The Amplify-and-Forward Half-Duplex Cooperative System: Pairwise Error Probability and Precoder Design

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***Abstract*— In this paper, an exact asymptotic pairwise error probability (PEP) is derived for a half-duplex cooperative system employing an amplify-and-forward (AF) protocol. When compared with the PEP of a traditional multiple-input multiple-output (MIMO) system, the "diversity gain" for the cooperative system is no longer just a simple exponential function of the signal-to-noise ratio (SNR), rather, it involves the logarithm of the SNR. The term *diversity gain function* is used to designate this characteristic of the PEP. The coding gain, on the other hand, is found similar to that for the MIMO system and is proportional to the determinant of the autocorrelation of the error matrix. Based on our analysis and observations, we propose a design of unitary precoder for the cooperative system to achieve the full diversity gain function. For the case of a 4-QAM signal being transmitted, we further optimize the coding gain, and arrive at a closed-form optimum precoder. Simulations indicate that our proposed precoder designs greatly improve the performance of the cooperative system.**

***Index Terms*— Cooperative system, half-duplex, amplify-and-forward (AF), pairwise error probability, diversity gain function, precoder.**

I. Introduction

Diversity techniques have been employed in practical wireless communication systems to overcome the effects of channel fading. Among the various forms of the diversities, spatial diversity which is often implemented by transmitting signals between geographically separated transmitting and receiving antennas is most commonly used since it can be readily combined with the other forms (such as time, frequency) of diversity. The gain in employing spatial diversity is usually measured by the product of the number of transmitter and receiver antennas. However, while having multiple transmitter and receiver antennas is often desirable to obtain higher diversity gain, this is often impractical in some applications such as mobile communications for which installing multiple antennas would increase the size and complexity of the wireless units. To overcome this limitation, another form of spatial diversity called *cooperative diversity*, has recently been proposed for mobile wireless communications. Here, a strategy is used such that the in-cell mobile users share the use of their antennas to create a virtual array through distributed transmission and signal processing. The fundamental idea of a cooperative system can be traced back to the literature of relay systems [1], [2]. Applications of the idea to wireless communication systems are, however, more recent (e.g. [3]–[10]). New cooperative protocols such as protocols with low-complexity and protocols achieving optimal diversity-multiplexing tradeoff [11] have been proposed [6]–[8]. These protocols can be generally classified into two types, Amplify-and-Forward (AF) and Decode-and-Forward (DF) protocol. In an AF protocol, the relay nodes retransmit a scaled version of the signal received from the source node to the destination node. In a DF protocol the relay nodes decode the message first, re-encode it and then transmit it to the destination.

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In this paper, we focus our consideration on an AF protocol over a half-duplex cooperative system, where the source and relay nodes either transmit or receive the signal, but do not do both at the same time. Such a system has a lower complexity and is easier to implement than a full-duplex system. The AF protocol we study was first proposed in [6]. In this paper, we analyze the pairwise error performance of the AF half-duplex relay system in which maximum likelihood (ML) detector is used. For the Alamouti coded AF protocol, an upper bound of the pairwise error probability (PEP) has already been presented in [5]. Here, we derive an *exact* expression for the asymptotic PEP. We observe that, unlike in the case of a traditional MIMO system, the "diversity gain" of the AF half-duplex cooperative system is not simply an exponential function of signal to noise ratio (SNR) as it is in conventional MIMO systems. Rather, it involves the logarithm function of the SNR. We designate this characteristic the *diversity gain function* of the AF half-duplex cooperative system. On the other hand, for this AF system, the coding gain is found to be proportional to the determinant of the autocorrelation of the coding error matrices, which is similar to the case of conventional MIMO system. From the expression of the PEP, we design a *unitary* precoder to achieve the maximum diversity gain function. We then further optimize the precoder to maximize the minimum of the coding gain. Simulations indicate that our proposed precoders not only significantly improve the performance over the system for which no precoder is used, but also outperform the system for which Alamouti's code is employed.

By considering an orthogonal space-time modulated relay system equipped with 2 relays [12] and examining a system with multiple relays [13], [14], the $\log$SNR factor in the diversity gain function has also been observed. However, for the two systems considered in [12]–[14], transmission between the source and destination is not permitted, whereas such direct transmission is scheduled to occur in the substantially different

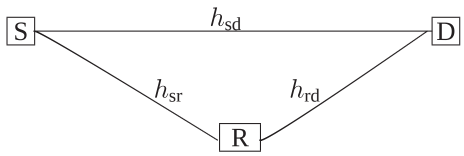

Fig. 1. A single relay system

configuration considered in this paper. It should be pointed out that the AF protocol [6] considered here has been shown to achieve the optimal diversity-multiplexing tradeoff [11] for a single antenna relay system if Gaussian codes of sufficient length is used. It should also be pointed out that this protocol has been extended to a multiple antenna relay system [15] for which space-time block codes are constructed to achieve the diversity-multiplexing tradeoff by the principle of non-vanishing determinant.

