

The α - μ Distribution: A Physical Fading Model for the Stacy Distribution

Michel Daoud Yacoub

Abstract—This paper introduces a fading model, which explores the nonlinearity of the propagation medium. It derives the corresponding fading distribution—the α - μ distribution—which is in fact a rewritten form of the Stacy (generalized Gamma) distribution. This distribution includes several others such as Gamma (and its discrete versions Erlang and central Chi-squared), Nakagami-m (and its discrete version Chi), exponential, Weibull, one-sided Gaussian, and Rayleigh. Based on the fading model proposed here, higher order statistics are obtained in closed-form formulas. More specifically, level-crossing rate, average fade duration, and joint statistics (joint probability density function, general joint moments, and general correlation coefficient) of correlated α - μ variates are obtained, and they are directly related to the physical fading parameters.

Index Terms—Generalized Gamma distribution, Nakagami-m distribution, Stacy distribution, Weibull distribution, α - μ distribution.

I. INTRODUCTION

A GREAT NUMBER of distributions exist that well describe the statistics of the mobile radio signal. The long-term signal variation is known to follow the Lognormal distribution, whereas the short-term signal variation is described by several other distributions such as Hoyt, Rayleigh, Rice, Nakagami-m, and Weibull. It is generally accepted that the path strength at any delay is characterized by the short-term distributions over a spatial dimension of a few hundred wavelengths, and by the Lognormal distribution over areas whose dimension is much larger [1]. Among the short-term fading distributions, Nakagami-m has been given a special attention for its ease of manipulation and wide range of applicability. Although, in general, it has been found that the fading statistics of the mobile radio channel may well be characterized by Nakagami-m, situations are easily found for which other distributions such as Hoyt, Rice, and Weibull yield better results [2]–[5]. More importantly, situations are encountered for which no distributions seem to adequately fit experimental data, although one or another may yield a moderate fitting. Some researches [4] even question the use of the Nakagami-m distribution because its tail does not seem to yield a good fitting to experimental data, better fitting being found around the mean or median. The well-known fading distributions have been derived assuming a homogeneous diffuse scatter-

ing field, resulting from randomly distributed point scatterers. The assumption of a homogeneous diffuse scattering field is certainly an approximation, because the surfaces are spatially correlated characterizing a nonlinear environment [6]. With the aim at exploring the nonlinearity of the propagation medium, a general fading distribution—the α - μ distribution—was proposed [7]. This distribution was thought to be new, but it is, in fact, a rewritten version of the generalized Gamma distribution (at the time unknown to the author), which was first proposed by Stacy [8]. In Stacy's own words [8], the aim of his proposal "concerns a generalization of the Gamma distribution," which "in essence... is accomplished by supplying a positive parameter included as an exponent in the exponential factor of the Gamma distribution." Stacy's work was connected neither with any specific application nor with any physical modeling of any given phenomenon. It was purely a mathematical problem in which some statistical properties of a generalized version of the Gamma distribution were investigated. The derivation of the α - μ distribution [7], in contrast, has as its base a fading model. Thence, its parameters are directly associated with the physical properties of the propagation medium. The Stacy (generalized Gamma) or α - μ distribution is general, flexible, and has easy mathematical tractability. It includes important distributions such as Gamma (and its discrete versions Erlang and central Chi-squared), Nakagami-m (and its discrete version Chi), exponential, Weibull, one-sided Gaussian, and Rayleigh. Its density, cumulative frequency, and moments appear in simple closed-form expressions. All these features combined make the Stacy or α - μ distribution very attractive. Using the fading model as proposed in [7], a deeper characterization of the Stacy or α - μ distribution can be achieved. This paper aims at revisiting the derivation of the α - μ distribution and showing how some of its higher order statistics may be attained. In particular, we shall derive statistics such as level-crossing rate, average fade duration, joint distribution, cross correlation, and correlation factor. Other higher order statistics may be derived which are not shown in this paper.

II. THE α - μ DISTRIBUTION—A REWRITTEN FORM OF THE STACY DISTRIBUTION

For a fading signal with envelope R , an arbitrary parameter $\alpha > 0$, and a α -root mean value $\hat{r} = \sqrt[\alpha]{E(R^\alpha)}$, the α - μ probability density function $f_R(r)$ of R is written as

$$f_R(r) = \frac{\alpha \mu^\mu r^{\alpha\mu-1}}{\hat{r}^{\alpha\mu} \Gamma(\mu)} \exp\left(-\mu \frac{r^\alpha}{\hat{r}^\alpha}\right) \quad (1)$$

Manuscript received November 11, 2004; revised October 21, 2005 and January 25, 2006. The review of this paper was coordinated by Prof. C. Xiao.

The author is with the School of Electrical and Computer Engineering, State University of Campinas, Campinas 130823-852, Brazil (e-mail: michel@decom.fee.unicamp.br).

Digital Object Identifier 10.1109/TVT.2006.883753

where $\mu > 0$ is the inverse of the normalized variance of R^α , i.e.,

$$\mu = \frac{E^2(R^\alpha)}{V(R^\alpha)} \quad (2)$$

$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) dt$ is the Gamma function, and $E(\cdot)$ and $V(\cdot)$ are, respectively, the expectation and variance operators. For a normalized envelope $P = R/\hat{r}$, the probability density function $f_P(\rho)$ of P is obtained as

$$f_P(\rho) = \frac{\alpha \mu^\mu \rho^{\alpha\mu-1}}{\Gamma(\mu) \exp(\mu \rho^\alpha)} \quad (3)$$

where $\mu > 0$ is given by

$$\mu = V^{-1}(P^\alpha). \quad (4)$$

The k th moment $E(P^k)$ is found as

$$E(P^k) = \frac{\Gamma(\mu + k/\alpha)}{\mu^{k/\alpha} \Gamma(\mu)}. \quad (5)$$

Of course, $E(R^k) = \hat{r}^k E(P^k)$. Now, defining $U = R^\alpha/\alpha$ as a hyperpower and $\bar{u} = E(U)$, the probability density function $f_U(u)$ of U is obtained as

$$f_U(u) = \frac{\mu^\mu u^{\mu-1}}{\bar{u}^\mu \Gamma(\mu)} \exp\left(-\mu \frac{u}{\bar{u}}\right). \quad (6)$$

