The Allocation of a Prize (Expanded)

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Abstract

Consider agents who undertake costly effort to produce stochastic outputs observable by a principal. The principal can award a prize deterministically to the agent with the highest output, or to all of them with probabilities that are proportional to their outputs. We show that, if there is sufficient diversity in agents' skills relative to the noise on output, then the proportional prize will, in a precise sense, elicit more output on average, than the deterministic prize. Indeed, assuming agents know each others' skills (the complete information case), this result holds when any Nash equilibrium selection, under the proportional prize, is compared with any individually rational selection under the deterministic prize. When there is incomplete information, the result is still true but now we must restrict to Nash selections for both prizes.

We also compute the optimal scheme, from among a natural class of probabilistic schemes, for awarding the prize; namely that which elicits maximal effort from the agents for the least prize. In general the optimal scheme is a monotonic step function which lies "between" the proportional and deterministic schemes. When the competition is over small fractional increments, as happens in the presence of strong contestants whose base levels of production are high, the optimal scheme awards the prize according to the "log of the odds", with odds based upon the proportional prize.

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1 Introduction

Consider agents who undertake costly effort to produce stochastic outputs that are observable, and valued, by a principal. The principal, in exchange, has[1] a "pot of gold" that is valued by the agents. The question is: how should the principal award the gold in order to elicit maximal expected output from the agents? Should he give the entire pot to the best performer? Or should he *a priori* divide the pot into k parts and award these as $1^{st}, 2^{nd}, ..., k^{th}$ prizes to the agents, based upon the rank-order of their outputs? Or is there something else the principal can do?

We propose the following simple scheme. Let the principal "market" the gold to the agents on the understanding that they must pay for it with the output they have produced. How the gold gets allocated is then left to market forces. Indeed, suppose that agents $1, ..., n$ have put up supplies of $x_1, ..., x_n$ units of output; and that the principal has put up y units of gold on the other side of the market. The only price p, of the output in terms of gold, which will "clear" the market is[2] $p = y/(x_1 + \cdots + x_n)$, and this is tantamount to handing out the gold y to the agents in proportion to the quantities they have put up[3].

Note that this scheme also makes sense when the pot is indivisible. In this event, what is being marketed is the *probability* of winning the entire pot y. We shall indeed couch our analysis in terms of the indivisible prize rather than the divisible pot of gold (the two are isomorphic). And, for this reason, when the entire pot goes to the highest output, we shall refer to it as the "deterministic scheme / prize", though it is deterministic only in the outputs, and not necessarily in the effort undertaken by the contestants, since output may be a random function of effort.

We first compare the proportional (marketed) prize π_P to the deteministic prize π_D, which in turn is often better than multiple *a priori* fixed prizes. (see (22), and also subsection 7.3). Our main result here is that, if there is sufficient diversity in agents' characteristics, then — in a sense about to be made precise — *the proportional prize elicits more expected total output from the agents than the deterministic prize.*

What is essential for our analysis is that agents' performance be susceptible to quantification in terms of some tangible output produced or, more generally, a "score". This often obtains in practice. For instance, a manager can consider total revenue earned as the criterion to award a badge of honor, or promotion to a higher

[1] To borrow the vision from (22)

[2] the total demand for gold is $px_1 + \cdots + px_n$ which must equal the supply y

[3] To continue the propaganda, the proportional scheme is the only one which is *non-manipulable* in the following sense: if an agent pretends to be several agents by splitting his output to be sent out in different names, this can be of no benefit to him; nor can several agents benefit by merging their outputs and pretending to be one agent (see M.A.de Frutos (1999)).

echelon, to the best salesman of the year. In a race, the time taken for completion comes naturally to mind. Sometimes scores are of a more subtle structure: in a gymnastics contest each member of a jury gives subjective scores to different aspects of performance which are then aggregated to come up with final scores. (The reader can no doubt think of many other examples.) One upshot of assigning numerical scores, and perhaps the reason why they are so prevalent, is that they enable us to judge not only who beat whom, but by how much. Was the race keenly contested or one-sided? What was the margin of victory? These are questions that are often not without meaning, and amenable to plausible answers, which is reflected in the way scores get defined in practice.

Turning to prizes based on scores, the use of the deterministic prize π_D is an established tradition, and it has been well studied by economic theorists (see the literature survey in subsection 1.1 below). However, in principle, the prize could be given with different probabilities to the contestants based upon the scores that they achieve, opening up for consideration a wide class of schemes (see section 9), of which π_D is but one. The proportional prize π_P, which we first focus on and juxtapose with π_D, is equivalent to putting up "lottery tickets" at the market, which the contestants can "buy" with their scores. The use of lotteries to award prizes is also extremely widespread, but it has not received much attention from theorists, except in the context of lobbying (see, again, Section 1.1).

The proportional scheme π_P is our proxy for awarding the prize in a manner that is less drastic than the deterministic π_D, and more commensurate with performance. Any scheme close to π_P (in the bounded variation norm) will inherit its properties. So, for our purposes, the precision with which probabilities of winning the prize are defined does not really matter, so long as they do not stray too far from proportionality; and, in the same vein, minor differences in the delineation of the scores do not disturb our conclusions (see subsection 7.2.) Needless to say, if performances are incapable of being sensibly quantified by scores, and can only be ranked, then the proportional scheme has no meaning and only ordinal schemes (i.e., π_D and its variants with multiple deterministic prizes) make sense. (For an excellent treatment of the ordinal case, see (22).) In our model here, as in much of the literature, the principal is presumed to be maximizing the total score (output) of all the agents, so a fortiori he can observe the individual scores that make up the total. It is not so much a matter of observability, but that the cost of observation is small enough to be ignored. This assumption underlies our entire analysis.

We further assume that outputs are *all* that the principal can observe. He does not have knowledge of agents' characteristics (*i.e.*, productive skill, cost of effort, valuation of the prize), nor even of their precise population distribution. Our purpose is to design a *robust* scheme, based on observable outputs alone[4], which does well over a wide range of possible distributions. Both the deterministic and the proportional schemes are robust but, as was said, the proportional scheme inspires better performance.

The intuition for this result is simple and best brought out with two agents who have complete information about each other's characteristics. (We show, in section 8, that our results are not marred when there is incomplete information, *i.e.*, each agent is informed only of his own characteristics and has a probability distribution over those of his rivals.) Suppose the deterministic prize π_D is in use and that the two agents' skills are sufficiently disparate so that the weak cannot produce more than the strong, with any significant probability, even if he works hard and the other slackens. Since effort is costly, the upshot is an equilibrium at which both agents undertake low effort, so that total output is also low. In contrast, the proportional prize π_P generates better incentives to work. By increasing effort and producing more output, the weak agent is able to achieve a decent increment in his probability of winning the prize, even when his output always lags behind his rival's. Therefore he is inspired to work and creates the competition which also spurs his rival to work, culminating in an equilibrium where effort and output are high. That an egalitarian scheme, which distributes rewards commensurate with output produced, will often generate better incentives to work than an elitist scheme in which the rewards are reserved for the top few — this, in our view, is a theme of wide-ranging application in the presence of heterogeneous agents, and it runs like a leitmotif in the design of mechanisms in different contexts (see, *e.g.,* (13), (12), (11)).

On the other hand, when skills are similar (think of athletic stars competing in the Olympics), π_D will clearly elicit more effort than π_P. For if both work, they come out with nearly equal probabilities of winning the prize under either scheme. But if anyone slackens, his probability drops sharply under π_D, and less so under π_P. Thus there is more to lose by slackening when π_D is in use.

Now if agents' skills are picked at random from a sufficiently "diverse" set, and the noise on output is not so large as to overwhelm skills and make them count for little, then the probability that agents are similar will tend to be low. Therefore the average output will go up when π_P replaces π_D. In fact we show that this is the case when any Nash Equilibrium (NE) selection under π_P is compared with

[4]Vague as this may sound, it is also desirable that the scheme be *simple,* which is a "feel" one gets about both π_D and π_P. The restriction to schemes that are based on outputs alone, does help to put a lid on their complexity. In general (appealing to the "revelation principle") one could require agents to report their characteristics and base the allocation of the prize on these reports, truthful or not. But the authors could not see tractable schemes in this direction.

any individually rational (IR) selection under π_D. Furthermore, when π_P replaces π_D, an impoverished majority of non-elite agents, who were idle before but are now incentivized to work, are made better off at the expense of the elite coterie (see subsection 7.1). Were the principal to ask for a vote, π_P would win with a thumping majority over π_D. And indeed why would he not ask, seeing that π_P elicits so much more output for him?

In Section 8, we show that our theme remains intact when there is incomplete information among the agents: the NE-selection under π_P elicits more output compared to the NE-selection under π_D, as long as the noise on output is not too large compared to the diversity of agents' skills. (We write "the NE" because, in the binary games of Section 8, NE's do turn out to be unique.)

So far the scheme (π_P or π_D) was taken to be fixed and the behavior (NE or IR) induced by it was examined. In Section 9, we adopt the reverse approach: behavior is fixed at maximal effort and our focus is on schemes that implement it[5] as NE. More precisely, we consider a natural class of probabilistic schemes for handing out the prize, which includes the deterministic and proportional schemes as special cases. Then, fixing an arbitrary domain of agents'characteristics, for each scheme there is a threshold (possibly infinity) such that the scheme will implement maximal effort as an NE on the domain if, and only if, the value of the prize exceeds the threshold. Thus schemes may be ranked via their thresholds, and the one with the smallest threshold will be optimal: it will Nash-implement maximal effort whenever *any* other scheme does so.[6] There is clearly no problem regarding the *existence* of such an optimal — or, at least, nearly optimal — scheme. The challenge is to uncover its *structure.* We do so for two special domains. The first is a binary set-up with two agents and two effort levels (low, high), in which agents' skills can be ordered so as to exhibit "decreasing, or increasing, returns". The optimal scheme turns out to be a monotonic step function, whose graph lies in between those of the proportional and the deterministic schemes. Next we analyse the binary model with the added

[5] Implementing maximal effort is consistent with maximization of expected total output if we make the implicit "background" assumption that the principal values outputs sufficiently highly compared to the prize he must hand out to compensate agents for their effort.

There is, however, an alternative interpretation one might consider. The agents will clearly work hard provided they value the prize sufficiently highly. In the context of the infinitely many worlds defined by the variation of agents' characteristics, including their valuation of the prize, we say that π is optimal if it induces maximal effort most frequently,i.e., *whenever* any other scheme does so.

[6] In contrast, in the earlier approach (of fixed schemes and variable behavior), two schemes may well become incomparable on account of the multiplicity of NE: either may elicit more output than the other, depending on *which* pair of NE is examined.

proviso that agents' base skills are so strong (think again of champions, or stars, or experts) that the *percentage* gain in output, when an agent switches from low to high effort, is small (even though, on the absolute scale, these gains may be substantial enough to enable meaningful comparisons between the two agents). In this scenario we show that the optimal scheme awards the prize according to the "log of the odds", with odds based upon the proportional scheme. Moreover the optimal scheme does *not* depend on the distribution of skills of the agents, except insofar as they exhibit decreasing or increasing returns.

Finally let us note that a precursor to this paper is (10).

Related Literature. There is a literature on lobbying, where agents put up bids of money and are awarded the prize either via the proportional scheme or the deterministic scheme (called often "lottery" or "all-pay auctions", respectively). See, e.g., (28),(19),(14),(26),(27),(3),(4), (8),(9),(24),(15) and the references therein. In much of this literature agents are assumed to have complete information about each other, and in all of it there is no issue of "moral hazard", i.e., the bids submitted by the agents are perfectly observable.

The literature on tournaments is vast and does often emphasize moral hazard, i.e., the setting in which observable outputs depend stochastically on unobservable effort ("bids"). However proportional prizes do not seem to have received attention there. For tournaments with a single prize, see (21),(18),(23),(25). Subsequent writers have considered multiple prizes whose number and sizes are fixed prior to the contest, and which are then awarded to the contestants based upon the rank-order of their performance ((17), (5), (1), (7), (20), (6), (2), (22)).

In both strands of literature the focus is on analyzing Nash Equilibria (NE), which are often unique and susceptible of being described by explicit formulae, given the special structural assumptions of the models.

What is new in our approach is that we compare the proportional and deterministic prizes in the presence of moral hazard. Our setting is sufficiently general so as neither to preclude multiple NE, nor to guarantee pure-strategy NE. No assumptions are made on disutility or productivity other than the fact that they are monotonic in effort in the appropriate sense; in particular they are *not* required to be concave or convex. Nevertheless we are able to show that the worst NE selection under the proportional prize elicits more output than the best NE under the deterministic prize. In fact we show more, since our comparison is based on "Weak Nash Strategies" (see subsection 5.1) and IR strategies, which are looser notions than NE (indeed IR is so mild a reqirement that *any* solution concept would be expected to satisfy it). To the extent that this constrains agents' behavior less, our comparison is that much stronger (more credible?). Of course, the price we pay for our generality is that we stop at this comparison, and are unable to discern any finer structure in agents' behavior, which would come to the fore were one to confine attention to NE, especially in simple scenarios where they are unique (as happens in some of the structured examples we study here).

The Numbering System. All definitions, axioms, lemmas, theorems are taken to constitute a *single* series, and enumerated in the order they first appear. Thus the reader will see, starting in the next section, Axiom 1, Axiom 2, Theorem 3, Axiom 4, Lemma 5 etc. Here Lemma 5 does not mean the "fifth" lemma, but the lemma whose "name" (or "marker") in the series is 5 .

2 The Model

Each **agent** in our model has access to a finite subset $E \subset [0,1]$ of effort levels. We assume $0 \in E$ and $1 \in E$. These represent no effort and maximal effort respectively.

An agent may choose any effort $x \in E$. In doing so, he incurs disutility $\delta(x) \geq 0$ and produces stochastic output given by a non-negative random variable $\tau(x)$ with finite mean $\mu(x)$. (We allow for the possibility that the range of $\tau(x)$ is discrete, even finite.) Effort 0 incurs disutility $\delta(0) = 0$ and produces output $\tau(0) = 0$ with certainty: it is just a proxy for "not participating" in the game.

Agents are driven to work by the lure of an indivisible prize, which is handed out to them by a prinicpal. If an agent places valuation $v > 0$ on the prize, and is awarded it with probability p, this yields him expected utility pv. (See, however, the subsection 7.2, where it is shown that the tenor of our results remains unchanged for a wider class of utilities.)

The triple (δ, τ, v) characterizes an agent. We make *throughout* the following monotonicity and boundedness assumptions on the space[7] X of possible **characteristics** (δ, τ, v):

Axiom 1 *Both δ, μ are weakly monotonic in x and there exist universal positive constants c, C, d, D such that, for all $x \in E \setminus \{0\}$,*

$$cx < \delta(x) < Cx \tag{1}$$

[7]This space X is defined after fixing the domain and range of τ. It will shortly be taken to be measurable. One can confine attention to random variables τ which are characterized by finitely many parameters, so that (δ, τ, v) is a finite-dimensional vector; and then the Euclidean space generates the Borel sets. In this case the space X consists of all (δ, τ, v) that satisfy (1) and (2) of Axiom1 below, along with the aforesaid finiteness restrictions on τ . More generally, without such restrictions, the Levy-Prokhorov metric on the random variables τ is understood to define the Borel sets.

and

$$dx < \mu(x) < Dx \tag{2}$$

(Note that, on account of *weak* monotonicity, there is no loss of generality in supposing that all agents have the same set E of effort levels. The case of an arbitrary allocation of subsets of E across agents is automatically included, provided that 0 and 1 belong to each agent's set.)

