto \varphi(x)(x), \\ v^0(\psi)\colon M\to H_0(\mathbf{Z}_2;\, A,\varepsilon);\ x\mapsto \psi(x)(x). \end{cases}$$

An ε-symmetric form (M, φ) is *even* if

$$\begin{aligned}\varphi \in \ker(\hat{v}_0 \colon Q^\varepsilon(M) \to \mathrm{Hom}_A(M, \hat{H}^0(\mathbf{Z}_2; A, \varepsilon)))\\ = Q\langle v_0\rangle^\varepsilon(M)\\ = \mathrm{im}((1+T_\varepsilon) \colon Q_\varepsilon(M) \to Q^\varepsilon(M)) \subseteq Q^\varepsilon(M),\end{aligned}$$

that is if it is the ε-symmetrization $(M, (1+T_\varepsilon)\psi)$ of an ε-quadratic form (M, ψ).

An ε-symmetric form over A $(M, \varphi \in Q^\varepsilon(M))$ is the same as a sesquilinear ε-symmetric pairing on a f.g. projective A-module M

$$\lambda \colon M \times M \to A \,;\ (x, y) \mapsto \lambda(x, y) = \varphi(x)(y)$$

such that

$$\begin{aligned}\lambda(x, y+y') &= \lambda(x, y) + \lambda(x, y'),\\ \lambda(x, ay) &= a\lambda(x, y),\\ \lambda(x, y) &= \varepsilon\overline{\lambda(y, x)} \qquad (x, y \in M,\ a \in A).\end{aligned}$$

The above definition of an ε-quadratic form $(M, \psi \in Q_\varepsilon(M))$ over an arbitrary ring with involution A is a generalization due to Wall [**26**] of the definition due to Tits [**23**] for division rings A, which itself goes back to the work of Klingenberg and Witt [**6**] on the invariant of Arf [**1**] for $A = \mathbf{Z}_2$. (In fact, the Arf invariant had been previously obtained by Dickson in [**5**, § 199]—I am indebted to William Pardon for this reference.) As shown by Wall in [**26**] this definition is equivalent to that given by Wall in [**25**, § 5], as a triple

$$(M, \lambda \colon M \times M \to A, \mu \colon M \to H_0(\mathbf{Z}_2; A, \varepsilon))$$

such that (M, λ) is an ε-symmetric pairing and μ satisfies

$$\begin{aligned}\lambda(x, x) &= \mu(x) + \varepsilon\overline{\mu(x)} \in H^0(\mathbf{Z}_2; A, \varepsilon),\\ \lambda(x, y) &= \mu(x+y) - \mu(x) - \mu(y) \in H_0(\mathbf{Z}_2; A, \varepsilon),\\ \mu(ax) &= a\mu(x)\bar{a} \in H_0(\mathbf{Z}_2; A, \varepsilon) \quad (x, y \in M,\ a \in A).\end{aligned}$$

The transformation $(M, \psi) \mapsto (M, \lambda, \mu)$ is given by

$$\begin{aligned}\lambda(x, y) &= \psi(x)(y) + \varepsilon\overline{\psi(y)(x)} \in A,\\ \mu(x) &= v^0(\psi)(x) = \psi(x)(x) \in H_0(\mathbf{Z}_2; A, \varepsilon).\end{aligned}$$

This definition of ε-quadratic form is also equivalent to that of Ranicki [**13**].

PROPOSITION 2.1. *The homotopy equivalence classes of* 0*-dimensional (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes over A are in a natural one–one correspondence with the isomorphism classes of (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ forms over A. Poincaré complexes correspond to non-singular forms.*

Proof. For any f.g. projective A-module M we can identify the ε-duality involution

$$T_\varepsilon: \mathrm{Hom}_A(M, M^*) \to \mathrm{Hom}_A(M, M^*);\ \varphi \mapsto \varepsilon\varphi^*$$

with the ε-transposition involution

$$T_\varepsilon: M^{*t}\otimes_A M^* \to M^{*t}\otimes_A M;\ f\otimes g \mapsto g\otimes \varepsilon f,$$

using the slant map isomorphism

$$\backslash : M^{*t}\otimes_A M^* \to \mathrm{Hom}_A(M, M^*);\ f\otimes g \mapsto (x \mapsto (y \mapsto g(y).\overline{f(x)})).$$

Thus for any 0-dimensional A-module chain complex C we can identify

$$\begin{cases} Q^0(C,\varepsilon) = Q^\varepsilon(H^0(C)), \\ Q_0(C,\varepsilon) = Q_\varepsilon(H^0(C)). \end{cases}$$

Given a 0-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex over A $(C, \varphi \in Q^0(C,\varepsilon))$ $\{(C, \psi \in Q_0(C,\varepsilon))\}$ there is defined an ε-symmetric $\{\varepsilon$-quadratic$\}$ form over A $(H^0(C), \varphi \in Q^\varepsilon(H^0(C)))$ $\{(H^0(C), \psi \in Q_\varepsilon(H^0(C)))\}$, such that the 0th ε-symmetric $\{\varepsilon$-quadratic$\}$ Wu class

$$v_0(\varphi): H^0(C) \to H^0(\mathbf{Z}_2; A, \varepsilon) \quad \{v^0(\psi): H^0(C) \to H_0(\mathbf{Z}_2; A, \varepsilon)\}$$

of the complex (C, φ) $\{(C, \psi)\}$ is just the Wu class of the form $(H^0(C), \varphi)$ $\{(H^0(C), \psi)\}$. In particular, we have that (C, φ) is an even ε-symmetric complex if and only if $(H^0(C), \varphi)$ is an even ε-symmetric form.

Conversely, an ε-symmetric $\{\varepsilon$-quadratic$\}$ form over A (M, φ) $\{(M, \psi)\}$ can be considered as a fixed point $\{$an orbit space$\}$ of the ε-duality involution T_ε on $\mathrm{Hom}_A(M, M^*)$, corresponding to a $\mathbf{Z}_2$-cohomology $\{\mathbf{Z}_2$-homology$\}$ class $\varphi \in H^0(\mathbf{Z}_2; \mathrm{Hom}_A(M, M^*))$ $\{\psi \in H_0(\mathbf{Z}_2; \mathrm{Hom}_A(M, M^*))\}$, and there is defined a strictly 0-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex over A $(C, \varphi \in Q^0(C,\varepsilon))$ $\{C, \psi \in Q_0(C,\varepsilon))\}$ with $C_0 = M^*$. Now apply Proposition 1.5.

Given an ε-symmetric $\{\varepsilon$-quadratic$\}$ form over A $(M, \varphi \in Q^\varepsilon(M))$ $\{(M, \psi \in Q_\varepsilon(M))\}$ and a submodule L of M define the *annihilator of* L to be the submodule

$$\begin{cases} L^\perp = \ker(j^*\varphi: M \to L^*) \\ L^\perp = \ker(j^*(\psi + \varepsilon\psi^*): M \to L^*) \end{cases}$$

of M, with $j \in \mathrm{Hom}_A(L, M)$ the inclusion. A *sublagrangian* of (M, φ) $\{(M, \psi)\}$ is a direct summand L of M such that $j^*\varphi j = 0 \in Q^\varepsilon(M)$ $\{j^*\psi j = 0 \in Q_\varepsilon(M)\}$ and $j^*\varphi \in \mathrm{Hom}_A(M, L^*)$ $\{j^*(\psi + \varepsilon\psi^*) \in \mathrm{Hom}_A(M, L^*)\}$ is onto, so that the annihilator $L^\perp$ is a direct summand of M containing L,

$$L \subseteq L^\perp.$$

A lagrangian is a sublagrangian L such that

$$L = L^{\perp},$$

that is, such that there is defined an exact sequence

$$\begin{cases} 0 \longrightarrow L \xrightarrow{\;j\;} M \xrightarrow{\;j^*\varphi\;} L^* \longrightarrow 0, \\ 0 \longrightarrow L \xrightarrow{\;j\;} M \xrightarrow{\;j^*(\psi+\varepsilon\psi^*)\;} L^* \longrightarrow 0. \end{cases}$$

An ε-symmetric $\{\varepsilon$-quadratic$\}$ form is *hyperbolic* if it admits a lagrangian, in which case it is non-singular. (Hyperbolic symmetric forms were termed 'metabolic' by Knebusch [7], but a uniform terminology for the ε-symmetric and ε-quadratic cases seems preferable here.)

Given an ε-symmetric form over A $(M, \varphi \in Q^{\varepsilon}(M))$ {a f.g. projective A-module L} define the *standard* hyperbolic ε-symmetric {even ε-symmetric, ε-quadratic} form over A

$$\begin{cases} H^{\varepsilon}(M,\varphi) = \left(M^* \oplus M, \begin{pmatrix} 0 & 1 \\ \varepsilon & \varphi \end{pmatrix} \in Q^{\varepsilon}(M^* \oplus M)\right), \\ H^{\varepsilon}(L) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \in Q\langle v_0\rangle^{\varepsilon}(L \oplus L^*)\right), \\ H_{\varepsilon}(L) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q_{\varepsilon}(L \oplus L^*)\right). \end{cases}$$

The various standard hyperbolic forms are related to each other by

$$(1+T_{\varepsilon})H_{\varepsilon}(L) = H^{\varepsilon}(L) = H^{\varepsilon}(L^*, 0).$$

Every hyperbolic form is isomorphic to a standard hyperbolic form, by the following generalization of a theorem of Witt [**29**].

PROPOSITION 2.2. *The inclusion of a sublagrangian L in an ε-symmetric {even ε-symmetric, ε-quadratic} form over A is a morphism of forms*

$$\begin{cases} j\colon (L,0) \to (M,\varphi) \\ j\colon (L,0) \to (M,\psi+\varepsilon\psi^*) \\ j\colon (L,0) \to (M,\psi) \end{cases}$$

which extends to an isomorphism

$$\begin{cases} f\colon H^{\varepsilon}(L^*,\theta) \oplus (L^{\perp}/L, \varphi^{\perp}/\varphi) \to (M,\varphi), \\ f\colon H^{\varepsilon}(L) \oplus (L^{\perp}/L, \psi^{\perp}/\psi + \varepsilon(\psi^{\perp}/\psi)^*) \to (M,\psi+\varepsilon\psi^*), \\ f\colon H_{\varepsilon}(L) \oplus (L^{\perp}/L, \psi^{\perp}/\psi) \to (M,\psi). \end{cases}$$

Proof. A morphism of ε-symmetric $\{\varepsilon$-quadratic$\}$ forms

$$\begin{cases} g\colon (N,\nu) \to (N',\nu') \\ g\colon (N,\chi) \to (N',\chi') \end{cases}$$

with (N,ν) $\{(N,\chi)\}$ non-singular extends to an isomorphism

$$\begin{cases} (g \quad g^{\perp})\colon (N,\nu) \oplus (N^{\perp},\nu^{\perp}) \to (N',\nu') \\ (g \quad g^{\perp})\colon (N,\chi) \oplus (N^{\perp},\chi^{\perp}) \to (N',\nu') \end{cases}$$

with $g^{\perp} \in \mathrm{Hom}_A(N^{\perp},N')$ the inclusion of

$$\begin{cases} N^{\perp} = \ker(g^*\nu'\colon N' \to N^*) \\ N^{\perp} = \ker(g^*(\chi'+\varepsilon\chi'^*)\colon N' \to N^*) \end{cases}$$

and $\nu^{\perp} = (g^{\perp})^*\nu' g^{\perp} \in Q^{\varepsilon}(N')$ $\{\chi^{\perp} = (g^{\perp})^*\chi' g^{\perp} \in Q_{\varepsilon}(N')\}$, since the exact sequence

$$\begin{cases} 0 \longrightarrow N^{\perp} \xrightarrow{g^{\perp}} N' \xrightarrow{g^*\nu'} N^* \longrightarrow 0 \\ 0 \longrightarrow N^{\perp} \xrightarrow{g^{\perp}} N' \xrightarrow{g^*(\chi'+\varepsilon\chi'^*)} N^* \longrightarrow 0 \end{cases}$$

is split by $g\nu^{-1} \in \mathrm{Hom}_A(N^*,N')$ $\{g(\chi+\varepsilon\chi^*)^{-1} \in \mathrm{Hom}_A(N^*,N')\}$.

The inclusion of a sublagrangian L in an ε-symmetric $\{$even ε-symmetric, ε-quadratic$\}$ form over A, $(M,\varphi \in Q^{\varepsilon}(M))$ $\{(M,\psi+\varepsilon\psi^* \in Q\langle v_0\rangle^{\varepsilon}(M))$, $(M,\psi \in Q_{\varepsilon}(M))\}$ extends to a morphism of forms

$$\begin{cases} g = (j\ k)\colon H^{\varepsilon}(L^*,k^*\varphi k) = \left(L\oplus L^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & k^*\varphi k \end{pmatrix}\right) \to (M,\varphi) \\ g = (j\ (k-\bar{\varepsilon}jk^*\psi k))\colon H^{\varepsilon}(L) = \left(L\oplus L^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}\right) \to (M,\psi+\varepsilon\psi^*) \\ g = (j\ (k-\bar{\varepsilon}jk^*\psi k))\colon H_{\varepsilon}(L) = \left(L\oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \to (M,\psi) \end{cases}$$

with $k \in \mathrm{Hom}_A(L^*,M)$ any A-module morphism such that

$$\begin{cases} j^*\varphi k = 1 \in \mathrm{Hom}_A(L^*,L^*) \\ j^*(\psi+\varepsilon\psi^*)k = 1 \in \mathrm{Hom}_A(L^*,L^*), \\ j^*(\psi+\varepsilon\psi^*)k = 1 \in \mathrm{Hom}_A(L^*,L^*). \end{cases}$$

Now $H^{\varepsilon}(L^*,k^*\varphi k)$ $\{H^{\varepsilon}(L), H_{\varepsilon}(L)\}$ is non-singular, so that g extends to an isomorphism of forms

$$\begin{cases} f = (g\ h)\colon H^{\varepsilon}(L^*, k^*\varphi k) \oplus (L^{\perp}/L, \varphi^{\perp}/\varphi) \to (M, \varphi), \\ f = (g\ h)\colon H^{\varepsilon}(L) \oplus (L^{\perp}/L, \psi^{\perp}/\psi + \varepsilon(\psi^{\perp}/\psi)^*) \to (M, \psi + \varepsilon\psi^*), \\ f = (g\ h)\colon H_{\varepsilon}(L) \oplus (L^{\perp}/L, \psi^{\perp}/\psi) \to (M, \psi). \end{cases}$$

An *(even) ε-symmetric* {*ε-quadratic*} *formation over* A $(M, \varphi; F, G)$ $\{(M, \psi; F, G)\}$ is a non-singular (even) ε-symmetric {ε-quadratic} form over A (M, φ) $\{(M, \psi)\}$ together with a lagrangian F and a sublagrangian G. The formation is *non-singular* if G is a lagrangian. An *isomorphism* of (even) ε-symmetric {ε-quadratic} formations

$$\begin{cases} f\colon (M, \varphi; F, G) \to (M', \varphi'; F', G') \\ f\colon (M, \psi; F, G) \to (M', \psi'; F', G') \end{cases}$$

is an isomorphism of forms

$$\begin{cases} f\colon (M, \varphi) \to (M', \varphi') \\ f\colon (M, \psi) \to (M', \psi') \end{cases}$$

such that

$$f(F) = F', \quad f(G) = G'.$$

A *stable isomorphism* of (even) ε-symmetric {ε-quadratic} formations

$$\begin{cases} [f]\colon (M, \varphi; F, G) \to (M', \varphi'; F', G') \\ [f]\colon (M, \psi; F, G) \to (M', \psi'; F', G') \end{cases}$$

is an isomorphism of the type

$$\begin{cases} f\colon (M, \varphi; F, G) \oplus (H^{\varepsilon}(P); P, P^*) \to (M', \varphi'; F', G') \oplus (H^{\varepsilon}(P'); P', P'^*) \\ f\colon (M, \psi; F, G) \oplus (H_{\varepsilon}(P); P, P^*) \to (M', \psi'; F', G') \oplus (H_{\varepsilon}(P'); P', P'^*) \end{cases}$$

for some f.g. projective A-modules P, P'.

Formations first appeared (as 'pairs of subkernels') in the work of Wall [**24**] on the classification of quadratic forms on finite abelian groups (with $A = \mathbf{Z}$). The obstruction to surgery on odd-dimensional compact manifolds was obtained by Wall in [**25**, § 6] as an equivalence class of the matrix of an automorphism of a hyperbolic $\pm$ quadratic form $\alpha\colon H \pm (L) \to H \pm (L)$ with L a based f.g. free A-module. The work of Novikov [**12**] made apparent that only the structure of the non-singular $\pm$ quadratic formation $(H \pm (L); L, \alpha(L))$ was relevant. Moreover, the obstruction to proper surgery on odd-dimensional paracompact manifolds (Maumary [**8**]) is an equivalence class of non-singular $\pm$ quadratic formations $(H \pm (F); F, G)$ with f.g. projective lagrangians F, G for which there may be no automorphism $\alpha\colon H \pm (F) \to H \pm (F)$ such that $\alpha(F) = G$.

(See also Pedersen and Ranicki [**30**].) Thus formations cater for a wider range of surgery obstructions than automorphisms of hyperbolic forms. The algebraic properties of $\pm$ quadratic formations were studied by Ranicki in [**13**].

We shall now relate formations to 1-dimensional algebraic Poincaré complexes. It is convenient to treat the ε-symmetric and ε-quadratic cases separately.

The *Wu class* of an ε-symmetric formation over A $(M,\varphi;\, F,G)$ is the quadratic function

$$v_0(M,\varphi;\, F,G) = [v_0(\varphi)]\colon M/(F+G) \to \hat{H}^0(\mathbf{Z}_2;\, A,\varepsilon);$$
$$[x] \mapsto \varphi(x)(x) \quad (x \in M).$$

Note that $(M,\varphi;\, F,G)$ is even if and only if $v_0(M,\varphi;\, F,G) = 0$.

A 1-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex over A

$$(C,\varphi \in Q^1(C,\varepsilon)) \quad \{(C,\psi \in Q_1(C,\varepsilon))\}$$

is *connected* if $H_0(\varphi_0\colon C^{1-*} \to C) = 0$ $\{H_0((1+T_\varepsilon)\psi_0\colon C^{1-*} \to C) = 0\}$. In particular, Poincaré complexes are connected.

PROPOSITION 2.3. *The homotopy equivalence classes of connected 1-dimensional (even) ε-symmetric complexes over A are in a natural one–one correspondence with the stable isomorphism classes of (even) ε-symmetric formations over A. Poincaré complexes correspond to non-singular formations.*

Proof. By Proposition 1.5 it suffices to show that the stable isomorphism classes of connected strictly 1-dimensional (even) ε-symmetric complexes over A are in a natural one–one correspondence with the stable isomorphism classes of (even) ε-symmetric formations over A.

Let $(C,\varphi \in Q^1(C,\varepsilon))$ be a strictly 1-dimensional ε-symmetric complex over A. The $\mathbf{Z}_2$-hypercohomology class $\varphi \in Q^1(C,\varepsilon)$ is represented by a cycle $\varphi \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \mathrm{Hom}_A(C^*,C))_1$, as defined by A-module morphisms

$$\varphi_0\colon C^0 \to C_1, \quad \tilde{\varphi}_0\colon C^1 \to C_0, \quad \varphi_1\colon C^1 \to C_1,$$

such that

$$d\varphi_0 + \tilde{\varphi}_0 d^* = 0\colon C^0 \to C_1,$$
$$d\varphi_1 - \tilde{\varphi}_0 + \varepsilon\varphi_0^* = 0\colon C^1 \to C_0,$$
$$\varphi_1 - \varepsilon\varphi_1^* = 0\colon C^1 \to C_1.$$

The algebraic mapping cone $C(\varphi_0\colon C^{1-*} \to C)$ is given by

$$C(\varphi_0)\colon 0 \longrightarrow C^0 \xrightarrow{\begin{pmatrix}\bar{\varepsilon}\varphi_0\\ d^*\end{pmatrix}} C_1 \oplus C^1 \xrightarrow{(\varepsilon d \quad \tilde{\varphi}_0) = (\varepsilon\varphi_0^* \quad d)\begin{pmatrix}0 & 1\\ \varepsilon & \varphi_1\end{pmatrix}} C_0 \longrightarrow 0.$$

If (C,φ) is connected then $(\varepsilon\varphi_0^*\ d)\colon C^1\oplus C_1\to C_0$ is onto, and the dual

$$\begin{pmatrix}\bar{\varepsilon}\varphi_0\\ d^*\end{pmatrix}\colon C^0\to C_1\oplus C^1$$

is a split monomorphism, so that there is defined an ε-symmetric formation over A

$$\begin{aligned}(M,\theta;\,F,G) &= (H^{\varepsilon}(C^1,\varphi_1);\ C_1,C^0)\\ &= \left(C_1\oplus C^1,\begin{pmatrix}0&1\\ \varepsilon&\varphi_1\end{pmatrix};\ C_1,\mathrm{im}\left(\begin{pmatrix}\bar{\varepsilon}\varphi_0\\ d^*\end{pmatrix}\colon C^0\to C_1\oplus C^1\right)\right).\end{aligned}$$

The exact sequence of A-modules

$$0\longrightarrow H^0(C)\xrightarrow{\varphi_0}H_1(C)\longrightarrow H_1(\varphi_0)\longrightarrow H^1(C)\xrightarrow{\varphi_0}H_0(C)\longrightarrow 0$$

can be identified with the exact sequence

$$0\to F\cap G\to F\cap G^{\perp}\to G^{\perp}/G\to M/(F+G)\to M/(F+G^{\perp})\to 0,$$

and the 0th ε-symmetric Wu class of (C,φ) is the Wu class of $(M,\theta;\,F,G)$,

$$v_0(\varphi)=v_0(M,\theta;\,F,G)\colon H^1(C)=M/(F+G)\to\hat{H}^0(\mathbf{Z}_2;\,A,\varepsilon).$$

It follows that the complex (C,φ) is a Poincaré (respectively even) complex if and only if the formation $(M,\theta;\,F,G)$ is non-singular (respectively even). Moreover (C,φ) is contractible if and only if $(M,\theta;\,F,G)$ is isomorphic to $(H^{\varepsilon}(F);\,F,F^*)$.

Let (C,φ), (C',φ') be isomorphic connected strictly 1-dimensional ε-symmetric complexes over A. Given an isomorphism of ε-symmetric complexes

$$f\colon(C,\varphi)\to(C',\varphi')$$

we have an isomorphism of chain complexes

$$\begin{tikzcd}
C\colon \dots \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] & C_1 \arrow[r, "d"] \arrow[d, "f"'] & C_0 \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \dots \\
C'\colon \dots \arrow[r] & 0 \arrow[r] & C'_1 \arrow[r, "d'"] & C'_0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Choosing representative cycles

$$\varphi\in\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W,\mathrm{Hom}_A(C^*,C))_1,\quad \varphi'\in\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W,\mathrm{Hom}_A(C'^*,C'))_1$$

we have that

$$f^{\%}(\varphi)-\varphi'=d\chi\in\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W,\mathrm{Hom}_A(C'^*,C'))_1$$

for some chain $\chi \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \mathrm{Hom}_A(C'^*, C'))_2$, which is represented by an A-module morphism $\chi_0 \in \mathrm{Hom}_A(C'^1, C_1')$ such that

$$\begin{aligned} f\varphi_0\tilde{f}^* - \varphi_0' &= -\chi_0 d'^*: C'^0 \to C_1',\\ \tilde{f}\tilde{\varphi}_0 f^* - \varphi_0' &= d'\chi_0: C'^1 \to C_0',\\ f\varphi_1 f^* - \varphi_1' &= \chi_0 + \varepsilon\chi_0^*: C^{1\prime} \to C_1'. \end{aligned}$$

The A-module isomorphism

$$h = \begin{pmatrix} f & \bar{\varepsilon}\chi_0(f^*)^{-1} \\ 0 & (f^*)^{-1} \end{pmatrix}: C_1 \oplus C^1 \to C_1' \oplus C'^1$$

defines an isomorphism of the associated ε-symmetric formations

$$h: \left(H^\varepsilon(C^1, \varphi_1);\, C_1, \mathrm{im}\left(\begin{pmatrix} \bar{\varepsilon}\varphi_0 \\ d^* \end{pmatrix}: C^0 \to C_1 \oplus C^1\right)\right)$$

$$\to \left(H^\varepsilon(C'^1, \varphi_1');\, C_1', \mathrm{im}\left(\begin{pmatrix} \bar{\varepsilon}\varphi_0' \\ d'^* \end{pmatrix}: C'^0 \to C_1' \oplus C'^1\right)\right).$$

Every ε-symmetric formation is isomorphic to one of the type $(H^\varepsilon(F^*, \lambda);\, F, G)$ (by Proposition 2.2) and so determines a connected strictly 1-dimensional ε-symmetric complex $(C, \varphi \in Q^1(C, \varepsilon))$, as follows. Write the inclusion of G in $F \oplus F^*$ as

$$\begin{pmatrix} \gamma \\ \mu \end{pmatrix}: G \to F \oplus F^*,$$

and let

$$\begin{aligned} d &= \mu^*: C_1 = F \to C_0 = G^*,\\ \varphi_0 &= \varepsilon\gamma\, \mathrm{Hom}_A(G, F),\\ \tilde{\varphi}_0 &= \gamma^* + \mu^*\lambda \in \mathrm{Hom}_A(F^*, G^*),\\ \varphi_1 &= \lambda \in \mathrm{Hom}_A(F^*, F). \end{aligned}$$

Given an isomorphism of ε-symmetric formations

$$h: (H^\varepsilon(F^*, \lambda);\, F, G) \to (H^\varepsilon(F'^*, \lambda');\, F', G')$$

write the restrictions of h to the (sub)lagrangians as

$$\alpha = h|: F \to F', \quad \beta = h|: G \to G',$$

and define an isomorphism of the associated A-module chain complexes

$$f: C \to C'$$

by

$$\begin{tikzcd}
C: \dots \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & F \arrow[r, "\mu^*"] \arrow[d, "\alpha"] & G^* \arrow[r] \arrow[d, "\beta^{*-1}"] & 0 \arrow[r] & \dots \\
C': \dots \arrow[r] & 0 \arrow[r] & F' \arrow[r, "\mu'^*"] & G'^* \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Then

$$f\colon (C,\varphi) \to (C',\varphi')$$

is an isomorphism of the associated strictly 1-dimensional ε-symmetric complexes.

Given a 1-dimensional ε-quadratic complex over A $(C,\psi \in Q_1(C,\varepsilon))$ with C a f.g. projective A-module chain complex

$$C: \dots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{\ d\ } C_0 \longrightarrow 0 \longrightarrow \dots$$

we have that the $\mathbf{Z}_2$-hyperhomology class $\psi \in Q_1(C,\varepsilon)$ is represented by A-module morphisms

$$\psi_0\colon C^0 \to C_1,\quad \tilde{\psi}_0\colon C^1 \to C_0,\quad \psi_1\colon C^0 \to C_0$$

such that

$$d\psi_0 + \tilde{\psi}_0 d^* + \psi_1 - \varepsilon\psi_1^* = 0\colon C^0 \to C_0.$$

The algebraic mapping cone $C((1+T_\varepsilon)\psi_0\colon C^{1-*} \to C)$ can be expressed as

$$0 \longrightarrow C^0 \xrightarrow{\begin{pmatrix}\bar{\varepsilon}\psi_0+\tilde{\psi}_0^* \\ d^*\end{pmatrix}} C_1 \oplus C^1$$

$$\xrightarrow{(\varepsilon d\quad (\tilde{\psi}_0+\varepsilon\psi_0^*)) = ((\bar{\varepsilon}\psi_0+\tilde{\psi}_0^*)^*\quad d)\begin{pmatrix}0 & 1\\ \varepsilon & 0\end{pmatrix}} C_0 \longrightarrow 0.$$

Thus if (C,ψ) is connected there is defined an ε-quadratic formation

$$(H_\varepsilon(C_1);\, C_1, C^0) = \left(C_1\oplus C^1, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix};\, C_1, \mathrm{im}\left(\begin{pmatrix}\bar{\varepsilon}\psi_0+\tilde{\psi}_0^* \\ d^*\end{pmatrix}\colon C^0 \to C_1\oplus C^1\right)\right),$$

which is non-singular if and only if (C,ψ) is a Poincaré complex. The formation $(H_\varepsilon(C_1);\, C_1, C^0)$ does not involve ψ_1. In Proposition 2.4 below we shall show that homotopy equivalence classes of connected 1-dimensional ε-quadratic complexes correspond to the stable isomorphism classes of ε-quadratic formations together with the extra structure afforded by ψ_1, the 'split ε-quadratic formations'.

Given a f.g. projective A-module M and a direct summand L define the abelian group

$$Q_\varepsilon(M,L) = \frac{\{(\psi,\theta)\in \mathrm{Hom}_A(M,M^*)\oplus \mathrm{Hom}_A(L,L^*)\,|\, j^*\psi j = \theta-\varepsilon\theta^*\}}{\{(\chi-\varepsilon\chi^*, j^*\chi j+\nu+\varepsilon\nu^*)\,|\,(\chi,\nu)\in \mathrm{Hom}_A(M,M^*)\oplus\mathrm{Hom}_A(L,L^*)\}},$$

so that there is defined an exact sequence

$$Q_\varepsilon(M,L)\xrightarrow{\partial} Q_\varepsilon(M)\xrightarrow{j_\%} Q_\varepsilon(L)$$

with $j\in\mathrm{Hom}_A(L,M)$ the inclusion and

$$\partial\colon Q_\varepsilon(M,L)\to Q_\varepsilon(M);\ (\psi,\theta)\mapsto\psi,\quad j_\%\colon Q_\varepsilon(M)\to Q_\varepsilon(L);\ \psi\mapsto j^*\psi j.$$

A *hessian* for a sublagrangian L of an ε-quadratic form $(M,\psi\in Q_\varepsilon(M))$ is a choice of lift of $\psi\in Q_\varepsilon(M)$ to an element $(\psi,\theta)\in Q_\varepsilon(M,L)$ such that $\partial(\psi,\theta)=\psi\in Q_\varepsilon(M)$. Every sublagrangian admits hessians, since

$$j^*\psi j = 0\in Q_\varepsilon(L),$$

but they are not unique. A connected 1-dimensional ε-quadratic complex $(C,\psi\in Q_1(C,\varepsilon))$ (as above) determines an ε-quadratic formation

$$(H_\varepsilon(C_1);\ C_1, C^0)$$

together with a hessian

$$\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, -(\psi_1+d\psi_0)\right)\in Q_\varepsilon(C_1\oplus C^1, C^0)\quad\text{for } C^0 \text{ in } H_\varepsilon(C_1).$$

A *split ε-quadratic formation over* A $(F,G)=\left(F,\left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta\right)G\right)$ is an ε-quadratic formation over A $\left(H_\varepsilon(F);\ F,\mathrm{im}\left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix}\colon G\to F\oplus F^*\right)\right)$ $\left(\text{with } \begin{pmatrix}\gamma\\ \mu\end{pmatrix}\colon G\to F\oplus F^* \text{ the inclusion}\right)$ together with a $(-\varepsilon)$-quadratic form $\theta\in Q_{-\varepsilon}(G)$ such that

$$\gamma^*\mu = \theta-\varepsilon\theta^*\in\mathrm{Hom}_A(G,G^*).$$

This determines a hessian $\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\theta\right)\in Q_\varepsilon(F\oplus F^*, G)$ for G in $H_\varepsilon(F)$, since $\gamma^*\mu\in\mathrm{Hom}_A(G,G^*)$ is the composite

$$\gamma^*\mu\colon G\xrightarrow{\begin{pmatrix}\gamma\\ \mu\end{pmatrix}} F\oplus F^*\xrightarrow{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}} F^*\oplus F\xrightarrow{(\gamma^*\ \ \mu^*)} G^*.$$

An *isomorphism* of split ε-quadratic formations over A

$$(\alpha, \beta, \psi): (F, G) \to (F', G')$$

is a triple

(isomorphism $\alpha \in \mathrm{Hom}_A(F, F')$, isomorphism $\beta \in \mathrm{Hom}_A(G, G')$,

$(-\varepsilon)$-quadratic form over A $(F^*, \psi \in Q_{-\varepsilon}(F^*))$)

such that

(i) $\alpha\gamma + \alpha(\psi - \varepsilon\psi^*)^*\mu = \gamma'\beta \in \mathrm{Hom}_A(G, F')$,

(ii) $\alpha^{*-1}\mu = \mu'\beta \in \mathrm{Hom}_A(G, F'^*)$,

(iii) $\theta + \mu^*\psi\mu = \beta^*\theta'\beta \in Q_{-\varepsilon}(G)$.