For a normalized hyperpower $Y = U/\bar{u}$, the probability density function $f_Y(v)$ of Y is found to be

$$f_Y(v) = \frac{\mu^\mu v^{\mu-1}}{\Gamma(\mu) \exp(\mu v)}. \quad (7)$$

The probability distribution function $F_W(w)$ of an α - μ variate W can be found in a closed-form formula. In particular, $F_R(r)$ for the envelope R is given by

$$F_R(r) = \frac{\Gamma(\mu, \mu r^\alpha / \hat{r}^\alpha)}{\Gamma(\mu)} \quad (8)$$

where $\Gamma(z, y) = \int_0^y t^{z-1} \exp(-t) dt$ is the incomplete Gamma function. Equivalently, it is

$$F_P(\rho) = \frac{\Gamma(\mu, \mu \rho^\alpha)}{\Gamma(\mu)}. \quad (9)$$

III. THE α - μ PHYSICAL MODEL

The fading model for the α - μ distribution considers a signal composed of clusters of multipath waves propagating in a nonhomogeneous environment. Within any one cluster, the phases of the scattered waves are random and have similar delay times with delay-time spreads of different clusters being relatively large. The clusters of multipath waves are assumed to have the scattered waves with identical powers. The resulting envelope is obtained as a nonlinear function of the modulus of the sum of the multipath components. Such a nonlinearity is manifested in terms of a power parameter, so that the resulting

signal intensity is obtained not simply as the modulus of the sum of the multipath components, but as this modulus to a certain given exponent. The author of this paper does not try to explain why or how such a nonlinearity occurs or even if it indeed occurs. What the author conjectures about is that the resulting effect on the received signal propagated in a certain medium is manifested in terms of a nonlinearity. The area of nonlinear phenomena of the propagation medium is rather old and well explored. It was probably inaugurated in 1930 by the first reported Luxemburg effect. At that opportunity, a Dutch scientist listening to the signal from a transmitter located in Beromünster was surprised to hear at the same time a different radio station located in Luxemburg (cross modulation). In spite of the fact that a lot has been done in this area, it is still open for investigation and has incited the interest for recent researches (e.g., [9]). In [9], a radio measurement facility has been specially set up in order to explore the nonlinearity of the medium. The author of [9] maintains that “a number of surprising effects” (due to nonlinearity) have been observed and the investigations “continue to generate new discoveries and scientific output.” In addition to the phenomenon related to the propagation medium, the nonlinearity expressed in the α - μ distribution may also account for the practical limitations of the detection process of the receiver.

IV. DERIVATION OF THE α - μ DISTRIBUTION

Assume that at a certain given point, the received signal encompasses an arbitrary number n of multipath components, and the propagation environment is such that the resulting signal envelope is observed as a nonlinear function of the modulus of the sum of these components. Suppose that such a nonlinearity is in the form of a power parameter $\alpha > 0$ so that the resulting envelope R is

$$R^\alpha = \sum_{i=1}^n (X_i^2 + Y_i^2) \quad (10)$$

where X_i and Y_i are mutually independent Gaussian processes, with $E(X_i) = E(Y_i) = 0$ and $E(X_i^2) = E(Y_i^2) = \hat{r}^\alpha/2n$. Departing from (10) and following the standard procedure of transformation of variables in probability theory, the density $f_R(r)$ of R is found as

$$f_R(r) = \frac{\alpha n^n r^{\alpha n-1}}{\hat{r}^{\alpha n} \Gamma(n)} \exp\left(-n \frac{r^\alpha}{\hat{r}^\alpha}\right). \quad (11)$$

From (10), it can be shown that $E(R^\alpha) = \hat{r}^\alpha$ and $E(R^{2\alpha}) = \hat{r}^{2\alpha}(n+1)/n$. The inverse of the normalized variance of R^α , defined as μ , is then obtained from (2) as $\mu = n$. Note that the variable n is totally expressed in terms of a physical parameter, namely the normalized variance of R^α , i.e., $\mu = n = V^{-1}(P)$. Note also that whereas this physical parameter is of a continuous nature, n is of a discrete nature. It is plausible to presume that if such a parameter is to be obtained by field measurements, figures departing from the exact n will certainly occur. Several reasons exist for this. One of them, which probably is the most meaningful one, is that, although the model proposed here is

general, it is in fact an approximate solution to the so-called random phase problem, as all the other well-known fading models are approximate solution to the random phase problem. The limitation of the model can be made less stringent by defining $\mu > 0$ as the real extension of n . Noninteger values of the parameter μ may account for 1) nonzero correlation among the clusters of multipath components; 2) nonzero correlation between the in-phase and quadrature components; and 3) non-Gaussianity of the in-phase and quadrature components of the fading signal; and others. Noninteger values of multipath clusters have been found in practice, and this issue is extensively reported in the literature. (See, for instance, [10] and the references therein.) In addition, of course, scattering occurs continuously throughout the surface and not at discrete points [6], [11]. Now, replacing n by μ and using $\hat{r} = \sqrt[n]{E(R^\alpha)}$, (1) results.

V. THE α - μ DISTRIBUTION AND THE OTHER FADING DISTRIBUTIONS

The α - μ distribution is a general fading distribution that includes the Gamma (and its discrete versions Erlang and central Chi-squared), Nakagami- m (and its discrete version Chi), exponential, Weibull, one-sided Gaussian, and Rayleigh. The Weibull distribution can be obtained from the α - μ distribution by setting $\mu = 1$. From the Weibull distribution, by setting $\alpha = 2$, the Rayleigh distribution results. Still from the Weibull distribution, the negative exponential distribution is obtained by setting $\alpha = 1$. The Nakagami- m distribution can be obtained from the α - μ distribution by setting $\alpha = 2$. From the Nakagami- m distribution, by setting $\mu = 1$, the Rayleigh distribution results. Still from the Nakagami- m distribution, the one-sided Gaussian distribution is obtained by setting $\mu = 1/2$. Let $R_{\text{Distribution}}$ denote the variate of the corresponding distribution. Then, in accordance with the model proposed here, the following hold:

$$R_{\alpha-\mu}^\alpha = R_{\text{Nakagami-}m}^2 = R_{\text{Gamma}}. \quad (12)$$

For half integer values of μ

$$R_{\alpha-\mu}^\alpha = R_{\text{Chi}}^2 = R_{\text{Chi-Squared}}. \quad (13)$$

For integer values of μ

$$\begin{aligned} R_{\alpha-\mu}^\alpha &= \sum_{i=1}^{\mu} R_{\text{Exponential-}i} \\ &= \sum_{i=1}^{\mu} R_{\text{Rayleigh-}i}^2 \\ &= \sum_{i=1}^{2\mu} R_{\text{Gaussian-}i}^2. \end{aligned} \quad (14)$$

Of course, the same relations are valid for their corresponding normalized variates.

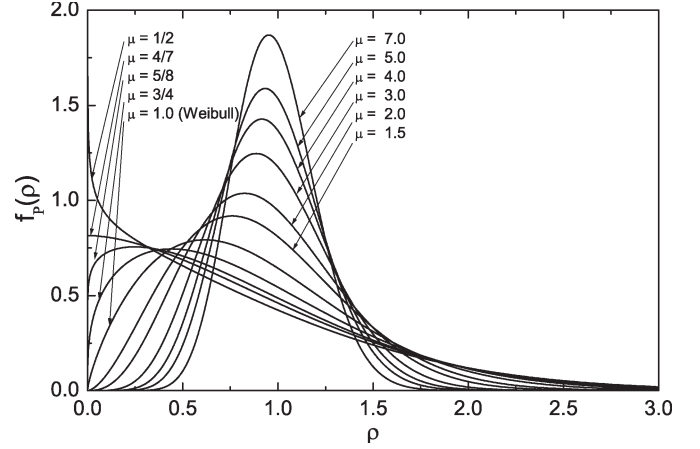


Fig. 1. Various shapes of the α - μ density function for $\alpha = 7/4$.