Suppose now that we have a finite set N of agents with characteristics $(\delta^n, \tau^n, v^n)_{n\in N}$. The **principal** cannot observe these characteristics, or the effort levels $(e^n)_{n\in N}$ that the agents might have undertaken; all he can see are the realizations $t = (t^n)_{n\in N}$ of the random outputs $(\tau^n(e^n))_{n\in N}$. Thus his **allocation** π **of the prize** is given by a function $\mathbf{R}_+^N \setminus \{0\} \xrightarrow{\pi} \Delta^N$ where Δ^N is the unit simplex in $\mathbf{R}^N$; the component $\pi^n(t)$, of the vector $\pi(t)$, denotes the probability with which $n \in N$ is allocated the prize. We further assume that $\pi^n(t) = 0$ for all $n \in N$ if $t = 0$, otherwise agents would be rewarded for not participating in the game.

The principal is risk-neutral and cares only about the expected total output produced by the agents. To this end he can devise different allocation schemes π. A full class Π of such schemes will be considered later in section 9. For the present, we focus on two particular schemes.

The **deterministic scheme** π_D shares the prize equally among the winners $W(t) = \{k \in N : t^k = max\{t^n : n \in N\}\}$, i.e., $\pi_D^n(t) = 1/|W(t)|$ if $n \in W(t)$ and $t \neq 0$; and is 0 if $t = 0$.

(Note that π_D is deterministic only in the outputs, not necessarily in the effort levels.)

The **proportional scheme** π_P awards the prize to each agent in proportion to his output,i.e., $\pi_P^n(t) = t^n/(\sum_{k\in N} t^k)$ if $t \neq 0$; and is 0 if $t = 0$.

3 The Strategic Game of Complete Information

As was said, the principal does not know agents' characteristics, nor even the distribution of their characteristics. He wishes to compare π_D versus π_P over a *large* class of distributions. As for the agents, we at first take them to be well informed. We suppose that, in addition to knowing $\pi = \pi_D$ or π_P, the agents also know each others' characteristics $(\delta^n, \tau^n, v^n)_{n\in N}$. This seems to be a tenable hypothesis if agents compete in close proximity with one another. (In Section 8 we consider the case when an agent knows his own characteristics but is unsure about those of his rivals.)

Given $(\delta^n, \tau^n, v^n)_{n\in N}$, a strategic game is induced among the agents by the principal's choice of an allocation scheme π. The set of pure strategies of each agent

$n \in N$ is E. Any N-tuple of pure strategies $e = (e^n)_{n \in N}$ gives rise to a random vector $\tilde{t} = \tilde{t}\,(e) = (\tau^n(e^n))_{n \in N}$ of outputs. The expected value p^k of $\pi^k(\tilde{t})$ represents the probability of k winning the prize and we define k's payoff to be $F^k(e) = p^k v^k - \delta^k(e^k)$.

Denote by Γ the mixed extension of this game; and by Σ^k the set of (mixed) strategies of k in Γ, i.e. Σ^k is just the set of probability distributions on E. (Without confusion, $F^k(\sigma)$ will continue to denote k's payoff, when the mixed strategy N-tuple $\sigma \in \prod_{n \in N} \Sigma^n \equiv \Sigma$ is played in Γ.)

First we recall three standard concepts. For any $\sigma \equiv (\sigma^n)_{n \in N} \in \Sigma^N$, denote $\sigma^{-n} \equiv \left(\sigma^k\right)_{k \in N \setminus \{n\}} \in \Sigma^{-n} \equiv \prod_{k \in N \setminus \{n\}} \Sigma^k$.

The choice $\sigma \in \Sigma^N$ is **individually rational** (IR) in Γ if

$$F^n(\sigma) \geq \max_{u \in \Sigma^n} \min_{w \in \Sigma^{-n}} F^n(u, w)$$

for all $n \in N$.

The choice $\sigma \in \Sigma^N$ is a **Nash Equilibrium** (NE) of Γ if

$$F^n(\sigma) = \max_{\tilde{\sigma}^n \in \Sigma^n} F^n(\tilde{\sigma}^n, \sigma^{-n})$$

for all $n \in N$.

The choice $\sigma^n \in \Sigma^n$ is **strictly dominant** (SD) for n in Γ if

$$F^n(\sigma^n, v) > F^n(u, v)$$

for all $u \in \Sigma^n \setminus \{\sigma^n\}$ and all $v \in \Sigma^{-n}$.

Finally we introduce a weakening of the notion of NE which will be relevant for us. The idea is to restrict the set of unilateral deviations available to an agent n by only allowing him to shift probabilities (to whatever extent he wishes) from his current strategy σ^n onto maximal effort 1. More precisely, denote

$$\Sigma^n(\sigma^n) = \{\tilde{\sigma}^n \in \Sigma^n : \tilde{\sigma}^n(e) \leq \sigma^n(e) \text{ for all } e \in E \setminus \{1\}\}$$

Then we say that $\sigma \equiv (\sigma^n)_{n \in N}$ is a **weak Nash strategy-tuple** (WNS) if

$$F^n(\sigma) = \max_{\tilde{\sigma}^n \in \Sigma^n(\sigma^n)} F^n(\tilde{\sigma}^n, \sigma^{-n})$$

for all $n \in N$. If the above holds with $\{\sigma^n, 1\}$ in place of $\Sigma^n(\sigma^n)$, we say that σ is a **very weak Nash strategy-tuple** (VWNS). Here the agent n is only permitted to shift all the probabilities from σ^n abruptly onto 1. (Notice that maximal effort 1 is the anchor for both these notions. Indeed $\mathbf{1} \equiv \{1, ..., 1\}$ is always a WNS in any game and hence also a VWNS.)

Let us denote by IR(Γ), NE(Γ), SD(Γ), WNS(Γ), VWNS(Γ), the set of all strategies that are IR, NE, SD, WNS, VWNS in the game Γ. It is evident that

$$SD(\Gamma) \subset NE(\Gamma) \subset IR(\Gamma)$$

and that

$$NE(\Gamma) \subset WNS(\Gamma) \subset VWNS(\Gamma)$$

reflecting the progressively stringent requirements of the definition as we go from IR to NE to SD, or from VWNS to WNS to NE. (Note also that, obviously, SD(Γ)$\neq \emptyset$ implies SD(Γ) = NE(Γ) = a singleton set.)

4 Spaces of Games

Suppose characteristics $\boldsymbol{\chi} \equiv (\delta^n, \tau^n, v^n)_{n\in N}$ are picked from $X \times \cdots \times X \equiv \mathbf{X}$ according to some probability distribution ξ on $\mathbf{X}$. (Throughout, as was said, we assume that the underlying set X satisfies Axiom 1; and that X is a Borel space as explained in footnote 4, so that ξ is a measure on the Borel sets of $\mathbf{X}$,using the product topology from X.) Fix an allocation scheme π. Then any $\boldsymbol{\chi} \in \mathbf{X}$ induces a mixed-strategy game among the agents (as discussed in section 3), which we shall denote $\Gamma_\pi(\boldsymbol{\chi})$. We wish to extend our solution concepts to the space of games specified by ξ. Our focus will be on what happens for *almost all* $\boldsymbol{\chi}$ according to ξ, denoted $a.a.\boldsymbol{\chi}(\xi)$, i.e., for all $\boldsymbol{\chi}$ except perhaps for those in a set of ξ-measure zero.

Let $f : \mathbf{X} \to \Sigma$ be a measurable function. For each $\boldsymbol{\chi} \in \mathbf{X}$, note that $f(\boldsymbol{\chi})$ is an N-tuple of mixed strategies. Denoting $f(\boldsymbol{\chi}) \equiv (\sigma^n)_{n\in N}$, the total output at $\boldsymbol{\chi}$ is

$$T(f, \boldsymbol{\chi}) \equiv \sum_{n\in N} \sum_{x\in E} \sigma^n(x)\mu^n(x). \tag{3}$$

and integrating over $\mathbf{X}$ according to ξ, the **expected total output** is

$$T(f) \equiv \int_{\mathbf{X}} T(f, \boldsymbol{\chi})\, d\xi(\boldsymbol{\chi}) \tag{4}$$

Given a prize scheme π we will say that $f : \mathbf{X} \to \Sigma$ is an ξ-**NE selection under** π if f is measurable and $f(\boldsymbol{\chi})$ is an NE of the game $\Gamma_\pi(\boldsymbol{\chi})$ for $a.a.\boldsymbol{\chi}(\xi)$. The notion of a ξ-Φ **selection under** π is defined similarly (where $\Phi \equiv$ IR or NE or SD or WNS or VWNS).

5 Proportional Prize: Expected Total Output from Nash Equilibria

It is clear *a priori* that, for any $\boldsymbol{\chi} \in \mathbf{X}$ and any scheme π, the total expected output in $\Gamma_\pi(\boldsymbol{\chi})$, at any $\boldsymbol{\sigma} \in \Sigma$, cannot exceed $|N|D$ since no agent produces more than D when he chooses maximal effort 1 (see Axiom 1). Also[8], supposing $v^n = v$ for all $n \in N$, the total expected disutility incurred by the agents at any individually rational strategy selection cannot exceed v, otherwise some agent is incurring negative utility and would be better off not participating in the game. But then expected total output (see, again, Axiom 1) is at most Dv/c. Thus, the most this output can be is "of the order of" $\min(v, |N|)$, since D and c are constants of our model.

This is the flavor of our estimate in Theorem 3 below, showing that the proportional prize elicits a "decent quantum" of output from the agents. However the theorem requires an additional assumption, which we now describe.

For $\boldsymbol{\chi} = (\delta^n, \tau^n, v^n)_{n \in N}$ denote $\underline{v}(\boldsymbol{\chi}) = \min\{v^n : n \in N\}$ and define $\underline{v}$ to be the essential infimum of $\underline{v}(\boldsymbol{\chi})$ with respect to ξ.

Axiom 2 *(Minimum valuation)* $\underline{v} > DC/d$

This basically says that, for any two individuals picked from the population, if both work at maximal effort and are awarded the prize *proportionately,* then neither will have incentive to unilaterally quit the game — each values the prize sufficiently highly to want to stay in. Indeed, by Axiom 1 the most disadvantaged such individual produces d, incurs disutility C, and values the prize at $\underline{v}$ (while his rival produces D).Thus his reward is $\underline{v}d/(d+D)$ which must exceed C.Our Axiom 2 is somewhat milder.

We now show that Weak Nash Strategies (WNS) elicit a decent quantum of output under the proportional prize.

Theorem 3 *Suppose Axioms 1 and 2 hold. Denote* $e_{\min} \equiv \min\{x : x \in E \setminus \{0\}\}$*. Let* f *be a* ξ*-WNS function under* π_P*. Write* $a \equiv |N| d e_{min}$ *and* $b \equiv (d\underline{v}/C) - D$*, and let* $H \equiv 2ab/(a+b)$ *denote their harmonic mean. Then*

$$T(f) \geq H/2$$

where $T(f)$ *is the expected total output as in (4).*

[8]Given $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$, and a vector $\alpha \equiv (\alpha^n)_{n \in N} >> 0$ of positive scalars, let $\chi(\alpha) \equiv (\alpha^n \delta^n, \tau^n, \alpha^n v^n)$. Then the games $\Gamma_\pi(\chi)$ and $\Gamma_\pi(\chi(\alpha))$ are "strategically equivalent" and all our solution concepts remain the same for them. So w.l.o.g., scaling utilities appropriately, one could imagine $v^n = v$ for all $n \in N$.

(The proof is in the Appendix.)

A variant of theorem 3 for *Very* Weak Nash Strategies (VWNS) may also be of interest.

Axiom 4 *Same as axiom 2 , substituting* $2D$ *for* D*.*

Theorem 5 *Suppose Axioms 1 and 4 hold.The conclusion of theorem 3 holds with VWNS in place of WNS, and* $H_* \equiv 2ab_*/(a+b_*)$*, where* $b_* \equiv (d\underline{v}/C) - 2D$*, in place of* H*.*

(The proof is in the Appendix.)

It might help to see what Theorem 3 implies when the number of players increases. The following immediate Corollary asserts that the expected total output, elicited via WNS-strategy selections by the proportional prize, grows as fast as the minimum value of the prize or the number of players, whichever is smaller (modulo the very minor requirement, given in Axiom 2, that no one values the prize *too* low).

Corollary 6 ***(to Theorem 3)*** *Suppose the set of players is increasing, i.e.,* $|N| \to \infty$*, and the corresponding spaces* (X^N, ξ_N) *satisfy Axioms 1 and 2 with* $\underline{v}(N)$ *in place of* $\underline{v}$ *, and* ξ_N *in place of* ξ*. For each* N*, let* f_N *be a* ξ_N*-WNS-selection under* π_P*. Then*

$$T(f_N) \geq O(min\{|N|, \underline{v}(N)\})$$

Proof: Obvious from Theorem 3.■

5.1 Highly Valued Prizes

In Theorem 3, the maximum value $\bar{v} = max\{v^n : n \in N\}$ of the prize is allowed to be quite small, and then – as was already said – it is not possible to get too many agents to put in significant work under any allocation scheme π, simply because the disutility incurred jointly by them cannot exceed $\bar{v}$. But the value of the prize lies in the eyes of its beholders. Since we are speculating about populations of agents with highly variable characteristics, who will compete under the scheme π_P "for generations to come", we may imagine the scenario when all the agents are of a mind to place high valuations on the prize. Alternatively we can think of the scheme π_P being used to disburse a vast number of different indivisible prizes to the same population of agents, and then focus on the case when the prize is such that it happens to be valued highly by everyone.

In either setting, the mathematical analysis is the same. For $\boldsymbol{\chi} = (\delta^n, \tau^n, v^n)_{n \in N}$ recall that $\underline{v}(\boldsymbol{\chi}) = min\{v^n : n \in N\}$. We will show that, for sufficiently high values of $\underline{v}(\boldsymbol{\chi})$, maximal effort $\mathbf{1} \equiv (1, ..., 1)$ can be implemented in a progressively stronger manner : first as an NE, then as a unique WNS and finally as an "almost-SD" of the game $\Gamma_{\pi_P}(\boldsymbol{\chi})$. Put another way: in order to gain more certainty that agents will work hard, one must incur the cost of enhancing the prize.

For the analysis (see Theorems 8 and 9 below), we need to put an additional constraint on the distribution ξ of agents' characteristics. (Recall that $\mu^n(e)$ denotes the mean of $\tau^n(e)$.)

Axiom 7 *There exist universal positive constants β and $\Delta > 0$ such that for a.a.$\chi(\xi)$, if $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$, then*

$$\mu^n(1) - \mu^n(e) > \Delta$$

for all $e \in E \setminus \{1\}$ and all $n \in N$; and

$$\tau^n(e) < \beta$$

for all $e \in E$ and all $n \in N$.

Theorem 8 *Suppose Axioms 1 and 7 hold. Then there exist thresholds v_* and v^* such that for a.a.$\boldsymbol{\chi}(\xi)$:*

$$\mathbf{1} \text{ is an NE of } \Gamma_{\pi_{\mathrm{P}}}(\chi) \tag{5}$$

whenever $\underline{v}(\boldsymbol{\chi}) > v_*$; and

$$\mathbf{1} \text{ is the unique WNS, hence also the unique NE, of } \Gamma_{\pi_{\mathrm{P}}}(\chi) \tag{6}$$

whenever $\underline{v}(\boldsymbol{\chi}) > v^*$.