A *stable isomorphism* of split ε-quadratic formations over A

$$[\alpha, \beta, \psi]: (F, G) \to (F', G')$$

is an isomorphism of the type

$$(\alpha, \beta, \psi): (F, G) \oplus (P, P^*) \to (F', G') \oplus (P', P'^*),$$

for some f.g. projective A-modules P, P' with $(P, P^*) = \left(P, \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0\right) P^*\right)$.

The choice of hessian $\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \theta\right) \in Q_\varepsilon(F \oplus F^*, G)$ is the only difference between a split ε-quadratic formation (F, G) and an arbitrary ε-quadratic formation $(M, \psi; F, G)$ up to isomorphism, since every ε-quadratic formation $(M, \psi; F, G)$ is isomorphic to one of the type $(H_\varepsilon(F); F, G)$ (by Proposition 2.2) and the following result holds:

PROPOSITION 2.4. *A (stable) isomorphism of split ε-quadratic formations*

$$[\alpha, \beta, \psi]: (F, G) \to (F', G')$$

determines a (stable) isomorphism of the underlying ε-quadratic formations

$$[f]: (H_\varepsilon(F); F, G) \to (H_\varepsilon(F'); F', G').$$

Conversely, every (stable) isomorphism of ε-quadratic formations $[f]$ can be lifted to a (stable) isomorphism of split ε-quadratic formations $[\alpha, \beta, \psi]$.

Proof. Given an isomorphism of split ε-quadratic formations over A

$$(\alpha, \beta, \psi): (F, G) \to (F', G')$$

define an A-module isomorphism

$$f = \begin{pmatrix} \alpha & \alpha(\psi - \varepsilon\psi^*)^* \\ 0 & \alpha^{*-1} \end{pmatrix}: F \oplus F^* \to F' \oplus F'^*.$$

This fits into a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\beta} & G' \\ {\scriptstyle \begin{pmatrix}\gamma\\ \mu\end{pmatrix}}\Big\downarrow & & \Big\downarrow{\scriptstyle \begin{pmatrix}\gamma'\\ \mu'\end{pmatrix}} \\ F\oplus F^* & \xrightarrow{f} & F'\oplus F'^* \end{array}$$

so that there is defined an isomorphism of ε-quadratic formations over A

$$f\colon (H_\varepsilon(F);\, F, G) \to (H_\varepsilon(F');\, F', G').$$

Stable isomorphisms can be dealt with similarly.

Conversely, suppose given a stable isomorphism of ε-quadratic formations over A

$$[f]\colon (H_\varepsilon(F);\, F, G) \to (H_\varepsilon(F');\, F', G'),$$

as defined by an isomorphism

$$f\colon (H_\varepsilon(F);\, F, G)\oplus (H_\varepsilon(P);\, P, P^*) \to (H_\varepsilon(F');\, F', G')\oplus (H_\varepsilon(P');\, P', P'^*)$$

for some f.g. projective A-modules P, P'. The restrictions of f are A-module isomorphisms

$$\alpha = \begin{pmatrix} a & a_1 \\ a_2 & a_3\end{pmatrix}\colon F\oplus P \to F'\oplus P',\quad \beta = \begin{pmatrix} b & b_1 \\ b_2 & b_3\end{pmatrix}\colon G\oplus P^* \to G'\oplus P'^*.$$

The isomorphism f can be expressed as

$$f = \begin{pmatrix} \alpha & \alpha(\psi - \varepsilon\psi^*)^* \\ 0 & \alpha^{*-1}\end{pmatrix}\colon (F\oplus P)\oplus (F^*\oplus P^*) \to (F'\oplus P')\oplus (F'^*\oplus P'^*)$$

for some A-module morphism

$$\psi = \begin{pmatrix} s & s_1 \\ s_2 & s_3\end{pmatrix}\colon F^*\oplus P^* \to F\oplus P,$$

and there is defined a commutative diagram

$$\begin{array}{ccc} G\oplus P^* & \xrightarrow{\beta} & G'\oplus P'^* \\ {\scriptstyle \begin{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}\mu & 0\\ 0 & 1\end{pmatrix}\end{pmatrix}}\Big\downarrow & & \Big\downarrow{\scriptstyle \begin{pmatrix}\begin{pmatrix}\gamma' & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}\mu' & 0\\ 0 & 1\end{pmatrix}\end{pmatrix}} \\ (F\oplus P)\oplus (F^*\oplus P^*) & \xrightarrow{f} & (F'\oplus P')\oplus (F'^*\oplus P'^*) \end{array}$$

It follows that

$$\begin{pmatrix}\gamma^*\mu & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\mu^* & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}s-\varepsilon s^* & s_1-\varepsilon s_2^*\\ s_2-\varepsilon s_1^* & s_3-\varepsilon s_3^*\end{pmatrix}\begin{pmatrix}\mu & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}b^* & b_2^*\\ b_1^* & b_3^*\end{pmatrix}\begin{pmatrix}\gamma'^*\mu' & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}b & b_1\\ b_2 & b_3\end{pmatrix}: G\oplus P^*\to G^*\oplus P,$$

and in particular

$$s_3-\varepsilon s_3^* = b_1^*\gamma'^*\mu' b_1 : P^*\to P.$$

Choose a hessian $\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\theta'\right)\in Q_\varepsilon(F'\oplus F'^*,G')$ for G' in $H_\varepsilon(F')$, and define

$$\theta = b^*\theta' b-\mu^* s\mu\in Q_{-\varepsilon}(G).$$

The isomorphism of split ε-quadratic formations

$$(\alpha,\beta,\tilde{\psi})=\left(\begin{pmatrix}a & a_1\\ a_2 & a_3\end{pmatrix},\begin{pmatrix}b & b_1\\ b_2 & b_3\end{pmatrix},\begin{pmatrix}s & s_1\\ s_2 & b_1^*\theta' b_1\end{pmatrix}\right):$$
$$\left(F,\left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta\right)G\right)\oplus(P,P^*)\to\left(F',\left(\begin{pmatrix}\gamma'\\ \mu'\end{pmatrix},\theta'\right)G'\right)\oplus(P',P'^*)$$

defines a stable isomorphism of split ε-quadratic formations

$$[\alpha,\beta,\tilde{\psi}]: (F,G)\to(F',G')$$

covering the stable isomorphism of ε-quadratic formations $[f]$.

An *ε-quadratic {split ε-quadratic} homotopy equivalence* of 1-dimensional ε-quadratic complexes over A

$$f: (C,\psi)\to(C',\psi')$$

is a chain equivalence

$$f: C\to C'$$

such that

$$\begin{cases} f_\%(\psi)-\psi' = H(\theta)\in Q_1(C',\varepsilon)\\ f_\%(\psi)-\psi' = 0\in Q_1(C',\varepsilon)\end{cases}$$

for some Tate $\mathbf{Z}_2$-hypercohomology class $\theta\in\hat{Q}^2(C',\varepsilon)$ such that

$$\hat{v}_1(\theta)=0: H^1(C')\to\hat{H}^1(\mathbf{Z}_2; A,\varepsilon).$$

(A split ε-quadratic homotopy equivalence is the same as a homotopy equivalence.)

PROPOSITION 2.5. *The (split) ε-quadratic homotopy equivalence classes of connected 1-dimensional ε-quadratic complexes over A are in a natural one–one correspondence with the stable isomorphism classes of (split) ε-quadratic* *formations over A. Poincaré complexes correspond to non-singular formations.*

Proof. Given a connected 1-dimensional ε-quadratic complex

$$(C,\psi \in Q_1(C,\varepsilon))$$

with C a f.g. projective complex of the type

$$C:\ \ldots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \ldots$$

choose a cycle representative $\psi \in (W\otimes_{\mathbf{Z}[\mathbf{Z}_2]}\mathrm{Hom}_A(C^*,C))_1$ for $\psi \in Q_1(C,\varepsilon)$, and define a split ε-quadratic formation

$$(C_1,C^0) = \left(C_1,\left(\begin{pmatrix}\bar{\varepsilon}\psi_0+\tilde{\psi}_0^*\\ d^*\end{pmatrix},-(\psi_1+d\psi_0)\right)C^0\right) = \left(F,\left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta\right)G\right).$$

(The ε-quadratic Wu class v^1 of (C,ψ) is then given by

$$v^1(\psi)\colon H^0(C) = F\cap G = \ker(\mu\colon G\to F^*)\to \hat{H}^0(\mathbf{Z}_2;A,\varepsilon);\ x\mapsto \theta(x)(x).)$$

If $\psi' = \psi+H(\theta)\in Q_1(C,\varepsilon)$ for some $\theta\in\hat{Q}^2(C,\varepsilon)$ choose a cycle representative $\theta\in\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(\hat{W},\mathrm{Hom}_A(C^*,C))_2$, as given by A-module morphisms

$$\theta_0\colon C^1\to C_1,\quad \theta_{-1}\colon C^0\to C_1,\quad \tilde{\theta}_{-1}\colon C^1\to C_0,\quad \theta_{-2}\colon C^0\to C_0$$

such that

$$\theta_0+\varepsilon\theta_0^* = 0,\quad d\theta_0-\tilde{\theta}_{-1}-\varepsilon\theta_{-1}^* = 0,\quad \theta_0d^*+\theta_{-1}+\varepsilon\tilde{\theta}_{-1}^* = 0,$$
$$d\theta_{-1}+\tilde{\theta}_{-1}d^*+\theta_{-2}-\varepsilon\theta_{-2}^* = 0.$$

Thus

$$\psi_0' = \psi_0+\theta_{-1}\colon C^0\to C_1,$$
$$\tilde{\psi}_0' = \tilde{\psi}_0+\tilde{\theta}_{-1}\colon C^1\to C_0,$$
$$\psi_1' = \psi_1+\theta_{-2}\colon C^0\to C_0.$$

If $\hat{v}_1(\theta) = 0$ then $(C^1,\theta_0\in Q^{-\varepsilon}(C^1))$ is an even $(-\varepsilon)$-symmetric form over A, and there is defined an isomorphism of the ε-quadratic formations associated to the ε-quadratic homotopy equivalent complexes (C,ψ), (C,ψ')

$$\begin{pmatrix}1 & \theta_0^*\\ 0 & 1\end{pmatrix}\colon \left(H_\varepsilon(C_1);\ C_1,\mathrm{im}\left(\begin{pmatrix}\bar{\varepsilon}\psi_0+\tilde{\psi}_0^*\\ d^*\end{pmatrix}\colon C^0\to C_1\oplus C^1\right)\right)$$
$$\to \left(H_\varepsilon(C_1);\ C_1,\mathrm{im}\left(\begin{pmatrix}\bar{\varepsilon}\psi_0'+\tilde{\psi}_0'^*\\ d^*\end{pmatrix}\colon C^0\to C_1\oplus C^1\right)\right).$$

Conversely, given a split ε-quadratic formation $\left(F,\left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta\right)G\right)$ define a connected 1-dimensional ε-quadratic complex $(C,\psi\in Q_1(C,\varepsilon))$ by

$$\begin{aligned} C_1=F,\quad C_0=G^*,\quad d=\mu^*\in \operatorname{Hom}_A(C_1,C_0),\quad C_r=0\quad (r\neq 0,1),\\ \psi_0=\varepsilon\gamma\in \operatorname{Hom}_A(C^0,C_1),\\ \tilde{\psi}_0=0\in \operatorname{Hom}_A(C^1,C_0),\\ \psi_1=-\theta\in \operatorname{Hom}_A(C^0,C_0),\end{aligned}$$

for any representative $\theta\in \operatorname{Hom}_A(G,G^*)$ of $\theta\in Q_{-\varepsilon}(G)$. Every ε-quadratic formation is isomorphic to one of the type $(H_\varepsilon(F);F,G)$ (by Proposition 2.2), and choosing a hessian $\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\theta\right)\in Q_\varepsilon(F\oplus F^*,G)$ for G in $H_\varepsilon(F)$ we obtain a split ε-quadratic formation (F,G) such that the ε-quadratic homotopy equivalence class of the associated complex (C,ψ) depends only on the stable isomorphism class of $(H_\varepsilon(F);F,G)$.

The detailed verification that the (split) ε-quadratic homotopy equivalence classes of connected 1-dimensional ε-quadratic complexes correspond to the stable isomorphism classes of (split) ε-quadratic formations is omitted, as it is so similar to the ε-symmetric case (Proposition 2.3).

The hessian $\theta\in Q_{-\varepsilon}(G)$ in a non-singular split ε-quadratic formation $\left(F,\left(\begin{pmatrix}\gamma\\ \mu\end{pmatrix},\theta\right)G\right)$ does not affect the cobordism class (surgery obstruction) of the associated 1-dimensional ε-quadratic Poincaré complex defined in § 3 below.

3. Algebraic Poincaré cobordism

We define now an equivalence relation on algebraic Poincaré complexes which is analogous to the cobordism of manifolds, and which we shall also call cobordism. In § 4 we shall analyse algebraic cobordism by a method analogous to surgery on manifolds, and which we shall also call surgery. The cobordism classes of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes over A define an abelian group $L^n(A,\varepsilon)$ $\{L_n(A,\varepsilon)\}$, for $n\geqslant 0$, with respect to the direct sum $\oplus$. The symmetric L-groups $L^n(A)=L^n(A,1)$ are the 'algebraic Poincaré bordism' groups of Mishchenko [**10**]. In §§ 4 and 5 we shall show that the ε-quadratic L-groups are 4-periodic, $L_n(A,\varepsilon)=L_{n+4}(A,\varepsilon)$, and that the quadratic L-groups $L_n(A)=L_n(A,1)$ are the surgery obstruction groups of Wall [**25**]. In § 10 we shall construct an example to show that the ε-symmetric L-groups are not 4-periodic in general, $L^n(A,\varepsilon)\neq L^{n+4}(A,\varepsilon)$. In Part II we shall relate geometric cobordism and surgery to their algebraic analogues—in particular, the surgery obstruction of an n-dimensional normal map $(f\colon M \to X,\ b\colon \nu_M \to \nu_X)$ will be identified with the quadratic Poincaré cobordism class $\sigma_*(f,b) \in L_n(\mathbf{Z}[\pi_1(X)])$ of an n-dimensional quadratic Poincaré complex over $\mathbf{Z}[\pi_1(X)]$ naturally associated to (f,b). We shall also define the cobordism groups of n-dimensional even ε-symmetric Poincaré complexes over A $L\langle v_0\rangle^n(A,\varepsilon)$ $(n \geqslant 0)$, which we shall use in § 6 to define lower ε-symmetric L-groups $L^n(A,\varepsilon)$ $(n \leqslant -1)$.

Given an A-module chain map

$$f\colon C \to D$$

define the *relative Q-groups*

$$\begin{cases} Q^{n+1}_{[i,j]}(f,\varepsilon) = H_{n+1}(\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W[i,j], C(f^t\otimes_A f))) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (-\infty \leqslant i \leqslant j \leqslant \infty,\ n \in \mathbf{Z}) \\ Q_{n+1}^{[i,j]}(f,\varepsilon) = H_{n+1}(W[i,j]\otimes_{\mathbf{Z}[\mathbf{Z}_2]} C(f^t\otimes_A f)) \end{cases}$$

with $C(f^t\otimes_A f)$ the algebraic mapping cone of the $\mathbf{Z}[\mathbf{Z}_2]$-module chain map $f^t\otimes_A f\colon C^t\otimes_A C \to D^t\otimes_A D$, taking $T \in \mathbf{Z}_2$ to act by the ε-transposition T_ε. An element $(\delta\varphi,\varphi) \in Q^{n+1}_{[i,j]}(f,\varepsilon)$ $\{(\delta\psi,\psi) \in Q_{n+1}^{[i,j]}(f,\varepsilon)\}$ is represented by a collection of chains

$$\begin{cases} \{(\delta\varphi,\varphi)_s = (\delta\varphi_s,\varphi_s) \in (D^t\otimes_A D)_{n+s+1} \oplus (C^t\otimes_A C)_{n+s} \mid i \leqslant s \leqslant j\} \\ \{(\delta\psi,\psi)_s = (\delta\psi_s,\psi_s) \in (D^t\otimes_A D)_{n-s+1} \oplus (C^t\otimes_A C)_{n-s} \mid i \leqslant s \leqslant j\} \end{cases}$$

such that

$$\begin{cases} d(\delta\varphi,\varphi)_s \\ \quad \equiv (d_{D^t\otimes_A D}(\delta\varphi_s) + (-)^{n+s}(\delta\varphi_{s-1} + (-)^s T_\varepsilon\delta\varphi_{s-1}) + (-)^n (f^t\otimes_A f)(\varphi_s), \\ \qquad\qquad\qquad\qquad d_{C^t\otimes_A C}(\varphi_s) + (-)^{n+s-1}(\varphi_{s-1} + (-)^s T_\varepsilon\varphi_{s-1})) \\ \quad = 0 \in (D^t\otimes_A D)_{n+s} \oplus (C^t\otimes_A C)_{n+s-1} \quad (i \leqslant s \leqslant j,\ \delta\varphi_{i-1} = 0,\ \varphi_{i-1} = 0), \\ d(\delta\psi,\psi)_s \\ \quad \equiv (d_{D^t\otimes_A D}(\delta\psi_s) + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1} T_\varepsilon\delta\psi_{s+1}) + (-)^n (f^t\otimes_A f)(\psi_s), \\ \qquad\qquad\qquad\qquad d_{C^t\otimes_A C}(\psi_s) + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T_\varepsilon\psi_{s+1})) \\ \quad = 0 \in (D^t\otimes_A D)_{n-s} \oplus (C^t\otimes_A C)_{n-s-1} \quad (i \leqslant s \leqslant j,\ \delta\psi_{j+1} = 0,\ \psi_{j+1} = 0). \end{cases}$$

For $\varepsilon = 1 \in A$ we shall write

$$\begin{cases} Q^{n+1}_{[i,j]}(f,1) = Q^{n+1}_{[i,j]}(f), \\ Q_{n+1}^{[i,j]}(f,1) = Q_{n+1}^{[i,j]}(f). \end{cases}$$

Proposition 3.1. *For any A-module chain map $f\colon C \to D$ there is defined a long exact sequence of Q-groups*

$$\begin{cases} \ldots \longrightarrow Q^{n+1}_{[i,j]}(f,\varepsilon) \longrightarrow Q^n_{[i,j]}(C,\varepsilon) \\ \qquad \xrightarrow{f^{\%}} Q^n_{[i,j]}(D,\varepsilon) \longrightarrow Q^n_{[i,j]}(f,\varepsilon) \longrightarrow \ldots, \\ \ldots \longrightarrow Q^{[i,j]}_{n+1}(f,\varepsilon) \longrightarrow Q^{[i,j]}_n(C,\varepsilon) \\ \qquad \xrightarrow{f_{\%}} Q^{[i,j]}_n(D,\varepsilon) \longrightarrow Q^{[i,j]}_n(f,\varepsilon) \longrightarrow \ldots, \end{cases}$$

with

$$\begin{cases} Q^{n+1}_{[i,j]}(f,\varepsilon) \to Q^n_{[i,j]}(C,\varepsilon);\ (\delta\varphi,\varphi) \mapsto \varphi, \\ Q^n_{[i,j]}(D,\varepsilon) \to Q^n_{[i,j]}(f,\varepsilon);\ \delta\varphi \mapsto (\delta\varphi, 0), \\ Q^{[i,j]}_{n+1}(f,\varepsilon) \to Q^{[i,j]}_n(C,\varepsilon);\ (\delta\psi,\psi) \mapsto \psi, \\ Q^{[i,j]}_n(D,\varepsilon) \to Q^{[i,j]}_n(f,\varepsilon);\ \delta\psi \mapsto (\delta\psi, 0). \end{cases}$$

An $(n+1)$*-dimensional ε-symmetric* $\{\varepsilon$*-quadratic*$\}$ *pair over* A

$$(f\colon C \to D, (\delta\varphi,\varphi)) \quad \{(f\colon C \to D, (\delta\psi,\psi))\},$$

for $n \geqslant 0$, is a chain map $f\colon C \to D$ from an n-dimensional A-module chain complex C to an $(n+1)$-dimensional A-module chain complex D, together with a relative $\mathbf{Z}_2$-hypercohomology $\{\mathbf{Z}_2$-hyperhomology$\}$ class

$$\begin{cases} (\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon) = Q^{n+1}_{[0,\infty]}(f,\varepsilon), \\ (\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon) = Q^{[0,\infty]}_{n+1}(f,\varepsilon), \end{cases}$$

and it is a *Poincaré* pair if the relative homology class

$$(\delta\varphi_0,\varphi_0) \in H_{n+1}(f^t \otimes_A f) \quad \{((1+T_\varepsilon)\delta\psi_0, (1+T_\varepsilon)\psi_0) \in H_{n+1}(f^t \otimes_A f)\}$$

induces A-module isomorphisms

$$H^r(D,C) \equiv H^r(f) \to H_{n+1-r}(D) \quad (0 \leqslant r \leqslant n+1)$$

(Poincaré–Lefschetz duality) via the slant product

$$\begin{aligned} \backslash\colon & H^r(f) \otimes_{\mathbf{Z}} H_{n+1}(f^t \otimes_A f) \to H_{n+1-r}(D); \\ & (g,h) \otimes (u \otimes v, x \otimes y) \mapsto \overline{g(u)}v + \overline{h(x)}\,f(y) \\ & ((g,h) \in D^r \oplus C^{r-1},\ u \otimes v \in (D^t \otimes_A D)_{n+1},\ x \otimes y \in (C^t \otimes_A C)_n). \end{aligned}$$

The *boundary* of an $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré pair over A

$$(f\colon C \to D, (\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon)) \quad \{(f\colon C \to D, (\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon))\}$$

is the n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex over A $(C, \varphi \in Q^n(C,\varepsilon))$ $\{(C, \psi \in Q_n(C,\varepsilon))\}$. A *cobordism* of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes $(C,\varphi), (C',\varphi')$ $\{(C,\psi), (C',\psi')\}$ is an $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré pair with boundary $(C,\varphi)\oplus(C',-\varphi')$ $\{(C,\psi)\oplus(C',-\psi')\}$, say

$$\begin{cases} ((f\ \ f')\colon C\oplus C'\to D, (\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((f\ \ f'),\varepsilon)), \\ ((f\ \ f')\colon C\oplus C'\to D, (\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((f\ \ f'),\varepsilon)). \end{cases}$$

In Proposition 3.2 below we shall prove that cobordism is an equivalence relation on algebraic Poincaré complexes, such that the cobordism classes define abelian groups under the direct sum $\oplus$. The verification of the transitivity of cobordism requires the following algebraic glueing operation.

In § 6 of Part II we shall show that geometric cobordisms give rise to algebraic Poincaré cobordisms.

Define the *union* of adjoining ε-symmetric $\{\varepsilon$-quadratic$\}$ cobordisms

$$\begin{cases} c=((f_C\ \ f_{C'})\colon C\oplus C'\to D, (\delta\varphi,\varphi\oplus-\varphi')\in Q^{n+1}((f_C\ \ f_{C'}),\varepsilon)), \\ c=((f_C\ \ f_C)\colon C\oplus C'\to D, (\delta\psi,\psi\oplus-\psi')\in Q_{n+1}((f_C\ \ f_{C'}),\varepsilon)), \end{cases}$$

$$\begin{cases} c'=((f'_{C'}\ \ f'_{C''})\colon C'\oplus C''\to D', (\delta\varphi',\varphi'\oplus-\varphi'')\in Q^{n+1}((f'_{C'}\ \ f'_{C''}),\varepsilon)), \\ c'=((f'_{C'}\ \ f'_{C''})\colon C'\oplus C''\to D', (\delta\psi',\psi'\oplus-\psi'')\in Q_{n+1}((f'_{C'}\ \ f'_{C''}),\varepsilon)), \end{cases}$$

to be the ε-symmetric $\{\varepsilon$-quadratic$\}$ cobordism

$$\begin{cases} c\cup c'=((f''_C\ \ f''_{C''})\colon C\oplus C''\to D'', (\delta\varphi'',\varphi\oplus-\varphi'')\in Q^{n+1}((f''_C\ \ f''_{C''}),\varepsilon)) \\ c\cup c'=((f''_C\ \ f''_{C''})\colon C\oplus C''\to D'', (\delta\psi'',\psi\oplus-\psi'')\in Q_{n+1}((f''_C\ \ f''_{C''}),\varepsilon)) \end{cases}$$

given by

$$d_{D''}=\begin{pmatrix} d_D & (-)^{r-1}f_{C'} & 0 \\ 0 & d_{C'} & 0 \\ 0 & (-)^{r-1}f'_{C'} & d_{D'} \end{pmatrix}\colon$$

$$D''_r=D_r\oplus C'_{r-1}\oplus D'_r\to D''_{r-1}=D_{r-1}\oplus C'_{r-2}\oplus D'_{r-1},$$

$$f''_C=\begin{pmatrix} f_C \\ 0 \\ 0 \end{pmatrix}\colon C_r\to D''_r=D_r\oplus C'_{r-1}\oplus D'_r,$$

$$f''_{C''}=\begin{pmatrix} 0 \\ 0 \\ f'_{C''} \end{pmatrix}\colon C''_r\to D''_r=D_r\oplus C'_{r-1}\oplus D'_r,$$

$$\begin{cases} \delta\varphi''_s = \begin{pmatrix} \delta\varphi_s & 0 & 0 \\ (-)^{n-r}\varphi'_s f^*_{C'} & (-)^{n-r+s+1}T_s\varphi'_{s-1} & 0 \\ 0 & (-)^s f'_{C'}\varphi'_s & \delta\varphi'_s \end{pmatrix}: \\ \qquad D''^{n-r+s+1} = D^{n-r+s+1}\oplus C'^{n-r+s}\oplus D'^{n-r+s+1} \\ \qquad\qquad \to D''_r = D_r\oplus C'_{r-1}\oplus D'_r \quad (s \geqslant 0), \\ \delta\psi''_s = \begin{pmatrix} \delta\psi_s & 0 & 0 \\ (-)^{n-r}\psi'_s f^*_{C'} & (-)^{n-r-s}T_s\psi'_{s+1} & 0 \\ 0 & (-)^s f'_{C}\psi'_s & \delta\psi'_s \end{pmatrix}: \\ \qquad D''^{n-r-s+1} = D^{n-r-s+1}\oplus C'^{n-r-s}\oplus D'^{n-r-s+1} \\ \qquad\qquad \to D''_r = D_r\oplus C'_{r-1}\oplus D'_r \quad (s \geqslant 0). \end{cases}$$

We shall normally write

$$D'' = D\cup_{C'} D', \quad \begin{cases} \delta\varphi'' = \delta\varphi\cup_{\varphi'}\delta\varphi', \\ \delta\psi'' = \delta\psi\cup_{\psi'}\delta\psi'. \end{cases}$$

PROPOSITION 3.2. *Cobordism is an equivalence relation on n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes over A, such that homotopy equivalent complexes are cobordant. The cobordism classes define an abelian group, the n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ L-group of A $L^n(A,\varepsilon)$ $\{L_n(A,\varepsilon)\}$, for $n \geqslant 0$, with addition and inverses by*

$$\begin{cases} (C,\varphi)+(C',\varphi') = (C\oplus C',\varphi\oplus\varphi'), \quad -(C,\varphi) = (C,-\varphi) \in L^n(A,\varepsilon), \\ (C,\psi)+(C',\psi') = (C\oplus C',\psi\oplus\psi'), \quad -(C,\psi) = (C,-\psi) \in L_n(A,\varepsilon). \end{cases}$$

Proof. Given a homotopy equivalence of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes over A

$$\begin{cases} f\colon (C,\varphi) \to (C',\varphi'), \\ f\colon (C,\psi) \to (C',\psi'), \end{cases}$$

let $\varphi \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C^t\otimes_A C)_n$ $\{\psi \in (W\otimes_{\mathbf{Z}[\mathbf{Z}_2]}(C^t\otimes_A C))_n\}$ be a cycle representing $\varphi \in Q^n(C,\varepsilon)$ $\{\psi \in Q_n(C,\varepsilon)\}$, so that

$$\begin{cases} \varphi' = f^{\%}(\varphi) \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C'^t\otimes_A C')_n \\ \psi' = f_{\%}(\psi) \in (W\otimes_{\mathbf{Z}[\mathbf{Z}_2]}(C'^t\otimes_A C'))_n \end{cases}$$

is a cycle representing $\varphi' \in Q^n(C',\varepsilon)$ $\{\psi' \in Q_n(C',\varepsilon)\}$. There is then defined a cobordism

$$\begin{cases} ((f \;\; 1)\colon C\oplus C' \to C', (0,\varphi\oplus -\varphi') \in Q^{n+1}((f \;\; 1),\varepsilon)) \\ ((f \;\; 1)\colon C\oplus C' \to C', (0,\psi\oplus -\psi') \in Q_{n+1}((f \;\; 1),\varepsilon)) \end{cases}$$

from (C,φ) to (C',φ') $\{(C,\psi)$ to $(C',\psi')\}$. This verifies that homotopy equivalent algebraic Poincaré complexes are cobordant, and in particular that cobordism is reflexive.

If

$$\begin{cases} ((f\ \ f')\colon C\oplus C' \to D, (\delta\varphi, \varphi\oplus -\varphi') \in Q^{n+1}((f\ \ f'),\varepsilon)) \\ ((f\ \ f')\colon C\oplus C' \to D, (\delta\psi, \psi\oplus -\psi') \in Q_{n+1}((f\ \ f'),\varepsilon)) \end{cases}$$

is a cobordism, then so is

$$\begin{cases} ((f'\ \ f)\colon C'\oplus C \to D, (-\delta\varphi, \varphi'\oplus -\varphi) \in Q^{n+1}((f'\ \ f),\varepsilon)), \\ ((f'\ \ f)\colon C'\oplus C \to D, (-\delta\psi, \psi'\oplus -\psi) \in Q_{n+1}((f'\ \ f),\varepsilon)), \end{cases}$$

thus verifying the symmetry of cobordism.

The union operation ensures that cobordism is transitive.

The correspondence between low dimensional $(n = 0, 1)$ algebraic Poincaré complexes and forms and formations of § 2 will be extended to the L-groups in § 5 below, and $L^0(A,\varepsilon)$ (respectively $L^1(A,\varepsilon)$) $\{L_0(A,\varepsilon)$ (respectively $L_1(A,\varepsilon))\}$ will be identified with the Witt group of non-singular ε-symmetric $\{\varepsilon$-quadratic$\}$ forms (respectively formations) over A.

We shall denote the symmetric {quadratic} L-groups by

$$L^n(A,1) = L^n(A) \quad \{L_n(A,1) = L_n(A)\}.$$

The symmetric L-groups $L^n(A)$ are the 'algebraic Poincaré bordism' groups $\Omega_n(A)$ of Mishchenko [**10**], except that $\Omega_n(A)$ was defined using only f.g. free (rather than f.g. projective) A-module chain complexes—the difference this makes will be studied in § 9 below.