Notations: Bold upper- and lower-case letters denote respectively matrices and column vectors, with $(\cdot)^T$ and $(\cdot)^H$ denoting their transpose and conjugate-transpose respectively. A length P vector $\mathbf{s}$ is expressed as $\mathbf{s} = [s(1),\ s(2),\ \cdots\ , s(P)]^T$, with $\|\mathbf{s}\|$ standing for its 2-norm. $\mathbf{I}_K$ denotes a $K \times K$ identity matrix and $\mathbf{0}$ stands for an all-zero matrix of appropriate dimensions. $\mathbb{E}[\cdot]$ is the expectation operator. Notation $f(x) \triangleq O\big(g(x)\big),\ g(x) > 0$ denotes that there exists a positive constant c such that $|f(x)| \leq cg(x)$ when x is large.

II. System Model

In this paper, we focus on the AF protocol proposed in [6], which is also referred as a *non-orthogonal AF protocol*. We first introduce the system model in this section.

A. Single relay

In a single relay system as represented in Fig 1, the relay node R assists the transmission from the source node S to the destination node D. We assume that all the nodes are equipped with one transmitter antenna. The channel gain from the source to destination is denoted by h_{sd} whereas those from the source to the relay and from the relay to the destination are denoted by h_{sr} and h_{rd} respectively. We consider a symmetric relay network, where all channel gains are assumed to be independent and identically distributed (IID) zero mean circularly Gaussian with unit variance, and remain unchanged during the period of observation. We assume that the original information signals are equally probable from a constellation set $\mathcal{S}$ composed of quadrature amplitude modulation (QAM) signals which are processed by a *unitary* precoder before being transmitted from the source node. The transmission of the signal is carried out block by block, each block being of length $2P,\ P \geq 1$. Therefore, there is also a data receiving period for the relay node before it forwards the received data block to the destination. We denote the original data block by $\mathbf{s} = [\mathbf{s}_{\text{I}}^T \quad \mathbf{s}_{\text{II}}^T]^T$, where $\mathbf{s}_{\text{I}} = [s(1), \cdots, s(P)]^T$, $\mathbf{s}_{\text{II}} = [s(P+1), \cdots, s(2P)]^T$, with $s(i)$ being the original information symbol at the ith time instant, $i = 1, \cdots, 2P$.

TABLE I
Scheme 1: An AF Protocol fitted with a Single Relay

Time slots	Operation
1st P-time-slots	S $\rightarrow$ R, D
2nd P-time-slots	S $\rightarrow$ D, R $\rightarrow$ D

The covariance of $\mathbf{s}$ is assumed to be an identity matrix, i.e. $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathbf{I}_{2P}$. The precoded data block $\mathbf{x} = [\mathbf{x}_{\text{I}}^T \quad \mathbf{x}_{\text{II}}^T]^T$, $\mathbf{x}_{\text{I}} = [x(1), \cdots, x(P)]^T$, and $\mathbf{x}_{\text{II}} = [x(P+1), \cdots, x(2P)]^T$, can be expressed as

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_{\text{I}} \\ \mathbf{x}_{\text{II}} \end{pmatrix} = \mathbf{F} \begin{pmatrix} \mathbf{s}_{\text{I}} \\ \mathbf{s}_{\text{II}} \end{pmatrix} \tag{1}$$

where $\mathbf{F}$ is a $2P \times 2P$ unitary matrix representing the precoder.

For an AF half-duplex system with a single relay, several transmission schemes have been proposed [5]–[7]. However, they can be considered under the general description of the non-orthogonal AF protocol the operation of which is shown in Table 1. The source transmits $\mathbf{x}_{\text{I}}$ to both destination and relay node during the first P-time-slots (the shortest relay length is $P = 1$), and in the second P-time-slots, the source transmits $\mathbf{x}_{\text{II}}$ to the destination and the relay node simply amplifies and forwards what it receives from the first P-time-slots to the destination. This single relay transmission method is referred to as Scheme 1 in this paper. The input-output relation can be expressed as

$$r(p) = \sqrt{E_{\text{p}}} h_{\text{sd}}\ x(p) + v(p) \tag{2}$$

$$r(P+p) = \sqrt{E_{\text{p}}} h_{\text{sd}}\ x(P+p) + h_{\text{rd}} b \underbrace{\left(\sqrt{E_{\text{p}}} h_{\text{sr}}\ x(p) + w(p)\right)}_{\text{received at relay}} + v(P+p) \tag{3}$$

