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THE ALMOST SURE SKOROKHOD REPRESENTATION FOR SUBSEQUENCES IN NONMETRIC SPACES[1)]

Показано, что для широкого класса топологических пространств любая равномерно плотная последовательность случайных элементов, содержит подпоследовательность, которая допускает обычное п.н. представление Скорохода на интервале Лебега.

Ключевые слова и фразы: п.н. представление Скорохода, сходимость по распределению на неметрических топологических пространствах, теорема Прохорова.

1. The a.s. Skorokhod representation

Let $(\mathcal{X}, \rho)$ be a Polish space and let $X_1, X_2, \ldots$ be random elements taking values in $\mathcal{X}$ and converging in distribution to X_0:

$$X_n \xrightarrow[\mathcal{D}]{} X_0. \tag{1}$$

In his famous paper [11], Skorokhod proved that there exist $\mathcal{X}$-valued random elements $Y_0, Y_1, Y_2, \ldots$, defined on the unit interval $([0,1], \mathcal{B}_{[0,1]})$ equipped with the Lebesgue measure ℓ, such that

$$\text{the laws of } X_n \text{ and } Y_n \text{ coincide for } n = 0, 1, 2, \ldots, \tag{2}$$

$$\rho\Big(Y_n(\omega), Y_0(\omega)\Big) \longrightarrow 0, \quad \text{as } n \to +\infty, \text{ for each } \omega \in [0,1]. \tag{3}$$

Later, Dudley [2] extended Skorokhod's ideas to separable metric spaces, and Wichura [13] and Fernandez [5] proved the existence of the Skorokhod representation in nonseparable metric spaces, for limits with separable range (see also [3]). The price to be paid was larger space required by the definition of the representation.

It may be worth to emphasize that if we restrict our attention to convergence in distribution of random elements with *tight* (or *Radon*) distributions then even in arbitrary metric spaces the a.s. Skorokhod representation exists in its original shape (on $[0,1]$). This is an easy consequence of the fact that each σ-compact metric space can be *homeomorphically* imbedded into a Polish space, and of Le Cam's theorem [7] asserting that in metric spaces any sequence $\{\mu_n\}$ of *tight* probability measures weakly convergent to a *tight* measure μ_0 is *uniformly tight*, i.e. for every $\varepsilon > 0$ there exists a compact subset K_ε such that

$$\mu_n(K_\varepsilon) > 1 - \varepsilon, \qquad n = 1, 2, \ldots. \tag{4}$$

When we leave the safe area of metrisable spaces no positive result on the a.s. Skorokhod representation seems to be known.

Let us consider, for example, the weak topology $\tau_w = \sigma(H, H)$ on the infinite dimensional separable Hilbert space $(H, \langle,\rangle)$. Suppose that X_n, $n = 0, 1, \ldots$, take values in $(H, \mathcal{B}_{\tau_w})$ and that $X_n \longrightarrow_{\mathcal{D}} X_0$ in this space, i.e.

$$\mathbf{E} f(X_n) \longrightarrow \mathbf{E} f(X_0), \qquad \text{as } n \to +\infty, \tag{5}$$

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for each bounded and *weakly* continuous function $f\colon H \to \mathbf{R}^1$.

In general, *there is no* a.s. Skorokhod representation for $\{X_n\}$. This may be seen by the following chain of arguments. First, if $x_n \longrightarrow_{\tau_w} x_0$, then $\sup_n \|x_n\| < +\infty$. It follows that $Y_n(\omega) \longrightarrow_{\tau_w} Y_0(\omega)$ a.s. implies $\sup_n \|Y_n(\omega)\| < +\infty$ a.s., hence uniform tightness of $\{Y_n\}$ in (H, τ_w). On the other hand, Fernique [6, p. 24–25] gave an example of a sequence satisfying (5) with $X_0 \equiv 0$, and such that

$$\liminf_n \mathbf{P}\{\|X_n\| > K\} = 1, \qquad \text{for every} \quad K > 0.$$

This sequence has no subsequence which is uniformly tight on (H, τ_w), so no subsequence which admits the a.s. Skorokhod representation.