In § 4 we shall establish that

$$L_n(A,\varepsilon) = L_{n+2}(A,-\varepsilon) = L_{n+4}(A,\varepsilon) \quad (n \geqslant 0).$$

The quadratic L-groups $L_n(A)$ are thus the analogues of the surgery obstruction groups of Wall [**25**], defined using f.g. projective (rather than based f.g. free) A-modules—the difference this makes will also be studied in § 9 below.

The *ε-symmetrization* of an $(n+1)$-dimensional ε-quadratic (Poincaré) pair over A $(f\colon C\to D, (\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon))$ is the $(n+1)$-dimensional ε-symmetric (Poincaré) pair

$$(1+T_\varepsilon)(f\colon C\to D, (\delta\psi,\psi)) = (f\colon C\to D, (1+T_\varepsilon)(\delta\psi,\psi) \in Q^{n+1}(f,\varepsilon)),$$

where

$$(1+T_\varepsilon)(\delta\psi,\psi)_s = \begin{cases} ((1+T_\varepsilon)\delta\psi_0, (1+T_\varepsilon)\psi_0) \in (D^t\otimes_A D)_{n+1} \oplus (C^t\otimes_A C)_n & \text{if } s = 0, \\ (0,0) \in (D^t\otimes_A D)_{n+s+1} \oplus (C^t\otimes_A C)_{n+s} & \text{if } s \geqslant 1. \end{cases}$$

The ε-symmetrization of a null-cobordant ε-quadratic Poincaré complex is thus a null-cobordant ε-symmetric Poincaré complex, and there are defined ε-symmetrization maps in the L-groups

$$(1+T_\varepsilon)\colon L_n(A,\varepsilon) \to L^n(A,\varepsilon);\ (C,\psi) \mapsto (C,(1+T_\varepsilon)\psi).$$

We shall prove that these are isomorphisms modulo 8-torsion in § 8 below.

The *skew-suspension* of an $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ (Poincaré) pair over A

$$(f\colon C \to D, (\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon)) \quad \{(f\colon C \to D, (\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon))\}$$

is the $(n+3)$-dimensional $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic$\}$ (Poincaré) pair over A

$$\begin{cases} \bar{S}(f\colon C \to D, (\delta\varphi,\varphi)) = (Sf\colon SC \to SD, \bar{S}(\delta\varphi,\varphi) \in Q^{n+3}(Sf,-\varepsilon)), \\ \bar{S}(f\colon C \to D, (\delta\psi,\psi)) = (Sf\colon SC \to SD, \bar{S}(\delta\psi,\psi) \in Q_{n+3}(Sf,-\varepsilon)), \end{cases}$$

with $\bar{S}\colon Q^{n+1}(f,\varepsilon) \to Q^{n+3}(Sf,-\varepsilon)$ $\{\bar{S}\colon Q_{n+1}(f,\varepsilon) \to Q_{n+3}(Sf,-\varepsilon)\}$ the relative version of the isomorphism defined in the absolute case in § 1. Thus the skew-suspension of a null-cobordant n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex is a null-cobordant $(n+2)$-dimensional $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic$\}$ Poincaré complex, and there are defined skew-suspension maps in the L-groups

$$\begin{cases} \bar{S}\colon L^n(A,\varepsilon) \to L^{n+2}(A,-\varepsilon);\ (C,\varphi) \mapsto (SC,\bar{S}\varphi) \\ \bar{S}\colon L_n(A,\varepsilon) \to L_{n+2}(A,-\varepsilon);\ (C,\psi) \mapsto (SC,\bar{S}\psi) \end{cases} \quad (n \geqslant 0).$$

In § 4 below we shall prove that $\bar{S}\colon L_n(A,\varepsilon) \to L_{n+2}(A,-\varepsilon)$ is an isomorphism for all $A,\varepsilon,n \geqslant 0$ (Proposition 4.3). It will also be proved that $\bar{S}\colon L^n(A,\varepsilon) \to L^{n+2}(A,-\varepsilon)$ is an isomorphism if A is noetherian of finite global dimension m and $n+2 \geqslant 2m$ (Proposition 4.5).

Define the *0th Wu class* of an $(n+1)$-dimensional ε-symmetric pair over A $(f\colon C \to D, (\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon))$ to be the function

$$\begin{aligned} v_0(\delta\varphi,\varphi)\colon H^{n+1}(f) &\to \hat{H}^0(\mathbf{Z}_2; A,\varepsilon); \\ (y,x) &\mapsto (\delta\varphi_{n+1}(y\otimes y) + (-)^n\varphi_n(x\otimes x)) \\ &\qquad (x \in C^n,\ y \in D^{n+1},\ n \geqslant 0). \end{aligned}$$

It is possible to define higher Wu classes for ε-symmetric pairs, as well as Wu classes for ε-quadratic pairs, generalizing the absolute Wu classes of § 1. However, we shall only need the relative v_0.

An $(n+1)$-dimensional ε-symmetric pair over A

$$(f\colon C \to D, (\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon))$$

is *even* if

$$v_0(\delta\varphi,\varphi) = 0\colon H^{n+1}(f) \to \hat{H}^0(\mathbf{Z}_2; A,\varepsilon).$$

Then $(C, \varphi \in Q^n(C,\varepsilon))$ is an n-dimensional even ε-symmetric complex over A, since

$$v_0(\varphi): H^n(C) \xrightarrow{\quad\delta\quad} H^{n+1}(f) \xrightarrow{v_0(\delta\varphi,\varphi)=0} \hat{H}^0(\mathbf{Z}_2; A,\varepsilon).$$

The *n-dimensional even ε-symmetric L-group of A* $L\langle v_0\rangle^n(A,\varepsilon)$ $(n \geqslant 0)$ is the abelian group with respect to the direct sum $\oplus$ of the cobordism classes of n-dimensional even ε-symmetric Poincaré complexes over A

$$(C, \varphi \in Q\langle v_0\rangle^n(C,\varepsilon) = \ker(\hat{v}_0: Q^n(C,\varepsilon) \to \mathrm{Hom}_A(H^n(C), \hat{H}^0(\mathbf{Z}_2; A,\varepsilon))),$$

where the cobordisms are required to be $(n+1)$-dimensional even ε-symmetric Poincaré pairs.

The ε-symmetrization of an ε-quadratic complex (respectively pair) is an even ε-symmetric complex (respectively pair), so that the ε-symmetrization map factors through the even ε-symmetric L-groups

$$1+T_\varepsilon: L_n(A,\varepsilon) \xrightarrow{1+T_\varepsilon} L\langle v_0\rangle^n(A,\varepsilon) \longrightarrow L^n(A,\varepsilon) \quad (n \geqslant 0).$$

The skew-suspension of an n-dimensional ε-symmetric complex (respectively pair) is an $(n+2)$-dimensional even $(-\varepsilon)$-symmetric complex (respectively pair), so that the skew-suspension map in the ε-symmetric L-groups factors through the even ε-symmetric L-groups

$$\bar{S}: L^n(A,\varepsilon) \xrightarrow{\bar{S}} L\langle v_0\rangle^{n+2}(A,-\varepsilon) \longrightarrow L^{n+2}(A,-\varepsilon) \quad (n \geqslant 0).$$

In Proposition 4.4 below it will be shown that the skew-suspension maps $\bar{S}: L^n(A,\varepsilon) \to L\langle v_0\rangle^{n+2}(A,-\varepsilon)$ $(n \geqslant 0)$ are isomorphisms. Thus we shall be mainly concerned with $L\langle v_0\rangle^n(A,\varepsilon)$ for $n = 0, 1$.

If 2 is invertible in A the various types of L-groups coincide

$$L_n(A,\varepsilon) = L\langle v_0\rangle^n(A,\varepsilon) = L^n(A,\varepsilon) \quad (n \geqslant 0).$$

More generally:

PROPOSITION 3.3. *If A is such that $\hat{H}^*(\mathbf{Z}_2; A,\varepsilon) = 0$ then the natural maps*

$$1+T_\varepsilon: L_n(A,\varepsilon) \to L\langle v_0\rangle^n(A,\varepsilon), \quad L\langle v_0\rangle^n(A,\varepsilon) \to L^n(A,\varepsilon) \quad (n \geqslant 0)$$

are isomorphisms. In particular, this is the case if there exists a central element $a \in A$ such that $a+\bar{a} = 1 \in A$ (for example, $a = \frac{1}{2} \in A$).

Proof. If $\hat{H}^*(\mathbf{Z}_2; A,\varepsilon) = 0$ then $\hat{Q}^*(C,\varepsilon) = 0$ for any finite-dimensional A-module chain complex C (by Proposition 1.4(i)), so that the ε-symmetrization map in the Q-groups $(1+T_\varepsilon): Q_*(C,\varepsilon) \to Q^*(C,\varepsilon)$ is an isomorphism (Proposition 1.2), and there are natural identifications

$$\begin{aligned}(\varepsilon\text{-quadratic complexes over } A) &= (\text{even } \varepsilon\text{-symmetric complexes over } A)\\ &= (\varepsilon\text{-symmetric complexes over } A).\end{aligned}$$

Similarly for the relative Q-groups, and hence for the L-groups.

We shall now define the notion of homotopy equivalence appropriate to algebraic Poincaré pairs. It turns out that the homotopy equivalence classes of ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré pairs are in a natural one–one correspondence with the homotopy equivalence classes of certain ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes (Proposition 3.4). This allows for considerable conceptual simplification, giving the ε-symmetric $\{\varepsilon$-quadratic$\}$ L-groups an expression entirely in terms of ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes. In § 4 we shall use this expression to establish the 4-periodicity in the ε-quadratic L-groups, $L_n(A,\varepsilon) = L_{n+4}(A,\varepsilon)$, and in § 5 we shall use it to identify the low dimensional L-groups with Witt groups of forms and formations.

Define a *homotopy equivalence of* n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ pairs over A

$$\begin{cases} (g,h;\,k)\colon (f\colon C \to D, (\delta\varphi,\varphi) \in Q^n(f,\varepsilon)) \\ \qquad\qquad\qquad \to (f'\colon C' \to D', (\delta\varphi',\varphi') \in Q^n(f',\varepsilon)) \\ (g,h;\,k)\colon (f\colon C \to D, (\delta\psi,\psi) \in Q_n(f,\varepsilon)) \\ \qquad\qquad\qquad \to (f'\colon C' \to D', (\delta\psi',\psi') \in Q_n(f',\varepsilon)) \end{cases}$$

to be a triple $(g,h;\,k)$ consisting of chain equivalences

$$g\colon C \to C', \quad h\colon D \to D'$$

together with a chain homotopy

$$k\colon f'g \simeq hf\colon C \to D'$$

such that

$$\begin{cases} (g,h;\,k)^{\%}(\delta\varphi,\varphi) = (\delta\varphi',\varphi') \in Q^n(f',\varepsilon), \\ (g,h;\,k)_{\%}(\delta\psi,\psi) = (\delta\psi',\psi') \in Q_n(f',\varepsilon), \end{cases}$$

where

$$\begin{cases} (g,h;\,k)^{\%}(\delta\varphi,\varphi)_s \\ \qquad = ((h\otimes_A h)(\delta\varphi_s) + (-)^{n-1}(hf\otimes k)(\varphi_s) + (-)^p(k\otimes f'g)(\varphi_s) \\ \qquad\qquad + (-)^{n+p-1}(k\otimes k)(\varphi_{s-1}),\ (g\otimes g)(\varphi_s)) \\ \qquad\qquad\qquad\qquad \in (D'^t\otimes_A D')_{n+s} \oplus (C'^t\otimes_A C')_{n+s-1} \\ (g,h;\,k)_{\%}(\delta\psi,\psi)_s \\ \qquad = ((h\otimes_A h)(\delta\psi_s) + (-)^{n-1}(hf\otimes k)(\psi_s) + (-)^p(k\otimes f'g)(\psi_s) \\ \qquad\qquad + (-)^{n+p}(k\otimes k)(\psi_{s+1}),\ (g\otimes g)(\psi_s)) \\ \qquad\qquad\qquad\qquad \in (D'^t\otimes_A D')_{n-s} \oplus (C'^t\otimes_A C')_{n-s-1} \\ \qquad\qquad\qquad\qquad\qquad (s \geqslant 0,\ (D'^t\otimes_A D')_r = \sum_{p+q=r} D'^t_p\otimes_A D'_q). \end{cases}$$

An n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex $(C,\varphi\in Q^n(C,\varepsilon))$ $\{(C,\psi\in Q_n(C,\varepsilon))\}$ is *connected* if

$$\begin{cases} H_0(\varphi_0\colon C^{n-*}\to C)=0,\\ H_0((1+T_\varepsilon)\psi_0\colon C^{n-*}\to C)=0.\end{cases}$$

In particular, Poincaré complexes are connected. For $n=0$ 'connected' is the same as 'Poincaré'.

Define the *boundary* of a connected n-dimensional (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ complex over A $(C,\varphi\in Q^n(C,\varepsilon))$ $\{(C,\psi\in Q_n(C,\varepsilon))\}$, for $n\geqslant 1$, to be the $(n-1)$-dimensional (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex over A

$$\begin{cases} \partial(C,\varphi)=(\partial C,\partial\varphi\in Q^{n-1}(\partial C,\varepsilon))\\ \partial(C,\psi)=(\partial C,\partial\psi\in Q_{n-1}(\partial C,\varepsilon))\end{cases}$$

given by

$$d_{\partial C}=\begin{cases}\begin{pmatrix} d_C & (-)^r\varphi_0\\ 0 & (-)^r d_C^*\end{pmatrix}\\[2ex] \begin{pmatrix} d_C & (-)^r(1+T_\varepsilon)\psi_0\\ 0 & (-)^r d_C^*\end{pmatrix}\end{cases} : \partial C_r=C_{r+1}\oplus C^{n-r}\to\partial C_{r-1}=C_r\oplus C^{n-r+1},$$

$$\begin{cases}\partial\varphi_0=\begin{pmatrix}(-)^{n-r-1}T_\varepsilon\varphi_1 & (-)^{r(n-r-1)}\varepsilon\\ 1 & 0\end{pmatrix}:\\ \qquad\qquad\partial C^{n-r-1}=C^{n-r}\oplus C_{r+1}\to\partial C_r=C_{r+1}\oplus C^{n-r},\\[1ex] \partial\psi_0=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}: \partial C^{n-r-1}=C^{n-r}\oplus C_{r+1}\to\partial C_r=C_{r+1}\oplus C^{n-r},\end{cases}$$

$$\begin{cases}\partial\varphi_s=\begin{pmatrix}(-)^{n-r+s-1}T_\varepsilon\varphi_{s+1} & 0\\ 0 & 0\end{pmatrix}:\\ \qquad\partial C^{n-r+s-1}=C^{n-r+s}\oplus C_{r-s+1}\to\partial C_r=C_{r+1}\oplus C^{n-r}\quad(s\geqslant 1),\\[1ex] \partial\psi_s=\begin{pmatrix}(-)^{n-r-s}T_\varepsilon\psi_{s-1} & 0\\ 0 & 0\end{pmatrix}:\\ \qquad\partial C^{n-r-s-1}=C^{n-r-s}\oplus C_{r+s+1}\to\partial C_r=C_{r+1}\oplus C^{n-r}.\end{cases}$$

(Motivation: let M be an n-dimensional manifold with boundary ∂M, and let $\sigma^*(M,\partial M)=(f\colon C(\widetilde{\partial M})\to C(\tilde M),(\varphi,\partial\varphi)\in Q^n(f))$ be the n-dimensional symmetric Poincaré pair over $\mathbf{Z}[\pi_1(M)]$ associated to the universal cover $\tilde{M}$ in § 6 of Part II. The n-dimensional symmetric complex

$$(C,\theta) = (C(\tilde{M},\widetilde{\partial M}),(\varphi,\partial\varphi) \in Q^n(C(\tilde{M},\widetilde{\partial M})))$$

obtained from the pair $\sigma^*(M,\partial M)$ by collapsing

$$\sigma^*(\partial M) = (C(\widetilde{\partial M}),\partial\varphi \in Q^{n-1}(C(\widetilde{\partial M})))$$

has boundary $\partial(C,\theta)$ homotopy equivalent to $\sigma^*(\partial M)$. In § 5 we shall show that the boundary operations on ε-quadratic forms and formations defined in § 3 of Ranicki [**13**] are special cases of the boundary operation on ε-quadratic complexes.)

PROPOSITION 3.4. (i) *There is a natural one–one correspondence between the homotopy equivalence classes of n-dimensional (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré pairs over A and the homotopy equivalence classes of connected n-dimensional (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes which preserves boundaries. Poincaré pairs with contractible boundaries correspond to Poincaré complexes.*

(ii) *A connected n-dimensional (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ complex is a Poincaré complex if and only if its boundary is a contractible $(n-1)$-dimensional (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex.*

(iii) *An n-dimensional (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex is null-cobordant if and only if it is homotopy equivalent to the boundary of a connected $(n+1)$-dimensional (even) ε-symmetric $\{\varepsilon$-quadratic$\}$ complex.*

Proof. (i) Given a connected n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex $(C,\varphi \in Q^n(C,\varepsilon))$ $\{(C,\psi \in Q_n(C,\varepsilon))\}$ with boundary

$$\partial(C,\varphi) = (\partial C,\partial\varphi) \quad \{\partial(C,\psi) = (\partial C,\partial\psi)\}$$

there is defined an n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré pair

$$\begin{cases} (i_C\colon \partial C \to C^{n-*},(0,\partial\varphi) \in Q^n(i_C,\varepsilon)), \\ (i_C\colon \partial C \to C^{n-*},(0,\partial\psi) \in Q_n(i_C,\varepsilon)), \end{cases}$$

with

$$i_C = (0\ 1)\colon \partial C_r = C_{r+1} \oplus C^{n-r} \to (C^{n-*})_r = C^{n-r}.$$

Given a homotopy equivalence of connected n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes $f\colon (C,\varphi) \to (C',\varphi')$ $\{f\colon (C,\psi) \to (C',\psi')\}$ choose cycle representatives

$$\begin{cases} \varphi \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W,\mathrm{Hom}_A(C^*,C))_n, \varphi' \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W,\mathrm{Hom}_A(C'^*,C'))_n, \\ \psi \in (W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \mathrm{Hom}_A(C^*,C))_n, \psi' \in (W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \mathrm{Hom}_A(C'^*,C'))_n, \end{cases}$$

so that

$$\begin{cases} f^{\%}(\varphi) - \varphi' = d(\nu) \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W,\mathrm{Hom}_A(C'^*,C'))_n \\ f_{\%}(\psi) - \psi' = d(\chi) \in (W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \mathrm{Hom}_A(C'^*,C'))_n \end{cases}$$

for some chain

$$\nu \in \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, \mathrm{Hom}_A(C'^*, C'))_{n+1} \quad \{\chi \in (W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} \mathrm{Hom}_A(C'^*, C'))_{n+1}\},$$

with

$$\begin{cases} f\varphi_s f^* - \varphi'_s = d_{C'}\nu_s + (-)^r \nu_s d^*_{C'} + (-)^{n+s}(\nu_{s-1} + (-)^s T_\varepsilon \nu_{s-1}) \colon C'^{n-r+s} \to C'_r \\ f\psi_s f^* - \psi'_s = d_{C'}\chi_s + (-)^r \chi_s d^*_{C'} + (-)^{n-s}(\chi_{s+1} + (-)^{s+1} T_\varepsilon \chi_{s+1}) \colon \\ \hfill C'^{n-r-s} \to C'_r \\ \hfill (s \geqslant 0,\ \nu_{-1} = 0) \end{cases}$$

(taking C and C' to be f.g. projective, as we may do without loss of generality). Let $f' \colon C' \to C$ be a chain homotopy inverse for $f \colon C \to C'$, and let $g \colon f'f \simeq 1 \colon C \to C$ be a chain homotopy, with

$$f'f - 1 = d_C g + g d_C \colon C_r \to C_r \quad (g \in \mathrm{Hom}_A(C_r, C_{r+1})).$$

The A-module morphisms

$$\begin{cases} \partial f = \begin{pmatrix} f & -f\varphi_0 g^* + (-)^r \nu_0 f'^* \\ 0 & f'^* \end{pmatrix} \colon \partial C_r = C_{r+1} \oplus C^{n-r} \to \partial C'_r = C'_{r+1} \oplus C'^{n-r} \\ \partial f = \begin{pmatrix} f & -f(1+T_\varepsilon)\psi_0 g^* + (-)^r (1+T_\varepsilon)\chi_0 f'^* \\ 0 & f'^* \end{pmatrix} \colon \\ \hfill \partial C_r = C_{r+1} \oplus C^{n-r} \to \partial C'_r = C'_{r+1} \oplus C'^{n-r} \end{cases}$$

are such that

$$\begin{cases} (\partial f, 1; 0) \colon (i_C \colon \partial C \to C^{n-*}, (0, \partial\varphi)) \to (i_{C'} \colon \partial C' \to C'^{n-*}, (0, \partial\varphi')) \\ (\partial f, 1; 0) \colon (i_C \colon \partial C \to C^{n-*}, (0, \partial\psi)) \to (i_{C'} \colon \partial C' \to C'^{n-*}, (0, \partial\psi')) \end{cases}$$

is a homotopy equivalence of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré pairs over A. (The definition of the boundary $\partial(C, \varphi)$ $\{\partial(C, \psi)\}$ depends on a choice of cycle representative for $\varphi \in Q^n(C, \varepsilon)$ $\{\psi \in Q_n(C, \varepsilon)\}$. In particular, we have just shown that a different choice of representative defines an isomorphic complex.)

Conversely, given an n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré pair over A $(f \colon \partial C \to C, (\varphi, \partial\varphi) \in Q^n(f, \varepsilon))$ $\{(f \colon \partial C \to C, (\psi, \partial\psi) \in Q_n(f, \varepsilon))\}$ define a connected n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex

$(C', \varphi' \in Q^n(C', \varepsilon))$ $\{(C', \psi' \in Q_n(C', \varepsilon))\}$ by $C' = C(f)$ and

$$\begin{cases} \varphi'_s = \begin{pmatrix} \varphi_s & 0 \\ (-)^{n-r-1}\partial\varphi_s f^* & (-)^{n-r+s}T_\varepsilon\partial\varphi_{s-1} \end{pmatrix}: \\ \qquad C'^{n-r+s} = C^{n-r+s} \oplus \partial C^{n-r+s-1} \to C'_r = C_r \oplus \partial C_{r-1} \quad (s \geqslant 0), \\ \psi'_s = \begin{pmatrix} \psi_s & 0 \\ (-)^{n-r-1}\partial\psi_s f^* & (-)^{n-r-s-1}T_\varepsilon\partial\psi_{s+1} \end{pmatrix}: \\ \qquad C'^{n-r-s} = C^{n-r-s} \oplus \partial C^{n-r-s-1} \to C'_r = C_r \oplus \partial C_{r-1} \quad (s \geqslant 0). \end{cases}$$

(This is an algebraic analogue of the Thom complex construction in topology, being just the collapsing of the boundary $(\partial C, \partial\varphi)$ $\{(\partial C, \partial\psi)\}$.) There is defined a homotopy equivalence of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré pairs

$$\begin{cases} (\partial g, g; h)\colon (i_{C'}\colon \partial C' \to C'^{n-*}, (0, \partial\varphi')) \to (f\colon \partial C \to C, (\varphi, \partial\varphi)), \\ (\partial g, g; h)\colon (i_{C'}\colon \partial C' \to C'^{n-*}, (0, \partial\psi')) \to (f\colon \partial C \to C, (\psi, \partial\psi)), \end{cases}$$

with

$$g = \begin{cases} (\varphi_0 \quad f\partial\varphi_0) \\ ((1+T_\varepsilon)\psi_0 \quad f(1+T_\varepsilon)\partial\psi_0) \end{cases} : C'^{n-r} = C^{n-r} \oplus \partial C^{n-r-1} \to C_r,$$

$$\partial g = \begin{cases} (0 \quad 1 \quad 0 \quad \partial\varphi_0) \\ (0 \quad 1 \quad 0 \quad (1+T_\varepsilon)\partial\psi_0) \end{cases} : \partial C'_r = C_{r+1} \oplus \partial C_r \oplus C^{n-r} \oplus \partial C^{n-r-1} \to \partial C_r,$$

$$h = ((-)^r \quad 0 \quad 0 \quad 0)\colon \partial C'_r = C_{r+1} \oplus \partial C_r \oplus C^{n-r} \oplus \partial C^{n-r-1} \to C_{r+1}.$$

(ii) Given a connected n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex $(C, \varphi \in Q^n(C, \varepsilon))$ $\{(C, \psi \in Q_n(C, \varepsilon))\}$ we can identify $S\partial C = C(\varphi_0\colon C^{n-*} \to C)$ $\{S\partial C = C((1+T_\varepsilon)\psi_0\colon C^{n-*} \to C)\}$, so that ∂C is chain contractible if and only if (C, φ) $\{(C, \psi)\}$ is a Poincaré complex.

(iii) is immediate from (i).

4. Algebraic surgery

We shall now develop an algebraic surgery technique on algebraic Poincaré complexes, which is analogous to the familiar geometric technique of surgery on manifolds. Given an n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex over A (C, φ) $\{(C, \psi)\}$ and an $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ pair

$$(f\colon C \to D, (\delta\varphi, \varphi)) \quad \{(f\colon C \to D, (\delta\psi, \psi))\}$$

we shall construct a cobordant n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex (C', φ') $\{(C', \psi')\}$ 'by an algebraic surgery killing $\mathrm{im}(f^*\colon H^*(D) \to H^*(C))$'. In §7 of Part II we shall show that the chain level effect of a geometric oriented {framed} surgery is an algebraic symmetric {quadratic} surgery killing the A-module generated by a single cohomology class $x \in H^*(C)$. Here, we shall apply algebraic surgery to obtain the 4-periodicity in the ε-quadratic L-groups

$$L_n(A,\varepsilon) = L_{n+2}(A,-\varepsilon) = L_{n+4}(A,\varepsilon) \quad (n \geqslant 0).$$

An n-dimensional ε-symmetric {ε-quadratic} pair

$$\begin{cases} (f\colon C \to D, (\delta\varphi,\varphi) \in Q^n(f,\varepsilon)) \\ (f\colon C \to D, (\delta\psi,\psi) \in Q_n(f,\varepsilon)) \end{cases}$$

is *connected* if

$$\begin{cases} H_0\left(\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix}\colon D^{n-*} \to C(f)\right) = 0, \\ H_0\left(\begin{pmatrix} (1+T_\varepsilon)\delta\psi_0 \\ (1+T_\varepsilon)\psi_0 f^* \end{pmatrix}\colon D^{n-*} \to C(f)\right) = 0. \end{cases}$$

In particular, Poincaré pairs are connected.

Define as follows the connected n-dimensional ε-symmetric {ε-quadratic} complex $(C',\varphi' \in Q^n(C',\varepsilon))$ {$(C',\psi' \in Q_n(C',\varepsilon))$} obtained from a connected n-dimensional ε-symmetric {ε-quadratic} complex $(C,\varphi \in Q^n(C,\varepsilon))$ {$(C,\psi \in Q_n(C,\varepsilon))$} by *$\varepsilon$-symmetric {$\varepsilon$-quadratic} surgery* on a connected $(n+1)$-dimensional ε-symmetric {ε-quadratic} pair

$$(f\colon C \to D, (\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon))\ \{(f\colon C \to D, (\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon))\}.$$

In the ε-symmetric case let

$$d_{C'} = \begin{pmatrix} d_C & 0 & (-)^{n+1}\varphi_0 f^* \\ (-)^r f & d_D & (-)^r\delta\varphi_0 \\ 0 & 0 & (-)^r d_D^* \end{pmatrix}\colon$$
$$C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to C'_{r-1} = C_{r-1} \oplus D_r \oplus D^{n-r+2}$$

$$\varphi'_0 = \begin{pmatrix} \varphi_0 & 0 & 0 \\ (-)^{n-r} f T_\varepsilon \varphi_1 & (-)^{n-r} T_\varepsilon \delta\varphi_1 & (-)^{r(n-r)}\varepsilon \\ 0 & 1 & 0 \end{pmatrix}\colon$$
$$C'^{n-r} = C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1} \to C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1},$$

$$\varphi'_s = \begin{pmatrix} \varphi_s & 0 & 0 \\ (-)^{n-r} f T_\varepsilon \varphi_{s+1} & (-)^{n-r+s} T_\varepsilon \delta\varphi_{s+1} & 0 \\ 0 & 0 & 0 \end{pmatrix}\colon$$
$$C'^{n-r+s} = C^{n-r+s} \oplus D^{n-r+s+1} \oplus D_{r-s+1} \to C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1}$$
$$(s \geqslant 1).$$

In the ε-quadratic case let

$$d_{C'} = \begin{pmatrix} d_C & 0 & (-)^{n+1}(1+T_\varepsilon)\psi_0 f^* \\ (-)^r f & d_D & (-)^r(1+T_\varepsilon)\delta\psi_0 \\ 0 & 0 & (-)^r d_D^* \end{pmatrix}:$$

$$C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to C'_{r-1} = C_{r-1} \oplus D_r \oplus D^{n-r+2},$$

$$\psi'_0 = \begin{pmatrix} \psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}:$$

$$C'^{n-r} = C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1} \to C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1},$$

$$\psi'_s = \begin{pmatrix} \psi_s & (-)^{r+s}T_\varepsilon\psi_{s-1}f^* & 0 \\ 0 & (-)^{n-r-s+1}T_\varepsilon\delta\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}:$$

$$C'^{n-r-s} = C^{n-r-s} \oplus D^{n-r-s+1} \oplus D_{r+s+1} \to C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \quad (s \geqslant 1).$$

(We are using matrix notation as if C, D were f.g. projective chain complexes.)

It may be verified that performing ε-symmetric $\{\varepsilon$-quadratic$\}$ surgery using a different cycle representative of

$$(\delta\varphi, \varphi) \in Q^{n+1}(f, \varepsilon) \quad \{(\delta\psi, \psi) \in Q_{n+1}(f, \varepsilon)\}$$

leads to an isomorphic ε-symmetric $\{\varepsilon$-quadratic$\}$ complex (C', φ') $\{(C', \psi')\}$. Note that the ε-symmetrization of the ε-quadratic surgery on

$$(f\colon C \to D, (\delta\psi, \psi) \in Q_{n+1}(f, \varepsilon))$$

is the ε-symmetric surgery on $(f\colon C \to D, (1+T_\varepsilon)(\delta\psi, \psi) \in Q^{n+1}(f, \varepsilon))$.

Let $(C, \varphi \in Q^n(C, \varepsilon))$ $\{(C, \psi \in Q_n(C, \varepsilon))\}$ be a connected n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex. The $(n-1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex obtained from $(0, 0)$ by surgery on the connected n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ pair

$$(0\colon 0 \to C, (0, \varphi) \in Q^n(0, \varepsilon)) \quad \{(0\colon 0 \to C, (0, \psi) \in Q_n(0, \varepsilon))\}$$

is just the boundary $\partial(C, \varphi)$ $\{\partial(C, \psi)\}$, as defined in § 3 above. We can thus interpret Proposition 3.4(iii) as stating that an n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex is null-cobordant if and only if it can be obtained from $(0, 0)$ by surgery and homotopy equivalence. This is a special case of the following result: cobordism is the equivalence relation on algebraic Poincaré complexes generated by surgery and homotopy equivalence. There is an obvious analogy here with Theorem 1 of Milnor [**9**], which showed that compact oriented manifolds are cobordant if and only if one can be obtained from the other by a finite sequence of geometric surgeries.