VI. SAMPLE SHAPES OF THE α - μ DISTRIBUTION AND MOMENT-BASED PARAMETER ESTIMATORS

Some basic features of the α - μ probability density function can be observed directly from (3). For $\alpha\mu < 1$ (i.e., $\mu < 1/\alpha$), $f_P(\rho)$ tends to infinity as ρ approaches zero and $f_P(\rho)$ decreases monotonically with the increase of ρ . For $\alpha\mu = 1$ (i.e., $\mu = 1/\alpha$), $f_P(\rho) = \mu^{\mu-1}/[\Gamma(\mu) \exp(\mu \psi \rho)]$, and, at the origin, $f_P(0) = \mu^{\mu-1}/\Gamma(\mu)$, and it decreases toward zero with the increase of ρ . For $\alpha\mu > 1$ (i.e., $\mu > 1/\alpha$), $f_P(\rho)$ is nil at the origin; it increases with the increase of ρ to reach a maximum at $\rho = \sqrt[\alpha]{(\alpha\mu - 1)/\alpha\mu}$, then decreasing toward zero as ρ increases. Fig. 1 plots $f_P(\rho)$ versus ρ for $\alpha = 7/4$ and varying μ . Fig. 2 plots $f_P(\rho)$ versus ρ for $\mu = 4/7$ and varying α . Apparently, the effects of α and μ on $f_P(\rho)$ seem comparable but, in fact, the shapes of the curves vary substantially if they are observed more closely. Moreover, as will be shown later, their effects on the correlation properties are rather different. In order to investigate the flexibility of the α - μ distribution, it is interesting to examine the various shapes of the α - μ distribution for a fixed value of the Nakagami parameter m . To this end, we define $\beta\mu$ as the generalization of the parameter μ , such that

$$\beta\mu = \frac{E^2(R^\beta)}{E(R^{2\beta}) - E^2(R^\beta)}. \quad (15)$$

Now, using (3)

$$\beta\mu = \frac{\Gamma^2(\mu + \beta/\alpha)}{\Gamma(\mu)\Gamma(\mu + 2\beta/\alpha) - \Gamma^2(\mu + \beta/\alpha)}. \quad (16)$$

Of course, $\alpha\mu = \mu$, and $2\mu = m$. Note from (16) that an infinite number of pairs (α, μ) may be found for the same parameter m ($2\mu = m$). In other words, for the same Nakagami- m curve, an infinite number of α - μ curves can be plotted. This is illustrated in Figs. 3 and 4, in which the various shapes of the α - μ density and distribution functions, respectively, are depicted for the same Nakagami parameter $m = 0.5$. In these curves, the parameter μ has been chosen as 0.5, 0.75, 1, 1.5, 2, 5, 10, 50, and 100, and α has been calculated from the relation $2\mu = m$ (16) for the respective μ . The corresponding values for μ are 2.0, 1.6449, 1.4418, 1.2046, 1.0629, 0.71485, 0.52682,

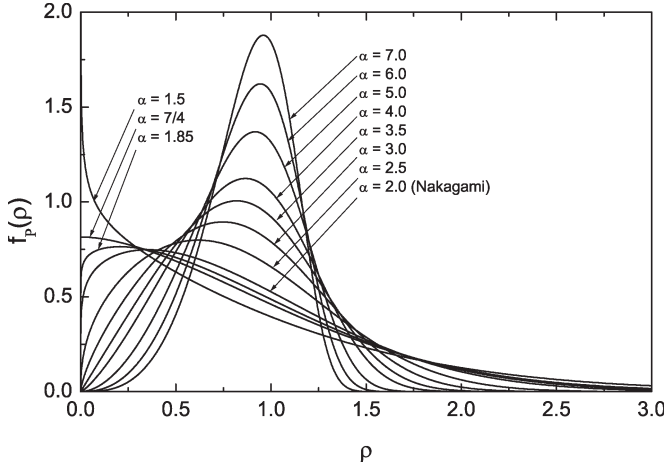


Fig. 2. Various shapes of the α - μ density function for $\mu = 4/7$.

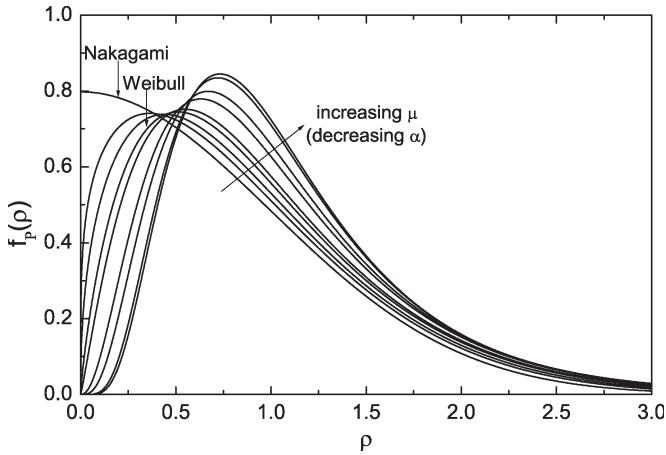


Fig. 3. Various shapes of the α - μ density function for the same Nakagami parameter $m = 0.5$.

0.25219, and 0.18166. We observe that an enormous variety of shapes can be found for the same Nakagami parameter m . Therefore, the α - μ distribution can be used to better adjust to field data. Equations (15) and (16) may be used as means of estimating the parameters of the distribution. From (15), for a given β , $\beta\mu$ can be obtained calculating the appropriate moments as indicated. Therefore, for β_1 and β_2 , chosen arbitrarily (say, $\beta_1 = 1$ and $\beta_2 = 2$), $\beta_1\mu$ and $\beta_2\mu$ are obtained and equated to (16) adequately. Therefore, two equations are set up to obtain the two unknowns (α and μ).