(The proof is in the Appendix)

Clearly there is a threshold $\tilde{v}$ (between v_* and v^*) above which **1** becomes the unique NE of $\Gamma_{\pi_P}(\boldsymbol{\chi})$. Moreover, there is another threshold above which it is possible to implement **1** almost as an SD. Fix $\epsilon > 0$ as well as $\chi = (\delta^n, \tau^n, v^n)_{n \in N}$. We shall say that **1** is "**strictly dominant 'up to error** ϵ" in the game $\Gamma_{\pi_P}(\chi)$ if maximal effort is a strictly dominant strategy for each player, conditional on the fact that his rivals' total output is at least ϵ, i.e.,

$$F^n(1|A) > F^n(\sigma^n|A)$$

for all $n \in N$ and all $\sigma^n \in \Sigma^n \setminus \{1\}$ and all $A > \epsilon$, where

$$F^n(\sigma^n|A) \equiv \sum_{e\in E} \sigma^n(e)[Exp_\tau(\frac{\tau^n(e)}{\tau^n(e)+A})v^n - \delta^n(e)]$$

Theorem 9 *Suppose Axioms 1 and 7 hold. Then for any $\epsilon > 0$, there exists $v^{**}(\epsilon)$ such that for a.a.$\boldsymbol{\chi}(\xi)$:*

$$\mathbf{1} \text{ is strictly dominant up to error } \epsilon \tag{7}$$

in the game $\Gamma_{\pi_P}(\boldsymbol{\chi})$, whenever $\underline{v}(\boldsymbol{\chi}) > v^{**}(\epsilon)$.

(The proof is in the Appendix)

5.2 Some Variations of Theorems 3 and 5

The presence of "$\mathrm{e_{min}}$" is a dampener on our lower bound in Theorems 3 and 5, but unavoidable given our extremely weak assumptions. Indeed there is nothing to preclude the scenario that every agent incurs sharply rising disutility of effort as he advances above $\mathrm{e_{min}}$,while his output hardly goes up; and then the best one can hope for is to inspire everyone to work at $\mathrm{e_{min}}$. Were we to strengthen our assumption on productivity, requiring output to go up in significant chunks as we go up the effort ladder from $\mathrm{e_{min}}$ to 1 (in the spirit of Axiom 7), sharper estimates could be reached by the methods of this paper. (We leave this to the reader). Incidentally notice that, in the special case of binary effort levels, i.e., $E = \{0, 1\}$, we automatically have $e_{min} = 1$ in Theorem 1 above, producing a sharp bound without further ado.

6 Deterministic Prize: Expected Output from Individually Rational Strategies

The following "Key Lemma" provides the crucial insight as to why the deterministic prize π_D elicits limited output. Indeed it shows that only the most productive agent, along with those who stand a chance of beating him, set the bound on the output at any individually rational strategy-tuple.

Fix $\boldsymbol{\chi} = (\delta^n, \tau^n, v^n)_{n\in N}$. Denote by h an agent (**the "hero"**) who has maximal mean output under effort level 1, i.e., for all $n \in N$, we have $\mu^h(1) \geq \mu^n(1)$ (where, recall again, $\mu^n(x)$ is the mean of $\tau^n(x)$). Define $K(\boldsymbol{\chi})$ to be the **set of "elite**

agents" whose outputs at effort 1 have a positive probability of exceeding that of h, i.e.,

$$K(\boldsymbol{\chi}) = \{n \in N : Pr[\tau^n(1) \geq \tau^h(1)] > 0\}$$

We shall show that the output under deterministic prize is commensurate with $|K(\boldsymbol{\chi})|$. First we need

Axiom 10

1. ***(Bounded relative valuations)*** *There exists a universal constant B such that for a.a.$\boldsymbol{\chi}(\xi)$, if $\boldsymbol{\chi} = (\delta^n, \tau^n, v^n)_{n \in N}$, then $v^n/v^k < B$ for all $n, k \in N$;*

2. ***(Stochastic dominance)*** *If $x > y$ in E then $\tau^n(x) \succeq \tau^n(y)$, where "$\succeq$" denotes first order stochastic dominance*[9]*.*

Lemma 11 *Suppose Axioms 1 and 10 hold. Let f be a ξ- IR-selection under π_D; then for a.a.$\boldsymbol{\chi}(\xi)$*

$$T(f, \boldsymbol{\chi}) \leq 2|K(\boldsymbol{\chi})|B^2CD/c$$

(The proof is in the Appendix.).

6.1 Estimation of the Average Size of the Elite Set $|K(\boldsymbol{\chi})|$

A natural scenario is that agents' characteristics are not correlated to be similar but are sufficiently "diverse" (e.g., drawn i.i.d. from a large set). We shall, in fact, require this diversity only on their productivities $(\tau^n(1))_{n \in N}$ under maximal effort. This is embodied in Axiom 13 below. First, a definition:

Definition 12 ***(Normalized Density)*** **.***Let Z be a random variable taking values in the n-cube $C_{|N|} = [d, D]^{|N|}$. Let λ denote the standard Lebesgue measure on $C_{|N|}$ scaled by $(D - d)^{-|N|}$. (so that $\lambda(C_{|N|}) = 1$). We say that Z has normalized density function ρ if ρ is Borel-measurable, nonnegative and $\Pr(Z \in A) = \int_A \rho(x) d\lambda(x)$ for all Borel sets $A \subset C_{|N|}$;and we define the upper bound of ρ to be the essential supremum of ρ on $C_{|N|}$.*

We are ready to state

Axiom 13 ***(Diversity of Skills)***

[9] Recall: $\tau^n(\tilde{e}) \succeq \tau^n(e)$ if $\text{Prob}\{\tau^n(\tilde{e}) \geq z\} \geq \text{Prob}\{\tau^n(e) \geq z\}$ for all z $\in$ Range $\tau^n(\tilde{e}) \cup$ Range $\tau^n(e)$

1. *There exists $\epsilon > 0$ such that, for a.a.$\boldsymbol{\chi}(\xi)$, if $\boldsymbol{\chi} = (\delta^n, \tau^n, v^n)_{n \in N}$, then for all $n \in N$: support $\tau^n(1) \subset [\mu^n(1) - \epsilon, \mu^n(1) + \epsilon]$*

2. *As we vary $\boldsymbol{\chi}$ on $\mathbf{X}$ according to ξ, the marginal distribution of the random variable*[10] *$(\mu^n(1))_{n \in N}$ has a normalized density function with finite upper bound β.*

Condition 2 of this assumption rules out the possibility that $(\mu^n(1))_{n \in N}$ is concentrated on the "diagonal" $\left\{(z, ..., z) \in C_{|N|} : d \leqq z \leqq D\right\}$ of the cube $C_{|N|}$. As the random variables $\mu^1(1), ..., \mu^N(1)$ go from being iid, with uniform density on $[d, D]$, to being concentrated on smaller and smaller neighbourhoods of the diagonal, β rises from 1 to ∞.In this scenario β is a measure of how likely it is that the agents are similar. We should expect a threshold β^*such that π_P outperforms π_D if $\beta < \beta^*$,and π_D outperforms π_P if $\beta > \beta^*$.This is not to say that high β is *necessarily* bad for π_P. Indeed if β were high in regions of $C_{|N|}$ where agents are *disparate* (e.g.,towards the northwest or southeast corners of the square, when $|N| = 2$), this would only accentuate the superiority of π_P over π_D We do not follow this general line of inquiry here , wherein β would be allowed to become unbounded in selective regions of $C_{|N|}$, and bound only where agents are similar. Instead we consider the restricted scenario where β is universally bounded on $C_{|N|}$, thereby only preventing agents from being similar (or dissimilar!) with high probability.

Returning to the iid case, we can think of ϵ as the size of the random noise on output, and then the “diversity" of agents' productive skills is reflected for us in how *small* the term $\beta\epsilon = \epsilon/(D - d)^{|N|}$ is. (Diversity in skills is dampened by the noise ϵ. Indeed suppose noise ϵ is symmetric across the two agents and let ϵ grow, keeping skills fixed. The two agents will become increasingly similar since their output will depend essentially on the identical noise term and their skills will count for little when ϵ is sufficiently large). Lemma 1 below shows that the average size of the elite set, is no more than $1 + \beta|N|\epsilon$ in the general setting of Axiom 13.

Lemma 14 *Suppose the distribution ξ satisfies Axiom 13. Then the expected size, under ξ, of the elite set $K(\boldsymbol{\chi})$ is at most $1 + \beta|N|\epsilon$.*

(The proof is in the Appendix.)

We are ready to state the main conclusion of this section.

Theorem 15 *Assume Axioms 1,10 and 13 hold. Let f be a ξ-IR-selection on $\mathbf{X}$ under π_D. Then*

[10] Recall that $(\mu^n(1))_{n \in N} \in C_{|N|}$ by (2).

$$T(f) \leq \frac{2B^2CD}{c}(1 + \beta|N|\epsilon)$$

Proof. Immediate from Lemmas 11 and 14. ■

7 Proportional Versus Deterministic Prizes

Theorems 3 and (**?**) enable an immediate comparison between the (expected total) outputs elicited by WNS, IR strategy selections under π_P, π_D respectively. Fix, for example, all the parameters $c, C, d, D, b, B, \underline{v}$ of the model and suppose that Axioms 1,2,10,13 hold. There exists a threshold $\bar{\epsilon}$ such that, if $\epsilon < \bar{\epsilon}$, then for large enough N and v, we have

$$T(f) > T(g)$$

for any ξ-WNS-selection f under π_P, and any ξ-IR-selection g under π_D.This is so because the lower bound on output given by Theorem 3 is independent of the noise ϵ, and rises with N, v ; while the upper bound given by Theorem 15 is independent of N, v and goes to $2B^2CD/c$ as ϵ goes to 0.

To get a better feel, it might help to consider a numerical example. Let $B = C = c = d = 1, D = 2$, $|N| = 7$, $\underline{v} = 30$, $\epsilon = 0.05$. Further let the set of effort levels be $E = \{0, 1\}$ so that $e_{\min} = 1$; and let the agents' skills be picked iid with uniform probability in the interval $[d, D] = [1, 2]$ so that $\beta = 1$.Thus the noise term is only 5% of the skill interval and does not dampen the diversity between the two agents.

By Theorem 3, the output is bounded *below* (noting $a = 7, b = 28$) by 5.6 at any NE-selection under the proportional prize. On the other hand, by Theorem 15, the output is bounded *above* by $(2B^2CD/c)(1 + \beta|N|\epsilon)) = 4(1 + 7(0.05)) = 5.4$ at any IR- selection under the deterministic prize. Thus the proportional prize outperforms the deterministic.

7.1 Welfare

For simplicity we take $\beta = 1/(D - d)^{|N|}$ in this section and the next section 8.3, i.e., the random variables $\mu^n(1)$ are iid with uniform distribution on $[d, D]$. When the deterministic prize is used, only the agents in the elite coterie $K(\boldsymbol{\chi})$ (whose size is $1 + [|N|\epsilon/(D - d)^{|N|}]$ on average) get the prize with significant probability under any IR strategy tuple. More precisely, the remaining agents in $N \setminus K(\boldsymbol{\chi})$ get the prize with probablity at most $\underline{v}(\boldsymbol{\chi})B\sum_{k\in K(\chi)}\delta^k(1)$ (See the proof of Lemma 11 in the appendix for this estimate.)

If the proportional prize is used then, at any WNS, not only does the expected total output go up for the principal as we just saw, but each agent in $N\setminus K(\boldsymbol{\chi})$ wins the prize with much greater probability than before (at least $de_{min}/|N|D \equiv O(1/|N|)$ each, provided $de_{min}\underline{v}(\boldsymbol{\chi})/|N|D > Ce_{min}$, i.e., provided $\underline{v}(\boldsymbol{\chi}) > C|N|D/d$). Thus, *provided the minimum valuation $\underline{v}(\boldsymbol{\chi})$ of the prize is large enough, all the agents in $N\setminus K(\boldsymbol{\chi})$, who constituted the impoverished majority under the deterministic scheme, suddenly find their prospects brighten when the proportional scheme is introduced and are able to become better off by working hard.* The elite coterie $K(\boldsymbol{\chi})$, of course, loses its status : the probabilities of winning the coveted prize drops from $O(1/|K(\boldsymbol{\chi})|)$ to $O(1/|N|)$ for each of its members, though they still must work so as to not lag behind the others. In short, the egalitarian distribution engendered by the proportional prize inspires all agents to work hard and considerably raises total output.

The principal and the impoverished majority $N\setminus K(\boldsymbol{\chi})$ should both applaud when π_P replaces π_D ; indeed, the principal can count on the unconditional support of the majority when he institutes π_P instead of π_D, and need only worry about having to brook the displeasure of the tiny elite coterie $K(\boldsymbol{\chi})$.

7.2 Bounded Deviation.

Suppose that, when an agent produces a fraction x of total output, he wins the prize with probability $h(x)$, with $h(0) = 0$ and $h(1) = 1$; and that h is of bounded deviation from the linear function π_D, i.e., $m(x-y) < h(x) - h(y) < M(x-y)$, for $y < x$ and positive constants m, M. Then a careful rereading of the proofs reveal that the estimates of Theorems 1 and 2 survive, though in slightly weakened form: lower bounds need to be diminished by a factor of m/M and upper bounds to be raised by a factor of M/m. In the same vein, an agent's utility from winning the prize with probability p could be $f(p)$ instead of the standard expected value $pf(1)$.If f is of bounded deviation from the linear expectation $pf(1)$, we can accomodate f just like h. Finally the quantification of output can be altered without disrupting our results, so long as the alteration is of bounded deviation.

7.3 Multiple Prizes.

One might wonder what happens when $l \leq |N|$ apriori fixed deterministic prizes are used instead of a single prize. When $|N| = 2$ it is evident that using two prizes is wasteful since the loser will always get the second prize for free. In general, if

$l << |N|$, then again the proportional prize will perform better. The reason is as follows. Assume everyone works hard. Define l "heroes" by the top l mean outputs (as in section 7); and then define the coterie K to consist of those agents whose outputs have a positive probability of overtaking the weakest hero. Arguing as in the proof of Lemma 11, the maximal effort in K will effectively bound the total IR output, regardless of the values of the l prizes. Also, as in the previous section, the expected size of K will be small. Thus the proportional prize will outperform l deterministic prizes when $l << |N|$. We leave the case of general l for future work.

7.4 Interdependent Production

The discerning reader will notice that our analysis remains valid even if the random output produced by an agent is influenced by the effort (possibly factored through output) of the *others.*Various assumptions will need to be recast (somewhat cumbersomely) but the same method of proof applies. We skip the details

7.5 More General Elite.

In our definition of the elite set, we need not rule out the possibility that the weakest agent can match the hero with small probability. This was done for ease of exposition. More generally say that $K(\boldsymbol{\chi})$ is an "$(1-\epsilon)-$ elite" set if the probability of any agent in $N \setminus K(\boldsymbol{\chi})$ producing output equalling or exceeding the hero's, is at most ϵ. (This probability is to be of course considered under the scenario that the agent and the hero are both at effort level 1; and, in the case of interdependent production, that everyone in $K(\boldsymbol{\chi})$ is also at effort level 1.) Then the Lemma 11 holds, replacing c by $c/(1-\epsilon)$ in the upper bound and so Theorem 15 , and hence also the comparison being carried out in this section, holds with the same amendment.

7.6 Regime Change (Two Agents with Variable Noise in Output)

We devote this section to a simple tutorial example which highlights the fact that if agents are "similar" then the deterministic prize elicits more output, otherwise that distinction goes to the proportional prize.