Suppose, however, that while checking (5) we applied the classical procedure based on the direct Prokhorov theorem. This means we were able to prove that for each $\varepsilon > 0$ there is a number $K_\varepsilon > 0$ such that

$$\mathbf{P}\Big\{\|X_n\| > K_\varepsilon\Big\} < \varepsilon, \qquad n = 1, 2, \dots, \tag{6}$$

(uniform τ_w-tightness) and then we identified the limiting distribution, via e.g.

$$\langle y,\, X_n\rangle \underset{\mathcal{D}}{\longrightarrow} \langle y,\, X_0\rangle, \qquad \text{as } n \to +\infty, \quad y \in H_0, \tag{7}$$

where H_0 is a dense subset of H. Consider the following theorem, which is a particular case of a much more general result proved in Section 2.

Theorem 1. *Let* $X_1, X_2, \dots$ *be uniformly* τ_w*-tight, i.e. satisfy* (6). *Then one can find a subsequence* $\{n_k\}$ *and* H*-valued random variables* $Y_0, Y_1, \dots$ *defined on* $([0,1], \mathcal{B}_{[0,1]}, \ell)$ *such that*

$$X_{n_k} \sim Y_k, \qquad k = 1, 2, \dots, \tag{8}$$

$$\langle y,\, Y_k(\omega)\rangle \longrightarrow \langle y,\, Y_0(\omega)\rangle, \qquad \textit{as} \quad k \to \infty, \quad \omega \in [0,1], \quad y \in H. \tag{9}$$

By the above theorem, if (5) and (6) hold, then *in every subsequence* $\{X_{n_k}\}_{k\in\mathbf{N}}$ *one can find a further subsequence* $\{X_{n_{k_l}}\}_{l\in\mathbf{N}}$, *for which the usual a.s. Skorokhod representation on the Lebesgue interval exists.* Let us say that $\{X_n\}_{n\in\mathbf{N}}$ possesses the a.s. Skorokhod representation *for subsequences.*

Notice that in practice the a.s. representation for subsequences is equally useful as the «full» representation. Typically one needs the Skorokhod representation to prove convergence in distribution of some functionals of the underlying processes (see [1] for standard examples). In the simplest case the functional is a measurable mapping, g say, which is a.s. continuous with respect to the limiting law $\mathcal{L}(X_0)$. But it follows from the very definition of the weak convergence of probability laws that $g(X_n) \longrightarrow_{\mathcal{D}} g(X_0)$ if and only if in every subsequence $\{g(X_{n_k})\}_{k\in\mathbf{N}}$ one can find a further subsequence $\{g(X_{n_{k_l}})\}_{l\in\mathbf{N}}$ converging in law to $g(X_0)$. Hence it is clear that the a.s. Skorokhod representation for subsequences is just what we need.

Our main general result, Theorem 2 (Section 2) is most suitable in cases when weak convergence does imply uniform tightness. For example, in spaces of distributions ($\mathcal{S}'$ and $\mathcal{D}'$) we can express weak convergence in terms of the a.s. Skorokhod representation (see Corollary 3 in Section 3) and it is the first result in this direction. In other cases our results may be applied every time we get weak convergence indirectly, i.e. when we first check relative compactness (via uniform tightness and the direct Prokhorov's theorem) and then identify limit by other tools. A special emphasis is given to nonmetric spaces.

2. Topological assumption and main theorem

Let $(\mathcal{X}, \tau)$ be a topological space. Denote by «$\longrightarrow_\tau$» convergence of sequences in the topology τ. The only assumption we impose on $(\mathcal{X}, \tau)$ is quite simple:

$$\text{There exists a countable family } \{f_i\colon \mathcal{X} \to [-1,1]\}_{i\in I} \text{ of } \tau\text{-continuous functions, which separate points of } \mathcal{X}. \tag{10}$$

This assumption gives us the mapping

$$\mathcal{X} \ni x \mapsto \tilde{f}(x) = (f_i(x))_{i\in I} \in [-1,1]^I, \tag{11}$$

which is *one-to-one* and *continuous*, but (in general) is not a homeomorphism of $\mathcal{X}$ onto a subspace of $[-1,1]^I$. In any case, $\tilde{f}$ defines another topology $\tau_{\tilde{f}}$ on $\mathcal{X}$, which is weaker (coarser) than the original one: $\tau \supset \tau_{\tilde{f}}$. Since $\tau_{\tilde{f}}$ is metrisable, it is Hausdorff (and so τ is Hausdorff as well). Moreover, by the well-known minimal property of compact topologies (see, for example, [4, Corollary 3.1.14, p. 126]), both topologies coincide on τ-compact sets, hence τ-compact sets are metrisable and $\tilde{f}$ is a homeomorphism, if restricted to each *τ-compact* subset $K \subset \mathcal{X}$. In particular, $\tilde{f}$ is a *measurable isomorphism*, if restricted to each σ-compact subspace of $(\mathcal{X}, \tau)$.