VII. LEVEL-CROSSING RATE AND AVERAGE FADE DURATION

Level-crossing rate $N_R(r)$ and average fade duration $T_R(r)$ are important second-order statistics, which are, respectively, defined as

$$N_R(r) = \int_0^\infty \dot{r} f_{\dot{R},R}(\dot{r}, r) d\dot{r} \quad (17)$$

$$T_R(r) = \frac{F_R(r)}{N_R(r)}. \quad (18)$$

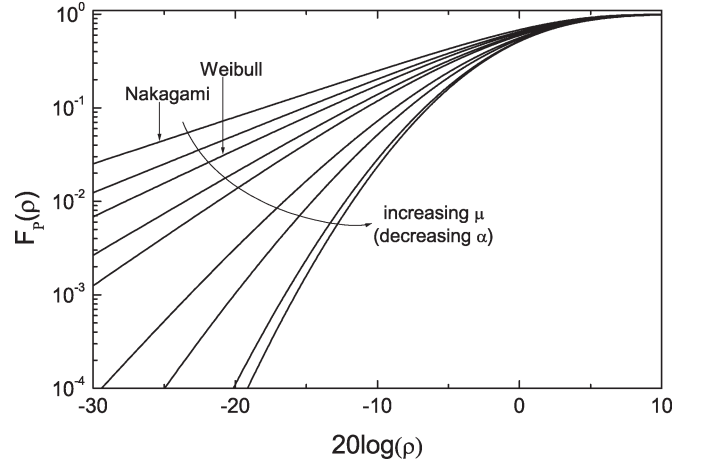


Fig. 4. Various shapes of the α - μ distribution function for the same Nakagami parameter $m = 0.5$.

In (17), the upper dot denotes time derivative and $f_{\dot{R},R}(\dot{r}, r)$ is the joint density of \dot{R} and R . These statistics can now be derived using any convenient relation between the α - μ and some other variate. In particular, we use that of (12), in which the α - μ variate is related to the Nakagami- m one. In order to simplify the notation, we define R and R_N as the α - μ and the Nakagami- m envelopes, respectively. Thus, from (12)

$$R^\alpha = R_N^2. \quad (19)$$

Deriving (19) with respect to time and denoting the resulting variates as \dot{R} and \dot{R}_N , then

$$\dot{R} = \frac{2}{\alpha} R^{1-\frac{\alpha}{2}} \dot{R}_N. \quad (20)$$

The variate \dot{R}_N is known to be zero-mean Gaussian distributed with variance $\dot{\sigma}_N^2$ [13]. Therefore, the probability density function $f_{\dot{R}|R}(\dot{r}|r)$ of \dot{R} given R is also zero-mean Gaussian distributed with variance $\dot{\sigma}^2$ found from (20) as $\dot{\sigma}^2 = (4/\alpha^2)r^{2-\alpha}\dot{\sigma}_N^2$. The joint density $f_{\dot{R},R}(\dot{r}, r)$ of \dot{R} and R is obtained as $f_{\dot{R},R}(\dot{r}, r) = f_{\dot{R}|R}(\dot{r}|r)f_R(r)$. Therefore

$$f_{\dot{R},R}(\dot{r}, r) = \frac{1}{\sqrt{2\pi\dot{\sigma}}} \exp\left(-\frac{\dot{r}^2}{2\dot{\sigma}^2}\right) \times \frac{\alpha\mu^\mu r^{\alpha\mu-1}}{\Gamma(\mu)} \exp\left(-\frac{r^\alpha}{\hat{r}^\alpha}\right). \quad (21)$$

It is noteworthy that $\dot{\sigma}$ depends on r , such dependence vanishing for $\alpha = 2$. Therefore, apart from this specific case, in which the α - μ distribution deteriorates into the Nakagami- m one, the variates \dot{R} and R are dependent random variables. Now, with (17), (18), and (21)

$$N_R(r) = \frac{\dot{\sigma} f_R(r)}{\sqrt{2\pi}} \quad (22)$$

$$T_R(r) = \frac{\sqrt{2\pi} F_R(r)}{\dot{\sigma} f_R(r)}. \quad (23)$$

For isotropic scattering, $\dot{\sigma}_N^2 = (\omega/2)^2 (E(R_N^2)/m)$ [13], where ω is the maximum Doppler shift in radians per second. Now, by means of (19), we find that $E(R_N^2) = E(R^\alpha)$, and

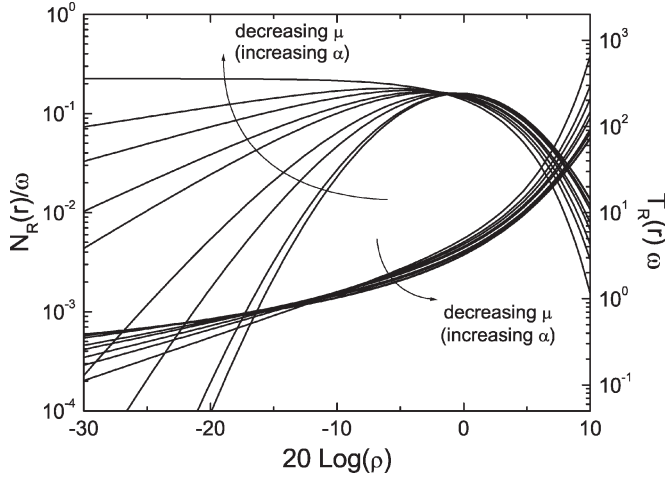


Fig. 5. Various shapes of the α - μ level-crossing rate and average fade duration for the same Nakagami parameter $m = 0.5$.

$E^2(R_N^2)/V(R_N^2) = E^2(R^\alpha)/V(R^\alpha)$. Therefore, $E(R_N^2) = \hat{r}^\alpha$ and $m = \mu$. Then, $\sigma_N^2 = (\omega/2)^2(\hat{r}^\alpha/\mu)$. Hence, $\hat{\sigma}^2 = (\omega/\alpha)^2(\hat{r}^\alpha r^{2-\alpha}/\mu)$, and

$$f_{\hat{R},R}(\hat{r},r) = \frac{\alpha^2 \mu^{\mu+0.5} r^{\alpha(\mu+0.5)-2}}{\sqrt{2\pi\omega\hat{r}^\alpha \mu+0.5}\Gamma(\mu)} \exp\left(-\frac{\mu\alpha^2 r^{\alpha-2}\hat{r}^2}{2\omega^2 \hat{r}^\alpha} - \frac{\mu r^\alpha}{\hat{r}^\alpha}\right). \quad (24)$$

Finally

$$N_R(r) = \frac{\omega \mu^{\mu-0.5} \rho^{\alpha(\mu-0.5)}}{\sqrt{2\pi}\Gamma(\mu) \exp(\mu\rho^\alpha)} \quad (25)$$

$$T_R(r) = \frac{\sqrt{2\pi}\Gamma(\mu, \mu\rho^\alpha) \exp(\mu\rho^\alpha)}{\omega \mu^{\mu-0.5} \rho^{\alpha(\mu-0.5)}} \quad (26)$$

where $\rho = r/\hat{r}$. For $\alpha = 2$ (Nakagami-m case), (25) and (26) reduce to [13, eqs. (17) and (21)], respectively. For $\mu = 1$ (Weibull case), (25) and (26) reduce to [14, eqs. (12) and (13)]. Fig. 5 illustrates these statistics for the same combination of the parameters as those used for Fig. 3 (and Fig. 4).