There are only two agents i.e., $N = \{1, 2\}$ and only two effort levels (besides the "0" which is tantamount to not participating in the game), i.e., $E = \{0, 1/2, 1\}$. For simplicity fix $\delta^1(1/2) = \delta^2(1/2) = 0$ (which is just a proxy for a very small positive number) and $\delta^1(1) = \delta^2(1) = \delta > 0$. Fix also two numbers $0 < a < b$. We shall vary the productive abilities τ_ϵ^n of $n = 1, 2$ with a parameter ϵ. For effort level $1/2$, both agents produce output uniformly in the interval $[0, \epsilon]$. For effort level 1, agent 1 produces uniformly in $[a, a + \epsilon]$ while agent 2 produces uniformly in $[b, b + \epsilon]$. Since $a < b$, agent 1 is weaker than agent 2, and the "dissimilarity" between them can be expressed by $\Delta(\epsilon) \equiv Prob\{\tau_\epsilon^2(1) > \tau_\epsilon^1(1)\}$. As ϵ increases from 0 to ∞, $\Delta(\epsilon)$ falls from 1 (complete disparity) to $1/2$ (complete similarity). We may think of the ϵ-spread a "noise" which, when large, overwhelms the intrinsic difference $b - a$ in the agents' abilities and makes them very similar.

Taking our cue from Theorem 8, our goal is to implement $\mathbf{1} = \{1, 1\}$ as an NE. For simplicity we suppose $v^1 = v^2 = v$ and inquire about the values[11] v of the prize for which $\pi = \pi_D$ or π_P implements $\mathbf{1}$ as an NE given ϵ. Indeed, since we have fixed δ^1 and δ^2, and are going to deduce v, the only exogenous variable is ϵ which defines the productivity functions $\tau_\epsilon^1(e), \tau_\epsilon^2(e)$. Thinking of $\boldsymbol{\chi} \equiv (\delta^n, \tau_\epsilon^n)_{n \in \{1,2\}}$ as the "*pre*characteristics" of the agents, the space from which $\boldsymbol{\chi}$ is chosen will be taken to be of the form $\mathbf{X}(\alpha, \beta) = \{(\tau_\epsilon^1, \tau_\epsilon^2) : \alpha \leq \epsilon \leq \beta\}$. (Notice that the same noise ϵ is used for each agent.)

For any given $\boldsymbol{\chi} \equiv (\delta^n, \tau_\epsilon^n)_{n \in \{1,2\}} \approx \epsilon$ and $v^1 = v^2 = v$, we have the game $\Gamma_\pi(\epsilon, v)$ where $\pi = \pi_D$ or π_P. A little reflection reveals that if $\mathbf{1}$ is an NE of $\Gamma_\pi(\epsilon, v)$, then $\mathbf{1}$ is also an NE of $\Gamma_\pi(\epsilon, \tilde{v})$ for all $\tilde{v} > v$. Thus we can measure the "efficacy" of π by the smallest value $v(\pi, \epsilon)$ of v for which π implements $\mathbf{1}$ as an NE, given precharacteristics ϵ. This is given by

$$v(\pi, \epsilon) = inf\{v \in R_+ : \mathbf{1} \in NE(\Gamma_\pi(\epsilon, v))\}$$

First let us restrict to the situation when $\alpha = \beta$, so that $X(\alpha, \beta) \equiv X(\epsilon)$ is a singleton. We shall show that there is a threshold ϵ^* (which depends on a,b) such that a "regime change" occurs there:

$$v(\pi_P, \epsilon) - v(\pi_D, \epsilon) = \begin{cases} \text{-tive if } \epsilon < \epsilon^* \\ \text{+tive if } \epsilon > \epsilon^* \end{cases}$$

i.e. the proportional prize π_P beats the deterministic prize π_D when the agents are in $[0, \epsilon^*)$, i.e., are sufficiently dissimilar, whereas it loses to π_D when similarity sets in for $\epsilon > \epsilon^*$. In our example, for $a = 2$ and $b = 3$, $\epsilon^* \approx 2.8$. Thus if one restricts noise so that the output of "shirk" ($e = 1/2$) cannot overtake the output produced by the

[11] This is *not* to say that the principal can strategically vary the value v of the prize $-$ that value is not his to vary; it lies in the eyes of the agents who behold the prize. *We*, the analysts, vary v in order to pinpoint the population of agents (or, of prizes) for which a given π implements $\mathbf{1}$ as an NE.

strong agent ($n = 1$) when he "works" ($e = 1$), then we must have $\epsilon < 3$, implying that π_P beats π_D with probability $2 \cdot 8/3 \approx 0.93$ (assuming all ϵ in $[0, 3]$ to be equally likely); if the overtaking can occur with probability at most 0.2, then $\epsilon - 3 < 0.2\epsilon$, i.e., $\epsilon < 3/.8$, in which case π_P beats π_D with probability $2.8/(3/.8) \approx 0.7$.

Let us verify the existence of the threshold ϵ^*. For the game on $(N, X(\epsilon))$, let $\Delta\bar{\pi}^n_D(\epsilon)$ = increase in probability of winning the prize for n, when he switches from effort $e = 1/2$ to $e = 1$ (assuming that his rival is at $e = 1$, and that the deterministic prize π_D is being used). Similarly, define $\Delta\bar{\pi}^n_P$ for the proportional prize π_P. Then clearly

$$v(\pi_D, \epsilon) = \frac{\delta}{\min \{\Delta\bar{\pi}^1_D(\epsilon), \Delta\bar{\pi}^2_D(\epsilon)\}}$$

and

$$v(\pi_P, \epsilon) = \frac{\delta}{\min \{\Delta\bar{\pi}^1_P(\epsilon), \Delta\bar{\pi}^2_P(\epsilon)\}}$$

Denoting the two minima by $\min_D(\epsilon)$ and $\min_P(\epsilon)$ respectively, we see that

$$\begin{aligned} min_P(\epsilon) > min_D(\epsilon) &\Longleftrightarrow \pi_P \text{ beats } \pi_D \\ min_D(\epsilon) > min_P(\epsilon) &\Longleftrightarrow \pi_D \text{ beats } \pi_P \end{aligned}$$

It is easy to compute all these terms for our simple example. Indeed

$$\begin{aligned} \Delta\bar{\pi}^1_D(\epsilon) &= \frac{(\max \{\epsilon - b + a, 0\})^2 - (\max \{\epsilon - b, 0\})^2}{2\epsilon^2} \\ \Delta\bar{\pi}^2_D(\epsilon) &= 1 - \frac{(\max \{\epsilon - b + a, 0\})^2 - (\max \{\epsilon - a, 0\})^2}{2\epsilon^2} \end{aligned}$$

and

$$\begin{aligned} \Delta\bar{\pi}^1_P(\epsilon) &= \int_b^{b+\epsilon} \left[\int_a^{a+\epsilon} \frac{x}{x+y} dx - \int_0^{\epsilon} \frac{x}{x+y} dx \right] dy \\ \Delta\bar{\pi}^2_P(\epsilon) &= \int_a^{a+\epsilon} \left[\int_b^{b+\epsilon} \frac{y}{x+y} dy - \int_0^{\epsilon} \frac{y}{x+y} dy \right] dx \end{aligned}$$

Taking $b = a + 1$, and integrating by parts, yields

$$\Delta\, \bar{\pi}^1_P(\epsilon) = F(\epsilon, y) - F(0, y) + F(a, y) - F(a + \epsilon, y)\Big|_{y=a+1}^{y=a+1+\epsilon}$$

where $F(c, y) \equiv \frac{1}{2}(y^2 - c^2) \ln(y + c) - \frac{1}{4}y^2 + \frac{1}{2}cy$ $(= \int y \ln(c + y)dy)$. And $\Delta\bar{\pi}^2_P(\epsilon)$ is an identical expression, obtained by swapping a with $a + 1$.

We may now (with the help of MAPLE, and taking $a = 2$ and $b = 3$) plot $\min_D(\epsilon)$, $\min_P(\epsilon)$ and $\min_P(\epsilon) - \min_D(\epsilon)$ against ϵ to see that the threshold ϵ^* is ≈ 2.8. (We suppress the MAPLE plots here because they can be easily replicated.)

Turning to broader spaces $X(\alpha, \beta)$ with $\alpha < \beta$, first notice that $\Delta\bar{\pi}^1_D(\epsilon) = 0$ if $\epsilon \leq 1$ (for in this case agent 1 always produces below b, while agent 2 always produces above b with effort level 1). Thus $v(N, \pi_D, X(\alpha, \beta)) = \infty$ if $\alpha < 1$. Since $\Delta\pi^n_P(\epsilon) > 0$ for all ϵ and $n \in \{1, 2\}$, $v(N, \pi_P, X(\alpha, \beta)) < \infty$. It follows that π_P is better than π_D for all (α, β) if $\alpha < 1$. This is also true by our earlier discussion if $\beta < \epsilon^* \approx 2.8$.

The MAPLE plots further reveal that when π_P beats π_D, it does so most of the time by a large margin (e.g. by more than 0.1 for $0 < \epsilon < 2$); whereas when it loses to π_D, the margin of loss is small (≤ 0.1).

An alternative way in which to vary the productivities $\tau^1(1)$, $\tau^2(1)$ of agents $1, 2$ is as follows. Fix $0 < a < b$. Let $\bar{N}(\sigma^2)$ be the "truncated" Normal distribution:

$$\bar{N}(\sigma^2) = \frac{N\left((a+b)/2, \sigma^2\right)}{\left(N\left((a+b)/2, \sigma^2\right)\right)[a, b]}$$

where the numerator is the standard normal distribution with mean $(a+b)/2$ and variance σ^2 and the denominator is the probability of the interval $[a, b]$ under that distribution. In short, $\bar{N}(\sigma^2)$ is the probability distribution induced by $N\left((a+b)/2, \sigma^2\right)$, conditional on being in $[a, b]$.

Pick x i.i.d. according to $\bar{N}(\sigma^2)$ for each agent n, and let $\tau^n(1)$ be uniformly distributed in $(x - \epsilon, x + \epsilon)$ (where ϵ is suitably small and fixed). As we increase σ from 0 to ∞, the chances of "similarity" between the two agents fall from maximal to minimal. There will be a threshold σ^* such that π_P elicits more output than π_D if, and only if, $\sigma > \sigma^*$. The verification, being straightforward, is omitted.

8 The Strategic Game of Incomplete Information

Our main theme, namely that π_P is better for the principal than π_D when agents' characteristics are sufficiently diverse, has been established under the hypothesis that agents know each others' characteristics. Now we show that the theme remains intact even when an agent only knows his own characteristics and has a probability distribution on those of his rivals. This is the standard scenario of incomplete information. Our analysis will be in terms of an illustrative binary game, and not at the level of generality of the complete information case. But precisely because we work with a structured example, we are able to accomplish a little bit more. We show that there is.a threshold on the random noise, below which π_P outperforms π_D (as usual, from the principal's point-of-view!), and above which π_D does better. Thus our comparison of the two schemes is more "even-handed" in the context of our example. It points to the need for a more general study of the incomplete information case, and in particular the specification of conditions where π_P outperforms π_D, or vice versa.

Let $E = \{0, 1\}$ and $N = \{1, 2\}$. Let $\delta^n(1) = 1$ and[12] $v^n = v > 1$ for $n = 1, 2$; i.e., the uncertainty pertains only to the productivities τ^1, τ^2. Of course, $\tau^n_z(0) = 0$ as always, no matter what the "skill" z of agent n may be. Suppose that $\tau^n_z(1)$ is uniformly distributed on the interval $[z, z + \epsilon]$, where ϵ is a measure of the noise on the output. Furthermore suppose that the skills of the agents $n = 1, 2$ are drawn independently from the intervals $[a_1, b_1]$ and $[a_2, b_2]$, with uniform probability (and that all this is common knowledge to the agents).

Since agent n is informed of only his own skill, a strategy for him is given by a function $\sigma^n : [a_n, b_n] \to [0, 1]$
where $\sigma^n(x)$ is the probability with which n chooses effort 1 when his skill is x.

For any prize allocation scheme π, the game of incomplete information Γ^*_π is then defined in the standard manner. (It depends not only on π but also on the parameters $v, a_1, b_1, a_2, b_2, \epsilon$ which we suppress because they will be understood. Our focus is on $\pi = \pi_P$ or π_D which we keep track of in our notation.)

First suppose ex-ante symmetry between the agents and no noise: $[a_1, b_1] = [a_2, b_2] = [0, 1]$ (say), and $\epsilon = 0$

Let $F^n_\pi((p, \sigma')|x)$ denote the payoff of n in the game Γ^*_π, when he chooses effort 1 with probability p and his skill level is x, while his rival chooses the strategy σ'. (Thus, if n's strategy is σ, his payoff in Γ^*_π will be $F^n_\pi(\sigma, \sigma') = \int_0^1 F^n_\pi((\sigma(x), \sigma')|x)dx$.) Notice that $F^n_\pi((1, \sigma')|x)$ increases[13] in x (for fixed n, π, σ'), since n's disutility of effort stays constant at 1 while his probability of winning the prize goes up[14]. Thus n's best reply to σ' is to switch from 0 to 1 at some "threshold" skill c, which solves $F^n_\pi((1, \sigma')|c) = 0$ i.e., denoting by σ_c the strategy which assigs effort 1 if $x \geq c$ and effort 0 if $x < c$, we see that σ_c is a best reply to σ' in the game Γ^*_π if $F^n_\pi((1, \sigma')|c) = 0$. We conclude that (σ_c, σ_c) is a[15] (symmetric) NE in Γ^*_π if $F^n_\pi((1, \sigma_c)|c) = 0$. The unique $c(\pi)$ that solves this equation is computed rather easily for $\pi = \pi_P$ or π_D. Indeed we have, $F^n_{\pi_D}((1, \sigma_c)|c) = cv - 1$ and $F^n_{\pi_P}((1, \sigma_c)|c) = cv + \int_c^1 (\frac{cv}{x+c})dx - 1 =$

[12]If $v \leq 1$ then the only NE in $\Gamma^*_{\pi_D}$ or $\Gamma^*_{\pi_P}$ is that both agents never work (since effort 1 costs 1 which cannot be compensated by any probability of winning the prize)

[13]weakly in $\Gamma^*_{\pi_D}$ and strictly in $\Gamma^*_{\pi_P}$

[14]weakly in $\Gamma^*_{\pi_D}$ and strictly in $\Gamma^*_{\pi_P}$

[15]also "the", i.e., there is only one symmetric NE as the reader may easily verify.

$cv[1 + ln\frac{1+c}{2c}] - 1$,which gives (denoting $c(\pi_D) \equiv c_D$ and $c(\pi_P) \equiv c_P$)

$$c_D = \frac{1}{v} \tag{8}$$

and

$$v = \frac{1}{c_P[1 + ln(\frac{1+c_P}{2c_P})]} \tag{9}$$

When $c_P = 0$, the right hand side of (9) is infinity by L'Hospital's rule while at $c = 1$, it is 1. Since $v > 1$ the solution of (9) is $c_P < 1$, hence we have $ln(\frac{1+c_P}{2c_P}) > 0$. Thus, for any $v > 1$, we deduce that $c_P > c_D$. In short, more agent-types are working at NE under π_P than under π_D and hence π_P elicits more expected output.