PROPOSITION 4.1. (i) *Algebraic surgery preserves the homotopy type of the boundary, sending algebraic Poincaré complexes to algebraic Poincaré complexes.*

(ii) *The n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes $(C,\varphi),(C',\varphi')$ $\{(C,\psi),(C',\psi')\}$ are cobordant if and only if (C',φ') $\{(C',\psi')\}$ can be obtained from (C,φ) $\{(C,\psi)\}$ by surgery and homotopy equivalence.*

Proof. (i) Let

$$\begin{cases} (f\colon C\to D,(\delta\varphi,\varphi)\in Q^{n+1}(f,\varepsilon)),(C',\varphi'\in Q^n(C',\varepsilon)) \\ (f\colon C\to D,(\delta\psi,\psi)\in Q_{n+1}(f,\varepsilon)),(C',\psi'\in Q_n(C',\varepsilon)) \end{cases}$$

be as in the definition of ε-symmetric $\{\varepsilon$-quadratic$\}$ surgery. Then the A-module morphisms

$$h=\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ (-)^{r(n-r)}f & 0 \end{pmatrix}:$$

$$\partial C_r = C_{r+1}\oplus C^{n-r}\to \partial C'_r = C_{r+1}\oplus D_{r+2}\oplus D^{n-r}\oplus C^{n-r}\oplus D^{n-r+1}\oplus D_{r+1}$$

define a homotopy equivalence of the boundary $(n-1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes

$$\begin{cases} h\colon \partial(C,\varphi)\to\partial(C',\varphi'), \\ h\colon \partial(C,\psi)\to\partial(C',\psi'). \end{cases}$$

Proposition 3.4(ii) states that (C,φ) $\{(C,\psi)\}$ is a Poincaré complex if and only if $\partial(C,\varphi)$ $\{\partial(C,\psi)\}$ is contractible. It follows that (C,φ) $\{(C,\psi)\}$ is a Poincaré complex if and only if (C',φ') $\{(C',\psi')\}$ is a Poincaré complex.

(ii) Continuing with the above notation assume also that (C,φ) $\{(C,\psi)\}$ is a Poincaré complex, and define an $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré pair

$$\begin{cases} ((g\quad g')\colon C\oplus C'\to D',(0,\varphi\oplus-\varphi')\in Q^{n+1}((g\quad g'),\varepsilon)) \\ ((g\quad g')\colon C\oplus C'\to D',(0,\psi\oplus-\psi')\in Q_{n+1}((g\quad g'),\varepsilon)) \end{cases}$$

by

$$d_{D'} = \begin{cases} \begin{pmatrix} d_C & (-)^{n+1}\varphi_0 f^* \\ 0 & (-)^r d_D^* \end{pmatrix} \\ \begin{pmatrix} d_C & (-)^{n+1}(1+T_\varepsilon)\psi_0 f^* \\ 0 & (-)^r d_D^* \end{pmatrix} \end{cases} : D'_r = C_r \oplus D^{n-r+1} \to D'_{r-1} = C_{r-1} \oplus D^{n-r+2},$$

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \to D'_r = C_r \oplus D^{n-r+1},$$

$$g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to D'_r = C_r \oplus D^{n-r+1}.$$

We thus have a cobordism from (C,φ) $\{(C,\psi)\}$ to (C',φ') $\{(C',\psi')\}$, so that surgery preserves cobordism classes.

Conversely, suppose given a cobordism of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes

$$\begin{cases} ((f \;\; f') \colon C \oplus C' \to D, (\delta\varphi, \varphi \oplus -\varphi') \in Q^{n+1}((f \;\; f'), \varepsilon)), \\ ((f \;\; f') \colon C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi') \in Q_{n+1}((f \;\; f'), \varepsilon)). \end{cases}$$

Let (C'',φ'') $\{(C'',\psi'')\}$ be the n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex obtained from (C,φ) $\{(C,\psi)\}$ by ε-symmetric $\{\varepsilon$-quadratic$\}$ surgery on the connected $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ pair

$$(g \colon C \to D', (\delta\varphi', \varphi) \in Q^{n+1}(g,\varepsilon)) \quad \{(g \colon C \to D', (\delta\psi', \psi) \in Q_{n+1}(g,\varepsilon))\}$$

defined by

$$d_{D'} = \begin{pmatrix} d_D & (-)^{r-1} f' \\ 0 & d_{C'} \end{pmatrix} \colon$$
$$D'_r = D_r \oplus C'_{r-1} \to D'_{r-1} = D_{r-1} \oplus C'_{r-2} \quad (D' = C(f')),$$

$$g = \begin{pmatrix} f \\ 0 \end{pmatrix} : C_r \to D'_r = D_r \oplus C'_{r-1},$$

$$\begin{cases} \delta\varphi'_s = \begin{pmatrix} \delta\varphi_s & (-)^s f'\varphi'_s \\ 0 & (-)^{n-r+s} T_\varepsilon \varphi'_{s-1} \end{pmatrix} \colon \\ \qquad D'^{n-r+s+1} = D^{n-r+s+1} \oplus C'^{n-r+s} \to D'_r = D_r \oplus C'_{r-1} \\ \qquad\qquad (s \geqslant 0,\ \varphi'_{-1} = 0), \\ \delta\psi'_s = \begin{pmatrix} \delta\psi_s & (-)^s f'\psi'_s \\ 0 & (-)^{n-r-s+1} T_\varepsilon \psi'_{s+1} \end{pmatrix} \colon \\ \qquad D'^{n-r-s+1} = D^{n-r-s+1} \oplus C'^{n-r-s} \to D'_r = D_r \oplus C'_{r-1} \quad (s \geqslant 0). \end{cases}$$

The A-module morphisms

$$\begin{cases} h = (0 \quad 0 \quad 1 \quad 0 \quad 0) \colon C''_r = C_r \oplus D_{r+1} \oplus C'_r \oplus D^{n-r+1} \oplus C'^{n-r} \to C'_r \\ h = (0 \quad 0 \quad 1 \quad 0 \quad -T_\varepsilon \psi'_0) \colon C''_r = C_r \oplus D_{r+1} \oplus C'_r \oplus D^{n-r+1} \oplus C'^{n-r} \to C'_r \end{cases}$$

define a homotopy equivalence of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes

$$\begin{cases} h \colon (C'', \varphi'') \to (C', \varphi'), \\ h \colon (C'', \psi'') \to (C', \psi'). \end{cases}$$

Thus (C', φ') $\{(C', \psi')\}$ may be obtained from (C, φ) $\{(C, \psi)\}$ by an ε-symmetric $\{\varepsilon$-quadratic$\}$ surgery followed by a homotopy equivalence.

We shall prove that certain skew-suspension maps in the L-groups are isomorphisms using the following criterion.

PROPOSITION 4.2. *The skew-suspension map*

$$\bar{S} \colon L^n(A, \varepsilon) \to L^{n+2}(A, -\varepsilon) \quad \{\bar{S} \colon L_n(A, \varepsilon) \to L_{n+2}(A, -\varepsilon)\}$$

is onto (respectively one-to-one) if for every connected strictly $(n+2)$-(respectively $(n+3)$)-dimensional $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic$\}$ complex over A (C, φ) $\{(C, \psi)\}$ with a boundary $\partial(C, \varphi)$ $\{\partial(C, \psi)\}$ which is contractible (respectively a skew-suspension) it is possible to do $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic$\}$ surgery on (C, φ) $\{(C, \psi)\}$ to obtain a skew-suspension.

Proof. It is immediate from Propositions 1.5 and 4.1(ii) that the claimed condition for

$$\bar{S} \colon L^n(A, \varepsilon) \to L^{n+2}(A, -\varepsilon) \quad \{\bar{S} \colon L_n(A, \varepsilon) \to L_{n+2}(A, -\varepsilon)\}$$

to be onto is both necessary and sufficient.

Assume the claimed condition for

$$\bar{S} \colon L^n(A, \varepsilon) \to L^{n+2}(A, -\varepsilon) \quad \{\bar{S} \colon L_n(A, \varepsilon) \to L_{n+2}(A, -\varepsilon)\}$$

to be one-to-one, and let (C, φ) $\{(C, \psi)\}$ be a strictly n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex over A such that

$$\begin{cases} \bar{S}(C, \varphi) = 0 \in L^{n+2}(A, -\varepsilon), \\ \bar{S}(C, \psi) = 0 \in L_{n+2}(A, -\varepsilon). \end{cases}$$

By Proposition 3.4(iii) we have that $\bar{S}(C, \varphi)$ $\{\bar{S}(C, \psi)\}$ is homotopy equivalent to the boundary $\partial(D, \nu)$ $\{\partial(D, \chi)\}$ of a connected strictly $(n+3)$-dimensional $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic$\}$ complex (D, ν) $\{(D, \chi)\}$. By hypothesis, it is possible to do surgery on (D, ν) $\{(D, \chi)\}$ to obtain the skew-suspension $\bar{S}(D', \nu')$ $\{\bar{S}(D', \chi')\}$ of a connected $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex (D', ν') $\{(D', \chi')\}$. It now follows from Proposition 4.1(i) that (C, φ) $\{(C, \psi)\}$ is homotopy equivalent to the boundary $\partial(D',\nu')$ $\{\partial(D',\chi')\}$, and so

$$\begin{cases} (C,\varphi) = 0 \in L^n(A,\varepsilon), \\ (C,\psi) = 0 \in L_n(A,\varepsilon). \end{cases}$$

Therefore the stated condition is sufficient to ensure that the skew-suspension map is one-to-one.

As a first application of our algebraic surgery we shall establish the 4-periodicity in the ε-quadratic L-groups, $L_n(A,\varepsilon) = L_{n+4}(A,\varepsilon)$, by analogy with the familiar result that it is always possible to perform framed surgery below the middle dimension and the consequent 4-periodicity $L_n(\pi) = L_{n+4}(\pi)$ of Wall **[25]**.

PROPOSITION 4.3. *The i-fold skew-suspension map*

$$\bar{S}^i\colon L_{n-2i}(A,(-)^i\varepsilon) \to L_n(A,\varepsilon) \quad (n \geqslant 2i)$$

is an isomorphism for all A,ε. The inverse isomorphism

$$\Omega^i = (\bar{S}^i)^{-1}\colon L_n(A,\varepsilon) \to L_{n-2i}(A,(-)^i\varepsilon) \quad (n = 2i \text{ or } 2i+1)$$

sends the cobordism class of a strictly n-dimensional ε-quadratic Poincaré complex $(C,\psi \in Q_n(C,\varepsilon))$ to the cobordism class of the $(n-2i)$-dimensional $(-)^i\varepsilon$-quadratic Poincaré complex corresponding to the non-singular $(-)^i\varepsilon$-quadratic form $\{$formation$\}$

$$\Omega^i(C,\psi) = \begin{cases} \left(\operatorname{coker}\left(\begin{pmatrix} d^* & 0 \\ (-)^{i+1}(1+T_\varepsilon)\psi_0 & d \end{pmatrix}\colon\right.\right. \\ \qquad\qquad \left.\left. C^{i-1}\oplus C_{i+2} \to C^i \oplus C_{i+1}\right), \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}\right) & \text{if } n = 2i, \\ \left(H_{(-)^i\varepsilon}(C_{i+1});\, C_{i+1}, \operatorname{im}\left(\begin{pmatrix} (1+T_\varepsilon)\psi_0 & d \\ \varepsilon d^* & 0 \end{pmatrix}\colon\right.\right. \\ \qquad\qquad \left.\left. C^i \oplus C_{i+2} \to C_{i+1}\oplus C^{i+1}\right)\right) & \text{if } n = 2i+1. \end{cases}$$

Proof. Given a connected strictly n-dimensional ε-quadratic complex $(C,\psi \in Q_n(C,\varepsilon))$ let $(C',\psi' \in Q_n(C',\varepsilon))$ be the connected n-dimensional ε-quadratic complex obtained from (C,ψ) by surgery on the connected $(n+1)$-dimensional ε-quadratic pair $(f\colon C \to D, (0,\psi) \in Q_{n+1}(f,\varepsilon))$ defined by

$$f = \begin{cases} 1 \\ 0 \end{cases}\colon C_r \to D_r = \begin{cases} C_r & \text{if } r > n-i \\ 0 & \text{if } r \leqslant n-i \end{cases} \quad (d_D = d_C).$$

Now

$$H_r(C') = H_r((1+T_\varepsilon)\psi_0\colon C^{n-*} \to C) \quad (r < i),$$

$$H^r(C') = 0 \quad (r > n-i),$$

and for $n = 2i+1$ also

$$H_i((1+T_\varepsilon)\psi_0' : C'^{n-*} \to C') = 0.$$

Thus if the boundary $\partial(C,\psi)$ is contractible (respectively an i-fold skew-suspension) then (C',ψ') is the i-fold skew-suspension $\bar{S}^i(C'',\psi'')$ of an $(n-2i)$-dimensional $(-)^i\varepsilon$-quadratic complex (C'',ψ'') which is a Poincaré (respectively connected) complex. Applying Proposition 4.2 we have that $\bar{S}^i : L_{n-2i}(A,(-)^i\varepsilon) \to L_n(A,\varepsilon)$ is onto (respectively

$$\bar{S}^i : L_{n-2i-1}(A,(-)^i\varepsilon) \to L_{n-1}(A,\varepsilon)$$

is one–one). If $n = 2i$ or $2i+1$ and (C,ψ) is a Poincaré complex define as follows a strictly $(n-2i)$-dimensional $(-)^i\varepsilon$-quadratic Poincaré complex $(B, \theta \in Q_{n-2i}(B,(-)^i\varepsilon))$ and a homotopy equivalence

$$g : (B,\theta) \to (C'',\psi'').$$

In the case where $n = 2i$ let

$$B_0 = \operatorname{coker}\left(\begin{pmatrix} d^* & 0 \\ (-)^{i+1}(1+T_\varepsilon)\psi_0 & d \end{pmatrix} : C^{i-1} \oplus C_{i+2} \to C^i \oplus C_{i+1}\right)^*,$$

$$g : B_0 \xrightarrow{(\text{projection})^*} (C^i \oplus C_{i+1})^* = C_i \oplus C^{i+1} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}} C_0''$$
$$= C_i \oplus C_{i+1} \oplus C^{i+1},$$

with $\theta_0 \in \operatorname{Hom}_A(B^0, B_0)$ any representative of the $(-)^i\varepsilon$-quadratic form

$$\left(B^0, \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix} \in Q_{(-)^i\varepsilon}(B^0)\right).$$

In the case where $n = 2i+1$ let

$$d_B = \varepsilon[0 \quad 1]^* : B_1 = C_{i+1} \to B_0$$
$$= \operatorname{im}\left(\begin{pmatrix} (1+T_\varepsilon)\psi_0 & d \\ \varepsilon d^* & 0 \end{pmatrix} : C^i \oplus C_{i+2} \to C_{i+1} \oplus C^{i+1}\right)^*,$$

$$\theta_0 = \begin{cases} [1 \quad 0] : B^0 \to B_1, \\ 0 : B^1 \to B_0, \end{cases} \qquad \theta_1 = \begin{bmatrix} \psi_1 + d\psi_0 & 0 \\ 0 & 0 \end{bmatrix} : B^0 \to B_0,$$

$$g = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} : B_1 \to C_1'' = C_{i+1} \oplus C_{i+2}, \\ \begin{bmatrix} (1+T_\varepsilon)\psi_0 & d \\ \varepsilon d^* & 0 \end{bmatrix}^* : B_0 \to C_0'' = C_i \oplus C^{i+2}. \end{cases}$$

If $n = 2i$ $\{n = 2i+1\}$ the correspondence of Proposition 2.1 {Proposition 2.5} sends (B,θ) to the non-singular $(-)^i\varepsilon$-quadratic form {formation} $\Omega^i(C,\psi)$.

It is claimed by Mishchenko in [**10**] that the double skew-suspension maps in the symmetric L-groups

$$\bar{S}^2\colon L^n(A) \to L^{n+4}(A) \quad (n \geqslant 0)$$

are isomorphisms for every ring with involution A. Such is indeed the case if $\frac{1}{2} \in A$ when $L_n(A) = L^n(A)$, by Propositions 3.3 and 4.3. However, the algebraic surgery technique of § 4 of Mishchenko [**10**] used to establish this 4-periodicity breaks down if $\frac{1}{2} \notin A$: the new algebraic Poincaré pair $(\bar{C}, {}^0\bar{C}, \bar{d}, \bar{D})$ may not satisfy property (a′) of § 3, since the A-module morphism

$${}^0\bar{D}^{n-2i-1} = \begin{pmatrix} {}^0D^{n-2i-1} & D^{n-2i-1}\beta \\ 0 & 0 \end{pmatrix}:$$

$$\bar{C}^{n-i-1} = C^{n-i-1} \oplus A^* \to \bar{C}_{n-i} = C_{n-i} \oplus A$$

need not map $A^* \subset \bar{C}^{n-i-1} = C^{n-i-1} \oplus A^*$ into ${}^0\bar{C}_{n-i} = {}^0C_{n-i} \subset {}^0\bar{C}_{n-i}$ if $D^{n-2i-1}\beta \neq 0$. In Proposition 4.4 below we shall prove that the skew-suspension maps

$$\bar{S}\colon L^n(A,\varepsilon) \to L\langle v_0\rangle^{n+2}(A,-\varepsilon) \quad (n \geqslant 0)$$

are isomorphisms. This implies that if $\hat{H}^1(\mathbf{Z}_2; A,\varepsilon) = 0$ then the skew-suspension maps

$$\bar{S}\colon L^n(A,\varepsilon) \to L^{n+2}(A,-\varepsilon) \quad (n \geqslant 0)$$

are isomorphisms (Proposition 6.1). In Proposition 4.5 below we shall prove that if A is noetherian of finite global dimension m then the skew-suspension maps

$$\bar{S}\colon L^n(A,\varepsilon) \to L^{n+2}(A,-\varepsilon) \quad (n+2 \geqslant 2m)$$

are isomorphisms. In particular, for a noetherian ring A of global dimension at most 1 (such as a Dedekind ring) we have that the double skew-suspension maps

$$\bar{S}^2\colon L^n(A,\varepsilon) \to L^{n+4}(A,\varepsilon) \quad (n \geqslant 0)$$

are isomorphisms. The L-theory of Dedekind rings will be studied further in § 7. In § 10 we shall give examples of noetherian rings A of global dimension at least 2 for which the skew-suspension maps

$$\bar{S}\colon L^n(A,\varepsilon) \to L^{n+2}(A,-\varepsilon) \quad (n \geqslant 0)$$

are not all isomorphisms, and such that ε-symmetric L-groups are not 4-periodic.

PROPOSITION 4.4. *The skew-suspension maps*

$$\bar{S}\colon L^n(A,\varepsilon) \to L\langle v_0\rangle^{n+2}(A,-\varepsilon) \quad (n \geqslant 0)$$

are isomorphisms for all A, ε.

Proof. We shall define an inverse isomorphism

$$\Omega\colon L\langle v_0\rangle^{n+2}(A,-\varepsilon) \to L^n(A,\varepsilon)$$

by working exactly as in the proof of Proposition 4.3. Given a strictly $(n+2)$-dimensional even $(-\varepsilon)$-symmetric Poincaré complex over A $(C, \varphi \in Q\langle v_0\rangle^{n+2}(C,-\varepsilon))$ define a connected $(n+3)$-dimensional even $(-\varepsilon)$-symmetric pair $(f\colon C \to D, (0,\varphi) \in Q\langle v_0\rangle^{n+3}(f,-\varepsilon))$ by

$$f = \begin{cases} 1 \\ 0 \end{cases}\colon C_r \to D_r = \begin{cases} C_{n+2} & \text{if } r = n+2, \\ 0 & \text{if } r \neq n+2. \end{cases}$$

The $(n+2)$-dimensional even $(-\varepsilon)$-symmetric Poincaré complex obtained from (C,φ) by surgery on $(f\colon C \to D, (0,\varphi))$ is the skew-suspension $\bar{S}\Omega(C',\varphi')$ of an n-dimensional ε-symmetric Poincaré complex $\Omega(C',\varphi')$.

PROPOSITION 4.5. *Let A be a ring with involution which is noetherian of finite global dimension m. The skew-suspension map*

$$\bar{S}\colon L^n(A,\varepsilon) \to L^{n+2}(A,-\varepsilon) \quad (n \geqslant 0)$$

is an isomorphism if $n+2 \geqslant 2m$, and a monomorphism if $n+3 = 2m$. If $m = 0$ (that is, if A is semi-simple) then

$$L^{2k+1}(A,\varepsilon) = 0 \quad (k \geqslant 0).$$

Proof. In the first instance, note that the hypothesis on A is equivalent to the property that every f.g. A-module M has a f.g. projective A-module resolution of length not greater than m

$$0 \to P_m \to P_{m-1} \to \ldots \to P_1 \to P_0 \to M \to 0.$$

Let $p = 2$ if $n+2 \geqslant 2m$ (respectively $p = 3$ if $n+3 \geqslant 2m$). Given a connected strictly $(n+p)$-dimensional $(-\varepsilon)$-symmetric complex over A $(C, \varphi \in Q^{n+p}(C,-\varepsilon))$ with a boundary $\partial(C,\varphi)$ which is contractible (respectively a skew-suspension) we have that $H_0(C) = \operatorname{coker}(d\colon C_1 \to C_0)$ is a f.g. A-module, with a f.g. projective A-module resolution of length m

$$0 \longrightarrow D_m \xrightarrow{d} D_{m-1} \longrightarrow \ldots \longrightarrow D_1 \xrightarrow{d} D_0 \longrightarrow H_0(C) \longrightarrow 0.$$

Let $f\colon C \to D$ be an A-module chain map inducing

$$f_* = 1\colon H_0(C) \to H_0(D) = H_0(C),$$

and let $\varphi \in \mathrm{Hom}_{Z[Z_2]}(W, \mathrm{Hom}_A(C^*, C))_{n+p}$ be a cycle representing $\varphi \in Q^{n+p}(C, -\varepsilon)$, so that

$$f^{\%}(\varphi)_s = \begin{cases} f\varphi_0 f^*\colon D^m \to D_m & \text{if } n+p = 2m,\ r = m,\ s = 0, \\ 0\colon D^r \to D_{n+p-r+s} & \text{otherwise.} \end{cases}$$

In the case where $n+p = 2m$ consider the commutative diagram

$$\begin{tikzcd} D^m \arrow[r, "f^*"] \arrow[d] & C^m \arrow[r, "\varphi_0"] \arrow[d, "d^*"'] & C_m \arrow[r, "f"] \arrow[d, "d"] & D_m \arrow[d, "d"] \\ 0 \arrow[r] & C^{m+1} \arrow[r, "(-)^m\varphi_0"] & C_{m-1} \arrow[r, "f"] & D_{m-1} \end{tikzcd}$$

Now $d \in \mathrm{Hom}_A(D_m, D_{m-1})$ is one–one, and

$$d(f\varphi_0 f^*) = 0\colon D^m \xrightarrow{f\varphi_0 f^*} D_m \xrightarrow{d} D_{m-1},$$

so that

$$f\varphi_0 f^* = 0\colon D^m \to D_m.$$

Thus, if $n+p \geqslant 2m$,

$$f^{\%}(\varphi) = 0 \in \mathrm{Hom}_{Z[Z_2]}(W, \mathrm{Hom}_A(D^*, D))_{n+p},$$

and there is defined a connected $(n+p+1)$-dimensional $(-\varepsilon)$-symmetric pair $(f\colon C \to D, (0, \varphi) \in Q^{n+p+1}(f, -\varepsilon))$. Let $(C', \varphi' \in Q^{n+p}(C', -\varepsilon))$ be the $(n+p)$-dimensional $(-\varepsilon)$-symmetric complex obtained from (C, φ) by surgery on this pair. Now

$$H_0(C') = H_1(f) = 0,$$

$$H^{n+p}(C') = H_0(f\varphi_0\colon C^{n+p-*} \to D) = H_0(\varphi_0\colon C^{n+p-*} \to C) = 0,$$

so that (C', φ') is the skew-suspension of an $(n+p-2)$-dimensiona ε-symmetric complex. Applying the criterion of Proposition 4.2 we have that $\bar{S}\colon L^n(A, \varepsilon) \to L^{n+2}(A, -\varepsilon)$ is onto if $n+2 \geqslant 2m$ (respectively one–one if $n+3 \geqslant 2m$).

In particular, if A is a noetherian ring of global dimension 1 we have that all the skew-suspensions

$$\bar{S}\colon L^n(A, \varepsilon) \to L^{n+2}(A, -\varepsilon) \quad (n \geqslant 0)$$

are isomorphisms. If A is semi-simple ($m = 0$) then the above procedure associates to a strictly 1-dimensional ε-symmetric Poincaré complex over A $(C, \varphi \in Q^1(C, \varepsilon))$ a 2-dimensional ε-symmetric Poincaré pair over A $(f\colon C \to D, (0, \varphi) \in Q^2(f, \varepsilon))$, and so

$$L^1(A, \varepsilon) = 0.$$

(Moreover, if (C,φ) is even then so is $(f\colon C\to D,(0,\varphi))$, and we also have

$$L\langle v_0\rangle^1(A,\varepsilon)=0.$$

It has already been proved in [16] that for a semi-simple ring with involution A

$$L_{2k+1}(A,\varepsilon)=0 \quad (k\geqslant 0).)$$

We shall find the following application of algebraic surgery of use in § 5 below.

An n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex $(C,\varphi\in Q^n(C,\varepsilon))$ $\{(C,\psi\in Q_n(C,\varepsilon))\}$ is *well-connected* if $H_0(C)=0$, in which case it is connected, that is

$$\begin{cases} H_0(\varphi_0\colon C^{n-*}\to C)=0,\\ H_0((1+T_\varepsilon)\psi_0\colon C^{n-*}\to C)=0.\end{cases}$$

PROPOSITION 4.6. *The n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes $(C,\varphi),(C',\varphi')$ $\{(C,\psi),(C',\psi')\}$ are cobordant if and only if there exists a homotopy equivalence*

$$\begin{cases} f\colon (C,\varphi)\oplus\partial(D,\nu)\to(C',\varphi')\oplus\partial(D',\nu')\\ f\colon (C,\psi)\oplus\partial(D,\chi)\to(C',\psi')\oplus\partial(D',\chi')\end{cases}$$

for some well-connected $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes $(D,\nu),(D',\nu')$ $\{(D,\chi),(D',\chi')\}$.

Proof. In view of Proposition 3.4(iii) it is sufficient to prove that for every connected $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex (D,ν) $\{(D,\chi)\}$ there exists a homotopy equivalence

$$\begin{cases} f\colon \partial(D,\nu)\oplus\partial(D',\nu')\to\partial(D'',\nu'')\\ f\colon \partial(D,\chi)\oplus\partial(D',\chi')\to\partial(D'',\chi'')\end{cases}$$

for some well-connected $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes $(D',\nu'),(D'',\nu'')$ $\{(D',\chi'),(D'',\chi'')\}$. Define a chain map $g\colon D\to D'$ by

$$\begin{array}{ccccccccccc} D\colon \dots & \longrightarrow & D_{n+2} & \xrightarrow{d} & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \longrightarrow & \dots\\ g\big\downarrow & & \big\downarrow 1 & & \big\downarrow 1 & & \big\downarrow & & \big\downarrow & & \\ D'\colon \dots & \longrightarrow & D_{n+2} & \xrightarrow{d} & D_{n+1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

and let $\nu'=-g^{\%}(\nu)\in Q^{n+1}(D',\varepsilon)$ $\{\chi'=-g_{\%}(\chi)\in Q_{n+1}(D',\varepsilon)\}$. Let (D'',ν'') $\{(D'',\chi'')\}$ be the connected $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex obtained from $(D,\nu)\oplus(D',\nu')$ $\{(D,\chi)\oplus(D',\chi')\}$ by surgery on the connected $(n+2)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ pair

$$\begin{cases} ((g\quad 1)\colon D\oplus D' \to D', (0,\nu\oplus\nu') \in Q^{n+2}((g\quad 1),\varepsilon)), \\ ((g\quad 1)\colon D\oplus D' \to D', (0,\chi\oplus\chi') \in Q_{n+2}((g\quad 1),\varepsilon)). \end{cases}$$

Now (D',ν') $\{(D',\chi')\}$ and (D'',ν'') $\{(D'',\chi'')\}$ are well-connected, and Proposition 4.1(i) shows that

$$\begin{cases} \partial(D,\nu)\oplus\partial(D',\nu') = \partial(D'',\nu'') \\ \partial(D,\chi)\oplus\partial(D',\chi') = \partial(D'',\chi'') \end{cases}$$

up to homotopy equivalence.

Intuitively, ε-symmetric $\{\varepsilon$-quadratic$\}$ surgery on

$$\begin{cases} (f\colon C \to D, (\partial\varphi,\varphi) \in Q^{n+1}(f,\varepsilon)) \\ (f\colon C \to D, (\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon)) \end{cases}$$

kills $\operatorname{im}(f^*\colon H^*(D) \to H^*(C))$, whereas a geometric surgery only kills individual (co)homology classes. In §7 of Part II we shall show that the chain level effect of geometric surgery on an r-dimensional spherical homology class in an n-dimensional manifold is an algebraic surgery on a connected $(n+1)$-dimensional algebraic pair such that

$$H^s(D) = \begin{cases} A & \text{if } s = n-r, \\ 0 & \text{if } s \neq n-r. \end{cases}$$

We shall now break down a general algebraic surgery into a sequence of such elementary surgeries (subject to a necessary K-theoretic restriction).

An algebraic surgery on a connected $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ pair over A

$$(f\colon C \to D, (\delta\varphi,\varphi) \in Q^{n+1}(f,\varepsilon)) \quad \{(f\colon C \to D, (\delta\psi,\psi) \in Q_{n+1}(f,\varepsilon))\}$$

is *elementary of type* $(r, n-r-1)$ if $D = S^{n-r}A$, that is,

$$D_s = \begin{cases} A & \text{if } s = n-r, \\ 0 & \text{if } s \neq n-r. \end{cases}$$

Such a surgery will be said to *kill* the (co)homology class $f^*(1) \in H^{n-r}(C)$ $(= H_r(C)$ if (C,φ) $\{(C,\psi)\}$ is a Poincaré complex).

PROPOSITION 4.7. *Let* $(C, \varphi \in Q^n(C,\varepsilon))$ $\{(C,\psi \in Q_n(C,\varepsilon))\}$ *be a connected* n*-dimensional* ε*-symmetric* $\{\varepsilon$*-quadratic*$\}$ *complex over* A. *Then*

(i) *A cohomology class* $x \in H^{n-r}(C)$ *can be killed by an elementary* ε*-symmetric* $\{\varepsilon$*-quadratic*$\}$ *surgery of type* $(r, n-r-1)$ *if and only if its*

ε-symmetric $\{\varepsilon$-quadratic$\}$ Wu class vanishes, that is,

$$\begin{cases} v_r(\varphi)(x) = 0 \in H^{n-2r}(\mathbf{Z}_2;\, A, (-)^{n-r}\varepsilon), \\ v^r(\psi)(x) = 0 \in H_{2r-n}(\mathbf{Z}_2;\, A, (-)^{n-r}\varepsilon), \end{cases}$$

and, in the case where $r = n$, $x \in H^0(C)$ generates a direct summand of $H^0(C)$.