VIII. JOINT STATISTICS

This section aims at deriving the joint statistics for two α - μ variates. This is accomplished by capitalizing on some results already available in the literature for the Nakagami-m distribution and by the use of the relation as in (19) between the α - μ and Nakagami-m variates. Let 1) R_{N1} and R_{N2} be two Nakagami-m variates whose marginal statistics are respectively described by the parameters m_1 , $E(R_{N1}^2) = \Omega_1$ and m_2 , $E(R_{N2}^2) = \Omega_2$, $m_1 \leq m_2$; 2) R_1 and R_2 be two α - μ variates whose marginal statistics are respectively described by the parameters α_1 , μ_1 , \hat{r}_1 , and α_2 , μ_2 , \hat{r}_2 ; and 3) $0 \leq \delta \leq 1$ be a correlation parameter. (We postpone the discussion about this parameter to Section VIII-D.)

A. Joint Probability Density Function

The joint probability density function $f_{R_{N1},R_{N2}}(r_{N1},r_{N2})$ of two Nakagami-m variates R_{N1} and R_{N2} with marginal

statistics as described previously is given by [15, eq. (12)]. By means of (19), so that $R_1^{\alpha_1} = R_{N1}^2$ and $R_2^{\alpha_2} = R_{N2}^2$, we find that $\hat{r}_1^{\alpha_1} = \Omega_1$, $\hat{r}_2^{\alpha_2} = \Omega_2$, $\mu_1 = m_1$, $\mu_2 = m_2$. Now, with [15, eq. (12)] and the relations just given, the joint probability density function $f_{R_1,R_2}(r_1,r_2)$ of two α - μ variates R_1 and R_2 is found as $f_{R_1,R_2}(r_1,r_2) = |J|f_{R_{N1},R_{N2}}(r_{N1},r_{N2})$, in which J is the Jacobian of the transformation. Following the standard statistical procedure of transformation of variates and after several algebraic manipulations and simplifications, the joint probability density function $f_{P_1,P_2}(\rho_1,\rho_2)$ of the α - μ normalized envelopes $P_1 = R_1/\hat{r}_1$ and $P_2 = R_2/\hat{r}_2$ is found as

$$f_{P_1,P_2}(\rho_1,\rho_2) = \frac{\mu_1^{\mu_1} \mu_2^{\mu_2} \rho_1^{\alpha_1 \mu_1 - 1} \rho_2^{\alpha_2 \mu_2 - 1}}{(1-\delta)^{\mu_1} \Gamma(\mu_1)} \times \sum_{k=0}^{\infty} \frac{(\delta \mu_1 \mu_2 \rho_1^{\alpha_1} \rho_2^{\alpha_2})^k {}_1F_1\left(\mu_2 - \mu_1, \mu_2 + k; \frac{\mu_2 \rho_2^{\alpha_2}}{(1-\delta)}\right)}{k! \Gamma(\mu_2 + k) \exp\left(\frac{\mu_1 \rho_1^{\alpha_1} + \mu_2 \rho_2^{\alpha_2}}{1-\delta}\right)} \quad (27)$$

where ${}_1F_1(\cdot, \cdot; \cdot)$ is the confluent hypergeometric function [16, eq. 13.1.2]. For $\alpha_1 = \alpha_2 = 2$, (27) then reduces to [15, eq. (12)], i.e., for this condition, (27) yields the joint distribution of two Nakagami-m envelopes with arbitrary $m_1 = \mu_1$ and $m_2 = \mu_2$. In case $\mu_1 = \mu_2 = \mu$, then a much simpler result is found as

$$f_{P_1,P_2}(\rho_1,\rho_2) = \frac{\alpha_1 \alpha_2 \mu^{\mu+1} \rho_1^{\frac{\alpha_1}{2}(\mu+1)-1} \rho_2^{\frac{\alpha_2}{2}(\mu+1)-1}}{(1-\delta) \delta^{\frac{\mu-1}{2}} \Gamma(\mu)} \times \exp\left(-\mu \frac{\rho_1^{\alpha_1} + \rho_2^{\alpha_2}}{1-\delta}\right) I_{\mu-1}\left(\frac{2\mu \sqrt{\delta \rho_1^{\alpha_1} \rho_2^{\alpha_2}}}{1-\delta}\right) \quad (28)$$

where $I_\nu(\cdot)$ is the modified Bessel function of the first kind and order ν [16, eq. 9.6.18]. For $\alpha_1 = \alpha_2 = 2$, then (28) reduces to [12, eq. (126)], i.e., for this condition, (28) yields the joint distribution of two Nakagami-m envelopes with identical $m = \mu$.

B. Generalized Joint Moments

The joint moments $E(R_{N1}^k R_{N2}^l)$ of two Nakagami-m variates with marginal statistics as described previously is given by [15, eq. (16)]. Because $R_1^{\alpha_1} = R_{N1}^2$ and $R_2^{\alpha_2} = R_{N2}^2$, then the joint moments $E(R_1^{k\alpha_1/2} R_2^{l\alpha_2/2})$ is also identified as [15, eq. (16)]. Now, defining $p = k\alpha_1/2$ and $q = l\alpha_2/2$, and using the relations $\hat{r}_1^{\alpha_1} = \Omega_1$, $\hat{r}_2^{\alpha_2} = \Omega_2$, $\mu_1 = m_1$, $\mu_2 = m_2$, as before, the generalized moments $E(R_1^p R_2^q)$ of two α - μ variates R_1 and R_2 can be obtained from [15, eq. (16)]. The corresponding normalized moments, i.e., $E(P_1^p P_2^q)$, are then given by

$$E(P_1^p P_2^q) = \frac{\Gamma\left(\mu_1 + \frac{p}{\alpha_1}\right) \Gamma\left(\mu_2 + \frac{q}{\alpha_2}\right) {}_2F_1\left(-\frac{p}{\alpha_1}, \frac{q}{\alpha_2}; \mu_2; \delta\right)}{\mu_1^{\frac{p}{\alpha_1}} \mu_2^{\frac{q}{\alpha_2}} \Gamma(\mu_1) \Gamma(\mu_2)} \quad (29)$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is the Gauss hypergeometric function [16, eq. 15.1.1]. Of course, $E(R_1^p R_2^q) = \hat{r}_1^p \hat{r}_2^q E(P_1^p P_2^q)$.

For $\alpha_1 = \alpha_2 = 2$, then (29) reduces to [15, eq. (16)], i.e., for this condition, (29) yields the generalized joint moments of two Nakagami-m envelopes with arbitrary $m_1 = \mu_1$ and $m_2 = \mu_2$. For $\mu_1 = \mu_2 = \mu$, the simplification in (29) is not any substantial. For $\alpha_1 = \alpha_2 = 2$ and $\mu_1 = \mu_2 = m$, then (29) reduces to [12, eq. (137)].