Now let noise increase (from 0 to infinity), still maintaining the ex-ante symmetry of the agents (i.e., $[a_n, b_n] = [0, 1]$ for $n = 1, 2$). Arguing as before, it is evident that threshold strategies will once again constitute NE. But for ϵ large enough, the symmetry between agents will obtain even ex-post (to any desired level of accuracy) not just ex-ante, i.e., no matter what the realization of their respective skills, the two agents are nearly evenly matched since the large noise renders their skills irrelevant. In this event, corroborating our intuitition from the introduction, π_D will elicit more effort than π_P. Indeed it is easy to verify (and we omit the routine algebra) that there exists an $\tilde{\epsilon}$ such that $c_P(\epsilon) < c_D(\epsilon)$ if $\epsilon < \tilde{\epsilon}$ and $c_P(\epsilon) > c_D(\epsilon)$ if $\epsilon > \tilde{\epsilon}$;which asserts that, *unless the noise is so high as to make skills count for little π_P outperforms π_D in games of incomplete information.*

Next let us consider the effect of allowing for ex-ante asymmetry of the incomplete information. To this end, let $[a_2, b_2] = [\Delta, 1+\Delta]$ for $0 < \Delta < 1$[16] and $[a_1, b_1] = [0, 1]$, i.e., agent 2's skills are Δ-higher than 1's, so that Δ denotes the degree of asymmetry. For convenience, fix the noise $\epsilon = 0$. Arguing as in the ex-ante symmetric case, there again exist thresholds $c^n_D(\Delta), c^n_P(\Delta)$ such that $(\sigma^1_{c_D(\Delta)}, \sigma^2_{c_D(\Delta)}), (\sigma^1_{c_P(\Delta)}, \sigma^2_{c_P(\Delta)})$ constitute the symmetric NE of the games $\Gamma^*_{\pi_D}, \Gamma^*_{\pi_P}$ respectively; and, moreover,

$$c^n_P(\Delta) < c^n_D(\Delta)$$

for $n = 1, 2$ and all Δ (unless v is so small that no agent ever works in NE$-$ we implicitly eliminate such trivial NE by presuming v is high enough). Thus π_P always outperforms π_D and, as anticipated, *the superiority of π_p becomes more pronounced as the degree Δ of the asymmetry rises.*

[16] If $\Delta > 1$ then we have the trivial situation that the highest skill-type of 1 cannot beat the lowest skill type of 2 which renders the deterministic prize ineffective, while the proportional still continues to elicit effort.

The exact calculations for the asymmetric case emerge from the following lemma. Suppose an agent is informed that his rival's output is uniformly distributed in some interval $[z, z+\eta] \subset R_+$ and that his own skill is x. Fix x and think of z, η as variable. We can compute two critical values $z_D \equiv z_D(x,\eta)$, $z_P \equiv z_P(x,\eta)$ such that the expected payoff of the agent is zero in $\Gamma^*_{\pi_D}$, $\Gamma^*_{\pi_P}$ if he chooses effort 1 and if $z = z_D$, $z = z_P$ respectively. Since this payoff varies inversely in z, the agent's best reponse to the rival is to choose effort 1 if $z < z_D$ and effort 0 if $z > z_D$ in the game Γ_D (or, effort 1 if $z < z_P$ and 0 if $z > z_P$, in the game Γ_P). The critical values z_D, z_P are as follows .

Lemma 16 *The critcal z-values are* $z_D = x - \eta/v$ *and* $z_P = \frac{\eta}{\exp(\eta/vx)-1} - x$ *. Moreover we have* $x\,(v-1) - \eta \leq z_P \leq x\,(v-1)$*.*

(The proof is in the Appendix.)

We leave it to the reader to see how our results for the asymmetric case can be straightforwardly derived from this proposition. In fact, this proposition suffices also for the analysis of games of "partial information" which lie between what we, following others, have called games of "complete" and "incomplete" information. To be concrete suppose $[a_n, b_n]$ is partitioned into k (for simplicity, equal) subintervals $[a_n + i\Delta, a_n + (i+1)\Delta]$ where $\Delta = (b_n - a_n)/k$ and $i = 0, 1, 2, ...k-1$. (When $k = 1$ we have "incomplete" information and as $k \to \infty$ we converge to "complete" information.) Each agent is now informed of his own exact skill and of the subinterval of $[a_n, b_n]$ in which his rival's skill lies. This defines a game of partial information in the obvious way (from his initial probability distribution on $[a_n, b_n]$, the agent can infer conditional probabilities of his rival's skill given the subinterval of $[a_n, b_n]$ in which it lies).

We have not done the exact calculations, but it seems reasonably clear that π_P outperforms π_D for every k, not just for the two extreme points $k = \infty$ and $k = 1$ that have already been checked.

9 Optimal Prizes with Complete Information

Consider any class Π of prize allocation schemes (i.e., maps $\mathbf{R}^N_+ \setminus \{0\} \xrightarrow{\pi} \Delta^N$ and $\pi(0) = 0$), and any set $\mathbf{X}$ of **pre-characteristics**[17] $\boldsymbol{\chi} \equiv (\delta^n, \tau^n)_{n \in N}$ on which the disutilities of effort are universally bounded from above (as in the first part of Axiom

[17] In this section the symbol $\boldsymbol{\chi}$ will be reserved for pre-characteristics, even though it is used elsewhere for characteristics. Similarly $\mathbf{X}$ will denote a set of pre-characteristics.There will be no confusion.

1). Further assume that there exists a $\pi^* \in \Pi$ and a positive constant α such that: at every $\boldsymbol{\chi} \in \mathbf{X}$, if π^* is in use and all agents are working at maximal effort 1, and if any one of them unilaterally deviates to some effort $e < 1$, then the deviator's probability of winning the prize goes down by at least α. (In many examples, including the two about to be presented, the proportional prize π_D easily fulfils the role of such a π^*.) With this assumption, the *existence* of an "optimal" (or "nearly optimal") scheme in Π for $\mathbf{X}$ is automatic [18], as will become obvious from the definitions below. Its *structure*, however, is a delicate matter and will depend heavily on Π and $\mathbf{X}$.

The idea behind an optimal scheme in Π for $\mathbf{X}$ is that it should Nash-implement maximal effort $\mathbf{1} \equiv (1, ..., 1)$ on all of $\mathbf{X}$ for the least value of the prize, i.e., no other scheme in Π can implement $\mathbf{1}$ on $\mathbf{X}$ with a prize of smaller value.

More precisely,for $\boldsymbol{\chi} \equiv (\delta^n, \tau^n)_{n\in N}$, let $(\boldsymbol{\chi}, v)$ denote $(\delta^n, \tau^n, v)_{n\in N}$. Define $v(\pi, \boldsymbol{\chi}) = inf\{v \in R_+ : \mathbf{1} \in NE(\Gamma_\pi(\boldsymbol{\chi}, v))\}$,and $v(\pi) = sup\{v(\pi, \boldsymbol{\chi}) : \boldsymbol{\chi} \in \mathbf{X}\}$.Thus $v(\pi)$ is the smallest value $v = v^1 = ... = v^n$ of the prize which Nash-implements $\mathbf{1}$ uniformly over $\mathbf{X}$ when the scheme π is used. We define $\hat{\pi}$ to be **optimal** in Π for $\mathbf{X}$ if $v(\hat{\pi}) \leq v(\pi)$ for all $\pi \in \Pi$,in other words, if $v(\hat{\pi}) = \inf\{v(\pi) : \pi \in \Pi\}$. (And, in the same vein, we define $\hat{\pi}$ to be ϵ-**optimal**[19] in Π for $\mathbf{X}$ if $v(\hat{\pi}) \leq v(\pi) + \epsilon$ for all $\pi \in \Pi$.). An obviously equivalent definition would be: $\hat{\pi}$ is optimal if, whenever any $\pi \in \Pi$ Nash-implements $\mathbf{1}$ on X , so does $\hat{\pi}$.

Our goal in this section is to construct optimal schemes for two particular pairs Π, $\mathbf{X}$.

Let us restrict attention to the **class Π of all allocation schemes** which satisfy the following four conditions:

(i) (**Scale Invariance**) $\pi(rt) = \pi(t)$ for all scalars $r > 0$

(ii) (**Anonymity**) $\pi(\omega t) = \omega(\pi t)$ for any permutation $\omega : N \to N$

(iii)(**Monotonicity**) $\pi^n(t) \geq \pi^k(t)$ whenever $t^n \geq t^k$

(iv) (**Disbursal**) $\sum_{n\in N} \pi^n(t) = 1$ if $t \neq 0$,and is 0 otherwise

We shall examine the binary case of two agents (i.e., $N = \{1, 2\}$) with two effort levels and deterministic output. The effort levels are "shirk" ($e = 1/2$) and

[18] The extrema (infimum, supremum) in our definition of an "optimal scheme" are clearly finite, e.g., the scheme π^* always implements maximal effort for large enough v. Thus, even if the extrema are not attained but only approached, *approximately* optimal schemes will exist, to any degree of accuracy one may desire.

[19] Note that $\epsilon-$optimal schemes exist for every $\varepsilon > 0$, thanks that the fact that $v(\pi^*)$ is clearly finite under our assumptions.

"work" ($e = 1$), in addition of course to effort level 0 for not participating in the game. So $E = \{0, 1/2, 1\}$. The disutility of effort is constant across $\boldsymbol{\chi} \in \mathbf{X}$ (with[20] $\delta^n(1/2) = 0$ and $\delta^n(1) = \delta$ for $n = 1, 2$). What varies with $\boldsymbol{\chi} \in \mathbf{X}$ is the skill (productivity) of an agent. Let $\tau(e, s)$ denote the deterministic output of each agent when he exerts effort $e \in \{1/2, 1\}$ and and is endowed with "skill" $s \in [k, K]$ (Thus $\mathbf{X} \approx [k, K]^2$ here.)

For brevity, denote $\tau(1/2, s) \equiv \tau(s)$ and $\tau(1, s) \equiv \tau^*(s)$. We make some natural monotonicity assumptions on τ and τ^*, along with a form of "decreasing (or,later, increasing) returns to skill" :

Axiom 17 ***(Decreasing Returns to Skill)*** *Both* $\tau : [k, K] \longrightarrow R_+$, $\tau^* : [k, K] \longrightarrow R_+$ *are continuous and strictly monotonic; and* $\tau^*(s)/\tau(s) \leq \tau^*(s')/\tau(s')$ *if* $s' < s$. *Also* $\inf \{\tau^*(s) - \tau(s) : s \in [k, K]\} > 0$.

Axiom 17 says that the percentage gain in output, by switching from shirk to work, is a weakly decreasing function of the skill $s \in [k, K]$. (The case of increasing returns is entirely analogous; see Axiom 22 below.)

Our main result (see theorem 21 below) shows that, when Axiom 17 holds, *there exists an optimal scheme which takes the form of a monotonic step function.* The location of the jump points, and the sizes of the jumps, can be computed by an algorithm based on $r, R, \tilde{r},\ \tilde{R}$,i.e., on skill functions τ and τ^* restricted to the northeast boundary of the square $[k, K]^2$. And, graphically speaking, this optimal scheme lies " in between" the proportional scheme (whose graph is linear) and the deterministic scheme (whose graph has a single jump from 0 to 1 at 1/2).

To establish this result, first note that axiom 17 simplifies the analysis considerably, on account of:

Lemma 18 *Assume Axiom 17 holds. Let* $s \in (k, K)$ *and* $t \in (k, K)$, *Then there exist* $s' \in [k, K]$ *and* $t' \in [k, K]$ *such that*

$$\frac{\tau^*(s')}{\tau^*(s') + \tau^*(t')} - \frac{\tau(s')}{\tau(s') + \tau^*(t')} \leq \frac{\tau^*(s)}{\tau^*(s) + \tau^*(t)} - \frac{\tau(s)}{\tau(s) + \tau^*(t)} \text{ and } \frac{\tau(s')}{\tau(s') + \tau^*(t')} = \frac{\tau(s)}{\tau(s) + \tau^*(t)}$$

and either $s' = K$ *or* $t' = K$

(The proof is in the Appendix.)

Lemma 18 implies that our goal — of incentivizing an agent (of skill s) to switch from shirk to work, assuming his rival (of skill t) is working — will be achieved for

[20] We take $\delta^n(1/2) = 0$ for simplicity (recall that δ^n is permitted to be *weakly* increasing). But our analysis remains intact if $\delta^n(1)$ is sufficiently larger than $\delta^n(1/2) > 0$ (as can easily be checked.)

every $(s,t) \in [k,K] \times [k,K)$ if it is achieved for (s,K) and (K,s) for all $s \in [k,K]$; in other words, we need only worry about incentivizing the agent in the following two extremal cases, corresponding to the north and east boundaries of the square $[k,K]^2$:

Case A His skill is $s \in [k,K]$ and his rival is working with skill K.

Case B His skill is K and his rival is working with skill $s \in [k,K]$

Denote

$$R(s) = \frac{\tau^*(s)}{\tau^*(s)+\tau^*(K)}, r(s) = \frac{\tau(s)}{\tau(s)+\tau^*(K)}, \tilde{R}(s) = \frac{\tau^*(K)}{\tau^*(K)+\tau^*(s)}, \tilde{R}(s) = \frac{\tau^*(K)}{\tau^*(K)+\tau^*(s)}$$

When an agent switches from shirk to work, his fractional output goes up from $r(s)$ to $R(s)$ in Case A, $\tilde{r}(s)$ to $\tilde{R}(s)$ in Case B. Denote $q(s) = 1 - \tilde{r}(s)$. It is clear from our assumptions that $q > R > r$ and that $R(s) = 1 - \tilde{R}(s), R(K) = \tilde{R}(K) = 1/2$

It will be useful to introduce one more function, which captures the simple form of $\pi \in \Pi$ when there are only two agents.

Definition 19 ***(Prize function)*** *A prize function is a weakly increasing function $p : [0,1] \to [0,1]$ satisfying $p(1-x) = 1 - p(x)$ for all x.The function p is said to be* effective *at prize level v, if $\mathbf{1} = (1,1)$ is a Nash equilibrium for any pair $(s,t) \in [0,K] \times [0,K]$ of skills of the two agents in the associated game.*

(Note that our assumptions on Π imply that, if $|N| = 2$ and $\pi \in \Pi$, then there exists a prize function p such that $\pi^n(\tau^1,\tau^2) = p(\tau^n/(\tau^1+\tau^2))$, for $n \in N$, whenever $\tau^1 + \tau^2 \neq 0$,justifying our name for p). The lemma below will be handy:

Lemma 20 *The prize function p is effective at level v iff for all $s \in [0,K]$ we have*

$$p(q(s)) - \delta/v \geq p(R(s)) \geq p(r(s)) + \delta/v$$

Proof. As discussed earlier, $p(x)$ is effective iff $p\left(\tilde{R}(s)\right) \geq p(\tilde{r}(s)) + \delta/v$ and $p(R(s)) \geq p(r(s)) + \delta/v$ for all $s \in [0,K]$ Since $p\left(\tilde{R}(s)\right) = 1 - p(R(s))$, $p(\tilde{r}(s)) = 1 - p(q(s))$, the first inequality becomes $p(q(s)) - \delta/v \geq p(R(s))$ which proves the result. ■

Define a sequence of points $0 = x_0, x_1, \ldots, x_l$ in $[0,1/2]$ by $x_i = R(0)$ for $i = 1$;and $x_i = \rho(x_{i-1})$ for $1 < i \leq l$.where $\rho(x) = \min\left(R\left(r^{-1}(x)\right), q\left(R^{-1}(x)\right)\right)$ and l is the smallest index i for which $r^{-1}(x_i)$ is undefined. Note that since q, R, r are all strictly increasing functions, so is ρ, and therefore $x_1, \ldots, x_l$ is an increasing sequence.