(ii) *If (C',φ') $\{(C',\psi')\}$ is obtained from (C,φ) $\{(C,\psi)\}$ by an elementary ε-symmetric $\{\varepsilon$-quadratic$\}$ surgery of type $(r, n-r-1)$ then (C,φ) $\{(C,\psi)\}$ is homotopy equivalent to an ε-symmetric $\{\varepsilon$-quadratic$\}$ complex obtained from (C',φ') $\{(C',\psi')\}$ by an elementary ε-symmetric $\{\varepsilon$-quadratic$\}$ surgery of type $(n-r-1, r)$.*

(iii) *If (C',φ') $\{(C',\psi')\}$ is obtained from (C,φ) $\{(C,\psi)\}$ by ε-symmetric $\{\varepsilon$-quadratic$\}$ surgery on*

$$(f\colon C \to D, (\delta\varphi, \varphi) \in Q^{n+1}(f,\varepsilon)) \quad \{(f\colon C \to D, (\delta\psi, \psi) \in Q_{n+1}(f,\varepsilon))\}$$

such that D has projective class

$$[D] \equiv \sum_{r=-\infty}^{\infty} (-)^r [D_r] = 0 \in \tilde{K}_0(A)$$

then (C',φ') $\{(C',\psi')\}$ may be obtained from (C,φ) $\{(C,\psi)\}$ by a finite sequence of elementary ε-symmetric $\{\varepsilon$-quadratic$\}$ surgeries.

Proof. (i) The vanishing of the Wu class is just the condition required to represent $x \in H^{n-r}(C)$ by an $(n+1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ pair

$$\begin{cases} (f\colon C \to S^{n-r}A, (\delta\varphi, \varphi) \in Q^{n+1}(f,\varepsilon)) \\ (f\colon C \to S^{n-r}A, (\delta\psi, \psi) \in Q_{n+1}(f,\varepsilon)) \end{cases}$$

such that $f^*(1) = x \in H^{n-r}(C)$. This pair is automatically connected if $r < n$, but for $r = n$ it is connected if and only if $x \in H^0(C)$ generates a direct summand.

(ii) and (iii) follow from the result below on the composition of algebraic surgeries:

LEMMA. *Let $(C,\varphi), (C',\varphi'), (C'',\varphi'')$ be connected n-dimensional ε-symmetric complexes over A such that (C',φ') (respectively (C'',φ'')) is obtained from (C,φ) (respectively (C',φ')) by surgery on a connected $(n+1)$-dimensional ε-symmetric pair $(f\colon C \to D, (\delta\varphi, \varphi))$ (respectively $(f'\colon C' \to D', (\delta\varphi', \varphi'))$). Then (C'',φ'') is obtained from (C,φ) by surgery on a connected $(n+1)$-dimensional ε-symmetric pair $(f''\colon C \to D'', (\delta\varphi'', \varphi))$ with $D'' = C(g)$ the algebraic mapping cone of a chain map $g\colon \Omega D \to D'$.*

Conversely, let (C'', φ'') be the connected n-dimensional ε-symmetric complex over A obtained from a complex (C, φ) by surgery on a pair

$$(f'': C \to D'', (\delta\varphi'', \varphi))$$

such that $D'' = C(g)$ is the algebraic mapping cone of a chain map $g: \Omega D \to D'$, for some $(n+1)$-dimensional A-module chain complexes D, D'. Then (C'', φ'') is isomorphic to the complex $(\tilde{C}'', \tilde{\varphi}'')$ obtained from (C', φ') by surgery on a pair $(f': C' \to D', (\delta\varphi', \varphi'))$, with (C', φ') the complex obtained from (C, φ) by surgery on a pair $(f: C \to D, (\delta\varphi, \varphi))$.

The ε-quadratic case is similar, with ψ's in place of φ's.

Proof (ε-symmetric case only). Given

$$(f: C \to D, (\delta\varphi, \varphi)), \quad (f': C' \to D', (\delta\varphi', \varphi'))$$

write

$$f' = (\tilde{f}' \quad g \quad \nu_0): C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to D'_r,$$

and define $(f'': C \to D'', (\delta\varphi'', \varphi) \in Q^{n+1}(f'', \varepsilon))$ by

$$f'' = \begin{pmatrix} f \\ \tilde{f}' \end{pmatrix}: C_r \to D''_r = D_r \oplus D'_r,$$

$$d'' = \begin{pmatrix} d & 0 \\ (-)^{r-1}g & d' \end{pmatrix}: D''_r = D_r \oplus D'_r \to D''_{r-1} = D_{r-1} \oplus D'_{r-1},$$

$$\delta\varphi''_s = \begin{pmatrix} \delta\varphi_s & (-)^s f(T_s\varphi_{s+1})\tilde{f}'^* - (T_s\delta\varphi_{s+1})g^* \\ 0 & \delta\varphi'_s \end{pmatrix}:$$

$$D''^{n-r+s+1} = D^{n-r+s+1} \oplus D'^{n-r+s+1} \to D''_r = D_r \oplus D'_r \quad (s \geqslant 0).$$

Conversely, given $(f'': C \to D'', (\delta\varphi'', \varphi)), g: \Omega D \to D'$ write

$$f'' = \begin{pmatrix} f \\ \tilde{f}' \end{pmatrix}: C_r \to D''_r = C(g)_r = D_r \oplus D'_r,$$

$$\delta\varphi''_s = \begin{pmatrix} \delta\varphi_s & \tilde{\nu}_s \\ \nu_s & \delta\varphi'_s \end{pmatrix}: D''^{n-r+s+1} = D^{n-r+s+1} \oplus D'^{n-r+s+1} \to D''_r = D_r \oplus D'_r \quad (s \geqslant 0),$$

and define $(f': C' \to D', (\delta\varphi', \varphi') \in Q^{n+1}(f', \varepsilon))$ by

$$f' = (\tilde{f}' \quad g \quad \nu_0): C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to D'_r,$$

$$\delta\varphi'_s = \widetilde{\delta\varphi'_s} + (-)^{r(n-r+s+1)}\varepsilon g \nu^*_{s+1}: D'^{n-r+s+1} \to D'_r \quad (s \geqslant 0).$$

The A-module isomorphisms

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & (-)^{r(n-r+1)}\varepsilon\nu_1^* \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}:$$

$$\tilde{C}''_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \oplus D'_{r+1} \oplus D'^{n-r+1}$$
$$\to C''_r = C_r \oplus D_{r+1} \oplus D'_{r+1} \oplus D^{n-r+1} \oplus D'^{n-r+1}$$

define an isomorphism of ε-symmetric complexes over A,

$$h\colon (\tilde{C}'', \tilde{\varphi}'') \to (C'', \varphi'').$$

5. Witt groups

We shall now identify the ε-symmetric {even ε-symmetric, ε-quadratic} L-group

$$L^0(A, \varepsilon) \quad \{L\langle v_0\rangle^0(A, \varepsilon), L_0(A, \varepsilon)\}$$

(respectively $L^1(A, \varepsilon)$ $\{L\langle v_0\rangle^1(A, \varepsilon), L_1(A, \varepsilon)\}$) with the Witt group of non-singular ε-symmetric {even ε-symmetric, ε-quadratic} forms (respectively formations) over A. It will then follow from the 4-periodicity $L_n(A, \varepsilon) = L_{n+4}(A, \varepsilon)$ that the quadratic L-groups $L_n(A)$ agree with the surgery obstruction groups of Wall [**25**]. We shall use this characterization of $L\langle v_0\rangle^1(A, \varepsilon)$ $\{L_1(A, \varepsilon)\}$ to prove that a non-singular even ε-symmetric {ε-quadratic} formation represents 0 in the Witt group if and only if it is stably isomorphic to the graph formation of a singular $(-\varepsilon)$-symmetric {even $(-\varepsilon)$-symmetric} form. This generalizes the 'normal form' for the split unitary group of Sharpe [**22**] (cf. Proposition 9.2 below).

The *Witt group of ε-symmetric {even ε-symmetric, ε-quadratic} forms over A* $L^\varepsilon(A)$ $\{L\langle v_0\rangle^\varepsilon(A), L_\varepsilon(A)\}$ is the abelian group of equivalence classes of non-singular ε-symmetric {even ε-symmetric, ε-quadratic} forms over A (M, φ) subject to the relation

$(M, \varphi) \sim (M', \varphi')$ if there exists an isomorphism

$$f\colon (M, \varphi) \oplus (N, \theta) \to (M', \varphi') \oplus (N', \theta')$$

for some hyperbolic forms $(N, \theta), (N', \theta')$.

Addition is by

$$(M, \varphi) + (M', \varphi') = (M \oplus M', \varphi \oplus \varphi').$$

Inverses are given by

$$-(M, \varphi) = (M, -\varphi),$$

since the diagonal $\Delta = \{(x,x) \in M \oplus M \mid x \in M\}$ is a lagrangian of $(M \oplus M,\ \varphi \oplus -\varphi)$. There are defined forgetful maps

$$L_\varepsilon(A) \to L\langle v_0\rangle^\varepsilon(A);\ (M, \psi \in Q_\varepsilon(M)) \mapsto (M, \psi + \varepsilon\psi^* \in Q\langle v_0\rangle^\varepsilon(M)),$$

$$L\langle v_0\rangle^\varepsilon(A) \to L^\varepsilon(A);\ (M, \varphi \in Q\langle v_0\rangle^\varepsilon(M)) \mapsto (M, \varphi \in Q^\varepsilon(M))$$

(such that $L_\varepsilon(A) \to L\langle v_0\rangle^\varepsilon(A)$ is onto).

PROPOSITION 5.1. *There are natural identifications of abelian groups*

$$\begin{cases} L^0(A,\varepsilon) = L^\varepsilon(A), \\ L\langle v_0\rangle^0(A,\varepsilon) = L\langle v_0\rangle^\varepsilon(A), \\ L_0(A,\varepsilon) = L_\varepsilon(A). \end{cases}$$

Proof. In view of Propositions 2.1, 2.2, and 4.6 it is sufficient to observe that if $(D, \nu \in Q^1(D,\varepsilon))$ $\{(D, \chi \in Q_1(D,\varepsilon))\}$ is a well-connected 1-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex then

$$\begin{cases} \nu \in Q^1(D,\varepsilon) = Q^\varepsilon(H^1(D)), \\ \chi \in Q_1(D,\varepsilon) = 0, \end{cases}$$

and the boundary 0-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex $\partial(D,\nu)$ $\{\partial(D,\chi)\}$ corresponds to the standard hyperbolic ε-symmetric $\{\varepsilon$-quadratic$\}$ form $H^\varepsilon(H^1(D),\nu)$ $\{H_\varepsilon(H_1(D))\}$. Moreover, if (D,ν) is an even ε-symmetric complex then $(H^1(D),\nu)$ is an even ε-symmetric form and $H^\varepsilon(H^1(D),\nu)$ is isomorphic to the standard hyperbolic even ε-symmetric form $H^\varepsilon(H_1(D))$, with $Q\langle v_0\rangle^1(D,\varepsilon) = Q^\varepsilon(H^1(D))$.

The *Witt group of ε-symmetric $\{$even ε-symmetric, ε-quadratic$\}$ formations over A* $M^\varepsilon(A)$ $\{M\langle v_0\rangle^\varepsilon(A), M_\varepsilon(A)\}$ is the abelian group of equivalence classes of non-singular ε-symmetric $\{$even ε-symmetric, ε-quadratic$\}$ formations over A $(M,\varphi;\ F,G)$ subject to the relation

$$(M,\varphi;\ F,G) \sim (M',\varphi';\ F',G')$$

if there exists a stable isomorphism of the type

$$\begin{aligned} [f]\colon &(M,\varphi;\ F,G) \oplus (N,\theta;\ H,K) \oplus (N,\theta;\ K,L) \oplus (N',\theta';\ H',L') \\ &\to (M',\varphi';\ F',G') \oplus (N',\theta';\ H',K') \oplus (N',\theta';\ K',L') \oplus (N,\theta;\ H,L), \end{aligned}$$

with addition and inverses by

$$(M,\varphi;\ F,G) + (M',\varphi';\ F',G') = (M \oplus M',\ \varphi \oplus \varphi';\ F \oplus F',\ G \oplus G'),$$

$$-(M,\varphi;\ F,G) = (M,\varphi;\ G,F).$$

There are defined forgetful maps

$$M_\varepsilon(A) \to M\langle v_0\rangle^\varepsilon(A);$$
$$(M, \psi \in Q_\varepsilon(M);\ F, G) \mapsto (M, \psi + \varepsilon\psi^* \in Q\langle v_0\rangle^\varepsilon(M);\ F, G),$$
$$M\langle v_0\rangle^\varepsilon(A) \to M^\varepsilon(A);$$
$$(M, \varphi \in Q\langle v_0\rangle^\varepsilon(M);\ F, G) \mapsto (M, \varphi \in Q^\varepsilon(M);\ F, G).$$

PROPOSITION 5.2. *There are natural identifications of abelian groups*

$$\begin{cases} L^1(A,\varepsilon) = M^\varepsilon(A), \\ L\langle v_0\rangle^1(A,\varepsilon) = M\langle v_0\rangle^\varepsilon(A), \\ L_1(A,\varepsilon) = M_\varepsilon(A). \end{cases}$$

Proof. Let (C,φ), (C',φ'), (C'',φ'') be the 1-dimensional ε-symmetric Poincaré complexes associated by Proposition 2.3 to non-singular ε-symmetric formations $(N,\theta;\ H,K)$, $(N,\theta;\ K,L)$, $(N,\theta;\ H,L)$. We have to prove that

$$(C,\varphi)\oplus(C',\varphi') = (C'',\varphi'') \in L^1(A,\varepsilon),$$

corresponding to the generic sum formula in the Witt group

$$(N,\theta;\ H,K)\oplus(N,\theta;\ K,L) = (N,\theta;\ H,L) \in M^\varepsilon(A),$$

in order to verify that there is a well-defined morphism of abelian groups

$$M^\varepsilon(A) \to L^1(A,\varepsilon);\ (N,\theta;\ H,K) \mapsto (C,\varphi).$$

Choosing a chain homotopy inverse $\varphi_0{}^{-1}\colon C \to C^{1-*}$ for $\varphi_0\colon C^{1-*} \to C$ define a $\mathbf{Z}_2$-hypercohomology class

$$\tilde\varphi = (\varphi_0{}^{-1})^\%(\varphi) \in Q^1(C^{1-*},\varepsilon),$$

so that there is defined a homotopy equivalence of 1-dimensional ε-symmetric Poincaré complexes over A

$$\varphi_0\colon (C^{1-*},\tilde\varphi) \to (C,\varphi).$$

(In fact $(C^{1-*},\tilde\varphi)$ corresponds to the formation $(N, -\theta;\ K,H)$, which is thus stably isomorphic to $(N,\theta;\ H,K)$ with an isomorphism

$$(N,\theta;\ H,K)\oplus(H^\varepsilon(K);\ K,K^*) \to (N,-\theta;\ K,H)\oplus(H^\varepsilon(H);\ H,H^*).$$

Thus inverses in the Witt group are also given by

$$-(N,\theta;\ H,K) = (N,-\theta;\ H,K) \in M^\varepsilon(A).)$$

Define a chain map $(f\ \ f')\colon C^{1-*}\oplus C' \to D$ by

$$\begin{array}{ccccccccccc} C^{1-*}\oplus C'\colon \dots & \longrightarrow & 0 & \longrightarrow & K\oplus K & \longrightarrow & H^*\oplus L^* & \longrightarrow & 0 & \longrightarrow & \dots \\ \Big\downarrow{\scriptstyle (f\ \ f')} & & & & \Big\downarrow{\scriptstyle (1\ \ 1)} & & \Big\downarrow & & & & \\ D\colon \dots & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

Now (C'',φ'') is homotopy equivalent to the 1-dimensional ε-symmetric Poincaré complex obtained from $(C^{1-*},\tilde{\varphi})\oplus(C',\varphi')$ by surgery on the connected 2-dimensional ε-symmetric pair

$$((f\ \ f')\colon C^{1-*}\oplus C'\to D,(0,\tilde{\varphi}\oplus\varphi')\in Q^2((f\ \ f'),\varepsilon)),$$

and so

$$(C,\varphi)\oplus(C',\varphi')=(C^{1-*},\tilde{\varphi})\oplus(C',\varphi')=(C'',\varphi'')\in L^1(A,\varepsilon)$$

by Proposition 4.1(i).

The correspondence of Proposition 2.3 can also be used to define a morphism

$$L^1(A,\varepsilon)\to M^{\varepsilon}(A);\ (C,\varphi)\mapsto(N,\theta;\,H,K)$$

inverse to $M^{\varepsilon}(A)\to L^1(A,\varepsilon)$. This is well defined provided we can show that the non-singular ε-symmetric formation $(N,\theta;\,H,K)$ associated to the boundary $\partial(D,\nu)$ of a well-connected 2-dimensional ε-symmetric complex $(D,\nu\in Q^2(D,\varepsilon))$ represents 0 in the Witt group

$$(N,\theta;\,H,K)=0\in M^{\varepsilon}(A)$$

(applying Proposition 4.6). Without loss of generality, it may be assumed that D is a f.g. projective A-module chain complex of the type

$$D\colon\ \dots\longrightarrow 0\longrightarrow D_2\xrightarrow{\ d\ }D_1\longrightarrow 0\longrightarrow\dots,$$

so that a cycle $\nu\in\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W,\mathrm{Hom}_A(D^*,D))_2$ is represented by A-module morphisms

$$\nu_0\colon D^1\to D_1,\quad \nu_1\colon D^1\to D_2,\quad \tilde{\nu}_1\colon D^2\to D_1,\quad \nu_2\colon D^2\to D_2$$

such that

$$\begin{aligned}\nu_0+\varepsilon\nu_0^*+d\nu_1-\tilde{\nu}_1d^*&=0\colon D^1\to D_1,\\ \nu_1+\varepsilon\tilde{\nu}_1^*-\nu_2d^*&=0\colon D^1\to D_2,\\ \tilde{\nu}_1+\varepsilon\nu_1^*-d\nu_2&=0\colon D^2\to D_1,\\ \nu_2-\varepsilon\nu_2^*&=0\colon D^2\to D_2.\end{aligned}$$

The boundary 1-dimensional ε-symmetric Poincaré complex $\partial(D,\nu)$ corresponds to the non-singular ε-symmetric formation

$$\begin{aligned}(N,\theta;\,H,K)=\Bigg(&H^{\varepsilon}\left(D_1\oplus D^2,\begin{pmatrix}0&0\\0&-\nu_2\end{pmatrix}\right);\,D^1\oplus D_2,\\ &\mathrm{im}\left(\begin{pmatrix}\bar{\varepsilon}&0\\ \tilde{\nu}_1^*&1\\ -\nu_0^*&-d\\ d^*&0\end{pmatrix}\colon D^1\oplus D_2\to D^1\oplus D_2\oplus D_1\oplus D^2\right)\Bigg).\end{aligned}$$

The A-module automorphism

$$f = \begin{pmatrix} \bar{\varepsilon} & 0 & 0 & 0 \\ \tilde{\nu}_1^* & 1 & 0 & 0 \\ -\nu_0^* & -d & \bar{\varepsilon} & \nu_1^* \\ d^* & 0 & 0 & 1 \end{pmatrix} : N = D^1 \oplus D_2 \oplus D_1 \oplus D^2 \to D^1 \oplus D_2 \oplus D_1 \oplus D^2$$

defines an isomorphism of non-singular ε-symmetric formations over A

$$f\colon (N, \theta; L, H) \to (N, \theta; L, K),$$

where $L = D_2 \oplus D_1 \subseteq N = D^1 \oplus D_2 \oplus D_1 \oplus D^2$. It follows that

$$\begin{aligned}(N, \theta; H, K) &= (N, \theta; H, L) \oplus (N, \theta; L, K) \\ &= (N, \theta; H, L) \oplus (N, \theta; L, H) = 0 \in M^\varepsilon(A).\end{aligned}$$

Restricting attention to even ε-symmetric complexes and formations we obtain in the same way inverse isomorphisms

$$M\langle v_0\rangle^\varepsilon(A) \to L\langle v_0\rangle^1(A, \varepsilon), \quad L\langle v_0\rangle^1(A, \varepsilon) \to M\langle v_0\rangle^\varepsilon(A).$$

Similarly, the correspondence of Proposition 2.5 can be used to define an abelian group morphism

$$L_1(A, \varepsilon) \to M_\varepsilon(A);\ (C, \psi) \mapsto (N, \theta; H, K),$$

with $(N, \theta; H, K)$ the non-singular ε-quadratic formation associated to a 1-dimensional ε-quadratic Poincaré complex (C, ψ). In order to prove that this is an isomorphism we need the ε-quadratic case of the following result. (The even ε-symmetric case is required later on.)

LEMMA. *A 1-dimensional even ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex over A $(C, \varphi \in Q\langle v_0\rangle^1(C, \varepsilon))$ $\{(C, \psi \in Q_1(C, \varepsilon))\}$ represents 0 in $L\langle v_0\rangle^1(A, \varepsilon)$ $\{L_1(A, \varepsilon)\}$ if and only if it is homotopy equivalent to the boundary $\partial(D, \nu)$ $\{\partial(D, \chi)\}$ of a connected 2-dimensional even ε-symmetric $\{\varepsilon$-quadratic$\}$ complex over A $(D, \nu \in Q\langle v_0\rangle^2(D, \varepsilon))$ $\{(D, \chi \in Q_2(D, \varepsilon))\}$ such that*

$$H_0(D) = 0, \quad H^2(D) = 0.$$

Proof. In view of Propositions 3.4(iii) and 4.1(i) it is sufficient to observe that for any connected 2-dimensional even ε-symmetric $\{\varepsilon$-quadratic$\}$ complex over A (D, ν) $\{(D, \chi)\}$ there is defined a connected 3-dimensional even ε-symmetric $\{\varepsilon$-quadratic$\}$ pair over A

$$\begin{cases} (f\colon D \to E, (0, \nu) \in Q\langle v_0\rangle^3(f, \varepsilon)) \\ (f\colon D \to E, (0, \chi) \in Q_3(f, \varepsilon)) \end{cases}$$

with

$$\begin{tikzcd}
D\colon \dots \arrow[r] \arrow[d, "f"'] & D_3 \arrow[r, "d"] \arrow[d, "1"'] & D_2 \arrow[r, "d"] \arrow[d, "1"'] & D_1 \arrow[r, "d"] \arrow[d] & D_0 \arrow[r] \arrow[d] & \dots \\
E\colon \dots \arrow[r] & D_3 \arrow[r, "d"] & D_2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

such that surgery on $(f\colon D \to E, (0,\nu))$ $\{(f\colon D \to E, (0,\chi))\}$ results in a connected 2-dimensional even ε-symmetric $\{\varepsilon$-quadratic$\}$ complex over A (D',ν') $\{(D',\chi')\}$ such that

$$H_0(D') = 0, \quad H^2(D') = 0.$$

Every non-singular ε-quadratic formation over A is isomorphic to one of the type $(H_\varepsilon(F); F, G)$, and choosing a hessian for G we obtain a non-singular split ε-quadratic formation over A (F, G), and hence (by Proposition 2.5) a 1-dimensional ε-quadratic Poincaré complex over A (C,ψ). It follows that $L_1(A,\varepsilon) \to M_\varepsilon(A)$ is onto. In order to show that this morphism is also one–one recall from Ranicki [**13**] the characterization of $M_\varepsilon(A)$ as the abelian group with respect to the direct sum $\oplus$ of equivalence classes of non-singular ε-quadratic formations over A $(M,\psi; F, G)$ subject to the relation

$$(M,\psi; F, G) \sim (M',\psi'; F', G')$$

if there exists a stable isomorphism

$$[f]\colon (M,\psi; F,G) \oplus \partial(N,\theta) \to (M',\psi'; F',G') \oplus \partial(N',\theta')$$

for some even $(-\varepsilon)$-symmetric forms over A $(N,\theta), (N',\theta')$ with $\partial(N,\theta)$ defined by

$$\partial(N,\theta) = (H_\varepsilon(N); N, \{(x,\theta(x)) \in N \oplus N^* \mid x \in N\}).$$

(Only the case where $\varepsilon = \pm 1 \in A$ was considered there, but the methods apply for all $\varepsilon \in A$.) Translating the result of the lemma into the language of forms and formations (using Propositions 2.1 and 2.5) we have that a 1-dimensional ε-quadratic Poincaré complex over A (C,ψ) represents 0 in $L_1(A,\varepsilon)$ if and only if the associated non-singular split ε-quadratic formation over A (F,G) is stably isomorphic to $\partial(N,\chi)$ for some $(-\varepsilon)$-quadratic form over A $(N, \chi \in Q_{-\varepsilon}(N))$, where

$$\partial(N,\chi) = \left(N, \left(\begin{pmatrix} 1 \\ \chi - \varepsilon\chi^* \end{pmatrix}, \chi\right) N\right).$$

By Proposition 2.4 this occurs precisely when the underlying non-singular ε-quadratic formation $(H_\varepsilon(F); F, G)$ is isomorphic to $\partial(N,\theta)$ for some even $(-\varepsilon)$-symmetric form $(N, \theta \in Q\langle v_0\rangle^{-\varepsilon}(N))$. It follows that $L_1(A, \varepsilon) \to M_\varepsilon(A)$ is one-to-one as well as onto, and hence an isomorphism.

The *boundary* of an (even) ε-symmetric $\{\varepsilon$-quadratic, split ε-quadratic$\}$ formation over A $(M, \varphi; F, G)$ $\{(M, \psi; F, G), (F, G)\}$ is the non-singular (even) ε-symmetric $\{\varepsilon$-quadratic, ε-quadratic$\}$ form over A

$$\begin{cases} \partial(M, \varphi; F, G) = (G^\perp/G, \varphi^\perp/\varphi), \\ \partial(M, \psi; F, G) = (G^\perp/G, \psi^\perp/\psi), \\ \partial(F, G) = (G^\perp/G, \psi^\perp/\psi)\ (= \partial(H_\varepsilon(F); F, G)). \end{cases}$$

The *boundary* of an ε-symmetric $\{$even ε-symmetric, ε-quadratic$\}$ form over A $(M, \varphi \in Q^\varepsilon(M))$ $\{(M, \psi + \varepsilon\psi^* \in Q\langle v_0\rangle^\varepsilon(M)), (M, \psi \in Q_\varepsilon(M))\}$ is the non-singular even $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic, split $(-\varepsilon)$-quadratic$\}$ formation over A

$$\begin{cases} \partial(M, \varphi) = (H^{-\varepsilon}(M);\, M, \{(x, \varphi(x)) \in M \oplus M^* \mid x \in M\}), \\ \partial(M, \psi + \varepsilon\psi^*) = (H_{-\varepsilon}(M);\, M, \{(x, (\psi + \varepsilon\psi^*)(x)) \in M \oplus M^* \mid x \in M\}), \\ \partial(M, \psi) = \left(M, \left(\begin{pmatrix} 1 \\ \psi + \varepsilon\psi^* \end{pmatrix}, \psi\right) M\right). \end{cases}$$

An n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complex $(C, \varphi \in Q^n(C, \varepsilon))$ $\{(C, \psi \in Q_n(C, \varepsilon))\}$ is *highly-connected* if:

$$H_r(C) = 0 \text{ for } r < i, \quad H^r(C) = 0 \text{ for } r > n - i,$$

with $n = 2i$ or $2i+1$, and for $n = 2i+1$ also

$$\begin{cases} H_i(\varphi_0\colon C^{2i+1-*} \to C) = 0, \\ H_i((1 + T_\varepsilon)\psi_0\colon C^{2i+1-*} \to C) = 0. \end{cases}$$

Then (C, φ) $\{(C, \psi)\}$ is connected, and the boundary $\partial(C, \varphi)$ $\{\partial(C, \psi)\}$ is a highly-connected $(n-1)$-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complex.

PROPOSITION 5.3. *The homotopy equivalence classes of highly-connected n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ complexes over A are in a natural one-to-one correspondence for $n = 2i$ (respectively $n = 2i+1$) with the isomorphism classes of $(-)^i\varepsilon$-symmetric $\{(-)^i\varepsilon$-quadratic$\}$ forms over A (respectively the stable isomorphism classes of $(-)^i\varepsilon$-symmetric $\{$split $(-)^i\varepsilon$-quadratic$\}$ formations over A). Highly-connected Poincaré complexes correspond to non-singular forms (respectively formations). The boundary operation on highly-connected complexes corresponds to the boundary operation on forms (respectively formations).*

Proof. The proof is immediate from Propositions 2.1, 2.3, and 2.5.

PROPOSITION 5.4. (i) *A non-singular* ε*-symmetric* {*even* ε*-symmetric,* ε*-quadratic*} *form over* A *represents* 0 *in the Witt group* $L^{\varepsilon}(A) = L^0(A,\varepsilon)$ $\{L\langle v_0\rangle^{\varepsilon}(A) = L\langle v_0\rangle^0(A,\varepsilon), L_{\varepsilon}(A) = L_0(A,\varepsilon)\}$ *if and only if it is isomorphic to the boundary* $\partial(M,\varphi;\, F,G)$ $\{\partial(M,\psi+\varepsilon\psi^*;\, F,G), \partial(M,\psi;\, F,G)\}$ *of an* ε*-symmetric* {*even* ε*-symmetric,* ε*-quadratic*} *formation over* A

$$\begin{cases} (M,\varphi \in Q^{\varepsilon}(M);\, F,G), \\ (M,\psi+\varepsilon\psi^* \in Q\langle v_0\rangle^{\varepsilon}(M);\, F,G), \\ (M,\psi \in Q_{\varepsilon}(M);\, F,G). \end{cases}$$

(ii) *A non-singular even* ε*-symmetric* {ε*-quadratic*} *formation over* A *represents* 0 *in the Witt group* $M\langle v_0\rangle^{\varepsilon}(A) = L\langle v_0\rangle^1(A,\varepsilon)$ $\{M_{\varepsilon}(A) = L_1(A,\varepsilon)\}$ *if and only if it is stably isomorphic to the boundary* $\partial(M,\varphi)$ $\{\partial(M,\psi-\varepsilon\psi^*)\}$ *of an* $(-\varepsilon)$*-symmetric* {*even* $(-\varepsilon)$*-symmetric*} *form over* A

$$(M,\varphi \in Q^{-\varepsilon}(M)) \quad \{(M,\psi-\varepsilon\psi^* \in Q\langle v_0\rangle^{-\varepsilon}(M))\}.$$

Proof. (i) This is immediate from Proposition 2.2.

(ii) This is immediate from Proposition 5.3 and the lemma used in the proof of Proposition 5.2.

6. Lower L-theory

There is algebraic evidence to suggest that the ε-symmetric L-groups $L^n(A,\varepsilon)$ $(n \geqslant 0)$ and the ε-quadratic L-groups $L_n(A,\varepsilon)$ $(n \geqslant 0)$ should be regarded as belonging to a single sequence of algebraic L-groups $\{L^n(A,\varepsilon) \mid n \in \mathbf{Z}\}$ (to be defined below), with many of the formal properties of the sequence of algebraic K-groups $\{K_n(A) \mid n \in \mathbf{Z}\}$. For example, given a morphism of rings with involution $f\colon A \to B$ there are defined relative L-groups $L^n(f,\varepsilon)$ $(n \in \mathbf{Z})$ to fit into a change of rings exact sequence

$$\ldots \longrightarrow L^n(A,\varepsilon) \xrightarrow{f} L^n(B,\varepsilon) \longrightarrow L^n(f,\varepsilon) \longrightarrow L^{n-1}(A,\varepsilon) \longrightarrow \ldots \qquad (n \in \mathbf{Z});$$

we shall deal with relative L-theory in a later part of the paper. At any rate, we shall find the lower L-groups useful in the remainder of this part. In § 8 below, we shall define products

$$\otimes\colon L^m(A,\varepsilon)\otimes_{\mathbf{Z}} L^n(B,\eta) \to L^{m+n}(A\otimes_{\mathbf{Z}} B,\, \varepsilon\otimes\eta) \quad (m,\, n \in \mathbf{Z}).$$

Define the *lower* ε*-quadratic* L*-groups of* A $L_n(A,\varepsilon)$ $(n \leqslant -1)$ by

$$L_n(A,\varepsilon) = L_{n+2i}(A,(-)^i\varepsilon) \quad (n+2i \geqslant 0),$$

extending the periodicity $L_n(A,\varepsilon) = L_{n+2}(A,-\varepsilon)$ $(n \geqslant 0)$ of Proposition 4.3.