C. Generalized Correlation Coefficient

Define a generalized correlation coefficient $\delta_{p,q}$ of two α - μ variates R_1 and R_2 , such that

$$\delta_{p,q} = \frac{C(R_1^p, R_2^q)}{\sqrt{V(R_1^p)V(R_2^q)}} = \frac{C(P_1^p, P_2^q)}{\sqrt{V(P_1^p)V(P_2^q)}} \quad (30)$$

where $C(\cdot, \cdot)$ is the covariance operator. By means of (5) and (29) and after some careful algebraic manipulations, (31) is formulated as shown at the bottom of the page. For $\alpha_1 = \alpha_2 = 2$, then (31) yields the generalized correlation coefficient of two Nakagami-m envelopes with arbitrary $m_1 = \mu_1$ and $m_2 = \mu_2$. For $\alpha_1 = \alpha_2 = 2$, $\mu_1 = \mu_2 = m$, and $p = q$, then (31) reduces to [12, eq. (139)].¹

D. Some Insight Into the Correlation Coefficient

Let $\delta_{p,q}^N$ denote the generalized correlation coefficient of the Nakagami-m distribution, such that

$$\delta_{p,q}^N = \frac{C(R_{N1}^p, R_{N2}^q)}{\sqrt{V(R_{N1}^p)V(R_{N2}^q)}} \quad (32)$$

where R_{N1} and R_{N2} are two Nakagami-m variates. Then, using (30), (32), and the relation as in (19), it is easy to see that $\delta_{\alpha_1, \alpha_2} = \delta_{2,2}^N$. Of course, similar relations may be found between the correlation coefficients of the α - μ distribution and the other distributions using the respective relations. It can be seen from (31) that $\delta_{\alpha_1, \alpha_2} = \sqrt{\mu_1/\mu_2} \delta$. Now, in case we have $\mu_1 = \mu_2$, then $\delta_{\alpha_1, \alpha_2|\mu_1=\mu_2} = \delta$. This is a very interesting result, which shows that the correlation parameter δ equals the correlation coefficient $\delta_{p,q}$ of the α - μ distribution when $p = \alpha_1$, $q = \alpha_2$, and $\mu_1 = \mu_2$. Therefore, we have $\delta = \delta_{\alpha_1, \alpha_2|\mu_1=\mu_2}$. Note that such a relation is valid for any $\alpha_1 > 0$, $\alpha_2 > 0$, and $\mu_1 = \mu_2 > 0$. Taking advantage of some results in the

literature, we may choose these parameters appropriately to obtain the α - μ correlation coefficient. In particular, we may use $\alpha_1 = \alpha_2 = 2$ and $\mu_1 = \mu_2 = 1$, which identify the Rayleigh case. Therefore, $\delta = \delta_{2,2}^R$, in which the superscript R denotes Rayleigh. It is possible to arrive at this same result by another means, as follows. In [12], it was shown that $\delta_{2,2|m_1=m_2}^N = \delta_{2,2}^R$. As shown before, $\delta_{\alpha_1, \alpha_2} = \delta_{2,2}^N$. Therefore

$$\delta = \delta_{\alpha_1, \alpha_2|\mu_1=\mu_2}^N = \delta_{2,2|m_1=m_2}^N = \delta_{2,2}^R. \quad (33)$$

This is indeed interesting, for the α - μ correlation coefficient, $\delta_{p,q}$ can now be expressed in terms of the Rayleigh correlation coefficient $\delta_{2,2}^R$. Hence, the results for the Rayleigh case can be directly used. The correlation coefficient $\delta_{2,2}^R$ for the Rayleigh fading can be obtained as

$$\delta_{2,2}^R = \frac{C(R_{R1}^2, R_{R2}^2)}{\sqrt{V(R_{R1}^2)V(R_{R2}^2)}} \quad (34)$$

where R_{R1} and R_{R2} are two Rayleigh envelopes. Now, we write the Rayleigh processes in terms of the Gaussian in-phase and quadrature processes as $R_{Ri}^2 = X_i^2 + Y_i^2$, $i = 1, 2$, in which X_i, Y_i are zero-mean Gaussian in-phase and quadrature components. For the Rayleigh process $E(X_i) = E(Y_i) = 0$, $\forall i, j$, $E(X_i X_j) = E(Y_i Y_j)$, $\forall i, j$, $E(X_i Y_j) = -E(X_j Y_i)$, $i \neq j$. For any two Gaussian processes G_i , $i = 1, 2$ for which $E(G_i) = 0$, then $E(G_i^4) = 3E(G_i^2)$ and $E(G_1^2 G_2^2) = E(G_1^2)E(G_2^2) + 2E^2(G_1 G_2)$. Using these in (34) and after algebraic manipulations

$$\delta = \delta_{2,2}^R = \frac{E^2(X_1 X_2) + E^2(X_1 Y_2)}{E(X_1^2)E(X_2^2)}. \quad (35)$$

Any fading model for which the above statistics (35) are known can be used in order to obtain $\delta = \delta_{2,2}^R$. In particular, for the Jakes model [17], we use [17, eqs. (1.5–11), (1.5–14), and (1.5–15)], such that (36) is formulated, shown at the bottom of the next page, where $D(\Theta)$ is the horizontal directivity pattern of the receiving antenna; Θ is a variate denoting the angle of the incident power; ω , as already defined, is the maximum Doppler shift; τ is the time difference between the two fading signals; $\Delta\omega$ is the frequency difference between these signals; and T is a variate denoting the time delay. For an isotropic scattering (i.e., uniform distribution in angle of the incident power), omnidirectional receiving antenna [$D(\Theta) = 1$], and exponentially distributed time delay [17]

$$\delta = \frac{J_0^2(\omega\tau)}{1 + (\Delta\omega t)^2} \quad (37)$$

¹While this paper was still under review, results similar to those presented in Sections VIII-A and VIII-B were published independently in T. Piboonungon, V. A. Aalo, C. D. Iskander, and G. P. Efthymoglou, "Bivariate generalised Gamma distribution with arbitrary fading parameters," *IEE Electronics Letters*, vol. 41, no. 12, pp. 49–50, Jun. 2005. On the other hand, (9) of the same reference is a special case of (31) ($p = q = 2$) obtained here.

$$\delta_{p,q} = \frac{\Gamma\left(\mu_1 + \frac{p}{\alpha_1}\right) \Gamma\left(\mu_2 + \frac{q}{\alpha_2}\right) \left({}_2F_1\left(-\frac{p}{\alpha_1}, -\frac{q}{\alpha_2}; \mu_2; \delta\right) - 1\right)}{\sqrt{\left(\Gamma(\mu_1) \Gamma\left(\mu_1 + \frac{2p}{\alpha_1}\right) - \Gamma^2\left(\mu_1 + \frac{p}{\alpha_1}\right)\right) \left(\Gamma(\mu_2) \Gamma\left(\mu_2 + \frac{2q}{\alpha_2}\right) - \Gamma^2\left(\mu_2 + \frac{q}{\alpha_2}\right)\right)}} \quad (31)$$