Condition (10) is not very restrictive and possesses several nice consequences, which we list below together with some comments.

- $K \subset \mathcal{X}$ is compact if and only if it is sequentially compact (and then it is metrisable, as we already learnt above).
- $(\mathcal{X}, \tau)$ is functional Hausdorff, but need not be regular. In particular, sequential spaces may satisfy (10), while they are not completely regular, or it is very dif and only ificult to check their regularity.
- The closure of a relatively compact subset consists of limits of its convergent subsequences, but still need not be compact. Therefore in the definition of uniform τ-tightness we cannot, in general, replace sequential compactness with measurability and relative compactness.

Since in the present paper we do not use these properties, we omit their proofs[2)].

Let us suppose that $X\colon (\Omega, \mathcal{F}) \to \mathcal{X}$ is such that

$$f_i(X)\colon (\Omega, \mathcal{F}) \to \Big([-1,1], \mathcal{B}_{[-1,1]}\Big) \tag{12}$$

is measurable, for each $i \in I$. In most cases of interest, $\sigma(f_i; i \in I) = \mathcal{B}_\tau$ and (12) means simply that $X\colon (\Omega, \mathcal{F}) \to (\mathcal{X}, \mathcal{B}_\tau)$ is Borel-measurable. But there are spaces for which $\sigma(f_i;\ i \in I)$ is strictly smaller than $\mathcal{B}_\tau$. We will show that it is unimportant as far as we deal with random elements with *tight* laws. Since for every τ-compact set K, $\tilde{f}(K)$ is compact in $[-1,1]^I$ and $K = \tilde{f}^{-1}(\tilde{f}(K))$, we have $K \in \sigma(f_i;\ i \in I)$ and the event $\{X \in K\}$ is measurable. In particular, on the basis of (12) we can ask whether the law of X is tight on $(\mathcal{X}, \tau)$.

So assume now that the law of X is tight:

$$\mathbf{P}\{X \in K_j\} > 1 - \frac{1}{j},$$

for some compact $K_j \subset \mathcal{X}$, $j \in \mathbf{N}$. Let $f\colon (\mathcal{X}, \tau) \to \mathbf{R}^1$ be a continuous function and let $B \in \mathcal{B}^1$. Since $\tau = \tau_{\tilde{f}}$ on K_j, we have $\{f(X) \in B,\ X \in K_j\} \in \mathcal{F}$ for each $j \in \mathbf{N}$. Since

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[2)]See, however, *Jakubowski A.* A unification of the Prokhorov's and Skorokhod's ideas: convergence in distribution in nonmetric spaces. Preprint, 1995.

also

$$\left\{f(X)\in B,\ X\notin\bigcup_{j=1}^{\infty}K_j\right\}\subset\left\{X\notin\bigcup_{j=1}^{\infty}K_j\right\}$$

and the latter set has probability zero, we obtain that $f(X)$ is measurable with respect to the **P**-completion of $\mathcal{F}$. In a similar way we check that the law of X can be extended in a unique way to the whole Borel σ-algebra $\mathcal{B}_\tau$. We conclude that the property (12) essentially does not depend on the choice of the separating family $\{f_i;\ i\in I\}$, *provided* we consider random elements with tight laws. Hence in what follows we will restrict our attention to random elements X such that $f_i(X)$, $i\in I$, are random variables and the law of X is tight.