Now define $p^* : [0,1] \to [0,1]$ as follows (where $i = 0, 1, ..., l$):

$$p^*(x) = \begin{cases} i/2l & \text{for} \quad x_i \le x < x_{i+1} \\ 1/2 & \text{for} \quad x_l \le x \le 1/2 \\ 1 - p^*(1-x) & \text{for} \quad 1/2 < x \le 1 \end{cases}$$

We are now ready to state and prove

Theorem 21

(i) Any effective scheme has prize level $\ge 2l\delta$; (ii) $x \to p^*(x)\,\delta$ is an effective scheme with prize $2l\delta$.

Proof. Let p be effective with prize level v. By Lemma 20 with $s = 0$, we get $p(x_1) = p(R(0)) \ge p(r(0)) + \delta/v \ge \delta/v$. Next let $s = r^{-1}(x)$ or $s = R^{-1}(x)$ according as $\rho(x) = R(r^{-1}(x))$ or $q(R^{-1}(x))$. Then, again by Lemma 20, we get $p(\rho(x)) \ge p(x) + \delta/v$ whenever $x, \rho(x) \in [0,1]$. Applying this formula repeatedly we get

$$1/2 = p(x_l) \ge p(x_{l-1}) + \delta/v \ge \cdots \ge p(x_1) + (l-1)\,\delta/v \ge l\delta/v$$

which proves (i). For (ii) we first show that, for any s, each of the two intervals $[r(s), R(s)]$ and $[R(s), q(s)]$ contains some "jump" point x_i. Indeed if $x = r(s)$ is in $[x_{i-1}, x_i)$, then $R(s) = R(r^{-1}(x)) \ge \rho(x) > \rho(x_{i-1}) = x_i$, hence $x_i \in [r(s), R(s)]$. The argument is similar for $[R(s), q(s)]$. Now by the definition of p^* it follows that

$$p^*(q(s)) - 1/2l \ge p^*(R(s)) \ge p^*(r(s)) + 1/2l,$$

which is precisely the condition of Lemma 20 with $v = 2l\delta$. ■

One might define "increasing returns" as in Axiom 17 , substituting " $s' > s$ " in place of "$s' < s$ "

Axiom 22 ***(Increasing Returns to Skill)*** *Both* $\tau : [k, K] \longrightarrow R_+$, $\tau^* : [k, K] \longrightarrow R_+$ *are continuous and strictly monotonic; and* $\tau^*(s)/\tau(s) \le \tau^*(s')/\tau(s')$ *if* $s' > s$. *Also* $\inf\{\tau^*(s) - \tau(s) : s \in [k, K]\} > 0$.

With Axiom 22 in place of Axiom 17, the natural variant of Lemma 18 holds, substituting k for K.

Lemma 23 *Assume Axiom 22 holds. Let $s \in (k, K)$ and $t \in (k, K)$, Then there exist $s' \in [k, K]$ and $t' \in [k, K]$ such that*

$$\frac{\tau^*(s')}{\tau^*(s') + \tau^*(t')} - \frac{\tau(s')}{\tau(s') + \tau^*(t')} \leq \frac{\tau^*(s)}{\tau^*(s) + \tau^*(t)} - \frac{\tau(s)}{\tau(s) + \tau^*(t)} \text{ and } \frac{\tau(s')}{\tau(s') + \tau^*(t')} = \frac{\tau(s)}{\tau(s) + \tau^*(t)}$$

and either $s' = k$ or $t' = k$.

(The proof of this is the same as the proof of Lemma 18 in the Appendix, with $s - \Delta, t - \Delta, k, s' < s$ in place of $s + \Delta, t + \Delta, K, s' > s$ respectively.)

Thus the whole analysis for optimal prizes can be replicated for this dual case, focusing on the southwest boundary of the square $[k, K]^2$, in place of the northeast boundary. We omit the details.

9.1 Optimal Prizes with Small Fractional Increments

There are many contests where the exertion of effort causes only a small fractional increase in output. This happens when all the contestants are very strong — experts, champions,stars — and their base levels of output (namely, the outputs at their lowest effort levels $e_{\min}$) are so high that incremental output by each contestant is a small fraction of his base, even though these increments may have large observable differences between them on an absolute scale, enabling us to meaningfully compare the contestants.

We model this situation, retaining for simplicity the deterministic binary scenario of the previous section. Here an agent's skill may be identified with his deterministic output when he shirks. Thus we assume that an agent of skill $t \in [k, K]$ produces t units of output if he shirks; and $\Psi(t) > t$ units if he works, where $\Psi(t)$ is non-decreasing and continuous.

Let $\alpha = \alpha(t, x), \beta = \beta(t, x)$ denote the fractions of total output produced by an agent of skill t when he works, shirks respectively, and his rival is of skill x and working. Given a prize function π, we define $I(\pi, t, x)$, the t-agent's **incentive to work** by

$$I(\pi, t, x) \equiv \pi(\alpha) - \pi(\beta)$$

The minimum fraction is $b_* = k/(k + \Psi(K))$ while the maximum fraction is $b^* = \Psi(K)/(k + \Psi(K))$. Thus, in our context, we assume $\pi : [b_*, b^*] \mapsto [0, 1]$, with $\pi(b_*) = 0$, $\pi(b^*) = 1$ (and, of course, $\pi(x) = \pi(1 - x)$). Let Π denote the class of all such π, and let Π^* denote the subclass of Π that consists of differentiable functions. For any $\pi \in \Pi$, the minimum prize that will incentivize agents to work at all realizations $(t,x) \in [k,K]$, is given by $V(\pi) = d/m$ where d is the disutility of work and

$$m = \min\{I(\pi,t,x) : (t,x) \in [k,K]\}$$

is the minimum incentive. Thus to minimize $V(\pi)$ we must maximize the minimum incentive over $\pi \in \Pi$. We shall seek a π that is "continuum- optimal" in Π^* and give a heuristic argument that, in fact, it is also "nearly optimal" in Π . Of course the words within quotes have to still be made precise. Let us fix $\epsilon > 0$ and define $\psi(t) = \psi_\epsilon(t) = [\Psi(t) - t]/\epsilon$. First observe that, for small enough ϵ,

$$\alpha - \beta \simeq \left(\frac{d}{du} \left(\frac{u}{u+x} \right) \Big|_{u=t} \right) \Delta u = \frac{x}{(t+x)^2}(\psi(t)\epsilon) = \frac{x\psi(t)}{(t+x)^2}\epsilon$$

So, if $\pi \in \Pi^*$,

$$I(\pi,t,x) \equiv \pi(\alpha) - \pi(\beta) \simeq \pi'(\beta)\frac{x\psi(t)}{(t+x)^2}\epsilon$$

This motivates our next definition (restoring the notation $\beta = \beta(t,x)$, and taking the domain of the prize functions to be $[k/(k+K), K/(k+K)]$ by supposing ϵ to be infinitesimal):

Definition 24 . *π is **continuum-optimal** in Π^* if*

$$\min\left\{\pi'(\beta(t,x))\frac{x\psi(t)}{(t+x)^2} : (t,x) \in [k,K]\right\} \geq \min\left\{\widehat{\pi}'(\beta(t,x))\frac{x\psi(t)}{(t+x)^2} : (t,x) \in [k,K]\right\}$$

for all $\widehat{\pi} \in \Pi^*$.

Although we have not formally verified this, intuition suggests that: if V^ϵ denotes the minimum prize required in Π^* to incentivize work (in the "ϵ-model" wherein the work output of the t-agent is given by $t + \psi(t)\epsilon$), and if $V(\pi)$ denotes the corresponding quantity for a continuum-optimal π in Π^*, then $V^\epsilon/\ V(\pi)$ converges to 1 as ϵ goes to 0. In this sense, a π that is (idealistically) continuum-optimal in Π^* is (realistically) nearly optimal in $\Pi^*$for small ϵ. This motivates Theorem 25 below. First recall

Strictly decreasing (increasing) returns to skill:

$\dfrac{t+\psi(t)}{t}$ is strictly decreasing (increasing) in t, i.e., $\dfrac{\psi(t)}{t}$is strictly decreasing (increasing) in t

Theorem 25 *Assume that ψ has strictly decreasing returns to skill. There is a unique π (that does not depend on ψ) that is continuum-optimal in Π^*; and it is given by:*

$$\pi(x) = \frac{1}{2} + B \ln \frac{x}{1-x}$$

where $1/2 \leq x \leq K/(k+K)$ (the rest of π being determined by reflection around $1/2$: $\pi(x) = \pi(1-x)$) and the constant B chosen to satisfy $\pi(K/(k+K)) = 1$. In the case of strictly increasing returns, an entirely analogous result holds with $1/2 \geq x \geq k/(k+K)$ in place of $1/2 \leq x \leq K/(k+K)$, and $\pi(k/(k+K)) = 0$ in place of $\pi(K/(k+K)) = 1$.

(The proof is in the Appendix. An examination of that proof makes it clear that jumps in the prize function π will raise $V(\pi)$, justifying our decision to ignore $\Pi \backslash \Pi^*$ in the search of an optimal scheme.)

9.1.1 Universality of the "Log Odds" Solution

The term $x/(1-x)$ gives the "odds" of winning for the agent who produces the fraction x of the total output (while his rival produces the fraction $1-x$), assuming that lotteries are handed out in proportion to the outputs. Thus in the upper (lower) half of its domain, the optimal π awards the prize through "log of the odds" for strictly decreasing (increasing) returns to skill, completing π on the complementary half by the requirement $\pi(x) + \pi(1-x) = 1$. *What is noteworthy is that, apart from the type of returns (decreasing or increasing) exhibited by Ψ, the solution is independent of the precise form of Ψ.*The solution is first convex and then concave for strictly decreasing returns, and the other way round for strictly increasing returns , changing shape at the midpoint 1/2. In fact these two solutions are mirror images of each other if we reflect around the diagonal.

Also worthy of note is the fact (easily verified , and left to the reader) that, for constant returns to skill, we get the strictly increasing returns solution.

9.1.2 Interpretation of the Model with Small Fractional Increments

The idea of an optimal scheme here is *not* that it maximizes expected total output. That would be much ado about nothing, since the output of each person goes up by only $\epsilon\%$ (at an extra disutility also of the order of $\epsilon\%$) when he switches from shirk to work. The emphasis instead is on maximal effort *without* regard to the ensuing output. We have here an interpretation in mind that is, quite bluntly, *non-economic.* Output corresponds to a "score" that measures performance of "players" in a "game" (think of the average score assigned by different judges to each person in a diving contest). The players, who are all of star quality, are being incentivized to put in the final extra burst of effort to perform to the best of their ability.They value the prize enormously compared to the disutility incurred for the extra effort (the fame of being winner, perhaps also the money that fame might bring in the future). The interest is in finding a scheme π that is optimal in the sense that it most frugally (hence, frequently[21]) *creates competition and inspires maximal effort,* for its *own* sake (for the glory of the human spirit, and the sport). The minimum value $V(\pi)$ of the prize (which implements maximal effort under π) does entail *significant* savings, even though output rises very little: the ratio $V(\pi')/V(\pi) >> 1$ when we compare the optimal log-odds π with arbitrary $\pi' \in \Pi$.

9.1.3 Optimal Prize for the Incomplete Information Binary Game of Section 8

It is evident that no agent will work if $V < 1$, no matter which prize function we select (recall that v is the value of the prize and that the disutility of working is 1).

Consider π^* which is defined as follows:

$$\pi^*(x) = \begin{cases} 0 & \text{for} \quad x = 0 \\ 1/2 & \text{for} \quad 0 < x < 1 \\ 1 & \text{for} \quad x = 1 \end{cases}$$

It is equally evident that, when $v \geq 2$, all agents will work[22] under π^*, except for the agent with skill 0; and that no other prize function can elicit more effort than π^*, so that π^* is optimal. In fact π^* is also optimal for $1 \leq v < 2$, as was pointed out by Mayank Goswami and Biligbaatar Tumendemberel at Stony Brook. We quote their argument. First observe that, for any $\pi \in \Pi$, it is obvious that exists a *largest* threshold c_π such that all agents whose skill is *strictly* below c_π will work (shirk); and, from the fact that the agent with skill c_π is indifferent between shirking and working, we must have:

$$c_\pi + \int_{c_\pi}^{1} \pi(\frac{c_\pi}{c_\pi + x})dx = \frac{1}{v}$$

[21] i.e., π inspires maximal effort whenever any other scheme in Π does so (as we vary disutility of effort and valuation of the prize)

[22] We assume throughout that if an agent is indifferent between shirking and working, he will work (the principal paying him a "sliver" more in the background). This takes care of the case $V = 2$.

Notice that, for any $\pi \in \Pi$ and $x > c_\pi$, we have $c_\pi/(c_\pi + x) < 1/2$, and hence (recalling that π is weakly monotonic and $\pi(1/2) = 1/2$ for any

$$\pi(\frac{c_\pi}{c_\pi + x}) \leq \pi(\frac{1}{2}) = \frac{1}{2}$$

which implies

$$\frac{1}{v} = c_\pi + \int_{c_\pi}^{1} \pi(\frac{c_\pi}{c_\pi + x})dx \leq c_\pi + \int_{c_\pi}^{1} \frac{1}{2}dx = \frac{1}{2}(1 + c_\pi)$$

i.e.,

$$c_\pi \geq \frac{2}{v} - 1$$

But a simple calculation reveals that, when $1 < v < 2$, the unique NE under π^* requires that every agent whose skill is $(2/v) - 1$ or more must work (recall our convention that, if indifferent between shirking and working, an agent works) while the rest must shirk. This shows that π^* elicits as much effort as any $\pi \in \Pi$ even when $1 \leq v < 2$.

The discussion of optimal π became so simple for this binary example precisely because we could infer, from observing a positive output, that an agent must have worked. (Recall that the output of shirk was taken to be 0.) This enabled us to reward him *precisely* when he worked, via π^*.Were we to more generally postulate (as was done in section 9, for games of complete information) that the output of shirk, work is given by two functions τ, τ^* – that map skills to positive outputs — with $\tau < \tau^*$, the situation would become much more intriguing. For now it would no longer be possible to tell from an output whether it has been produced by a low-skill agent who is working or a high-skill agent who is shirking (when the intersection of the ranges of τ, τ^* is nonempty). Computation (hence comparison) of NE under π_D and π_P, in this scenario, as well as the computation of optimal π, are both interesting problems, which we leave for future research.

Appendix

This section contains proofs that were postponed.