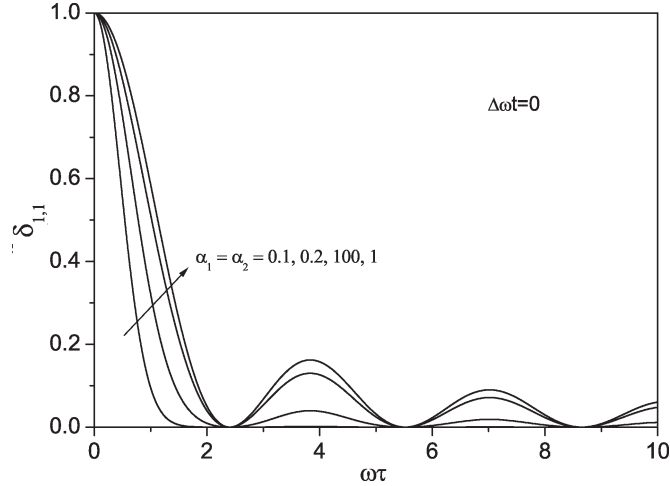


Fig. 6. Envelope correlation coefficient for the α - μ distribution as a function of $\omega\tau$ ($\mu = 2$).

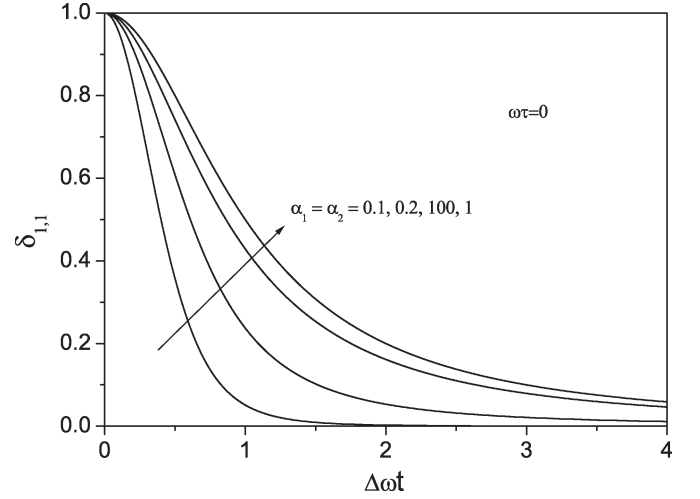


Fig. 7. Envelope correlation coefficient for the α - μ distribution as a function of $\Delta\omega t$ ($\mu = 2$).

in which t is the delay spread. The illustrations here consider an isotropic environment, $\mu_1 = \mu_2 = \mu$, and $p = q = 1$, i.e., envelope correlation. Fig. 6 depicts $\delta_{p,q}$ as a function of $\omega\tau$ for different values of the parameters $\alpha_1 = \alpha_2$ and $\Delta\omega t = 0$ ($\mu = 2$). Fig. 7 shows $\delta_{p,q}$ as a function of $\Delta\omega t$ for different values of the parameters $\alpha_1 = \alpha_2$ and $\omega\tau = 0$ ($\mu = 2$). Note, in both Figures, for $\alpha_1 = \alpha_2 > 1$, a large variation of these parameters, namely from $\alpha_1 = \alpha_2 = 1$ to $\alpha_1 = \alpha_2 = 100$, implies a small variation in the curves. In fact, it has been observed that the curve for which $\alpha_1 = \alpha_2 = 100$ is practically coincident with that for which $\alpha_1 = \alpha_2 \rightarrow \infty$. Therefore, we conclude that the correlation coefficient does not vary much for $\alpha_1 = \alpha_2 > 1$. The same does not hold for $\alpha_1 = \alpha_2 < 1$. In fact, for $\alpha_1 = \alpha_2 \rightarrow 0$, the correlation coefficient tends to an impulse at the origin. We note that the condition $\alpha_1 = \alpha_2 = 2$ corresponds to the Nakagami-m case. Therefore, the correlation properties for the fading environment are roughly those of the Nakagami-m ones in case the fading parameter is above one. Conversely, the correlation properties differ substantially from the Nakagami-m ones in case the fading parameter is below one. Interestingly, it has been observed through exhaustive plots that the correlation coefficient of the α - μ distribution is practically insensitive to the variation of μ . Therefore, these statistics are essentially the same as those for Weibull. The shapes of the autocorrelation are similar to those of the correlation coefficient, but the variations occur around a mean value, which increases with the increase of μ . These variations tend to be less prominent with the increase of μ (more deterministic scenario), as expected.

E. Multivariate Joint Probability Density Function

Let $R_i^{\alpha_i} = \sum_{k=1}^{\mu} (X_{ik}^2 + Y_{ik}^2)$, $1 \leq i \leq S$, be S α - μ variates, such that c_{ij} , the ij th element of the correlation matrix \mathbf{C}

whose inverse is tridiagonal with elements d_{ij} , is the correlation coefficient between Z_{ik} and Z_{jk} , $\forall k$, where $Z_{ik} = X_{ik}$ or $Z_{ik} = Y_{ik}$. Again, capitalizing on results available in the literature, a multivariate joint probability density function for S α - μ variates can be obtained. In [18], a fundamental result on generalized Rayleigh distributions was obtained, which was later on directly applied in [19] and [20] for a multivariate Nakagami-m distribution. Departing from [18, eq. (2.1)], or equivalently from [20, eq. (2)], and using the appropriate transformation of variables, as before, a multivariate α - μ distribution is obtained as

$$\begin{aligned} f_{P_1, P_2, \dots, P_S}(\rho_1, \rho_2, \dots, \rho_S) &= \frac{|\mathbf{C}^{-1}|^{\mu} \alpha_S \mu^{\mu+S-1} \rho_1^{\frac{\alpha_1}{2}(\mu-1)} \rho_S^{\frac{\alpha_S}{2}(\mu+1)-1}}{\Gamma(\mu) \exp(d_{S,S} \mu \rho_S^{\alpha_S})} \\ &\times \prod_{k=1}^{S-1} \frac{|d_{k,k+1}|^{1-\mu} \alpha_k \rho_k^{\alpha_k-1}}{\exp(d_{k,k} \mu \rho_k^{\alpha_k})} \\ &\times I_{\mu-1} \left(2|d_{k,k+1}| \mu \sqrt{\rho_k^{\alpha_k} \rho_{k+1}^{\alpha_{k+1}}} \right). \end{aligned} \quad (38)$$

For $S = 2$ and $\mathbf{C} = \begin{bmatrix} 1 & \sqrt{\delta} \\ \sqrt{\delta} & 1 \end{bmatrix}$, (38) reduces in an exact manner to (28).