Theorem 2. *Let $(\mathcal{X},\tau)$ be a topological space satisfying* (10) *and let $X_1, X_2,\ldots$ be $\mathcal{X}$-valued random elements. Suppose for each $\varepsilon>0$ there exists a compact subset $K_\varepsilon\subset\mathcal{X}$ such that*

$$\mathbf{P}\{X_n\in K_\varepsilon\}>1-\varepsilon,\qquad n=1,2,\ldots. \tag{13}$$

Then one can find a subsequence $\{X_{n_k}\}_{k\in\mathbf{N}}$ and $\mathcal{X}$-valued random elements $Y_0, Y_1, Y_2,\ldots$ defined on $([0,1],\mathcal{B}_{[0,1]},\ell)$ such that

$$X_{n_k}\sim Y_k,\qquad k=1,2,\ldots, \tag{14}$$

$$Y_k(\omega)\underset{\tau}{\longrightarrow}Y_0(\omega),\qquad \text{as}\ \ k\to\infty,\quad \omega\in[0,1]. \tag{15}$$

P r o o f. Let compact sets $K_m\subset\mathcal{X}$ be such that $K_m\subset K_{m+1}$, $m=1,2,\ldots$, and

$$\mathbf{P}\{X_n\in K_m\}>1-\frac{1}{m},\qquad n=1,2,\ldots. \tag{16}$$

Let $\tilde{f}$ be defined by (11). Set $\widetilde{\mu}_n=\mathcal{L}(\tilde{f}(X_n))$ and $\widetilde{K}_m=\tilde{f}(K_m)$. Define on $\mathbf{R}^I$ an integer-valued functional

$$\Phi(y):=\begin{cases}\min\{m\colon\ y\in\widetilde{K}_m\} & \text{if}\ \ y\in\bigcup_{m=1}^{\infty}\widetilde{K}_m,\\ +\infty & \text{otherwise.}\end{cases} \tag{17}$$

Clearly, Φ is lower semicontinuous, i.e.

$$\liminf_{n\to\infty}\Phi(y_n)\geqslant\Phi(y_0), \tag{18}$$

whenever y_n converges in $\mathbf{R}^I$ to y_0. Further, it follows from relation (16) that $\Phi<+\infty$ ($\widetilde{\mu}_n$-a.s.), for each $n\in\mathbf{N}$, and that $\{\widetilde{\mu}_n\circ\Phi^{-1}\}$ is a tight sequence of laws on $\mathbf{N}$. By the classical direct Prokhorov's theorem we may extract a subsequence $\{n_k\}_{k\in\mathbf{N}}$ such that on the space $\mathbf{R}^I\times\mathbf{N}$,

$$\widetilde{\mu}_{n_k}\circ\Psi^{-1}\Longrightarrow\nu_0,\qquad \text{as}\ \ k\to\infty,$$

where $\Psi(y)=(y,\Phi(y))$.

We need a slight refinement of the original Skorokhod construction [11, Lemma 3.1.1].

Lemma 1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Polish spaces, and let $\Phi\colon\ \mathcal{S}_1\to\mathcal{S}_2$ be measurable. Suppose*

$$\Big(X_n,\ \Phi(X_n)\Big)\underset{\mathcal{D}}{\longrightarrow}(X_0,Y_0)\quad on\ \ \mathcal{S}_1\times\mathcal{S}_2. \tag{19}$$

Then there exist random elements $X_0', X_1', X_2',\ldots$ (in $\mathcal{S}_1$) and Y_0' (in $\mathcal{S}_2$) defined on the standard probability space $([0,1],\mathcal{B}_{[0,1]},\ell)$ and such that

$$\mathcal{L}(X_0',\ Y_0')=\mathcal{L}(X_0,\ Y_0); \tag{20}$$

$$\mathcal{L}(X_n')=\mathcal{L}(X_n),\qquad n=1,2,\ldots; \tag{21}$$

$$\Big(X_n'(\omega),\ \Phi\big(X_n'(\omega)\big)\Big)\longrightarrow\Big(X_0'(\omega),\ Y_0'(\omega)\Big)\quad in\ \ \mathcal{S}_1\times\mathcal{S}_2, \tag{22}$$

for ℓ-almost all $\omega \in [0,1]$.

Proof of Lemma 1. We shall apply arguments similar to those as in [12]. Let (X_n', Y_n'), $n = 0,1,2,\ldots$, be the a.s. Skorokhod representation for (19). It suffices to check that

$$\Big(X_n,\, \Phi(X_n)\Big) \sim (X_n', Y_n') \tag{23}$$

implies $Y_n' = \Phi(X_n')$ ℓ-almost surely, for $n = 1,2,\ldots$. Let us apply to (23) the measurable functional

$$\mathcal{S}_1 \times \mathcal{S}_2 \ni (x,y) \mapsto (\Phi(x), y) \in \mathcal{S}_2 \times \mathcal{S}_2.$$