9.2 Theorem 3.

Proof. For brevity denote $Y(\boldsymbol{\chi}) = T(f, \boldsymbol{\chi})$ and $\bar{Y} = T(f)$. For $0 < p < 1$, consider the event $\mathbf{E} = \{Y(\boldsymbol{\chi}) < \bar{Y}/p\}$. It is evident that the probability $\xi(\mathbf{X} \setminus \mathbf{E}) \leq p$, hence

$\xi(\mathbf{E}) \geq 1 - p$. For $n \in N$, let $\mathbf{F}_n$ be the subset of $\boldsymbol{\chi} \in \mathbf{E}$ such that agent n chooses 0 effort with positive probablity in $f(\boldsymbol{\chi})$. If $\xi(\mathbf{F}_n) = 0$ for all n, then (by Axiom 1) every agent produces expected output at least $de_{\min}$ almost everywhere in $\mathbf{E}$, and so

$$\bar{Y} \geq (1-p)|N|de_{min} = (1-p)\, a > 0 \tag{10}$$

Now suppose $\xi(\mathbf{F}_n) > 0$ for some n. Fix $\boldsymbol{\chi} \in \mathbf{F}_n$, write $f(\boldsymbol{\chi}) = \left(\sigma^1, \ldots, \sigma^N\right)$, and let n unilaterally change his strategy by shifting his probability $\sigma^n(0)$ from effort 0 to effort 1. Since n gets the prize with probability 0 when he chooses 0, and gets it (again by Axiom 1) with probability at least $d/(Y(\boldsymbol{\chi}) + D) \geq d/(\bar{Y}/p + D)$ when he chooses effort 1, his gain in payoff is at least $\sigma^n(0)[\underline{v}d/(\bar{Y}/p + D) - C]$ at every $\boldsymbol{\chi} \in \mathbf{F}_n$. Since f is a ξ-WNS-function, we must have $\underline{v}d/(\bar{Y}/p+D) \leq C$, which gives

$$\bar{Y} \geq p\left(\frac{dv}{C} - D\right) = pb > 0 \tag{11}$$

(where $>$ holds by Axiom 2). Since either (10) or (11) must occur, we see that

$$\bar{Y} \geq \min\{(1-p)a, pb\}$$

for all $0 < p < 1$, and hence (by a straightforward calculation)

$$\bar{Y} \geq \max_{0<p<1} \min\{(1-p)a, pb\} = \frac{ab}{a+b} = \frac{H}{2}$$

■

Observe that the above proof does not work if f is a ξ-VWNS-selection. Denoting by $f^n_{\boldsymbol{\chi}}$ the mixed strategy assigned by f to agent n at $\boldsymbol{\chi}$, observe that if the unilaterally deviating agent n were to shift $f^n_{\boldsymbol{\chi}}(e)$ wholly onto 1 for *all* $e \in E \setminus \{1\}$, not just for $e = 0$, he may not stand to benefit because

- his increase in the probabilty of winning the prize when he switches from e to 1, may be miniscual whenever $e \neq 0$ (becasue the probability was already close to 1 when he chose e), while the cost $\delta^n(1) - \delta^n(e)$ may be significant
- at the same time $\sigma^n(0)$ may be very small compared to $\sum_{e \neq 0,1} \sigma^n(e)$, so the gain in switching from 0 to 1 is outweighed by all the losses entailed in switching from $e \neq 0, 1$ to 1.

Thus in analyzing VWNS, we need to make sure that $\sigma^n(0)$ is big enough (we will ensure that it is at least 1/2 in the variation of the proof of Theorem 1 given below).

9.3 Theorem 5

Proof. Let Y and $\bar{Y}$ be as in the proof of Theorem 3. Consider the event

$$E = \{\boldsymbol{\chi} \in \mathbf{X} : Y(\chi) < 2\bar{Y}\}$$

It is evident that $\xi(E) \geq 1/2$. Let $\sigma : \mathbf{X} \to \Sigma$ (instead of f) denote the VWNS-selection, and let $\sigma^n_{\boldsymbol{\chi}}$ denote the mixed strategy of agent $n \in N$ in Σ^n that is picked out by σ at $\boldsymbol{\chi} \in \mathbf{X}$. Denote

$$F = \{\boldsymbol{\chi} \in E : \exists k \in N s.t. \sigma^k_{\boldsymbol{\chi}}(0) > 1/2\}$$

If $\xi(F) = 0$, every agent produces expected output at least $(1/2)de_{min}$ almost everywhere in E, and so $\bar{Y} \geq (1/4)|N|de_{min}$, proving the theorem.

If $\xi(F) > 0$, then there is an agent n such that $\xi(F^n) > 0$ where

$$F^n = \{\boldsymbol{\chi} \in E : \sigma^n_{\boldsymbol{\chi}}(0) > 1/2\}$$

At each $\boldsymbol{\chi} \in F^n$, let agent n unilaterally change his strategy from σ^n_{χ} to 1. Since n gets the prize with probability 0 when he chooses 0, and gets it (see Axiom 1) with probability at least

$$\frac{d}{Y + D} \geq \frac{d}{2\bar{Y} + D}$$

when he chooses 1, his gain in payoff is at least

$$\frac{1}{2}.[\underline{v}(\frac{d}{2\bar{Y} + D})] - C$$

at every $\boldsymbol{\chi} \in F^n$. Since σ is a ξ-VWNS-selection, we must have

$$\underline{v}(\frac{d}{2\bar{Y} + D}) - 2C \leq 0$$

which gives

$$\bar{Y} \geq \frac{1}{4}(\frac{d\underline{v}}{C} - 2D) \tag{12}$$

proving the theorem. ■

9.4 Theorem 8

Proof. First let us a note an obvious fact which we shall use repeatedly. Let X be a nonnegative random variable, with upper bound $\tilde{B}$ and expectation $\tilde{M}$. Then, for any $\alpha \in (0,1)$ and $M \leq \tilde{M}$

$$Pr\{X > \alpha M\} > \frac{M - \alpha M}{\tilde{B} - \alpha M} \tag{13}$$

To see this, denote the LHS by p. Then $M \leq \tilde{M} \leq p\tilde{B} + (1-p)\alpha M$ which yields $p \geq (M - \alpha M)/(\tilde{B} - \alpha M)$.

We shall first establish (5) of Theorem 8. *Fix* throughout $\boldsymbol{\chi} = (\delta^n, \tau^n, v^n)_{n \in N}$ for which the bounds in Axiom 7 apply (such $\boldsymbol{\chi}$ occur with ξ-probability 1). For any $k \in N$, let $Y_{-k} \equiv \sum_{n \in N \backslash \{k\}} \tau^n(1)$ be the total output produced by the players in $N \backslash \{k\}$ when they all exert maximal effort. For brevity, denote $l \equiv |N| - 1 \equiv |N \backslash \{k\}|$. Then Exp $Y_{-k} \geq ld$ by (2). So, by Axiom 7 and (13)(taking $M = ld, \tilde{B} = l\beta, \alpha = 1/2$ and noting that $\beta > d$) we obtain

$$Pr(Y_{-k} \geq ld/2 \geq \frac{ld/2}{l\beta - (ld/2)}) > \frac{d}{2\beta} \tag{14}$$

Given any realization $A > 0$ of total output Y_{-k} produced by $N \setminus \{k\}$, let player k unilaterally deviate from effort $e \in E \setminus \{1\}$ to 1. Then k's probability of winning the prize goes up by (or, equivalently, others' probability of winning the prize goes down by)

$$\begin{aligned}
& Exp_\tau[\frac{A}{A + \tau^k(e)} - \frac{A}{A + \tau^k(1)}] \\
&= Exp_\tau[\frac{A(\tau^k(1) - \tau^k(e))}{(A + \tau^k(e))(A + \tau^k(1))}] \\
&\geq \frac{AExp_\tau(\tau^k(1) - \tau^k(e))}{|N|^2\beta^2} \\
&= \frac{A(\mu^k(1) - \mu^k(e))}{|N|^2\beta^2} \\
&\geq \frac{A\Delta}{|N|^2\beta^2}
\end{aligned} \tag{15}$$

(The inequalities here follow from Axiom 7.) But $A \geq ld/2$ with probability at least

d/β by (14). Thus k's gain in payoff, when he unilaterally deviates from $e \in E \setminus \{1\}$ to 1, is at least

$$\frac{d}{2\beta} \cdot \frac{ld}{2} \cdot \frac{\Delta}{|N|^2\beta^2} v^k \equiv Z v^k (\text{say})$$

On the other hand, his loss in payoff is at most $\delta^k(1) - \delta^k(e) \leq C$. Thus, if we choose $v_* > C/Z$, the gain outweighs the loss and we conclude that $\mathbf{1}$ is an NE of $\Gamma_{\pi_P}(\boldsymbol{\chi})$, proving (5). (Notice that, since $l \equiv |N| - 1$, we have $Z \approx 1/(|N|)$ which implies $v_* = O(|N|)$ as expected from Theorem 1 according to which the total expected output is $O(min(|N|, v_*))$.)

We now turn to the proof of (6). First let us establish that there exists v^+ such that, if $\min\{v^n : n \in N\} > v^+$, then at any NE σ of $\Gamma_{\pi_P}(\boldsymbol{\chi})$ we have

$$Exp_\tau \; Y_{-k} \geq ld/4 \tag{16}$$

for all $k \in N$. Suppose provisionally that (16) is false, i.e., $Exp_\tau \; Y_{-\bar{k}} < ld/4$ for some $\bar{k} \in N$. Then

$$Pr(Y_{-\bar{k}} < ld/2) > 1/2 \tag{17}$$

Clearly there exists $n \in N \setminus \{\bar{k}\}$ such that $\sigma^n(0) > 0$ (otherwise $Exp_\tau \; Y_{-\bar{k}} \geq ld$ contradicting our provisional hypothesis that $Exp_\tau \; Y_{-\bar{k}} < ld/4$.)

Let n shift probability $\sigma^n(0)$ from 0 to 1. His loss in utility, from the extra work is at most $\sigma^n(0)C$. On the other hand, from (17) and Assumption AII, we see that his probability of winning the prize goes up by at least

$$\sigma^n(0).[\frac{\Delta}{(ld/2) + \beta}].\frac{1}{2}$$

We choose v^+ to ensure that the gain outweighs the loss i.e.,

$$v^+.[\frac{\Delta}{(ld/2) + \beta}].\frac{1}{2} > C$$

contradicting that σ is a WNS of $\Gamma_{\pi_P}(\boldsymbol{\chi})$, and thus contradicting also (17), and thereby establishing (16)

Now by (13) and (16) (taking $M = ld/4, \alpha = 1/2, \tilde{B} = \beta l$ and noting that $\beta > d$ we derive

$$Pr(Y_{-k} > ld/8) \geq \frac{ld/8}{l\beta - (ld/8)} > \frac{d}{8\beta} \tag{18}$$

Consider any $k \in N$ and $e \in E \setminus \{1\}$. We shall show there exists v^* such that, if $v^k > v^*$, then k can improve his payoff by deviating from e to 1 (assuming of course that all the other players are producing some given amount $\tilde{A} > ld/8$). Indeed, in view of (18) and (15) (using now $ld/8$ as the lower bound for A in (15)), k's gain in payoff is at least

$$\frac{d}{8\beta}.\frac{ld}{8}.\frac{\Delta}{|N|^2\beta^2}.v^k \equiv \tilde{Z}v^k(say)$$

while his loss is at most C. Thus it suffices to choose $v^* > C/\tilde{Z}$. Since $\tilde{Z} > Z$, we have $v^* > v^+$, proving (6). ■ ■

9.5 Theorem 9

This is entirely analogous to the proof of Theorem 8 ■

9.6 Lemma 11

Proof. Since $\boldsymbol{\chi} \equiv (\delta^n, \tau^n, v^n)_{n\in N}$ is fixed, we shall suppress it and write $K \equiv K(\boldsymbol{\chi})$. Imagine the scenario when every agent in K chooses 1. In this scenario an $j \notin K$ has 0 probability of winning the prize at effort level 1 and hence, by the stochastic dominance condition of Axiom 10, at any effort level. This defines certain probabilities $\pi_*^k > 0$ for $k \in K$ to win the prize, and it is evident that $(i) \sum_{k\in K} \pi_*^k = 1$ and $(ii)\, \pi_*^k$ is independent of the mixed strategies chosen by the agents in $N \setminus K$. Furthermore for $k \in K$, again by stochastic dominance, the probability that k wins can only increase if any agents in $K \setminus \{k\}$ change to strategies other than 1. Hence we deduce that every agent $k \in K$ can *guarantee* himself the payoff $\pi_*^k v^k - \delta^k(1)$ by playing 1. Thus, if $\sigma \in IR(\Gamma_{\pi_D}(\boldsymbol{\chi}))$, the payoff $F^k(\sigma)$ of k at $\boldsymbol{\sigma}$ satisfies $F^k(\sigma) \geq \pi_*^k v^k - \delta^k(1)$ for all $k \in K$. But clearly $F^k(\sigma) \leq \bar{\pi}^k(\sigma)v^k$ (denoting $\bar{\pi}^k(\sigma) \equiv k$'s probability of winning the prize under $\boldsymbol{\sigma}$), so we have $\bar{\pi}^k(\sigma) \geq \pi_*^k - (\delta^k(1)/v^k)$ for all $k \in K$, which implies

$$\sum_{k\in K} \bar{\pi}^k(\sigma) \geq \sum_{k\in K} \pi_*^k - \sum_{k\in K} \frac{\delta^k(1)}{v^k} = 1 - \sum_{k\in K} \frac{\delta^k(1)}{v^k}$$

But then, putting $v \equiv v^1$ and observing $B^{-1}v \leq v^n \leq Bv$ for all $n \in N$ by part 1 of Axiom 10 , we have

$$\sum_{n \in N \setminus K} \bar{\pi}^n(\sigma) = 1 - \sum_{k \in K} \bar{\pi}^k(\sigma) \leq \sum_{k \in K} \frac{\delta^k(1)}{v^k} \leq \frac{B}{v} \sum_{k \in K} \delta_k(1)$$

So we obtain

$$\begin{aligned} \sum_{n \in N \setminus K} F^n(\sigma) &= \sum_{n \in N \setminus K} \left[\bar{\pi}^n(\sigma) v^n - \sum_{e \in E} \sigma^n(e) \delta^n(e) \right] \\ &\leq Bv \sum_{n \in N \setminus K} \bar{\pi}^n(\sigma) - \sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e) \delta^n(e) \\ &\leq B^2 \sum_{k \in K} \delta^k(1) - \sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e) \delta^n(e) \end{aligned}$$

But each $n \in N \setminus K$ can guarantee a payoff of at least 0 by choosing effort level 0, so each $F^n(\sigma)$ is non-negative since $\sigma \in IR(\Gamma_{\pi_D}(\boldsymbol{\chi}))$, and thus $\sum_{n \in N \setminus K} F^n(\sigma) \geq 0$. Combining the above two inequalities, we have

$$\sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e) \delta^n(e) \leq B^2 \sum_{k \in K} \delta^k(1)$$

Since $\delta^k(1) \leq C$ and $\delta^n(e) \geq ce$ by Axiom 1 , we get

$$\sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e) e \leq B^2 |K| \frac{C}{c}$$

Recalling also that $\mu^n(e) \leq De$ by Axiom 1, we obtain

$$\sum_{n \in N \setminus K} \sum_{e \in E} \sigma^n(e) \mu^n(e) \leq B^2 |K| \frac{C}{c} D$$

Clearly, by our definition of h and Axiom 1,

$$\sum_{k \in K} \sum_{e \in E} \sigma^n(e) \mu^k(e) \leq B^2 |K| \mu^h(1) \leq B^2 |K| \frac{C}{c} D$$

(using the fact that $C > c$ in the last inequality). The above two inequalities prove the Key Lemma. ■

9.7 Lemma 14

For the proof of Lemma 14, it will be useful to first establish some auxiliary results. First, some notation. Let $C = [0,1]^n$ be the unit cube in $\mathbb{R}^n$ and let $0 < \varepsilon < 1$ be fixed. For $x = (x_1, \ldots, x_n)$ in C we define

$$N_\varepsilon(x) = |\{i : x_i \in [M - \varepsilon, M)\}|\,, \text{ where } M = \max(x_i).$$