IX. CONCLUSION

This paper introduces the α - μ distribution, which is in fact a rewritten form of the Stacy (generalized Gamma) distribution put in terms of physical fading parameters. It includes several others such as Gamma (and its discrete versions Erlang and

$$\delta = \frac{E^2(D(\Theta) \cos(\omega\tau \cos \Theta - \Delta\omega T)) + E^2(D(\Theta) \sin(\omega\tau \cos \Theta - \Delta\omega T))}{E^2(D(\Theta))} \quad (36)$$

central Chi-squared), Nakagami-m (and its discrete version Chi), exponential, Weibull, one-sided Gaussian, and Rayleigh. Based on the fading model proposed here, the level-crossing rate, average fade duration, joint probability density function, general joint moments, and general correlation coefficient are obtained in closed-form formulas. These statistics are general, and they are written in terms of physical fading parameters. Therefore, the statistics of those distributions that are special cases of the α - μ distribution can be obtained directly from the formulations developed here by simply reducing these formulations into the respective particular case. It is noteworthy that, although the α - μ distribution has one more parameter than Nakagami-m or Weibull distributions, no additional mathematical difficulty is posed by this. The flexibility of the α - μ distribution conveys is outstanding and renders it suited to better adjust to field data.

ACKNOWLEDGMENT

The author would like to thank the anonymous reviewers for their valuable suggestions and J. R. Mendes, D. Benevides da Costa, J. C. S. S. Filho, G. Fraidenraich, and R. A. Amaral de Souza.

REFERENCES

- [1] G. L. Turin, "Introduction to spread-spectrum antmultipath techniques and their application to urban digital radio," *Proc. IEEE*, vol. 68, no. 3, pp. 328–353, Mar. 1980.
- [2] C.-X. Wang, N. Youssef, and M. Patzold, "Level-crossing rate and average duration of fades of deterministic simulation models for Nakagami-Hoyt fading channels," in *Proc. 5th Int. Symp. Wireless Pers. Multimedia Commun.*, Oct. 2002, vol. 1, pp. 272–276.
- [3] J. D. Parsons, *The Mobile Radio Channel*, 2nd ed, vol. 1. Chichester, U.K.: Wiley, 2000.
- [4] S. Stein, "Fading channel issues in system engineering," *IEEE J. Sel. Areas Commun.*, vol. SAC-5, no. 2, pp. 68–69, Feb. 1987.
- [5] G. Tzeremes and C. G. Christodoulou, "Use of Weibull distribution for describing outdoor multipath fading," in *Proc. IEEE Antennas Propag. Soc. Int. Symp.*, Jun. 2002, vol. 1, pp. 232–235.
- [6] W. R. Braun and U. Dersch, "A physical mobile radio channel model," *IEEE Trans. Veh. Technol.*, vol. 40, no. 2, pp. 472–482, May 1991.
- [7] M. D. Yacoub, "The α - μ distribution: A general fading distribution," in *Proc. IEEE Int. Symp. PIMRC*, Sep. 2002, vol. 2, pp. 629–633.
- [8] E. W. Stacy, "A generalization of the gamma distribution," *Ann. Math. Stat.*, vol. 33, no. 3, pp. 1187–1192, Sep. 1962.
- [9] T. D. Carozzi, "Radio waves in ionosphere: Propagation, generation, and detection," "Ph.D. dissertation," Swedish Inst. Space Phys., Uppsala Division, Uppsala, Sweden, Aug. 2000.
- [10] H. Asplund, A. F. Molisch, M. Steinbauer, and N. B. Mehta, "Clustering of scatterers in mobile radio channels—Evaluation and modeling in the COST259 directional channel model," in *Proc. IEEE ICC*, New York, Apr./May 2002, pp. 901–905.
- [11] B. I. Raju and M. A. Srinivasan, "Statistics of envelope of high-frequency ultrasonic backscatter from human skin," *IEEE Trans. Ultrason. Ferroelectr., Frequency Control*, vol. 49, no. 7, pp. 871–882, Jul. 2002.
- [12] M. Nakagami, "The m-distribution—A general formula of intensity distribution of rapid fading," in *Statistical Methods in Radio Wave Propagation*, W. C. Hoffman, Ed. Elmsford, New York: Pergamon, 1960.
- [13] M. D. Yacoub, J. E. V. Bautista, and L. Guerra de Rezende Guedes, "On higher order statistics of the Nakagami-m distribution," *IEEE Trans. Veh. Technol.*, vol. 48, no. 3, pp. 790–794, May 1999.
- [14] N. C. Sagias, D. A. Zogas, G. K. Karagiannidis, and G. S. Tombras, "Channel capacity and second-order statistics in Weibull fading," *IEEE Commun. Lett.*, vol. 8, no. 6, pp. 377–379, Jun. 2004.
- [15] J. Reig, L. Rubio, and N. Cardona, "Bivariate Nakagami-m distribution with arbitrary fading parameters," *Electron. Lett.*, vol. 38, no. 25, pp. 1715–1717, Dec. 5, 2002.
- [16] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1972.
- [17] *Microwave Mobile Communications*, W. C. Jakes, Jr., Ed. New York: Wiley, 1974.
- [18] L. E. Blumenson and K. S. Miller, "Properties of generalized Rayleigh distributions," *Ann. Math. Stat.*, vol. 34, pp. 903–910, 1963.
- [19] G. K. Karagiannidis, D. A. Zogas, and S. A. Kotsopoulos, "On the multivariate Nakagami-m distribution with exponential correlation," *IEEE Trans. Commun.*, vol. 51, no. 8, pp. 1240–1244, Aug. 2003.
- [20] —, "An efficient approach to multivariate Nakagami-m distribution using green's matrix approximation," *IEEE Trans. Wireless Commun.*, vol. 2, no. 5, pp. 883–889, Sep. 2003.



Michel Daoud Yacoub was born in Brazil in 1955. He received the B.S.E.E. and M.Sc. degrees from the School of Electrical and Computer Engineering of the State University of Campinas, UNICAMP, Campinas, SP, Brazil, in 1978 and 1983, respectively, and the Ph.D. degree from University of Essex, U.K., in 1988.

From 1978 to 1985, he worked as a Research Specialist in the development of the Tropic digital exchange family with the Research and Development Center of Telebrás, Brazil. In 1989, he joined the School of Electrical and Computer Engineering, UNICAMP, where he is currently a Full Professor. He consults for several operating companies and industries in the wireless communications area. He is the author of *Foundations of Mobile Radio Engineering* (Boca Raton, FL: CRC, 1993), *Wireless Technology: Protocols, Standards, and Techniques* (Boca Raton, FL: CRC, 2001), and the coauthor of *Telecommunications: Principles and Trends* (São Paulo, Brasil: Erica, 1997, in Portuguese). He holds two patents. His general research interests include wireless communications.