Define the *lower ε-symmetric L-groups of A* $L^n(A,\varepsilon)$ $(n \leqslant -1)$ by

$$L^n(A,\varepsilon) = \begin{cases} L\langle v_0\rangle^{n+2}(A,-\varepsilon) & (n = -1,-2), \\ L_n(A,\varepsilon) & (n \leqslant -3). \end{cases}$$

Define the *skew-suspension* maps

$$\bar{S}: L^n(A,\varepsilon) \to L^{n+2}(A,-\varepsilon) \quad (n \leqslant -1)$$

to be the $\pm\varepsilon$-quadratic skew-suspension isomorphisms for $n \leqslant -5$, and the forgetful maps if $-4 \leqslant n \leqslant -1$.

PROPOSITION 6.1. *If A,ε are such that*

$$\hat{H}^0(\mathbf{Z}_2;\, A,\varepsilon) \equiv \{a \in A \mid \varepsilon\bar{a} = a\}/\{b+\varepsilon\bar{b} \mid b \in A\} = 0$$

then the skew-suspension maps

$$\bar{S}: L^n(A,-\varepsilon) \to L^{n+2}(A,\varepsilon) \quad (n \in \mathbf{Z})$$

are isomorphisms.

Proof. The case where $n \leqslant -5$ has already been considered in Proposition 4.3. For $-4 \leqslant n \leqslant -1$ note that since $\hat{H}^0(\mathbf{Z}_2;\, A,\varepsilon) = 0$ the category of even ε-symmetric (respectively even $(-\varepsilon)$-symmetric) forms {formations} over A is equivalent to the category of ε-symmetric (respectively $(-\varepsilon)$-quadratic) forms {formations} over A. For $n \geqslant 0$ note that every $(n+2)$-dimensional ε-symmetric complex over A $(C, \varphi \in Q^{n+2}(C,\varepsilon))$ is even, since

$$\hat{v}_0(\varphi) = 0: H^{n+2}(C) \to \hat{H}^{n+2}(\mathbf{Z}_2;\, A,(-)^{n+2}\varepsilon) = \hat{H}^0(\mathbf{Z}_2;\, A,\varepsilon) = 0,$$

and similarly for pairs. Therefore we can identify

$$L\langle v_0\rangle^{n+2}(A,\varepsilon) = L^{n+2}(A,\varepsilon) \quad (n \geqslant 0),$$

and the skew-suspension maps

$$\bar{S}: L^n(A,-\varepsilon) \to L^{n+2}(A,\varepsilon) \quad (n \geqslant 0)$$

are isomorphisms by Proposition 4.4.

We shall need the following result in the computations of § 10.

PROPOSITION 6.2. (i) *The even ε-symmetrization map of Witt groups*

$$1+T_\varepsilon: L_0(A,\varepsilon) \to L\langle v_0\rangle^0(A,\varepsilon)$$

is onto, with kernel generated by the non-singular ε-quadratic forms over A of the type

$$\left(A \oplus A^*, \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \in Q_\varepsilon(A \oplus A^*)\right)$$

with $a,b \in \hat{H}^1(\mathbf{Z}_2;\, A,\varepsilon) = \{x \in A \mid x+\varepsilon\bar{x} = 0\}/\{y-\varepsilon\bar{y} \mid y \in A\}$.

(ii) *Let $A = \mathbf{Z}[\pi]$ be the group ring of a group π, with involution by $\bar{g} = g^{-1}$ $(g \in \pi)$. The skew-suspension maps*

$$\bar{S}: L^n(\mathbf{Z}[\pi]) \to L^{n+2}(\mathbf{Z}[\pi], -1) \quad (n \in \mathbf{Z})$$

are isomorphisms, and the skew-symmetrization map

$$1+T_{-1}: L_0(\mathbf{Z}[\pi], -1) \to L^0(\mathbf{Z}[\pi], -1) = L^{-2}(\mathbf{Z}[\pi])$$

is onto. If π has no 2-torsion there is defined a split short exact sequence

$$0 \longrightarrow L_0(\mathbf{Z}, -1) \longrightarrow L_0(\mathbf{Z}[\pi], -1) \xrightarrow{1+T_{-1}} L^0(\mathbf{Z}[\pi], -1) \longrightarrow 0.$$

Proof. (i) The map $L_0(A, \varepsilon) \to L\langle v_0\rangle^0(A, \varepsilon)$ is onto because every even ε-symmetric form is the ε-symmetrization of an ε-quadratic form. An element of the kernel is represented by a non-singular ε-quadratic form over A $(M, \psi \in Q_\varepsilon(M))$ such that

$$\psi + \varepsilon\psi^* = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}: M = L \oplus L^* \to M^* = L^* \oplus L$$

for a f.g. free A-module L, and (M, ψ) can be expressed as a direct sum of forms like $\left(A \oplus A^*, \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}\right)$.

(ii) This is immediate from Proposition 6.1 and (i), since

$$\hat{H}^0(\mathbf{Z}_2; \mathbf{Z}[\pi], -1) = 0$$

for any group π, and $\hat{H}^1(\mathbf{Z}_2; \mathbf{Z}[\pi], -1) = \hat{H}^1(\mathbf{Z}_2; \mathbf{Z}, -1)$ $(= \mathbf{Z}_2)$ if π has no 2-torsion. $\Big(L_0(\mathbf{Z}, -1) = \mathbf{Z}_2$, generated by the Arf form

$$\left(\mathbf{Z} \oplus \mathbf{Z}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in Q_-(\mathbf{Z} \oplus \mathbf{Z})\right).\Big)$$

7. Dedekind rings

In a later part of this paper we shall describe the L-theory exact sequence of a localization map of rings with involution $A \to S^{-1}A$ inverting a multiplicative subset S of A,

$$\ldots \to L^n(A, \varepsilon) \to L^n(S^{-1}A, \varepsilon) \to L^n(A, S, \varepsilon) \to L^{n-1}(A, \varepsilon) \to \ldots \quad (n \in \mathbf{Z}),$$

in which the relative terms $L^*(A, S, \varepsilon)$ are cobordism groups of algebraic Poincaré complexes over A which become contractible over $S^{-1}A$. In particular, such a localization sequence can be used to study the L-groups of a Dedekind ring A. Here, we shall only develop the L-theory of

Dedekind rings sufficiently far to compute $L^*(\mathbf{Z})$ and $L_*(\mathbf{Z})$, by reducing the computation to the well-known stable classification of symmetric and quadratic forms on finitely generated abelian groups. It should be noted that a Dedekind ring A is noetherian of global dimension 1, so that the skew-suspension maps

$$\bar{S}\colon L^n(A,\varepsilon) \to L^{n+2}(A,-\varepsilon) \quad (n \geqslant 0)$$

are isomorphisms (Proposition 4.5) and the ε-symmetric L-groups are 4-periodic

$$L^n(A,\varepsilon) = L^{n+4}(A,\varepsilon) \quad (n \geqslant 0).$$

Let A be a Dedekind ring, with quotient field $\mathbf{A} = (A-\{0\})^{-1}A$. Let M be a f.g. torsion A-module, that is a f.g. A-module such that

$$sM = 0$$

for some $s \in A-\{0\}$. Define the *Pontrjagin dual* of M to be the f.g. torsion A-module

$$M\hat{\ } = \mathrm{Hom}_A(M, \mathbf{A}/A)$$

with A acting by

$$A \times M\hat{\ } \to M\hat{\ };\ (a,f) \mapsto (x \mapsto f(x)\bar{a}),$$

so that

$$\bar{s}M\hat{\ } = 0.$$

The natural A-module isomorphism

$$M \to M\hat{\ }\hat{\ };\ x \mapsto (f \mapsto \overline{f(x)})$$

will be used as an identification. Given another f.g. torsion A-module N define Pontrjagin duality for A-module morphisms

$$\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(N\hat{\ }, M\hat{\ });\ f \mapsto (f\hat{\ }\colon g \mapsto (x \mapsto g(f(x)))).$$

Define the *ε-transposition* involution

$$T_\varepsilon\colon \mathrm{Hom}_A(M, M\hat{\ }) \to \mathrm{Hom}_A(M, M\hat{\ });\ \varphi \mapsto (\varepsilon\varphi\hat{\ }\colon x \mapsto (y \mapsto \varepsilon.\overline{\varphi(y)(x)})).$$

An *ε-symmetric linking form over A* (M,λ) is a f.g. torsion A-module M together with an element $\lambda \in \ker(1-T_\varepsilon\colon \mathrm{Hom}_A(M,M\hat{\ }) \to \mathrm{Hom}_A(M,M\hat{\ }))$, corresponding to an ε-symmetric pairing

$$\lambda\colon M \times M \to \mathbf{A}/A;\ (x,y) \mapsto \lambda(x)(y).$$

The *Wu class* of (M,λ) is the quadratic function

$$v_0(M,\lambda)\colon M \to H^1(\mathbf{Z}_2; A, \varepsilon);\ x \mapsto u - \varepsilon\bar{u}$$
$$(u \in \mathbf{A},\ \lambda(x)(x) = [u] \in \mathbf{A}/A).$$

An ε-symmetric linking form (M,λ) is *even* if $v_0(M,\lambda) = 0$, that is if each $\lambda(x)(x) \in \mathbf{A}/A$ $(x \in M)$ has a representative $u \in \mathbf{A}$ such that $\varepsilon\bar{u} = u$. An *ε-quadratic linking form over A* (M,λ,μ) is an even ε-symmetric linking form (M, λ) together with a function

$$\mu \colon M \to \mathbf{A}/\{a + \varepsilon\bar{a} \mid a \in A\}$$

such that

$$\mu(x+y) - \mu(x) - \mu(y) = \lambda(x)(y) + \varepsilon\overline{\lambda(x)(y)} \in \mathbf{A}/\{a+\varepsilon\bar{a} \mid a \in A\},$$
$$[\mu(x)] = \lambda(x)(x) \in \mathbf{A}/A,$$
$$\mu(bx) = b\mu(x)\bar{b} \in \mathbf{A}/\{a+\varepsilon\bar{a} \mid a \in A\} \quad (x, y \in M,\ b \in A).$$

An ε-symmetric $\{\varepsilon$-quadratic$\}$ linking form over A (M, λ) $\{(M, \lambda, \mu)\}$ is *non-singular* if $\lambda \in \mathrm{Hom}_A(M, M\hat{\ })$ is an isomorphism. A non-singular ε-symmetric $\{\varepsilon$-quadratic$\}$ linking form over A (M, λ) $\{(M, \lambda, \mu)\}$ is *hyperbolic* if there exists a submodule L of M such that

$$\lambda(L)(L) = 0 \quad \{\lambda(L)(L) = 0,\ \mu(L) = 0\},$$

and also

$$L = \ker(M \to L\hat{\ };\ x \mapsto (y \mapsto \lambda(x)(y))).$$

An *isomorphism* of ε-symmetric $\{\varepsilon$-quadratic$\}$ linking forms over A

$$\begin{cases} f \colon (M, \lambda) \to (M', \lambda') \\ f \colon (M, \lambda, \mu) \to (M', \lambda', \mu') \end{cases}$$

is an A-module isomorphism $f \in \mathrm{Hom}_A(M, M')$ such that

$$\lambda'(f(x))(f(y)) = \lambda(x)(y) \in \mathbf{A}/A \quad (x, y \in M)$$

and in the ε-quadratic case also

$$\mu'(f(x)) = \mu(x) \in \mathbf{A}/\{a+\varepsilon\bar{a} \mid a \in A\} \quad (x \in M).$$

An ε-symmetric form over A $(M, \varphi \in Q^{\varepsilon}(M))$ is *non-degenerate* if $\varphi \in \mathrm{Hom}_A(M, M^*)$ is one-to-one.

The *boundary* of a non-degenerate ε-symmetric $\{$even ε-symmetric$\}$ form over A $(M, \varphi \in Q^{\varepsilon}(M))$ $\{(M, \varphi \in Q\langle v_0\rangle^{\varepsilon}(M))\}$ is the non-singular even ε-symmetric $\{\varepsilon$-quadratic$\}$ linking form over A

$$\begin{cases} \partial(M, \varphi) = (\partial M, \lambda) \\ \partial(M, \varphi) = (\partial M, \lambda, \mu) \end{cases}$$

defined by

$$\partial M = \mathrm{coker}(\varphi \colon M \to M^*),$$
$$\lambda \colon \partial M \to \partial M\hat{\ };\ [x] \mapsto \left([y] \mapsto \frac{x(z)}{s}\right),$$
$$\mu \colon \partial M \to \mathbf{A}/\{a+\varepsilon\bar{a} \mid a \in A\};\ [y] \mapsto \frac{y(z)}{s}$$
$$(x, y \in M^*,\ z \in M,\ s \in A - \{0\},\ \varphi(z) = sy \in M^*).$$

It can be shown that there are natural one-to-one correspondences of equivalence classes

((even) ε-symmetric $\{\varepsilon$-quadratic$\}$ linking forms over A)

$\leftrightarrow$ (connected 1-dimensional (even) $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic$\}$ complexes over A which become contractible over $\mathbf{A}$)

$\leftrightarrow$ ((even) $(-\varepsilon)$-symmetric $\{(-\varepsilon)$-quadratic$\}$ formations over A $(M,\varphi;\,F,G)$ $\{(M,\psi;\,F,G)\}$ such that $F\cap G=\{0\}$ and $M/(F+G)$ is a torsion A-module)

with non-singular linking forms corresponding to Poincaré complexes and non-singular formations. The connection between linking forms and formations over a Dedekind ring A was first established (in the case where $A=\mathbf{Z}$) by Wall [**24**]. The boundary operations

$$\partial\colon (\text{non-degenerate forms}) \to (\text{non-singular linking forms})$$

agree with the boundary operations of § 5

$$\partial\colon (\text{forms}) \to (\text{non-singular formations}).$$

Given a finite-dimensional A-module chain complex C write $T_r(C)$ (respectively $T^r(C)$)

$$\{F_r(C)=H_r(C)/T_r(C) \text{ (respectively } F^r(C)=H^r(C)/T^r(C))\}$$

for the torsion submodule $\{$torsion-free quotient module$\}$ of $H_r(C)$ (respectively $H^r(C)$). The universal coefficient theorem gives natural A-module isomorphisms

$$T_r(C)\to T^{r+1}(C)\hat{\ }=\mathrm{Hom}_A(T^{r+1}(C),\mathbf{A}/A);\; x\mapsto \left(f\mapsto \overline{\frac{f(y)}{\bar{s}}}\right),$$

$$F_r(C)\to F^r(C)^*=\mathrm{Hom}_A(F^r(C),A);\; x\mapsto (g\mapsto \overline{g(x)})$$

$$(x\in C_r,\ y\in C_{r+1},\ s\in A-\{0\},\ sx=dy,\ f\in C^{r+1},\ g\in C^r).$$

PROPOSITION 7.1. (i) *The even-dimensional ε-symmetric $\{$even ε-symmetric, ε-quadratic$\}$ L-groups of a Dedekind ring A $L^{2k}(A,\varepsilon)$ $(k\geqslant 0)$ $\{L\langle v_0\rangle^{2k}(A,\varepsilon)$ $(k=0), L_{2k}(A,\varepsilon)$ $(k\geqslant 0)\}$ are isomorphic to the Witt groups*

$$L^{(-)^k\varepsilon}(A)\quad \{L\langle v_0\rangle^{(-)^k\varepsilon}(A), L_{(-)^k\varepsilon}(A)\}$$

of non-singular $(-)^k\varepsilon$-symmetric $\{$even $(-)^k\varepsilon$-symmetric, $(-)^k\varepsilon$-quadratic$\}$ forms over A. The cobordism class of a $2k$-dimensional ε-symmetric $\{$even ε-symmetric, ε-quadratic$\}$ Poincaré complex over A

$$(C,\varphi\in Q^{2k}(C,\varepsilon))\quad \{(C,\varphi\in Q\langle v_0\rangle^{2k}(C,\varepsilon)), (C,\psi\in Q_{2k}(C,\varepsilon))\}$$

corresponds to the Witt class of the non-singular $(-)^k\varepsilon$-symmetric $\{$even

$(-)^k\varepsilon$-symmetric, $(-)^k\varepsilon$-quadratic} form over A

$$\begin{cases} (F^k(C), \varphi_0 : F^k(C) \to F_k(C) = F^k(C)^*), \\ (F^k(C), \varphi_0), \\ (F^k(C), (1+T_\varepsilon)\psi_0, v^k(\psi) : F^k(C) \to H_0(\mathbf{Z}_2; A, (-)^k\varepsilon)). \end{cases}$$

(ii) *The odd-dimensional ε-symmetric {even ε-symmetric, ε-quadratic} L-groups of a Dedekind ring A $L^{2k+1}(A,\varepsilon)$ $(k \geqslant 0)$ $\{L\langle v_0\rangle^{2k+1}(A,\varepsilon)$ $(k=0)$, $L_{2k+1}(A,\varepsilon)$ $(k \geqslant 0)\}$ are isomorphic to the abelian groups with respect to the direct sum $\oplus$ of the equivalence classes of non-singular $(-)^{k+1}\varepsilon$-symmetric {even $(-)^{k+1}\varepsilon$-symmetric, $(-)^{k+1}\varepsilon$-quadratic} linking forms over A (M,λ) $\{(M,\lambda), (M,\lambda,\mu)\}$ subject to the equivalence relation $(M,\lambda) \sim (M',\lambda')$ $\{(M,\lambda) \sim (M',\lambda'), (M,\lambda,\mu) \sim (M',\lambda',\mu')\}$ if there exists an isomorphism*

$$\begin{cases} f : (M,\lambda) \oplus (M', -\lambda') \oplus (N,\nu) \to \partial(P,\theta) \oplus (N',\nu') \\ f : (M,\lambda) \oplus (M', -\lambda') \to \partial(P,\theta) \\ f : (M,\lambda,\mu) \oplus (M', -\lambda', -\mu') \to \partial(P,\theta) \end{cases}$$

for some non-degenerate $(-)^{k+1}\varepsilon$-symmetric $\{(-)^{k+1}\varepsilon$-symmetric, even $(-)^{k+1}\varepsilon$-symmetric} form over A (P,θ), with (N,ν), (N',ν') hyperbolic $(-)^{k+1}\varepsilon$-symmetric linking forms over A.

The cobordism class of a $(2k+1)$-dimensional ε-symmetric {even ε-symmetric, ε-quadratic} Poincaré complex over A

$$(C, \varphi \in Q^{2k+1}(C,\varepsilon)) \quad \{(C, \varphi \in Q\langle v_0\rangle^{2k+1}(C,\varepsilon)), (C, \psi \in Q_{2k+1}(C,\varepsilon))\}$$

corresponds to the equivalence class of the non-singular $(-)^{k+1}\varepsilon$-symmetric {even $(-)^{k+1}\varepsilon$-symmetric, $(-)^{k+1}\varepsilon$-quadratic} linking form over A

$$\begin{cases} \left(T^{k+1}(C), \varphi_0 : T^{k+1}(C) \to T_k(C) = T^{k+1}(C)^\wedge;\ [x] \mapsto \left([y] \mapsto \dfrac{\varphi_0(x)(z)}{s}\right)\right), \\ (T^{k+1}(C), \varphi_0), \\ \Big(T^{k+1}(C), (1+T_\varepsilon)\psi_0, \mu(\psi_0) : T^{k+1}(C) \to \mathbf{A}/\{a + (-)^{k+1}\varepsilon\bar{a} \mid a \in A\}; \\ \qquad\qquad\qquad\qquad [y] \mapsto \dfrac{(1+T_\varepsilon)\psi_0(y)(z)}{s}\Big) \\ \qquad\qquad (x, y \in C^{k+1},\ z \in C^k,\ s \in A - \{0\},\ d^*z = sy \in C^{k+1}). \end{cases}$$

It can be shown that a hyperbolic even ε-symmetric {ε-quadratic} linking form over A is isomorphic to the boundary $\partial(P,\theta)$ $\{\partial(P,\theta)\}$ of a non-degenerate ε-symmetric {even ε-symmetric} form over A $(P, \theta \in Q^\varepsilon(P))$ $\{(P, \theta \in Q\langle v_0\rangle^\varepsilon(P))\}$, so that hyperbolic linking forms represent 0 in the Witt-type groups defined in Proposition 7.1(ii).

In particular, we have that the ring of integers $\mathbf{Z}$ is a Dedekind ring, with the rationals $\mathbf{Q}$ as quotient field.

PROPOSITION 7.2. *The symmetric and quadratic L-groups of* $\mathbf{Z}$ *are given by*

$$L^n(\mathbf{Z}) = \begin{cases} \mathbf{Z} \\ \mathbf{Z}_2 \\ 0 \\ 0 \end{cases} \text{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \quad (n \geqslant 0),$$

$$L_n(\mathbf{Z}) = \begin{cases} \mathbf{Z} \\ 0 \\ \mathbf{Z}_2 \\ 0 \end{cases} \text{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \quad (n \in \mathbf{Z}),$$

$$L^n(\mathbf{Z}) = \begin{cases} 0 & \text{if } n = -1, -2, \\ L_n(\mathbf{Z}) & \text{if } n \leqslant -3. \end{cases}$$

The invariants are given by

$L^{4k}(\mathbf{Z}) \to \mathbf{Z};\ (C, \varphi \in Q^{4k}(C)) \mapsto$ signature $(F^{2k}(C), \varphi_0)$,

$L^{4k+1}(\mathbf{Z}) \to \mathbf{Z}_2;\ (C, \varphi \in Q^{4k+1}(C)) \mapsto$ de Rham invariant $(T^{2k+1}(C), \varphi_0)$,

$L_{4k}(\mathbf{Z}) \to \mathbf{Z};\ (C, \psi \in Q_{4k}(C)) \mapsto \frac{1}{8}($signature $(F^{2k}(C), (1+T)\psi_0))$,

$L_{4k+2}(\mathbf{Z}) \to \mathbf{Z}_2;$

$(C, \psi \in Q_{4k+2}(C)) \mapsto$ Arf invariant $(F^{2k+1}(C), (1+T)\psi_0, v^{2k+1}(\psi))$.

8. Products

We shall define now products in the L-groups

$$\left.\begin{aligned} L^m(A,\varepsilon) \otimes_{\mathbf{Z}} L^n(B,\eta) &\to L^{m+n}(A \otimes_{\mathbf{Z}} B,\ \varepsilon \otimes \eta) \\ L^m(A,\varepsilon) \otimes_{\mathbf{Z}} L_n(B,\eta) &\to L_{m+n}(A \otimes_{\mathbf{Z}} B,\ \varepsilon \otimes \eta) \end{aligned}\right. \quad (m, n \in \mathbf{Z}).$$

For $m = n = 0$ these are the usual products in the Witt groups of forms, induced by the tensor products of forms. In § 8 of Part II we shall use the products to obtain a formula for the surgery obstruction of a cartesian product of normal maps.

The tensor product of rings with involution A, B is a ring with involution $A \otimes_{\mathbf{Z}} B$, where

$$(\overline{a \otimes b}) = \bar{a} \otimes \bar{b} \in A \otimes_{\mathbf{Z}} B \quad (a \in A,\ b \in B).$$

The tensor product of an A-module chain complex C and a B-module chain complex D is an $A \otimes_{\mathbf{Z}} B$-module chain complex $C \otimes_{\mathbf{Z}} D$, with $A \otimes_{\mathbf{Z}} B$ acting by

$$A \otimes_{\mathbf{Z}} B \times C \otimes_{\mathbf{Z}} D \to C \otimes_{\mathbf{Z}} D;\ (a \otimes b,\ x \otimes y) \mapsto ax \otimes by.$$

If $\varepsilon \in A$, $\eta \in B$ are central units such that $\bar{\varepsilon} = \varepsilon^{-1} \in A$, $\bar{\eta} = \eta^{-1} \in B$ then $\varepsilon \otimes \eta \in A \otimes_{\mathbf{Z}} B$ is a central unit such that

$$\overline{(\varepsilon \otimes \eta)} = (\varepsilon \otimes \eta)^{-1} \in A \otimes_{\mathbf{Z}} B,$$

and there is a natural identification of $\mathbf{Z}[\mathbf{Z}_2]$-module chain complexes

$$(C^t \otimes_A C) \otimes_{\mathbf{Z}} (D^t \otimes_B D) = (C \otimes_{\mathbf{Z}} D)^t \otimes_{A \otimes_{\mathbf{Z}} B} (C \otimes_{\mathbf{Z}} D)$$

with $T \in \mathbf{Z}_2$ acting by $T_\varepsilon \otimes T_\eta$ on $(C^t \otimes_A C) \otimes_{\mathbf{Z}} (D^t \otimes_B D)$ and by $T_{\varepsilon \otimes \eta}$ on $(C \otimes_{\mathbf{Z}} D)^t \otimes_{A \otimes_{\mathbf{Z}} B} (C \otimes_{\mathbf{Z}} D)$. Let $\hat{W}$ be a complete resolution for $\mathbf{Z}_2$ obtained from a f.g. free $\mathbf{Z}[\mathbf{Z}_2]$-module resolution of $\mathbf{Z}$

$$W: \ldots \longrightarrow W_2 \xrightarrow{d} W_1 \xrightarrow{d} W_0 \xrightarrow{e} \mathbf{Z} \longrightarrow 0,$$

with

$$\hat{W}: \ldots \longrightarrow W_2 \xrightarrow{d} W_1 \xrightarrow{d} W_0 \xrightarrow{e^*e} W^0 \xrightarrow{d^*} W^1 \xrightarrow{d^*} W^2 \longrightarrow \ldots$$

$$(W^s = W_s^* = \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W_s, \mathbf{Z}[\mathbf{Z}_2])).$$

As in §4 of Chapter XII of Cartan and Eilenberg [4] it is possible to construct a diagonal chain map

$$\Delta \colon \hat{W} \to \hat{W} \otimes_{\mathbf{Z}} \hat{W}$$

(allowing infinite chains in $\hat{W} \otimes_{\mathbf{Z}} \hat{W}$), and so use the restriction

$$\begin{cases} \Delta \colon W \to W \otimes_{\mathbf{Z}} W \\ \Delta \colon W^{-*} \to W \otimes_{\mathbf{Z}} W^{-*} \end{cases}$$

to define a chain map

$$\begin{cases} \otimes \colon \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C^t \otimes_A C) \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, D^t \otimes_B D) \\ \qquad \to \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, (C \otimes_{\mathbf{Z}} D)^t \otimes_{A \otimes_{\mathbf{Z}} B} (C \otimes_{\mathbf{Z}} D)); \ \varphi \otimes \theta \mapsto (\varphi \otimes \theta)\Delta \\ \otimes \colon \mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W, C^t \otimes_A C) \otimes_{\mathbf{Z}} (W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} (D^t \otimes_B D)) \\ \qquad \mapsto W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} ((C \otimes_{\mathbf{Z}} D)^t \otimes_{A \otimes_{\mathbf{Z}} B} (C \otimes_{\mathbf{Z}} D)); \ \varphi \otimes \psi \mapsto (\varphi \otimes \psi)\Delta, \end{cases}$$

identifying $\mathrm{Hom}_{\mathbf{Z}[\mathbf{Z}_2]}(W^{-*}, -) = (W \otimes_{\mathbf{Z}[\mathbf{Z}_2]} -)$. The induced product in the Q-groups

$$\begin{cases} \otimes \colon Q^m(C, \varepsilon) \otimes_{\mathbf{Z}} Q^n(D, \eta) \to Q^{m+n}(C \otimes_{\mathbf{Z}} D, \varepsilon \otimes \eta) \\ \otimes \colon Q^m(C, \varepsilon) \otimes_{\mathbf{Z}} Q_n(D, \varepsilon) \to Q_{m+n}(C \otimes_{\mathbf{Z}} D, \varepsilon \otimes \eta) \end{cases}$$

is just the cup {cap} product

$$\begin{cases} \cup \colon (\mathbf{Z}_2\text{-hypercohomology}) \otimes (\mathbf{Z}_2\text{-hypercohomology}) \\ \qquad\qquad \to (\mathbf{Z}_2\text{-hypercohomology}), \\ \cap \colon (\mathbf{Z}_2\text{-hypercohomology}) \otimes (\mathbf{Z}_2\text{-hyperhomology}) \\ \qquad\qquad \to (\mathbf{Z}_2\text{-hyperhomology}). \end{cases}$$

In particular, for the standard $\mathbf{Z}[\mathbf{Z}_2]$-resolution W of $\mathbf{Z}$

$$W_s = \mathbf{Z}[\mathbf{Z}_2],\quad d\colon W_s \to W_{s-1};\ 1_s \mapsto 1_{s-1} + (-)^s T_{s-1} \quad (s \geqslant 0)$$

we can take

$$\Delta\colon \hat{W}_s \to (\hat{W} \otimes_{\mathbf{Z}} \hat{W})_s = \sum_{r=-\infty}^{\infty} \hat{W}_r \otimes_{\mathbf{Z}} \hat{W}_{s-r};\ 1_s \mapsto \sum_{r=-\infty}^{\infty} 1_r \otimes T^r_{s-r} \quad (s \in \mathbf{Z}),$$

giving explicit formulae

$$\begin{aligned}(\varphi\otimes\theta)_s = \sum_{r=0}^{s} (-)^{(m+r)s} \varphi_r \otimes T^r_\eta \theta_{s-r} &\in ((C\otimes_{\mathbf{Z}} D)^t \otimes_{A\otimes_{\mathbf{Z}} B} (C\otimes_{\mathbf{Z}} D))_{m+n+s} \\ &= \sum_{r=-\infty}^{\infty} (C^t \otimes_A C)_{m+r} \otimes_{\mathbf{Z}} (D^t \otimes_B D)_{n+s-r} \\ &\qquad (s \geqslant 0,\ \varphi \in Q^m(C,\varepsilon),\ \theta \in Q^n(D,\eta)),\end{aligned}$$

$$\begin{aligned}(\varphi\otimes\psi)_s = \sum_{r=0}^{\infty} (-)^{(m+r)s} \varphi_r \otimes T^r_\eta \psi_{s+r} &\in ((C\otimes_{\mathbf{Z}} D)^t \otimes_{A\otimes_{\mathbf{Z}} B} (C\otimes_{\mathbf{Z}} D))_{m+n-s} \\ &= \sum_{r=-\infty}^{\infty} (C^t \otimes_A C)_{m+r} \otimes_{\mathbf{Z}} (D^t \otimes_B D)_{n-s-r} \\ &\qquad (s \geqslant 0,\ \varphi \in Q^m(C,\varepsilon),\ \psi \in Q_n(D,\eta)).\end{aligned}$$

PROPOSITION 8.1. *There are defined natural products in the symmetric and quadratic L-groups*

$$\otimes\colon L^m(A,\varepsilon)\otimes_{\mathbf{Z}} L^n(B,\eta) \to L^{m+n}(A\otimes_{\mathbf{Z}} B,\ \varepsilon\otimes\eta),$$
$$\otimes\colon L^m(A,\varepsilon)\otimes_{\mathbf{Z}} L_n(B,\eta) \to L_{m+n}(A\otimes_{\mathbf{Z}} B,\ \varepsilon\otimes\eta),$$
$$\otimes\colon L_m(A,\varepsilon)\otimes_{\mathbf{Z}} L^n(B,\eta) \to L_{m+n}(A\otimes_{\mathbf{Z}} B,\ \varepsilon\otimes\eta),$$
$$\otimes\colon L_m(A,\varepsilon)\otimes_{\mathbf{Z}} L_n(B,\eta) \to L_{m+n}(A\otimes_{\mathbf{Z}} B,\ \varepsilon\otimes\eta),$$

for $m, n \in \mathbf{Z}$, which are related to each other by a commutative diagram

$$\begin{tikzcd}
L_m(A,\varepsilon)\otimes_{\mathbf{Z}} L_n(B,\eta) \arrow[r, "1\otimes(1+T_\eta)"] \arrow[d, "(1+T_\varepsilon)\otimes 1"'] \arrow[dr, "\otimes"] & L_m(A,\varepsilon)\otimes_{\mathbf{Z}} L^n(B,\eta) \arrow[d, "\otimes"] \\
L^m(A,\varepsilon)\otimes_{\mathbf{Z}} L_n(B,\eta) \arrow[r, "\otimes"] \arrow[d, "1\otimes(1+T_\eta)"'] & L_{m+n}(A\otimes_{\mathbf{Z}} B,\varepsilon\otimes\eta) \arrow[d, "1+T_{\varepsilon\otimes\eta}"] \\
L^m(A,\varepsilon)\otimes_{\mathbf{Z}} L^n(B,\eta) \arrow[r, "\otimes"] & L^{m+n}(A\otimes_{\mathbf{Z}} B,\varepsilon\otimes\eta)
\end{tikzcd}$$

Proof. Consider first the case where $m, n \geqslant 0$.