If X is a C-valued random variable with density $\rho(x)$, we write N_ε^ρ for the random variable

$$N_\varepsilon^\rho = N_\varepsilon(X)$$

If $\rho(x) \equiv 1$ then the x_i are iid with uniform density on $[0,1]$. In this case we will show that N_ε^1 is closely related to the binomial random variable B_ε, which counts the number of successes in n independent trials with individual success probability ε:

$$\Pr(B_\varepsilon = k) = \binom{n}{k}\varepsilon^k(1-\varepsilon)^{n-k}.$$

Lemma 26 *If* $\rho(x) \equiv 1$ *then*

$$\Pr\left(N_\varepsilon^1 = k\right) = \begin{cases} \Pr(B_\varepsilon = k) & \text{if } k < n-1 \\ \Pr(B_\varepsilon = n-1) + \Pr(B_\varepsilon = n) & \text{if } k = n-1 \end{cases}$$

Moreover

$$E\left(N_\varepsilon^1\right) \le n\varepsilon \tag{19}$$

Proof. It suffices to establish the first statement, since it implies that B_ε stochastically dominates N_ε^1, which in turn implies the second statement. For the proof of the first statement we note that the possible values of N_ε^1 are $0, 1, \ldots, n-1$, while those of B_ε are $0, 1, \ldots, n$. Therefore it suffices to prove that

$$\Pr\left(N_\varepsilon^1 = k\right) = \Pr(B_\varepsilon = k) \text{ for } k < n-1$$

Ignoring ties, which occur with probability 0, the event $N_\varepsilon^1 = k$ is a disjoint union of $n\binom{n-1}{k}$ events, corresponding to the choice of the maximum index (in n ways) and the choice of the next k indices (in$\binom{n-1}{k}$ ways). By symmetry, each of these events has probability $\Pr(E_k)$, where E_k is the event

$$E_k = \{x_1 \text{ is largest}\} \,\&\, \{x_2, \ldots, x_{k+1} \in (x_1 - \varepsilon, x_1)\} \,\&\, \{x_{k+2}, \ldots, x_n \in [0, x_1 - \varepsilon]\}$$

Thus its suffices to show that

$$\Pr\left(E_k\right) = \frac{\Pr\left(B_\varepsilon = k\right)}{n\binom{n-1}{k}} = \frac{\binom{n}{k}\varepsilon^k(1-\varepsilon)^{n-k}}{n\binom{n-1}{k}} = \frac{\varepsilon^k(1-\varepsilon)^{n-k}}{n-k}$$

Now writing $q\left(x\right) = \Pr\left(E_k | x_1 = x\right)$ we have

$$\Pr\left(E_k\right) = \int_0^1 q\left(x\right) dx$$

Since $x_2, \ldots, x_n$ are independent and uniform on $[0,1]$ we get

$$q\left(x\right) = \begin{cases} \varepsilon^k(x-\varepsilon)^{n-k-1} & \text{if } \; x > \varepsilon \\ 0 & \text{if } \; x \leq \varepsilon \end{cases}$$

Integrating over x, making a change of variable $y = x - \varepsilon$, we get, as desired

$$\Pr\left(E_k\right) = \int_\varepsilon^1 \varepsilon^k(x-\varepsilon)^{n-k-1}dx = \varepsilon^k \int_0^{1-\varepsilon} y^{n-k-1}dy = \frac{\varepsilon^k(1-\varepsilon)^{n-k}}{n-k}$$

■

Lemma 27 *Suppose $\rho\left(x\right)$ is bounded above by a constant β. Then we have*

$$E\left(N_\varepsilon^\rho\right) \leq \beta n \varepsilon.$$

Proof. Using (19) we get

$$E\left(N_\varepsilon^\rho\right) = \int_C N_\varepsilon\left(x\right) \rho\left(x\right) dx \leq \beta \int_C N_\varepsilon\left(x\right) dx = \beta E\left(N_\varepsilon^1\right) \leq \beta n \varepsilon$$

■

We can now prove Lemma 14

Proof. Transform Y, distributed uniformly on $[d,D]$, to $X = [Y-d]\,[D-d]^{-1}$ which is uniform on $[0,1]$. The average size of the elite set is unaffected by this transformation. Thus the result follows from Lemma 27 ■

9.8 Lemma 18

Proof. Since τ^* and τ are strictly monotonic and continuous, there exist $\Delta > 0$ and $\Delta' > 0$ such that $s' \equiv s + \Delta \in [k,K]$,and $t' \equiv t + \Delta' \in [k,K]$ and

$$\frac{\tau(s')}{\tau(s') + \tau^*(t')} = \frac{\tau(s)}{\tau(s) + \tau^*(t)} \tag{20}$$

Hence there exists a *maximal* pair Δ, Δ' satisfying (20), and then either $s' = K$ or $t' = K$ (otherwise both Δ and Δ' could be increased slightly, still maintaining (20), and contradicting the maximality of Δ, Δ').

In view of (20), to prove (b) it suffices to show that

$$\frac{\tau^*(s')}{\tau^*(s') + \tau^*(t')} \leq \frac{\tau^*(s)}{\tau^*(s) + \tau^*(t)} \tag{21}$$

which is equivalent to

$$\frac{\tau^*(t')}{\tau^*(s')} \geq \frac{\tau^*(t)}{\tau^*(s)} \tag{22}$$

as can be seen by dividing the numerator and the denominator of the LHS and RHS of (21) by $\tau^*(s'^*)$ and $\tau^*(s)$ respectively.

But a similar maneuver shows that (20) is equivalent to

$$\frac{\tau^*(t')}{\tau(s')} = \frac{\tau^*(t)}{\tau(s)} \tag{23}$$

And, since $s' > s$, decreasing returns (Assumption AIV) imply

$$\frac{\tau^*(s')}{\tau^*(s)} \leq \frac{\tau(s')}{\tau(s)} \tag{24}$$

From (23) and (24), we get

$$\frac{\tau^*(s')}{\tau^*(s)} \leq \frac{\tau(s')}{\tau(s)} = \frac{\tau^*(t')}{\tau(t)} \tag{25}$$

establishing (22), and thereby (21) ■

9.9 Lemma 16

Proof. First consider π_D. Then $z = z_D$ implies $x = z + \eta/v$, and thus the player wins if the opponent's output lies in the interval $[z, z + \eta/v]$. This event has probability $(\eta/v)/\eta = 1/v$ and gives expected payoff $v(1/v) - 1 = 0$.

Now consider π_P. The expected payoff is

$$\frac{1}{\eta}\int_z^{z+\eta} \left(\frac{xv}{x+y}\right) dy - 1 = \frac{xv}{\eta} \ln\left(\frac{x+\eta+z}{x+z}\right) - 1$$

Setting this equal to zero and solving for z we get

$$z = \frac{\eta}{\exp(\eta/xv) - 1} - x = z_P$$

For the bounds on z_P we note that for an opponent of skill exactly $y^* = x(v-1)$ the payoff under π_P is $\frac{xv}{x+y^*} - 1 = 0$. Thus if $z + \eta < y^*$ the payoff at each y in $[z, z+\eta]$ is ≥ 0, which implies $z_P \geq y^* - \eta$. Similarly if $z > y^*$, the payoffs in $[z, z+\eta]$ is ≤ 0, which implies $z_P \leq y^*$. ■

9.10 Theorem 25

Proof. First we focus on decreasing returns. Then, by Lemma 18, we need only consider the two cases below.

Case A. Agent is at t and the rival at K.Then

$$I(t, K) = \pi' \left(\frac{t}{t+K} \right) \frac{K\psi(t)}{(t+K)^2}$$

Case B. Agent is at K and the rival at t.Then

$$I(K, t) = \pi' \left(\frac{K}{t+K} \right) \frac{t\psi(K)}{(t+K)^2}$$

Since $\pi(x) = 1 - \pi(1-x)$ for all x, we get

$$\pi' \left(\frac{t}{t+K} \right) = \pi' \left(\frac{K}{t+K} \right)$$

which, in conjunction with $K\psi(t) > t\psi(K)$ (decreasing returns), implies $I(K, t) < I(t, K)$ for all $t \in [k, K]$. Thus it suffices to incentivize the $t-$ agent to switch from shirk to work in Case B (for all $t \in [k, K]$). Since we want to maximize the minimum incentive, we must arrange for $I(K, t) = \sigma$, for some constant σ, and for all $t \in [k, K]$ .To see this, denote

$$G(t) = \frac{t\psi(K)}{(t+K)^2}$$

and let π be a solution to the differential equation, with $\pi'(t/(t+K)) = \sigma/G(t)$ for all $t \in [k, K]$ .Suppose there is a scheme $\widetilde{\pi}$ which does *not* satisfy the differential equation. If $\widetilde{\pi}'(t_1/(t_1+K) > \sigma/G(t_1)$ for some $t_1 \in [k, K]$, then since $\int \pi'(y)dy =$

$\int \widetilde{\pi}'(y)dy = 1/2$ (writing $y = t/(t+K)$, and understanding the range of integration to be from $y = 1/2$ to $y = K/(k+K)$), we see at once that there exists $t_2 \in [k, K]$ such that $\widetilde{\pi}'(t_2/(t_2+K)) < \sigma/G(t_2)$.(Thus there always exists such a t_2 for $\widetilde{\pi}$.) But then the incentive (to work) at t_2 under $\widetilde{\pi}$, which is given by $\widetilde{\pi}'(t_2/(t_2+K))G(t_2)$, is strictly less than σ,which is the constant incentive under π at all $t \in [k, K]$. We conclude that the *minimum* incentive to work under $\widetilde{\pi}$ is less than that under π,establishing the superiority of π over $\widetilde{\pi}$. So an optimal scheme must satisfy the following differential equation (where $\widetilde{C}$ is another constant):

$$\pi'\left(\frac{K}{t+K}\right) = \widetilde{C}\frac{(t+K)^2}{t\psi(K)}, \text{i.e., } \pi'\left(\frac{K}{t+K}\right) = \frac{\widetilde{C}}{\psi(K)}\left(\frac{t+K}{t}\right)^2 t$$

For $x > 1/2$, let $x = K/(t+K)$, so $1-x = t/(t+K)$ and $t = K(1-x)/x$,enabling us to rewrite our differential equation:

$$\pi'(x) = \frac{\widetilde{C}}{\psi(K)}\left[\frac{1}{(1-x)^2}\right]\left[\frac{K(1-x)}{x}\right] = \frac{C}{x(1-x)}$$

where C is another constant and $1/2 \leq x \leq K/(k+K)$.The solution is

$$\pi(x) = A + B\ln\frac{x}{1-x}$$

where A, B are determined from the boundary conditions $\pi(1/2) = 1/2$ and $\pi(K/(k+K)) = 1$.(Thus $A = 1/2$.) Then, in the range $(k/(k+K)) \leq x < 1/2$,the value of π is determined by reflection around $1/2$, i.e., $\pi(x) = 1 - \pi(1-x)$.

The analysis for strictly increasing returns is entirely analogous. Indeed, by Lemma 23 for increasing returns, we need only consider two cases:

Case C. Agent is at t and the rival at k,where

$$I(t,k) = \pi'\left(\frac{t}{t+k}\right)\frac{k\psi(t)}{(t+k)^2}$$

Case D. Agent is at k and the rival at t,where

$$I(k,t) = \pi'\left(\frac{k}{t+k}\right)\frac{t\psi(k)}{(t+k)^2}$$

Strictly increasing returns imply $k\psi(t) > t\psi(k)$, hence $I(k,t) < I(t,k)$ for all $t \in [k, K]$, from which we derive as before that $\pi'(x) = C/(x(1-x))$ where C is another constant, $x = k/(t+k)$ and $1/2 \geq x \geq k/(k+K)$.The solution is

$$\pi(x) = A' + B'\ln\frac{x}{1-x}$$

for $1/2 \geq x \geq k/(k+K)$ and $1 - \pi(1-x)$ for $1/2 < K/(k+K)$, where A', B' are determined via the boundary conditions $\pi(k/(k+K)) = 0$ and $\pi(1/2) = 1/2$.. ■

References

[1] Anton, J., and Yao, D. (1992). Coordination in split award auctions. *Quarterly Journal of Economics* 107:681-701.

[2] Barut, Y. and Kovenock, D. (1998). The symmetric multiple prize all-pay auction with complete information. *European Journal of Political Economy.* 14:627-644.

[3] Baye, M., Kovenock, D. and De Vries, C.G. (1993). Rigging the lobbying process: An application of the all-pay auction. *American Economic Review* 83:289-294.

[4] Baye, M., Kovenock, D. and De Vries, C.G (1994). The solution to the Tullock rent-seeking game when R is greater than 2: Mixed strategy equilibria and mean dissipation rates. *Public Choice* 81:363-380.

[5] Broecker, T. (1990). Credit-worthiness tests and interbank competition. *Econometrica.* 58:429-452.

[6] Bulow, J., and Klemperer, P. (1999). The genenralized war of attrition. *American Economic Review.* 89:175-189.

[7] Clark, D., and Riis, C. (1998). Competition over more than one prize. *American Economic Review.* 88:276-289.

[8] Che, Y.K. and Gale, I. (1997). Rent dissipation when rent seekers are budget constrained. *Public Choice* 92:109-126.

[9] Che, Y.K. and Gale, I. (1998). Caps on political lobbying. *American Economic Review* 88:643-651.

[10] Dubey, P., and Sahi, S. (2009). The Allocation of a Prize, Working Paper, Department of Economics, Stony Brook University.

[11] Dubey, P., and Geanakoplos, J. (2010). Grading exams: 100,99,98 .... or A,B,C ? *Games and Economic Behavior*, Vol 69, Issue 1,pp 72-94, Special Issue in Honor of Robert Aumann

[12] Dubey, P., and Haimanko, O. (2003). Optimal scrutiny in multi-period promotion tournaments. *Games and Economic Behavior.* 42(1):1-24

[13] Dubey, P., and Wu, C. (2001). When less scrutiny induces more effort. *Journal of Mathematical Economics.* 36(4):311-336.

[14] Ellingsen, T. (1991). Strategic buyers and the social cost of monopoly. *American Economic Review* 81:648-657.

[15] Fang Hanming (2002) "Lottery versus All-pay Auction Models of Lobbying". Public Choice, pp 351-371

[16] M.A.de Frutos (1999) "Coalitional Manipulations in a Bankruptcy Problem", Review of Economic Design, Vol 4, No 3, pp 255-272.

[17] Glazer, A., and Hassin, R. (1988). Optimal contests. *Economic Inquiry.* 26:133-143.

[18] Green, J., and Stokey, N. (1983). A comparison of tournaments and contracts. *Journal of Political Economy.* 91(3):349-364.

[19] Hillman, A.L. and Riley, J.G. (1989) Politically contestable rents and transfers. *Economics and Politics* 1:17-39.

[20] Krishna, V., and Morgan, J. (1998). The winner-take-all principle in small tournaments. *Advances in Applied Microeconomics.* 7:61-74.

[21] Lazaer, E., and Rosen, S. (1981). Rank order tournaments as optimum labor contracts. *Journal of Political Economy.* 89:841-864.

[22] Moldovanu, B. and Sela, A. (2001). The optimal allocation of prizes in contests. *American Economic Review.* 91(3):542-558.

[23] Nalebuff, B., and Stiglitz, J. (1983). Prizes and incentives: Towards a general theory of compensation and competition. *Bell Journal of Economics.* 14:21-43.

[24] Nti, K.O. (1999) Rent-seeking with asymmetric valuations, *Public Choice, 98 (3-4), 415-430.*

[25] Rosen, S. (1986). Prizes and incentives in elimination tournaments. *American Economic Review.* 76:701-715.

[26] Rowley C.K. (1991) Gordon Tullock: Entrepeneur of public choice. *Public Choice* 71:149-169.

[27] Rowley C.K. (1993) *Public Choice Theory.* Edward Elgar Publishing.

[28] Tullock, G. (1975) On the efficient organization of trails. *Kyklos* 28:745-762.