It follows that $(\Phi(X_n'), Y_n') \sim (\Phi(X_n), \Phi(X_n))$, i.e.

$$\ell\Big(\Big(\Phi(X_n'),\, Y_n'\Big) \in \Big\{(x,\, x);\ x \in \mathcal{S}_2\Big\}\Big) = 1.$$

Proof of Theorem 2 (continued). By Lemma 1 we find an $\mathbf{R}^I$-valued representation X_k' such that

$$\Big(X_k'(\omega),\, \Phi\Big(X_k'(\omega)\Big)\Big) \longrightarrow \Big(X_0'(\omega),\, Y_0'(\omega)\Big) \quad (\ell\text{-a.s.}), \quad \text{as } k \to \infty, \tag{24}$$

and

$$\mathcal{L}(X_k') = \mathcal{L}\Big(\tilde{f}(X_{n_k})\Big), \qquad k = 1,2,\ldots. \tag{25}$$

Since $Y_0'(\omega) < +\infty$ (ℓ-a.s.), we have also

$$\sup_k \Phi\Big(X_k'(\omega)\Big) < +\infty \qquad (\ell\text{-a.s.}). \tag{26}$$

This implies that for ℓ-almost all ω, points $X_k'(\omega)$, $k = 1,2,\ldots$, remain inside a compact set $\widetilde{K}_{m(\omega)} = \tilde{f}(K_{m(\omega)})$. Hence also $X_0'(\omega) \in \widetilde{K}_{m(\omega)}$, and moreover,

$$\tilde{f}^{-1}\Big(X_k'(\omega)\Big) \underset{\tau}{\longrightarrow} \tilde{f}^{-1}\Big(X_0'(\omega)\Big).$$

Redefining (if necessary) X_k' on the set $\bigcup_{k=0}^{\infty}(X_k')^{-1}(\bigcap_{m=1}^{\infty}\widetilde{K}_m^c)$ of ℓ-measure 0, we obtain the desired Skorokhod representation for $\{X_{n_k}\}$ in the form $Y_k = \tilde{f}^{-1}(X_k')$, $k = 0,1,2,\ldots$.

Notice that the distribution of Y_0 is tight: since Φ is lower semicontinuous we have

$$\Phi\Big(X_0'(\omega)\Big) \leqslant Y_0' \quad \text{a.s.,}$$

and so

$$\begin{aligned}\mathbf{P}\{Y_0 \notin K_m\} &= \mathbf{P}\{\tilde{f}^{-1}(X_0') \notin K_m\}\\ &= \mathbf{P}\{\Phi(X_0') > m\} \leqslant \mathbf{P}\{Y_0' > m\} \to 0 \quad \text{as} \quad m \to +\infty.\end{aligned}$$

Remark 1. Let us notice that we have checked $Y_k(\omega) \longrightarrow_\tau Y_0(\omega)$ using the following property of spaces satisfying (10):

> If $\{x_n\} \subset \mathcal{X}$ is relatively compact, and for each $i \in I$
> $f_i(x_n)$ converges to some number α_i, then x_n τ-converges to some x_0
> and $f_i(x_0) = \alpha_i$, $i \in I$. (27)

Let us emphasize also that contrary to the metric case under (10) alone we do not know whether the set of convergence

$$\Big\{\omega:\ Y_k(\omega) \underset{\tau}{\longrightarrow} Y_0(\omega),\ \text{as } k \to \infty\Big\}$$

is measurable. What we know is measurability of sets of the form

$$C\Big(\{Y_k\},\, K\Big) = \Big\{\omega\colon\ Y_k(\omega) \underset{\tau}{\longrightarrow} Y_0(\omega),\ \text{as } k \to \infty\Big\} \bigcap \bigcap_{k=1}^{\infty} \Big\{\omega\colon\ Y_k(\omega) \in K\Big\}, \tag{28}$$

where $K \subset \mathcal{X}$ is compact. This becomes obvious when we observe that by property (27) we have

$$C\Big(\{Y_k\},\, K\Big) = \Big\{\omega\colon\ \tilde{f}(Y_k(\omega)) \longrightarrow \tilde{f}(Y_0(\omega)),\ \text{as } k \to \infty\Big\} \bigcap \bigcap_{k=1}^{\infty} \{\omega\colon\ Y_k(\omega) \in K\}.$$