The product of an m-dimensional ε-symmetric Poincaré complex over A $(C, \varphi \in Q^m(C, \varepsilon))$ and an n-dimensional η-symmetric Poincaré complex over B $(D, \theta \in Q^n(D, \eta))$ is an $(m+n)$-dimensional $(\varepsilon \otimes \eta)$-symmetric Poincaré complex over $A \otimes_{\mathbf{Z}} B$

$$(C, \varphi) \otimes (D, \theta) = (C \otimes_{\mathbf{Z}} D,\ \varphi \otimes \theta \in Q^{m+n}(C \otimes_{\mathbf{Z}} D,\ \varepsilon \otimes \eta)).$$

If $(f\colon D \to E, (\delta\theta, \theta) \in Q^{n+1}(f, \eta))$ is an $(n+1)$-dimensional η-symmetric Poincaré pair over B then the product

$$\begin{aligned}(C, \varphi) &\otimes (f\colon D \to E, (\delta\theta, \theta)) \\ &= (1 \otimes f\colon C \otimes_{\mathbf{Z}} D \to C \otimes_{\mathbf{Z}} E, (\varphi \otimes \delta\theta,\ \varphi \otimes \theta) \in Q^{m+n+1}(1 \otimes f,\ \varepsilon \otimes \eta))\end{aligned}$$

is an $(m+n+1)$-dimensional $(\varepsilon \otimes \eta)$-symmetric Poincaré pair over $A \otimes_{\mathbf{Z}} B$. Similarly for null-cobordisms of (C, φ), and also for products of other types of algebraic Poincaré complexes. Thus the Q-group products pass to the L-groups. The above diagram actually commutes on the Q-group level.

The product of an m-dimensional even ε-symmetric Poincaré complex over A $(C, \varphi \in Q\langle v_0\rangle^m(C, \varepsilon))$ and an n-dimensional η-symmetric Poincaré complex over B $(D, \theta \in Q^n(D, \eta))$ is an $(m+n)$-dimensional even $(\varepsilon \otimes \eta)$-symmetric Poincaré complex over $A \otimes_{\mathbf{Z}} B$ $(C, \varphi) \otimes (D, \theta)$. We thus obtain products

$$\otimes\colon L\langle v_0\rangle^m(A, \varepsilon) \otimes_{\mathbf{Z}} L^n(B, \eta) \to L\langle v_0\rangle^{m+n}(A \otimes_{\mathbf{Z}} B,\ \varepsilon \otimes \eta) \quad (m, n \geqslant 0).$$

Now Proposition 4.4 gives skew-suspension isomorphisms

$$\begin{aligned}&\bar{S}\colon L^{m-2}(A, -\varepsilon) \to L\langle v_0\rangle^m(A, \varepsilon),\\ &\bar{S}\colon L^{m+n-2}(A \otimes_{\mathbf{Z}} B,\ -\varepsilon \otimes \eta) \to L\langle v_0\rangle^{m+n}(A \otimes_{\mathbf{Z}} B,\ \varepsilon \otimes \eta),\end{aligned}$$

so that we can express these products as

$$\otimes\colon L^{m-2}(A, -\varepsilon) \otimes_{\mathbf{Z}} L^n(B, \eta) \to L^{m+n-2}(A \otimes_{\mathbf{Z}} B,\ -\varepsilon \otimes \eta) \quad (m, n \geqslant 0)$$

(agreeing with the products defined previously for $m-2 \geqslant 0$).

We shall define the remaining products using the identifications of § 5 of low-dimensional L-groups with Witt groups of forms and formations.

Define the products

$$\begin{aligned}\otimes\colon & L^{-2}(A, \varepsilon) \otimes_{\mathbf{Z}} L^{-2}(B, \eta) \to L^{-4}(A \otimes_{\mathbf{Z}} B,\ \varepsilon \otimes \eta) = L_0(A \otimes_{\mathbf{Z}} B,\ \varepsilon \otimes \eta);\\ &(M, \psi - \varepsilon\psi^* \in Q\langle v_0\rangle^{-\varepsilon}(M)) \otimes_{\mathbf{Z}} (N, \chi - \eta\chi^* \in Q\langle v_0\rangle^{-\eta}(N))\\ &\mapsto (M \otimes_{\mathbf{Z}} N,\ \psi \otimes (\chi - \eta\chi^*) = (\psi - \varepsilon\psi^*) \otimes \chi \in Q_{\varepsilon\otimes\eta}(M \otimes_{\mathbf{Z}} N))\\ &\hphantom{\mapsto (M \otimes_{\mathbf{Z}} N,\ \psi \otimes (\chi - \eta\chi^*) = (\psi - \varepsilon\psi^*) \otimes \chi} (\psi \in Q_{-\varepsilon}(M),\ \chi \in Q_{-\eta}(N)),\end{aligned}$$

$$\begin{aligned}\otimes\colon & L^{-1}(A, \varepsilon) \otimes_{\mathbf{Z}} L^{-2}(B, \eta) \to L^{-3}(A \otimes_{\mathbf{Z}} B,\ \varepsilon \otimes \eta) = L_1(A \otimes_{\mathbf{Z}} B,\ \varepsilon \otimes \eta);\\ &(M, \psi - \varepsilon\psi^*; F, G) \otimes (N, \chi - \eta\chi^*)\\ &\mapsto (M \otimes_{\mathbf{Z}} N,\ \psi \otimes (\chi - \eta\chi^*);\ F \otimes_{\mathbf{Z}} N,\ G \otimes_{\mathbf{Z}} N).\end{aligned}$$

It now only remains to define the product

$$\otimes: L^{-1}(A,\varepsilon)\otimes_{\mathbf{Z}} L^{-1}(B,\eta) \to L^{-2}(A\otimes_{\mathbf{Z}} B,\ \varepsilon\otimes\eta).$$

Given a non-singular even ε-symmetric formation over A $(H^{\varepsilon}(F); F, G)$ write the inclusion of the lagrangian as a morphism of even ε-symmetric forms

$$\begin{pmatrix}\gamma\\ \mu\end{pmatrix}: (G,0) \to H^{\varepsilon}(F) = \left(F\oplus F^*, \begin{pmatrix}0 & 1\\ \varepsilon & 0\end{pmatrix} \in Q\langle v_0\rangle^{\varepsilon}(F\oplus F^*)\right),$$

and let $(C, \varphi \in Q\langle v_0\rangle^1(C,\varepsilon))$ be the associated 1-dimensional even ε-symmetric Poincaré complex over A (obtained as in Proposition 2.3). Given also a non-singular even η-symmetric formation over B $(H^{\eta}(I); I, J)$ write the inclusion of the lagrangian as

$$\begin{pmatrix}\beta\\ \lambda\end{pmatrix}: (J,0) \to H^{\eta}(I) = \left(I\oplus I^*, \begin{pmatrix}0 & 1\\ \eta & 0\end{pmatrix} \in Q\langle v_0\rangle^{\eta}(I\oplus I^*)\right),$$

and let $(D, \theta \in Q\langle v_0\rangle^1(D,\eta))$ be the associated 1-dimensional even η-symmetric Poincaré complex over B. The product

$$(C,\varphi)\otimes(D,\theta) = (C\otimes_{\mathbf{Z}} D,\ \varphi\otimes\theta \in Q\langle v_0\rangle^2(C\otimes_{\mathbf{Z}} D,\ \varepsilon\otimes\eta))$$

is a 2-dimensional even $(\varepsilon\otimes\eta)$-symmetric Poincaré complex over $A\otimes_{\mathbf{Z}} B$ on which it is possible to do surgery to kill $H^2(C\otimes_{\mathbf{Z}} D)$, obtaining the skew-suspension $\bar{S}(E,\nu)$ of a 0-dimensional even $-(\varepsilon\otimes\eta)$-symmetric Poincaré complex over $A\otimes_{\mathbf{Z}} B$ $(E, \nu \in Q\langle v_0\rangle^0(E, -(\varepsilon\otimes\eta)))$. The complex (E,ν) corresponds to the non-singular even $-(\varepsilon\otimes\eta)$-symmetric form over $A\otimes_{\mathbf{Z}} B$,

$$\begin{aligned}
&(H^{\varepsilon}(F); F, G)\otimes(H^{\eta}(I); I, J)\\
&= \left(\operatorname{coker}\left(\begin{pmatrix}\mu\otimes 1\\ 1\otimes\lambda\\ \gamma\otimes\beta\end{pmatrix}: G\otimes_{\mathbf{Z}} J \to (F^*\otimes_{\mathbf{Z}} J)\oplus(G\otimes_{\mathbf{Z}} I^*)\oplus(F\otimes_{\mathbf{Z}} I)\right),\right.\\
&\qquad\qquad\left.\begin{bmatrix}0 & \varepsilon\gamma\otimes\beta^* & \varepsilon\otimes\eta\lambda^*\\ -\gamma^*\otimes\eta\beta & 0 & -\varepsilon\mu^*\otimes\eta\\ -1\otimes\lambda & \mu\otimes 1 & 0\end{bmatrix}\right).
\end{aligned}$$

This defines a product

$$\otimes: L\langle v_0\rangle^1(A,\varepsilon)\otimes_{\mathbf{Z}} L\langle v_0\rangle^1(B,\eta) \to L\langle v_0\rangle^0(A\otimes_{\mathbf{Z}} B,\ -(\varepsilon\otimes\eta)),$$

as required.

PROPOSITION 8.2. *The ε-symmetrization map*

$$1+T_\varepsilon\colon L_n(A,\varepsilon)\to L^n(A,\varepsilon)\quad(n\in\mathbf{Z})$$

is an isomorphism modulo 8*-torsion.*

Proof. Product with the generator $E_8 = 1 \in L_0(\mathbf{Z}) = \mathbf{Z}$ (Proposition 7.2) defines morphisms

$$E_8\otimes - : L^n(A,\varepsilon)\to L_n(A,\varepsilon)\quad(n\in\mathbf{Z})$$

such that both the composites with $(1+T_\varepsilon)$ are multiplication by 8, since $(1+T)E_8 = 8 \in L^0(\mathbf{Z}) = \mathbf{Z}$.

For a commutative ring A (with any involution) we can compose the products of Proposition 8.1 with the morphisms of L-groups induced by the morphism of rings with involution

$$A\otimes_{\mathbf{Z}}A\to A\,;\ a\otimes b\mapsto ab$$

to define internal products in the symmetric and quadratic L-groups

$$\begin{aligned}&L^m(A)\otimes_{\mathbf{Z}}L^n(A)\to L^{m+n}(A),\\&L^m(A)\otimes_{\mathbf{Z}}L_n(A)\to L_{m+n}(A),\\&L_m(A)\otimes_{\mathbf{Z}}L_n(A)\to L_{m+n}(A)\quad(m,n\in\mathbf{Z}).\end{aligned}$$

These make $L^*(A)$ into a graded ring with unit

$$1=(A,A\to A^*;\ a\mapsto(b\mapsto b\bar a))\in L^0(A),$$

and $L_*(A)$ into a graded $L^*(A)$-algebra. (The products in $L^*(A)$ and $L_*(A)$ are such that $xy = (-)^{|x||y|}yx$, where $|\ |$ denotes the grading.)

The suspension and skew-suspension operations $S,\bar S$ defined on the Q-groups in § 1 above can be expressed as products with universal classes defined over $\mathbf{Z}$. Specifically, define $\mathbf{Z}_2$-hypercohomology classes $\varphi\in Q^1(S\mathbf{Z})$, $\bar\varphi\in Q^2(S\mathbf{Z},-1)$ by

$$\begin{aligned}&\varphi_1 = 1\colon S\mathbf{Z}^1=\mathbf{Z}\to S\mathbf{Z}_1=\mathbf{Z},\\&\bar\varphi_0 = 1\colon S\mathbf{Z}^1=\mathbf{Z}\to S\mathbf{Z}_1=\mathbf{Z}.\end{aligned}$$

The suspension of an A-module chain complex C can be expressed as

$$SC = S\mathbf{Z}\otimes_{\mathbf{Z}}C,$$

and

$$\begin{cases}S=(S\mathbf{Z},\varphi)\otimes - : Q^n(C,\varepsilon)\to Q^{n+1}(SC,\varepsilon),\\ S=(S\mathbf{Z},\varphi)\otimes - : Q_n(C,\varepsilon)\to Q_{n+1}(SC,\varepsilon),\end{cases}$$

$$\begin{cases}\bar S=(S\mathbf{Z},\bar\varphi)\otimes - : Q^n(C,\varepsilon)\to Q^{n+2}(SC,-\varepsilon),\\ \bar S=(S\mathbf{Z},\bar\varphi)\otimes - : Q_n(C,\varepsilon)\to Q_{n+2}(SC,-\varepsilon).\end{cases}$$

9. Change of K-theory

We shall now consider the L-groups $L^*_X(A,\varepsilon)$ $\{L_*^X(A,\varepsilon)\}$ of cobordism classes of ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes over A (C,φ) $\{(C,\psi)\}$ with algebraic K-theory (such as the projective class $[C]\in\tilde{K}_0(A)$ or the torsion $\tau(\varphi_0\colon C^{n-*}\to C)\in\tilde{K}_1(A)$) restricted to lie in a prescribed subgroup X of $\tilde{K}_m(A)$ $(m=0,1)$.

Let X be a subgroup of the reduced projective class $\{$torsion$\}$ group $\tilde{K}_0(A)=\operatorname{coker}(K_0(\mathbf{Z})\to K_0(A))$ $\{\tilde{K}_1(A)=\operatorname{coker}(K_1(\mathbf{Z})\to K_1(A))\}$ which is setwise invariant under the duality involution

$$\begin{cases} *\colon \tilde{K}_0(A)\to\tilde{K}_0(A);\ [P]\mapsto[P^*],\\ *\colon \tilde{K}_1(A)\to\tilde{K}_1(A);\ \tau(f\colon P\to Q)\mapsto\tau(f^*\colon Q^*\to P^*),\end{cases}$$

denoting by $[P]$ $\{\tau(f\colon P\to Q)\}$ the projective class $\{$torsion$\}$ of a f.g. projective A-module P $\{$an isomorphism $f\in\operatorname{Hom}_A(P,Q)$ of based f.g. free A-modules $P,Q\}$. In dealing with based A-modules it is convenient (but not necessary, cf. Ranicki [**15**]) to assume that A is such that f.g. free A-modules have a well-defined rank. Also, we shall assume that $\tau(\varepsilon\colon A\to A)\in X\subseteq\tilde{K}_1(A)$.

The *intermediate ε-symmetric $\{\varepsilon$-quadratic$\}$ L-groups of A* $L^n_X(A,\varepsilon)$ $\{L_n^X(A,\varepsilon)\}$, for $n\in\mathbf{Z}$, are defined as follows. For $n\geqslant 0$, let $L^n_X(A,\varepsilon)$ $\{L_n^X(A,\varepsilon)\}$ be the cobordism group of n-dimensional ε-symmetric $\{\varepsilon$-quadratic$\}$ Poincaré complexes over A $(C,\varphi\in Q^n(C,\varepsilon))$ $\{(C,\psi\in Q_n(C,\varepsilon))\}$ with K-theory in X, meaning:

in the case where $X\subseteq\tilde{K}_0(A)$, C is a finite chain complex of f.g. projective A-modules such that

$$[C]\equiv\sum_{r=-\infty}^{\infty}(-)^r[C_r]\in X\subseteq\tilde{K}_0(A);$$

in the case where $X\subseteq\tilde{K}_1(A)$, C is a finite chain complex of based f.g. free A-modules such that

$$\begin{cases}\tau(\varphi_0\colon C^{n-*}\to C)\in X\subseteq\tilde{K}_1(A),\\ \tau((1+T_\varepsilon)\psi_0\colon C^{n-*}\to C)\in X\subseteq\tilde{K}_1(A).\end{cases}$$

Working exactly as in Proposition 4.3 we can show that the skew-suspension maps in the intermediate ε-quadratic L-groups

$$\bar{S}\colon L_n^X(A,\varepsilon)\to L_{n+2}^X(A,-\varepsilon)\quad(n\geqslant 0)$$

are isomorphisms, allowing the definition

$$L_n^X(A,\varepsilon)=L_{n+2i}^X(A,(-)^i\varepsilon)\quad(n\leqslant -1,\ n+2i\geqslant 0).$$

Furthermore, define $L\langle v_0\rangle^n_X(A,\varepsilon)$ $(n=0,1)$ to be the cobordism group of

n-dimensional even ε-symmetric Poincaré complexes over A with K-theory in X. Define $L^n_X(A,\varepsilon)$ $(n \leqslant -1)$ by

$$L^n_X(A,\varepsilon) = \begin{cases} L\langle v_0\rangle^{n+2}_X(A,-\varepsilon) & (n=-1,-2), \\ L^X_n(A,\varepsilon) & (n \leqslant -3). \end{cases}$$

All the results of §§1–8 above have obvious intermediate L-group analogues. In particular, $L^0_X(A,\varepsilon)$ $\{L\langle v_0\rangle^0_X(A,\varepsilon), L^X_0(A,\varepsilon)\}$ is the Witt group of non-singular ε-symmetric {even ε-symmetric, ε-quadratic} forms over A $(M, \varphi \in Q^\varepsilon(M))$ $\{(M,\varphi \in Q\langle v_0\rangle^\varepsilon(M)), (M,\psi \in Q_\varepsilon(M))\}$ with K-theory in X (meaning $[M] \in X$ if $X \subseteq \tilde{K}_0(A)$, and M based, $\tau(\varphi\colon M \to M^*) \in X$ if $X \subseteq \tilde{K}_1(A)$ with $\varphi = \psi + \varepsilon\psi^*$ in the ε-quadratic case), and $L^1_X(A,\varepsilon)$ $\{L\langle v_0\rangle^1_X(A,\varepsilon), L^X_1(A,\varepsilon)\}$ is the Witt group of non-singular ε-symmetric {even ε-symmetric, ε-quadratic} formations over A

$$\begin{cases} (M, \varphi \in Q^\varepsilon(M);\, F, G) \\ (M, \varphi \in Q\langle v_0\rangle^\varepsilon(M);\, F, G) \\ (M, \psi \in Q_\varepsilon(M);\, F, G) \end{cases}$$

with K-theory in X (meaning $[G]-[F^*] \in X$ if $X \subseteq \tilde{K}_0(A)$, and F, G based,

$$\tau(g^{-1}f\colon F \oplus F^* \to G \oplus G^*) \in X \quad \text{if} \quad X \subseteq \tilde{K}_1(A)$$

with $f\colon F \oplus F^* \to M$, $g\colon G \oplus G^* \to M$ any of the A-module isomorphisms extending the inclusions $F \to M$, $G \to M$ given by Proposition 2.2).

The L-groups considered so far have been the case where $X = \tilde{K}_0(A)$,

$$\begin{cases} L^n(A,\varepsilon) = L^n_{\tilde{K}_0(A)}(A,\varepsilon) \\ L_n(A,\varepsilon) = L^{\tilde{K}_0(A)}_n(A,\varepsilon) \end{cases} \quad (n \in \mathbf{Z}).$$

The intermediate symmetric {quadratic} L-groups $L^n_X(A,1)$ $\{L^X_n(A,1)\}$ will be denoted by $L^n_X(A)$ $\{L^X_n(A)\}$, for $n \in \mathbf{Z}$.

The groups $U_n(A)$, $V_n(A)$, $W_n(A)$ ($n(\mathrm{mod}\,4)$) of Ranicki **[13]** can be identified with the appropriate intermediate quadratic L-groups

$$\begin{cases} U_n(A) = L^{\tilde{K}_0(A)}_n(A) = L_n(A), \\ V_n(A) = L^{\{0\} \subseteq \tilde{K}_0(A)}_n(A) = L^{\tilde{K}_1(A)}_n(A), \\ W_n(A) = L^{\{0\} \subseteq \tilde{K}_1(A)}_n(A). \end{cases}$$

More generally, the intermediate quadratic L-groups $L^X_n(A)$ can be identified with the groups $U^X_n(A)$ $(X \subseteq \tilde{K}_0(A))$ and $V^X_n(A)$ $(X \subseteq \tilde{K}_1(A))$ defined for $n\,(\mathrm{mod}\,4)$ by Ranicki in **[15]** using $\pm$ quadratic forms and formations over A with K-theory in X. For a group ring $A = \mathbf{Z}[\pi]$ with the w-twisted involution $\bar{g} = w(g)g^{-1}$ $(g \in \pi)$ for some group morphism $w\colon \pi \to \mathbf{Z}_2$ there are thus obtained all the various geometric surgery obstruction groups. If $X \subseteq \tilde{K}_0(\mathbf{Z}[\pi])$ $\{X \subseteq \tilde{K}_1(\mathbf{Z}[\pi])\}$ is a $*$-invariant subgroup then $L_n^X(\mathbf{Z}[\pi])$ is the obstruction group for framed surgery on normal maps from compact manifolds to finitely-dominated {finite} geometric Poincaré complexes with fundamental group π, with all the Wall finiteness obstructions {Whitehead torsions} restricted to lie in

$$X \subseteq \tilde{K}_0(\mathbf{Z}[\pi]) \quad \{X/\pi \subseteq Wh(\pi) = \tilde{K}_1(\mathbf{Z}[\pi])/\{\pi\}\}.$$

(We recall that the finiteness obstruction of a finitely-dominated geometric Poincaré complex X is the projective class $[C(\tilde{X})] \in \tilde{K}_0(\mathbf{Z}[\pi_1(X)])$ of the chain complex $C(\tilde{X})$ of the universal cover $\tilde{X}$, and that the torsion of a finite geometric Poincaré complex X is $\tau([X] \cap - : C(\tilde{X})^{n-*} \to C(\tilde{X})) \in Wh(\pi_1(X))$.) In particular, we have the surgery obstruction groups

$$\begin{cases} L_n^s(\pi, w) = L_n^{\{\pi\} \subseteq \tilde{K}_1(\mathbf{Z}[\pi])}(\mathbf{Z}[\pi]) \\ L_n^h(\pi, w) = V_n(\mathbf{Z}[\pi]) \\ L_n^p(\pi, w) = U_n(\mathbf{Z}[\pi]) \end{cases}$$

considered by Wall [**25**] {Shaneson [**21**], Maumary [**8**], Pedersen and Ranicki [**30**]} (see also the discussion in [**25**, § 17D]). Intermediate surgery obstruction groups $L_n^X(\mathbf{Z}[\pi])$ $(X \subseteq \tilde{K}_1(\mathbf{Z}[\pi]))$ were first considered by Cappell [**3**].

We shall write $L_X^n(A, \varepsilon)$ $\{L_n^X(A, \varepsilon)\}$ as $U_X^n(A, \varepsilon)$ $\{U_n^X(A, \varepsilon)\}$ for $X \subseteq \tilde{K}_0(A)$, and as $V_X^n(A, \varepsilon)$ $\{V_n^X(A, \varepsilon)\}$ for $X \subseteq \tilde{K}_1(A)$, with

$$\begin{cases} U^n(A, \varepsilon) = U^n_{\tilde{K}_0(A)}(A, \varepsilon), \\ U_n(A, \varepsilon) = U_n^{\tilde{K}_0(A)}(A, \varepsilon), \end{cases} \quad \begin{cases} V^n(A, \varepsilon) = V^n_{\tilde{K}_1(A)}(A, \varepsilon) \\ V_n(A, \varepsilon) = V_n^{\tilde{K}_1(A)}(A, \varepsilon) \end{cases} \quad (n \in \mathbf{Z}),$$

extending the notation of Ranicki [**13**, **15**]. For $\varepsilon = 1 \in A$ the notation is contracted in the usual fashion, for example, $U^n(A, 1) = U^n(A)$. Similarly for the intermediate even ε-symmetric L-groups $L\langle v_0\rangle_X^n(A, \varepsilon)$ $(n = 0, 1)$.

Mishchenko [**10**] considered only symmetric Poincaré complexes over A (C, φ) in which C is a finite f.g. free A-module chain complex, so that the groups $\Omega_n(A)$ defined there are precisely $V^n(A)$ $(n \geqslant 0)$. The groups $\Omega_n(A) = V^n(A)$ differ from $L^n(A) = U^n(A)$ in at most 2-torsion—this is clear from the following exact sequence, since the reduced Tate $\mathbf{Z}_2$-cohomology groups $\hat{H}^n(\mathbf{Z}_2; G)$ are of exponent 2.

PROPOSITION 9.1. *The intermediate ε-symmetric $\{\varepsilon$-quadratic$\}$ L-groups of A associated to $*$-invariant subgroups $X \subseteq Y \subseteq \tilde{K}_m(A)$ $(m = 0$ or $1)$ are related by a long exact sequence of abelian groups*

$$\begin{cases} \ldots \to L_X^n(A, \varepsilon) \to L_Y^n(A, \varepsilon) \to \hat{H}^n(\mathbf{Z}_2;\, Y/X) \\ \qquad\qquad \to L_X^{n-1}(A, \varepsilon) \to L_Y^{n-1}(A, \varepsilon) \to \ldots \\ \ldots \to L_n^X(A, \varepsilon) \to L_n^Y(A, \varepsilon) \to \hat{H}^n(\mathbf{Z}_2;\, Y/X) \\ \qquad\qquad \to L_{n-1}^X(A, \varepsilon) \to L_{n-1}^Y(A, \varepsilon) \to \ldots \end{cases} \quad (n \in \mathbf{Z})$$

involving the reduced Tate $\mathbf{Z}_2$-cohomology groups of the involution on Y/X

$$\hat{H}^n(\mathbf{Z}_2;\ Y/X) = \{g \in Y/X \mid g^* = (-)^n g\}/\{h+(-)^n h^* \mid h \in Y/X\}.$$

Proof. The exact sequences in the ε-quadratic case have already been obtained, in Theorems 2.3 and 3.3 of Ranicki [**13**]—only $\varepsilon = \pm 1 \in A$ was considered there, but the methods apply for all $\varepsilon \in A$. The first such sequence was the Rothenberg exact sequence obtained by Shaneson [**21**]

$$\ldots \to L_n^s(\pi) \to L_n^h(\pi) \to \hat{H}^n(\mathbf{Z}_2;\ Wh(\pi)) \to L_{n-1}^s(\pi) \to L_{n-1}^h(\pi) \to \ldots.$$

The ε-symmetric case for $m = 0$, $n \geqslant 0$ proceeds as follows.

Given a f.g. projective A-module P define an $(n+1)$-dimensional ε-symmetric Poincaré pair over A,

$$(f(P,n)\colon C(P,n) \to \delta C(P,n), (\delta\varphi(P,n), \varphi(P,n)) \in Q^{n+1}(f(P,n),\varepsilon)),$$

with projective classes

$$[C(P,n)] = \sum_{r=-\infty}^{\infty} (-)^r [C(P,n)_r] = (-)^{n-i}([P]+(-)^n[P^*]) \in \tilde{K}_0(A),$$

$$[\delta C(P,n)] = \sum_{r=-\infty}^{\infty} (-)^r [\delta C(P,n)_r] = (-)^{n-i}[P] \in \tilde{K}_0(A) \quad (n = 2i \text{ or } 2i+1)$$

as follows:

if $n = 2i$

$$f(P,2i) = (1 \quad 0)\colon C(P,2i)_i = P \oplus P^* \to \delta C(P,2i)_i = P,$$

$$\varphi(P,2i)_0 = \begin{pmatrix} 0 & (-)^i\varepsilon \\ 1 & 0 \end{pmatrix}\colon C(P,2i)^i = P^* \oplus P \to C(P,2i)_i = P \oplus P^*,$$

$$C(P,2i)_r = \delta C(P,2i)_r = 0 \ (r \neq i), \quad \delta\varphi(P,2i) = 0;$$

if $n = 2i+1$

$$C(P,2i+1)_r = \begin{cases} P^* \text{ if } r = i, \\ P \ \text{ if } r = i+1, \\ 0 \ \text{ if } r \neq i, i+1, \end{cases} \qquad \delta C(P,2i+1)_r = \begin{cases} P \text{ if } r = i+1, \\ 0 \ \text{ if } r \neq i+1, \end{cases}$$

$$d = 0\colon C(P,2i+1)_{i+1} = P \to C(P,2i+1)_i = P^*, \quad \delta\varphi(P,2i+1) = 0,$$

$$\varphi(P,2i+1)_0 = \begin{cases} 1\colon C(P,2i+1)^i = P \to C(P,2i+1)_{i+1} = P, \\ \varepsilon\colon C(P,2i+1)^{i+1} = P^* \to C(P,2i+1)_i = P^*, \end{cases}$$

$$f(P,2i+1) = 1\colon C(P,2i+1)_{i+1} = P \to \delta C(P,2i+1)_{i+1} = P.$$

Define abelian group morphisms

$$\begin{aligned} &\beta\colon L_Y^n(A,\varepsilon) \to \hat{H}^n(\mathbf{Z}_2;\ Y/X);\ (C,\varphi) \mapsto [C] \\ &\gamma\colon L_X^n(A,\varepsilon) \to L_Y^n(A,\varepsilon);\ (C,\varphi) \mapsto (C,\varphi) \qquad (n \geqslant 0). \\ &\partial\colon \hat{H}^{n+1}(\mathbf{Z}_2;\ Y/X) \to L_X^n(A,\varepsilon);\ [P] \mapsto (C(P,n),\varphi(P,n)) \end{aligned}$$

The composite

$$\hat{H}^{n+1}(\mathbf{Z}_2;\ Y/X) \xrightarrow{\partial} L^n_X(A,\varepsilon) \xrightarrow{\gamma} L^n_Y(A,\varepsilon) \tag{I}$$

is 0, since $(f(P,n)\colon C(P,n) \to \delta C(P,n), (\delta\varphi(P,n), \varphi(P,n)))$ is a null-cobordism of $\gamma\partial[P] = (C(P,n), \varphi(P,n))$ with

$$[\delta C(P,n)] = (-)^{n-i}[P] \in Y \subseteq \tilde{K}_0(A).$$

Given $(C,\varphi) \in \ker\gamma$ there exists a null-cobordism $(f\colon C \to \delta C, (\delta\varphi, \varphi))$ with $[\delta C] \in Y \subseteq \tilde{K}_0(A)$, and

$$(C,\varphi) = (-)^{n-i}\partial[\delta C] \in \operatorname{im}(\partial\colon \hat{H}^{n+1}(\mathbf{Z}_2;\ Y/X) \to L^n_X(A,\varepsilon))$$

so that (I) is exact.