Now suppose for each $\varepsilon > 0$ there is a compact set K_ε such that

$$\mathbf{P}\Big\{C(\{Y_k\},\, K_\varepsilon)\Big\} > 1 - \varepsilon. \tag{29}$$

Then the set of convergence contains a measurable set of full probability and one can say that Y_k *converges to* Y_0 *almost surely «in compacts»*. In particular we have

Corollary 1. *Convergence almost surely «in compacts» implies uniform tightness.*

It is clear from the proof that the a.s. convergence (15) has been established exactly the way described above. If the representation $Y_0, Y_1, Y_2, \ldots$ satisfies (14) and the convergence (15) is strengthened to the almost sure convergence «in compacts», then we will call it *«the strong a.s. Skorokhod representation»*. Using this terminology we may rewrite Theorem 2 in the following form:

Theorem 3. *Let* $(\mathcal{X}, \tau)$ *be a topological space satisfying* (10) *and let* $\{\mu_n\}_{n\in\mathbf{N}}$ *be a uniformly tight sequence of laws on* $\mathcal{X}$. *Then there exists a subsequence* $\mu_{n_1}, \mu_{n_2}, \ldots$ *which admits the strong a.s. Skorokhod representation defined on* $([0,1], \mathcal{B}_{[0,1]}, \ell)$.

3. Some examples

Clearly, Theorem 1 is an example of application of Theorem 2. One can go further in this direction, using the full generality offered by the Banach–Alaoglu Theorem (see e.g. [9, p. 66]).

Theorem 4. *Let* E *be a separable linear topological space and let* E' *be its topological dual. Let* $X_1, X_2, \ldots$ *be* E'*-valued random elements. If for each* $\varepsilon > 0$ *there is an open set* $V_\varepsilon \subset E$ *which contains* $0 \in E$ *and is such that*

$$\inf_n \mathbf{P}\Big\{\sup_{x\in V_\varepsilon} |\langle x,\, X_n\rangle| \leqslant 1\Big\} \geqslant 1 - \varepsilon, \tag{30}$$

then along some subsequence $\{n_k\}$ *there exists the strong a.s. Skorokhod representation* Y_k, $k = 0, 1, 2, \ldots$. *In particular,*

$$\Big\langle x,\, Y_k(\omega)\Big\rangle \longrightarrow \Big\langle x,\, Y_0(\omega)\Big\rangle, \qquad \text{as}\quad k \to \infty,\quad x \in E,\quad \omega \in [0,1]. \tag{31}$$

R e m a r k 2. If E is a normed space, then condition (30) takes the form:

$$\lim_{K\to+\infty} \sup_n \mathbf{P}\Big\{\|X_n\|_{E'} > K\Big\} = 0. \tag{32}$$

Somewhat dif and only iferent results arise when we consider $\mathcal{S}'$-valued (or $\mathcal{D}'$-valued) random elements or, more generally, random elements with values in the topological dual to a Frechét nuclear space (or to the strict inductive limit of a sequence of Frechét nuclear spaces).

For the sake of brevity we will formulate here results for the simpler case only. Let Φ be a Frechét nuclear space (see e.g. [10]). Let $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \cdots$ be an increasing sequence of Hilbertian seminorms defining the topology on Φ. Denote by $(\Phi_p, \|\cdot\|_p)$ the

Hilbert space arising by completion of the quotient space $\Phi/\|\cdot\|_p$ and by $(\Phi'_{-p}, \|\cdot\|_{-p})$ the topological dual of $(\Phi_p, \|\cdot\|_p)$. After obvious identification, Φ'_{-p} is a subset of Φ' and $\Phi' = \bigcup_{p=1}^{\infty} \Phi'_{-p}$. Φ' is equipped with the strong topology β, which on every Φ_{-p} is *strictly* weaker than the Hilbert topology of the norm $\|\cdot\|_{-p}$. The point is that the convergence of sequences in topology β may be defined in the following way:

$$x_n \underset{\beta}{\longrightarrow} x_0 \quad \text{if and only if} \quad \|x_n - x_0\|_{-p} \to 0, \quad \text{as } n \to +\infty, \quad \text{for some} \quad p \in \mathbf{N}. \tag{33}$$