The composite

$$L^n_X(A,\varepsilon) \xrightarrow{\gamma} L^n_Y(A,\varepsilon) \xrightarrow{\beta} \hat{H}^n(\mathbf{Z}_2;\ Y/X) \tag{II}$$

is 0. Given $(C,\varphi) \in \ker\beta$ there exists a f.g. projective A-module P such that

$$[C] + (-)^{n-i}([P] + (-)^n[P^*]) = 0 \in \tilde{K}_0(A), \quad [P] \in X \subseteq \tilde{K}_0(A),$$

and

$$(C,\varphi) = (C,\varphi) \oplus \partial[P] \in \operatorname{im}(\gamma\colon L^n_X(A,\varepsilon) \to L^n_Y(A,\varepsilon)),$$

so that (II) is exact.

The composite

$$L^{n+1}_Y(A,\varepsilon) \xrightarrow{\beta} \hat{H}^{n+1}(\mathbf{Z}_2;\ Y/X) \xrightarrow{\partial} L^n_X(A,\varepsilon) \tag{III}$$

is 0, for if $(C,\varphi) \in L^{n+1}_Y(A,\varepsilon)$ and P is a f.g. projective A-module such that $[P] = (-)^{n-i+1}[C] \in \tilde{K}_0(A)$ then

$$(f(P,n) \oplus 0\colon C(P,n) \to \delta C(P,n) \oplus C, (\delta\varphi(P,n) \oplus \varphi, \varphi(P,n)))$$

is a null-cobordism of $\partial\beta(C,\varphi) = \partial[P] = (C(P,n), \varphi(P,n))$ such that $[\delta C(P,n) \oplus C] \in X \subseteq \tilde{K}_0(A)$. Given $[P] \in \ker\partial$ let

$$(g\colon C(P,n) \to D, (\theta, \varphi(P,n)) \in Q^{n+1}(g,\varepsilon))$$

be a null-cobordism of $\partial[P] = (C(P,n), \varphi(P,n))$ such that $[D] \in X \subseteq \tilde{K}_0(A)$. The union

$$\begin{aligned}(f(P,n)\colon C(P,n) &\to \delta C(P,n), (\delta\varphi(P,n), \varphi(P,n)))\\ &\cup (g\colon C(P,n) \to D, (\theta, \varphi(P,n))) = (D', \theta' \in Q^{n+1}(D',\varepsilon))\end{aligned}$$

is an $(n+1)$-dimensional ε-symmetric Poincaré complex over A such that

$$[D'] = [D] - [C(P,n)] + [\delta C(P,n)] = [P] \in \hat{H}^{n+1}(\mathbf{Z}_2;\ Y/X).$$

Thus $[P] = \beta(D', \theta') \in \operatorname{im}(\beta\colon L^{n+1}_{Y}(A,\varepsilon) \to \hat{H}^{n+1}(\mathbf{Z}_2; Y/X))$, and (III) is exact.

The cases where $m = 1$, $n \leqslant -1$ may be treated similarly.

Define the *stable ε-symmetric* {*ε-quadratic*} *unitary group of A* to be the direct limit

$$\begin{cases} \mathscr{U}^*(A,\varepsilon) = \varinjlim_m \operatorname{Aut} H^\varepsilon(A^m) \\ \mathscr{U}_*(A,\varepsilon) = \varinjlim_m \operatorname{Aut} H_\varepsilon(A^m) \end{cases}$$

of the automorphism groups of the standard hyperbolic even ε-symmetric {ε-quadratic} forms over A,

$$\begin{cases} H^\varepsilon(A^m) = \left(A^m \oplus (A^m)^*, \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \in Q\langle v_0\rangle^\varepsilon(A^m \oplus (A^m)^*)\right) \\ H_\varepsilon(A^m) = \left(A^m \oplus (A^m)^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q_\varepsilon(A^m \oplus (A^m)^*)\right) \end{cases} \quad (m \geqslant 1),$$

the limit being taken with respect to the inclusions

$$\begin{cases} \operatorname{Aut} H^\varepsilon(A^m) \to \operatorname{Aut} H^\varepsilon(A^{m+1});\ u \mapsto u \oplus 1 \\ \operatorname{Aut} H_\varepsilon(A^m) \to \operatorname{Aut} H_\varepsilon(A^{m+1});\ u \mapsto u \oplus 1 \end{cases} \quad (m \geqslant 1).$$

(The hyperbolic forms $\{H^\varepsilon(A^m) \mid m \geqslant 1\}$ $\{\{H_\varepsilon(A^m) \mid m \geqslant 1\}\}$ are a cofinal family of objects in the category of non-singular even ε-symmetric {ε-quadratic} forms over A.)

Define the *elementary ε-symmetric* {*ε-quadratic*} *unitary group of A* $\mathscr{E}\mathscr{U}^*(A,\varepsilon)$ $\{\mathscr{E}\mathscr{U}_*(A,\varepsilon)\}$ to be the subgroup of $\mathscr{U}^*(A,\varepsilon)$ $\{\mathscr{U}_*(A,\varepsilon)\}$ generated by the elements of type

(i) $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}$, for any automorphism $\alpha \in \operatorname{Aut}_A(A^m, A^m)$,

(ii) $\begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}$, for any $(-\varepsilon)$-symmetric {even $(-\varepsilon)$-symmetric} form over A $(A^m, \varphi \in Q^{-\varepsilon}(A^m))$ $\{(A^m, \varphi \in Q\langle v_0\rangle^{-\varepsilon}(A^m))\}$,

(iii) $\sigma_m = \begin{pmatrix} 0 & \gamma_m^{-1} \\ \varepsilon\gamma_m & 0 \end{pmatrix}$, where $(A^m, \gamma_m \in Q^+(A^m))$ is the non-singular symmetric form given by

$$\gamma_m\colon A^m \to (A^m)^*;\ (a_1, a_2, \ldots, a_m) \mapsto \left((b_1, b_2, \ldots, b_m) \mapsto \sum_{i=1}^m b_i \bar{a}_i\right).$$

Given a $*$-invariant subgroup $X \subseteq \tilde{K}_1(A)$ define a subgroup of $\mathscr{U}^*(A,\varepsilon)\{\mathscr{U}_*(A,\varepsilon)\}$

$$\begin{cases} \mathscr{U}^*_X(A,\varepsilon) = \ker(\tau\colon \mathscr{U}^*(A,\varepsilon) \to \tilde{K}_1(A)/X), \\ \mathscr{U}^X_*(A,\varepsilon) = \ker(\tau\colon \mathscr{U}_*(A,\varepsilon) \to \tilde{K}_1(A)/X), \end{cases}$$

and let $\mathscr{E}\mathscr{U}^*_X(A,\varepsilon)$ $\{\mathscr{E}\mathscr{U}^X_*(A,\varepsilon)\}$ be the subgroup of $\mathscr{E}\mathscr{U}^*(A,\varepsilon)$ $\{\mathscr{E}\mathscr{U}_*(A,\varepsilon)\}$ obtained by restricting the generators of type (i) to be such that $\tau(\alpha) \in X \subseteq \tilde{K}_1(A)$.

PROPOSITION 9.2. *Let $X \subseteq \tilde{K}_1(A)$ be a $*$-invariant subgroup.*

(i) *The elementary subgroup $\mathscr{E}\mathscr{U}^*_X(A,\varepsilon)$ $\{\mathscr{E}\mathscr{U}^X_*(A,\varepsilon)\}$ contains the commutator subgroup of the stable unitary group*

$$[\mathscr{U}^*_X(A,\varepsilon), \mathscr{U}^*_X(A,\varepsilon)] \quad \{[\mathscr{U}^X_*(A,\varepsilon), \mathscr{U}^X_*(A,\varepsilon)]\}.$$

(ii) *There are natural identifications of abelian groups*

$$\begin{cases} V\langle v_0\rangle^1_X(A,\varepsilon) = \mathscr{U}^*_X(A,\varepsilon)/\mathscr{E}\mathscr{U}^*_X(A,\varepsilon), \\ V^X_1(A,\varepsilon) = \mathscr{U}^X_*(A,\varepsilon)/\mathscr{E}\mathscr{U}^X_*(A,\varepsilon). \end{cases}$$

(iii) *Every element of $\mathscr{E}\mathscr{U}^*_X(A,\varepsilon)$ $\{\mathscr{E}\mathscr{U}^X_*(A,\varepsilon)\}$ is represented by an automorphism $u\colon H^\varepsilon(A^m) \to H^\varepsilon(A^m)$ $\{u\colon H_\varepsilon(A^m) \to H_\varepsilon(A^m)\}$ for some $m \geqslant 0$ such that, for some $n \geqslant 0$,*

$$u \oplus \sigma_n = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \theta^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta'^* \\ 0 & 1 \end{pmatrix}$$

for some automorphism $\alpha \in \mathrm{Hom}_A(A^p, A^p)$ with $\tau(\alpha) \in X \subseteq \tilde{K}_1(A)$, and some $(-\varepsilon)$-symmetric {even $(-\varepsilon)$-symmetric} forms over A $((A^p)^*, \theta)$, (A^p, φ), $((A^p)^*, \theta')$ $(p = m+n)$.

Proof. A unitary automorphism

$$u\colon H^\varepsilon(A^m) \to H^\varepsilon(A^m) \quad \{u\colon H_\varepsilon(A^m) \to H_\varepsilon(A^m)\}$$

such that $\tau(u) \in X$ determines a non-singular even ε-symmetric {ε-quadratic} formation over A $(H^\varepsilon(A^m); A^m, u(A^m))$ $\{(H_\varepsilon(A^m); A^m, u(A^m))\}$ with K-theory in X. The based analogue of Proposition 5.2 identifies $V\langle v_0\rangle^1_X(A,\varepsilon)$ $\{V^X_1(A,\varepsilon)\}$ with the Witt group of non-singular even ε-symmetric {ε-quadratic} formations over A with K-theory in X. Given unitary automorphisms $u, v\colon H^\varepsilon(A^m) \to H^\varepsilon(A^m)$ there is defined an isomorphism of formations

$$u\colon (H^\varepsilon(A^m); A^m, v(A^m)) \to (H^\varepsilon(A^m); u(A^m), uv(A^m)).$$

Thus

$$\begin{aligned}(H^\varepsilon(A^m); A^m, uv(A^m)) &= (H^\varepsilon(A^m); A^m, u(A^m)) \oplus (H^\varepsilon(A^m); u(A^m), uv(A^m)) \\ &= (H^\varepsilon(A^m); A^m, u(A^m)) \oplus (H^\varepsilon(A^m); A^m, v(A^m)) \\ &\qquad \in V\langle v_0\rangle^1_X(A,\varepsilon),\end{aligned}$$

and we have a well-defined group morphism

$$f^*\colon \mathscr{U}^*_X(A,\varepsilon) \to V\langle v_0\rangle^1_X(A,\varepsilon);$$
$$(u\colon H^\varepsilon(A^m) \to H^\varepsilon(A^m)) \mapsto (H^\varepsilon(A^m);\, A^m, u(A^m)).$$

Similarly, there is a well-defined group morphism

$$f_*\colon \mathscr{U}^X_*(A,\varepsilon) \to V^X_1(A,\varepsilon);$$
$$(u\colon H_\varepsilon(A^m) \to H_\varepsilon(A^m)) \mapsto (H_\varepsilon(A^m);\, A^m, u(A^m)).$$

By Proposition 2.2 every non-singular even ε-symmetric $\{\varepsilon$-quadratic$\}$ formation over A with K-theory in X is isomorphic to $(H^\varepsilon(A^m);\, A^m, u(A^m))$ $\{(H_\varepsilon(A^m);\, A^m, u(A^m))\}$, for some unitary automorphism u, so that f^* $\{f_*\}$ is onto. The given generators of $\mathscr{E}\mathscr{U}^*_X(A,\varepsilon)$ $\{\mathscr{E}\mathscr{U}^X_*(A,\varepsilon)\}$ are sent to 0 by f^* $\{f_*\}$, so that $\mathscr{E}\mathscr{U}^*_X(A,\varepsilon) \subseteq \ker(f^*)$ $\{\mathscr{E}\mathscr{U}^X_*(A,\varepsilon) \subseteq \ker(f_*)\}$. The based analogue of Proposition 5.4(ii) characterizes the unitary automorphisms

$$u\colon H^\varepsilon(A^m) \to H^\varepsilon(A^m) \quad \{u\colon H_\varepsilon(A^m) \to H_\varepsilon(A^m)\}$$

such that $u \in \ker(f^*)$ $\{u \in \ker(f_*)\}$ as precisely those for which the formation $(H^\varepsilon(A^m);\, A^m, u(A^m))$ $\{(H_\varepsilon(A^m);\, A^m, u(A^m))\}$ is stably isomorphic to the boundary $\partial(A^{m'},\varphi)$ of a $(-\varepsilon)$-symmetric $\{$even $(-\varepsilon)$-symmetric$\}$ form over A $(A^{m'},\varphi)$, or equivalently those for which $u \oplus \sigma_n$ for some $n \geqslant 0$ admits a product decomposition as in (iii). Such unitary automorphisms belong to $\mathscr{E}\mathscr{U}^*_X(A,\varepsilon)$ $\{\mathscr{E}\mathscr{U}^X_*(A,\varepsilon)\}$, and so

$$\begin{cases} [\mathscr{U}^*_X(A,\varepsilon), \mathscr{U}^*_X(A,\varepsilon)] \subseteq \ker(f^*) = \mathscr{E}\mathscr{U}^*_X(A,\varepsilon), \\ [\mathscr{U}^X_*(A,\varepsilon), \mathscr{U}^X_*(A,\varepsilon)] \subseteq \ker(f_*) = \mathscr{E}\mathscr{U}^X_*(A,\varepsilon). \end{cases}$$

The original definition of the odd-dimensional surgery obstruction groups of Wall [**25**] was given by

$$L^s_{2i+1}(\pi,w) = \mathscr{U}^{\{\pi\}}_*(\mathbf{Z}[\pi],(-)^i)/\mathscr{E}\mathscr{U}^{\{\pi\}}_*(\mathbf{Z}[\pi],(-)^i) \quad (i \ (\mathrm{mod}\ 2)),$$

using the w-twisted involution on the group ring $\mathbf{Z}[\pi]$. The inclusion $[\mathscr{U}^*,\mathscr{U}^*] \subseteq \mathscr{E}\mathscr{U}^*$ $\{[\mathscr{U}_*,\mathscr{U}_*] \subseteq \mathscr{E}\mathscr{U}_*\}$ was first obtained by Wasserstein [**28**] $\{$Wall [**25**]$\}$ using explicit matrix identities. (The quotient $\mathscr{U}^*/[\mathscr{E}\mathscr{U}^*,\mathscr{E}\mathscr{U}^*]$ $\{\mathscr{U}_*/[\mathscr{E}\mathscr{U}_*,\mathscr{E}\mathscr{U}_*]\}$ is generated by σ_1, so has order at most 2.) The 'Bruhat decomposition' of $\mathscr{E}\mathscr{U}_*$ given by Proposition 9.2(iii) is the improvement due to Wall [**27**] on the 'normal form' of Sharpe [**22**]. The extra structure carried by a 'split unitary automorphism' $u\colon H_\varepsilon(F) \to H_\varepsilon(F)$ in the sense of Sharpe [**22**] corresponds in our terminology to a choice of hessian $\theta \in Q_{-\varepsilon}(G)$ for the lagrangian $G = u(F)$ in the non-singular ε-quadratic formation $(H_\varepsilon(F);\, F, G)$, and so determines a split ε-quadratic formation (F,G).

10. Laurent extensions

The *Laurent extension* $A[z,z^{-1}]$ of a ring with involution A is the ring of finite polynomials $\sum_{j=-\infty}^{\infty} a_j z^j$ $(a_j \in A)$ in a central invertible indeterminate z over A with involution by $\bar{z} = z^{-1}$, that is

$$-: A[z,z^{-1}] \to A[z,z^{-1}];\ \sum_j a_j z^j \mapsto \sum_j \bar{a}_j z^{-j}.$$

PROPOSITION 10.1. *The free ε-quadratic L-groups $V_*(A[z,z^{-1}],\varepsilon)$ of the Laurent extension $A[z,z^{-1}]$ of a ring with involution A are such that*

$$V_n(A[z,z^{-1}],\varepsilon) = V_n(A,\varepsilon) \oplus U_{n-1}(A,\varepsilon) \quad (n \in \mathbf{Z}),$$

with $U_*(A,\varepsilon)$ the projective ε-quadratic L-groups.

Proof. A splitting theorem of this type was first obtained for the surgery obstruction groups

$$L_n^s(\pi \times \mathbf{Z}) = L_n^s(\pi) \oplus L_{n-1}^h(\pi)$$

by Shaneson [**21**], using geometric methods. A splitting theorem for the quadratic L-groups of arbitrary rings A was then obtained by Novikov [**12**] (modulo 2-primary torsion, with $\frac{1}{2} \in A$) and Ranicki [**14**], using purely algebraic methods. In particular, in [**14**] there were defined natural isomorphisms

$$(\bar{e}\ \ \bar{B}): V_n(A,\varepsilon) \oplus U_{n-1}(A,\varepsilon) \to V_n(A[z,z^{-1}],\varepsilon) \quad (n \in \mathbf{Z})$$

with $\bar{e}$ the split injection induced functorially by the split injection of rings with involution

$$\bar{e}: A \to A[z,z^{-1}];\ a \mapsto a.$$

Only the case where $\varepsilon = \pm 1 \in A$ was considered there, but the methods apply for all $\varepsilon \in A$.

In Part II we shall associate an n-dimensional geometric Poincaré complex over $\mathbf{Z}[\pi_1(X)]$, $\sigma^*(X) = (C(\tilde{X}), \varphi \in Q^n(C(\tilde{X})))$, to the universal cover $\tilde{X}$ of an n-dimensional geometric Poincaré complex X. In particular, for the circle $X = S^1$ we have the 1-dimensional symmetric Poincaré complex over $\mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[z,z^{-1}]$

$$\sigma^*(S^1) = (C, \varphi \in Q^1(C))$$

defined by

$$C_r = \begin{cases} \mathbf{Z}[\mathbf{Z}] & \text{if } r = 0,1, \\ 0 & \text{if } r \neq 0,1, \end{cases} \qquad d = 1-z: C_1 \to C_0,$$

$$\varphi_0 = \begin{cases} 1: C^1 \to C_0, \\ z^{-1}: C^0 \to C_1, \end{cases} \qquad \varphi_1 = 1: C^1 \to C_1,$$

corresponding to the non-singular symmetric formation over $\mathbf{Z}[\mathbf{Z}]$,

$$\sigma^*(S^1) = (H^+(\mathbf{Z}[\mathbf{Z}]^*, 1);\ \mathbf{Z}[\mathbf{Z}], \{(x, (z-1)x) \in \mathbf{Z}[\mathbf{Z}] \oplus \mathbf{Z}[\mathbf{Z}]^* \mid x \in \mathbf{Z}[\mathbf{Z}]\}).$$

The split injections appearing in Proposition 10.1 are precisely the products

$$\bar{B} = \sigma^*(S^1) \otimes - : U_{n-1}(A, \varepsilon) \to V_n(A[z, z^{-1}], \varepsilon) \quad (n \in \mathbf{Z}),$$

identifying

$$\mathbf{Z}[\mathbf{Z}] \otimes_{\mathbf{Z}} A = A[z, z^{-1}].$$

For any ring with involution A let us write $\bar{T} = T_{-1}$ and

$$\begin{cases} \bar{L}^n(A) = L^n(A, -1) \\ \bar{L}_n(A) = L_n(A, -1)\ (= L_{n+2}(A)) \end{cases} \quad (n \in \mathbf{Z}).$$

PROPOSITION 10.2. *The $(k+1)$-fold skew-suspension maps*

$$\bar{S}^{k+1}\colon \bar{L}^0(\mathbf{Z}[\mathbf{Z}^2]) \to L^{2k+2}(\mathbf{Z}[\mathbf{Z}^2], (-)^k) \quad (k \geqslant 0)$$

are not isomorphisms, with $\bar{S}^k(\sigma^*(S^1) \otimes \sigma^*(S^1)) \notin \operatorname{im}(\bar{S}^{k+1})$.

Proof. Consider first the case where $k = 0$. (Here, we can interpret the result as stating that it is not possible to make the symmetric Poincaré complex of the torus $\sigma^*(S^1 \times S^1) = \sigma^*(S^1) \otimes \sigma^*(S^1)$ highly-connected by algebraic surgery.) The products of Proposition 8.1 fit into a commutative diagram

$$\begin{array}{ccc} \bar{L}_0(\mathbf{Z}[\mathbf{Z}^2]) \otimes_{\mathbf{Z}} \bar{L}_0(\mathbf{Z}) & \xrightarrow{1 \otimes (1+\bar{T})} & \bar{L}_0(\mathbf{Z}[\mathbf{Z}^2]) \otimes_{\mathbf{Z}} \bar{L}^0(\mathbf{Z}) \\ \Big\downarrow{\scriptstyle (1+\bar{T}) \otimes 1} & & \Big\downarrow{\scriptstyle \otimes} \\ \bar{L}^0(\mathbf{Z}[\mathbf{Z}^2]) \otimes_{\mathbf{Z}} \bar{L}_0(\mathbf{Z}) & \xrightarrow{\otimes} & L_0(\mathbf{Z}[\mathbf{Z}^2]) \\ \Big\downarrow{\scriptstyle \bar{S} \otimes 1} & & \Big\downarrow{\scriptstyle \bar{S}} \\ L^2(\mathbf{Z}[\mathbf{Z}^2]) \otimes_{\mathbf{Z}} \bar{L}_0(\mathbf{Z}) & \xrightarrow{\otimes} & \bar{L}_2(\mathbf{Z}[\mathbf{Z}^2]) \end{array}$$

The skew-symmetrization map $(1+\bar{T})\colon \bar{L}_0(\mathbf{Z}[\mathbf{Z}^2]) \to \bar{L}^0(\mathbf{Z}[\mathbf{Z}^2])$ is onto, by Proposition 6.2. Thus if $\sigma^*(S^1 \times S^1) \in \operatorname{im}(\bar{S}\colon \bar{L}^0(\mathbf{Z}[\mathbf{Z}^2]) \to L^2(\mathbf{Z}[\mathbf{Z}^2]))$ there exists an element $x \in \bar{L}_0(\mathbf{Z}[\mathbf{Z}^2])$ such that

$$\sigma^*(S^1 \times S^1) = \bar{S}(1+\bar{T})(x) \in L^2(\mathbf{Z}[\mathbf{Z}^2]).$$

The Arf invariant 1 element $c \in \bar{L}_0(\mathbf{Z}) = \mathbf{Z}_2$ is such that

$$\sigma^*(S^1 \times S^1) \otimes c = \bar{B}^2(c) = (0, 0, 0, 1) \neq 0 \in \bar{L}_2(\mathbf{Z}[\mathbf{Z}^2]) = \mathbf{Z} \oplus 0 \oplus 0 \oplus \mathbf{Z}_2$$

by Proposition 10.1. On the other hand, it follows from the above diagram that

$$\sigma^*(S^1 \times S^1) \otimes c = \bar{S}(x \otimes (1+\bar{T})(c)) = 0 \in \bar{L}_2(\mathbf{Z}[\mathbf{Z}^2]),$$

since

$$(1+\bar{T})(c) = 0 \in \bar{L}^0(\mathbf{Z}) = 0.$$

This is a contradiction, so there is no such $x \in \bar{L}_0(\mathbf{Z}[\mathbf{Z}^2])$ and

$$\sigma^*(S^1 \times S^1) \notin \operatorname{im}(\bar{S}).$$

For general $k \geqslant 0$ observe that $\bar{S}\colon L^2(\mathbf{Z}[\mathbf{Z}^2]) \to \bar{L}^4(\mathbf{Z}[\mathbf{Z}^2])$ is an isomorphism by Proposition 6.2, and that

$$\bar{S}^j\colon \bar{L}^4(\mathbf{Z}[\mathbf{Z}^2]) \to L^{4+2j}(\mathbf{Z}[\mathbf{Z}^2], (-)^{j+1}) \quad (j \geqslant 1)$$

is an isomorphism by Proposition 4.5, since $\mathbf{Z}[\mathbf{Z}^2]$ is noetherian of global dimension 3.

The method of Proposition 10.2 can be used to obtain another failure of 4-periodicity in the symmetric L-groups, involving the deRham invariant $d \in L^1(\mathbf{Z}) = \mathbf{Z}_2$ instead of the Arf invariant $c \in L_2(\mathbf{Z}) = \mathbf{Z}_2$, as follows.

PROPOSITION 10.3. *The $(k+1)$-fold skew-suspension map*

$$\bar{S}^{k+1}\colon \bar{L}^0(\mathbf{Z}[\mathbf{Z}]) = 0 \to L^{2k+2}(\mathbf{Z}[\mathbf{Z}], (-)^k) \quad (k \geqslant 0)$$

is not onto, with $\bar{S}^k\bar{B}(d) \notin \operatorname{im}(\bar{S}^{k+1})) = 0$.

Proof. Let $\mathbf{Z}[\mathbf{Z}^-]$ denote the ring $\mathbf{Z}[z, z^{-1}]$ with involution

$$\bar{z} = -z^{-1}.$$

By Theorem V.1 of Morgan [**11**] we have that the product

$$d \otimes - : L_3(\mathbf{Z}[\mathbf{Z}^-]) = \mathbf{Z}_2 \to L_4(\mathbf{Z}[\mathbf{Z}^-]) = \mathbf{Z}_2$$

is an isomorphism. It follows that the product by

$$\bar{S}^k\bar{B}(d) \in L^{2k+2}(\mathbf{Z}[\mathbf{Z}], (-)^k),$$

$$\bar{S}^k\bar{B}(d) \otimes - : L_3(\mathbf{Z}[\mathbf{Z}^-]) = \mathbf{Z}_2 \to L_{2k+5}(\mathbf{Z}[\mathbf{Z} \times \mathbf{Z}^-], (-)^k) = \mathbf{Z}_2$$

is an isomorphism for each $k \geqslant 0$, and hence that $\bar{S}^k\bar{B}(d) \neq 0$.

The symmetrization functor $1+T$ embeds the category of quadratic forms over a group ring $\mathbf{Z}[\pi]$ with the untwisted involution (which is the same as the category of even symmetric forms over $\mathbf{Z}[\pi]$) in the category of symmetric forms over $\mathbf{Z}[\pi]$. Nevertheless, it need not be the case that the symmetrization map of Witt groups $(1+T)\colon L_0(\mathbf{Z}[\pi]) \to L^0(\mathbf{Z}[\pi])$ is one-to-one, as shown by the following example.

PROPOSITION 10.4. *The symmetrization map*

$$1+T\colon L_0(\mathbf{Z}[\mathbf{Z}^2]) \to L^0(\mathbf{Z}[\mathbf{Z}^2])$$

is not one–one, with $\bar{B}^2(c) \in \ker(1+T)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} L^2(\mathbf{Z}[\mathbf{Z}^2])\otimes_{\mathbf{Z}}\bar{L}_0(\mathbf{Z}) & \xrightarrow{\otimes} & \bar{L}_2(\mathbf{Z}[\mathbf{Z}^2]) & \xleftarrow{\bar{S}} & L_0(\mathbf{Z}[\mathbf{Z}^2]) \\ \Big\downarrow{\scriptstyle 1\otimes(1+\bar{T})} & & \Big\downarrow{\scriptstyle 1+\bar{T}} & & \Big\downarrow{\scriptstyle 1+T} \\ L^2(\mathbf{Z}[\mathbf{Z}^2])\otimes_{\mathbf{Z}}\bar{L}^0(\mathbf{Z}) & \xrightarrow{\otimes} & \bar{L}^2(\mathbf{Z}[\mathbf{Z}^2]) & \xleftarrow{\bar{S}} & L^0(\mathbf{Z}[\mathbf{Z}^2]) \end{array}$$

in which the skew-suspensions $\bar{S}$ are isomorphisms (by Propositions 4.3 and 6.2(ii)), and $\bar{L}^0(\mathbf{Z}) = 0$. The Arf invariant element $c \in \bar{L}_0(\mathbf{Z}) = \mathbf{Z}_2$ is such that

$$(1+T)\bar{S}^{-1}\bar{B}^2(c) = \bar{S}^{-1}(1+\bar{T})\bar{B}^2(c) = \bar{S}^{-1}\bar{B}^2(1+\bar{T})(c) = 0 \in L^0(\mathbf{Z}[\mathbf{Z}^2]).$$

Write the product by $\sigma^*(S^1) \in L^1(\mathbf{Z}[\mathbf{Z}])$ on the ε-symmetric L-groups as

$$\bar{B} = \sigma^*(S^1)\otimes - : U^n(A,\varepsilon) \to V^{n+1}(A[z,z^{-1}],\varepsilon) \quad (n \in \mathbf{Z}).$$

Conjecture. The natural maps

$$(\bar{e} \quad \bar{B}) \colon V^n(A,\varepsilon)\oplus U^{n-1}(A,\varepsilon) \to V^n(A[z,z^{-1}],\varepsilon)$$

are isomorphisms for all $A,\varepsilon,n \in \mathbf{Z}$.

Proposition 10.1 verifies the conjecture in the ε-quadratic range $n \leqslant -3$. The methods of Ranicki [**14**] can be extended to prove the conjecture in the range $-2 \leqslant n \leqslant 1$ when the L-groups can be expressed in terms of forms and formations. In particular, combined with the computation of $L^*(\mathbf{Z})$ (Proposition 7.2) the conjecture would give

$$L^2(\mathbf{Z}[\mathbf{Z}^2]) = L^2(\mathbf{Z})\oplus L^1(\mathbf{Z})\oplus L^1(\mathbf{Z})\oplus L^0(\mathbf{Z}) = 0\oplus \mathbf{Z}_2\oplus \mathbf{Z}_2\oplus \mathbf{Z}$$

with $\sigma^*(S^1\times S^1) = (0,0,0,1) \in L^2(\mathbf{Z}[\mathbf{Z}^2])$. Propositions 6.2 and 10.1 give that

$$\bar{L}^0(\mathbf{Z}[\mathbf{Z}^2]) = \bar{L}^0(\mathbf{Z})\oplus L_1(\mathbf{Z})\oplus L_1(\mathbf{Z})\oplus L_0(\mathbf{Z}) = 0\oplus 0\oplus 0\oplus 8\mathbf{Z},$$

where $8\mathbf{Z}$ denotes the subgroup $L_0(\mathbf{Z}) = 8\mathbf{Z} \subset L^0(\mathbf{Z}) = \mathbf{Z}$. The conjecture relates the failure of periodicity $\bar{L}^0(\mathbf{Z}[\mathbf{Z}^2]) \neq L^2(\mathbf{Z}[\mathbf{Z}^2])$ described by Proposition 10.2 to the familiar inequality $L_0(\mathbf{Z}) \neq L^0(\mathbf{Z})$. For the situation described by Proposition 10.3 the conjecture would give that $\bar{S}^k\bar{B}(d) \in L^{2k+2}(\mathbf{Z}[\mathbf{Z}],(-)^k) = \mathbf{Z}_2$ $(k \geqslant 0)$ is the generator. For the situation of Proposition 10.4 we have

$$L_0(\mathbf{Z}[\mathbf{Z}^2]) = L_0(\mathbf{Z})\oplus \bar{L}_1(\mathbf{Z})\oplus \bar{L}_1(\mathbf{Z})\oplus \bar{L}_0(\mathbf{Z}) = 8\mathbf{Z}\oplus 0\oplus 0\oplus \mathbf{Z}_2,$$

$$L^0(\mathbf{Z}[\mathbf{Z}^2]) = L^0(\mathbf{Z})\oplus \bar{L}^1(\mathbf{Z})\oplus \bar{L}^1(\mathbf{Z})\oplus \bar{L}^0(\mathbf{Z}) = \mathbf{Z}\oplus 0\oplus 0\oplus 0,$$

with $\bar{B}^2(c) = (0,0,0,1) \in L_0(\mathbf{Z}[\mathbf{Z}^2])$.

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