Theorem 5. *Let Φ' be the topological dual of a Frechét nuclear space Φ and let $X_1, X_2, \ldots$ be random elements with values in Φ'. Suppose that for every $\varphi \in \Phi$ random variables $\langle\varphi, X_n\rangle$, $n = 1, 2, \ldots$, are uniformly tight. Then there exists a subsequence n_k and the Skorokhod representation $Y_0, Y_1, Y_2, \ldots$ for this subsequence such that for each $\omega \in [0,1]$ one can find a number $p(\omega) \in \mathbf{N}$ with the property that*

$$\left\| Y_k(\omega) - Y_0(\omega) \right\|_{-p(\omega)} \longrightarrow 0, \qquad \text{as} \quad k \to \infty. \tag{34}$$

P r o o f. Standard arguments of the Minlos–Sazonov-type (see e.g. [8] or [6]) show that in Φ' «weak uniform tightness» implies usual uniform tightness: for each $\varepsilon > 0$ there are numbers $q_\varepsilon \in \mathbf{N}$ and $K_\varepsilon > 0$ such that

$$\mathbf{P}\Big\{\|X_n\|_{-q_\varepsilon} \leqslant K_\varepsilon\Big\} > 1 - \varepsilon. \tag{35}$$

Corollary 3. *Let X_n, $n = 0, 1, 2, \ldots$, be random elements in $\mathcal{S}'$ or $\mathcal{D}'$. Then $X_n \longrightarrow_{\mathcal{D}} X_0$ if and only if in every subsequence $\{X_{n_k}\}$ one can find a further subsequence $\{X_{n_{k_l}}\}$ which admits the strong a.s. Skorokhod representation (with $Y_0 \sim X_0$) (i.e. $\{X_n\}$ admits the strong a.s. Skorokhod representation for subsequences).*

P r o o f. It is proved in [6] that on $\mathcal{S}'$ or $\mathcal{D}'$ relative compactness (in distribution) is equivalent to uniform tightness.

Notice that our Theorem 2 may be viewed as the strong version of the direct Prokhorov's theorem. Indeed, if $f\colon \mathcal{X} \to \mathbf{R}^1$ is bounded and continuous and $Y_0, Y_1, Y_2, \ldots$ form the Skorokhod representation for $\{X_{n_k}\}$, then

$$\mathbf{E}f(X_{n_k}) = \mathbf{E}f(Y_k) \longrightarrow \mathbf{E}f(Y_0), \qquad \text{as } n \to +\infty, \tag{36}$$

and so $\mathcal{L}(X_{n_k})$ weakly converges to $\mathcal{L}(Y_0)$ in the classical sense. But (36) holds also for all *sequentially* continuous and bounded f! It means that in the nonmetric case the direct Prokhorov's theorem may give relative compactness in the stronger topology than the original one. Similar observation can be found in [6] where it was proved that convergence in distribution on $\mathcal{D}'$ equipped with the weak topology coincides with convergence in distribution with respect to the strong topology. This is not surprising in view of the fact that convergence *of sequences* in the weak topology on $\mathcal{D}'$ (and $\mathcal{S}'$) implies convergence in the strong topology.

The above remarks may also suggest that identifying convergence in distribution with weak convergence of laws is not completely justified for some quite good spaces. We refer to the paper mentioned in Footnote 2) for further discussion on this topic.

Finally let us mention that one of the main motivations to prove Theorem 2 was to deal with «really» sequential topology on the Skorokhod space $\mathbf{D}$. The reader may find information on this non-Skorokhod and nonmetric topology in the author's preprint «A non-Skorokhod topology on the Skorokhod space», which will be published in «Electronic journal of Probability».

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REMARKS ON THE PAPER « ON A PROBLEM OF A KHINCHIN-TYPE DECOMPOSITION THEOREM FOR EXTREME VALUES» BY E. PANCHEVA

(Theory of Probability and its Application, 1994, vol. 39, № 2, p. 329–336)

1. The paper contains an elegant approach for eliminating the problems caused by idempotents in the arithmetics of probability distribution functions over $\mathbf{R}^d$ where the operation is the coordinatewise maximum of the corresponding random vectors. This method is in fact an application of the general approach in Ruzsa and Székely [1]. In this book the notion of «hair» = «maximal idempotent divisor» $H(s)$ for any element s in abstract semigroups S was introduced and a new semigroup $H(s)S$ was considered to settle several problems much easier than directly in S, e.g. the characterization of the class

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