where for $p = 1, 2, \cdots, P$, $x(\cdot)$ and $r(\cdot)$ denote respectively the transmitted signal at the source and the received data at the destination, $v(\cdot)$ and $w(\cdot)$ denote respectively the IID zero-mean circularly Gaussian noise with variance σ^2 received at the destination and at the relay node, E_{p} is the average power for transmitting a symbol at each node, and b is the amplification coefficient at the relay node. If the channel gain h_{sr} is known at the relay node, then b is chosen [6], [7] to be $b = \sqrt{\frac{E_{\text{p}}}{|h_{\text{sr}}|^2 E_{\text{p}} + \sigma^2}}$. On the other hand, if instead of true knowledge of h_{sr}, only the second order statistics of h_{sr} is known at the relay node, b can be chosen as $b = \sqrt{\frac{E_{\text{p}}}{\mathbb{E}[|h_{\text{sr}}|^2] E_{\text{p}} + \sigma^2}} = \sqrt{\frac{E_{\text{p}}}{E_{\text{p}} + \sigma^2}}$, with h_{sr} being Gaussian having zero-mean and unit-variance. We assume the latter constraint in the paper. Writing (2) and (3) in a matrix form, we have

$$\begin{aligned} \mathbf{r}_1 &= \sqrt{E_{\text{p}}} \begin{pmatrix} h_{\text{sd}} \mathbf{I}_P & \mathbf{0} \\ b h_{\text{sr}} h_{\text{rd}} \mathbf{I}_P & h_{\text{sd}} \mathbf{I}_P \end{pmatrix} \begin{pmatrix} \mathbf{x}_{\text{I}} \\ \mathbf{x}_{\text{II}} \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ b h_{\text{rd}} \mathbf{I}_P & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{w}_{\text{I}} \\ \mathbf{w}_{\text{II}} \end{pmatrix} + \begin{pmatrix} \mathbf{v}_{\text{I}} \\ \mathbf{v}_{\text{II}} \end{pmatrix} \\ &= \sqrt{E_{\text{p}}} \mathbf{H}_1 \mathbf{x} + \mathbf{n}_1 \end{aligned} \tag{4}$$

where $\mathbf{r}_1 = [r(1),\ r(2),\ \cdots,\ r(2P)]^T$, $\mathbf{x} = \mathbf{F}\mathbf{s} = \left[\ \mathbf{x}_{\text{I}}^T \quad \mathbf{x}_{\text{II}}^T\ \right]^T$, the noise vectors $\mathbf{v}_{\text{I}}, \mathbf{v}_{\text{II}}$ and $\mathbf{w}_{\text{I}}, \mathbf{w}_{\text{II}}$ are respectively the IID zero-mean Gaussian noise in the direct and relay paths at the 1st and 2nd P-time-slots, and the

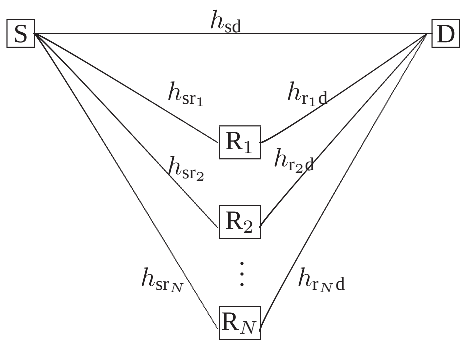

Fig. 2. Scheme M: A multiple relay system

subscript $_1$ indicates that the quantities are associated with Scheme 1. This renders the equivalent noise at the destination $\mathbf{n}_1$ being the sum of the two components such that $\mathbf{n}_1 = [\mathbf{v}_{\rm I}^T \quad (bh_{\rm rd}\mathbf{w}_{\rm I} + \mathbf{v}_{\rm II})^T]^T$ and $\mathbf{n}_1 \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{\Sigma}_1)$ with

$$\mathbf{\Sigma}_1 = \begin{pmatrix} \mathbf{I}_P & \mathbf{0} \\ \mathbf{0} & (1+b^2|h_{\rm rd}|^2)\mathbf{I}_P \end{pmatrix} \quad (5)$$

and

$$\mathbf{H}_1 = \begin{pmatrix} h_{\rm sd}\ \mathbf{I}_P & \mathbf{0} \\ bh_{\rm sr}h_{\rm rd}\mathbf{I}_P & h_{\rm sd}\mathbf{I}_P \end{pmatrix} \quad (6)$$

being the channel matrix. For convenience in the analysis of PEP, we rewrite $\mathbf{H}_1\mathbf{x}$ as

$$\mathbf{H}_1\mathbf{x} = \mathbf{X}_1\mathbf{h}_1 \quad (7)$$

where $\mathbf{X}_1$ is the signal matrix and $\mathbf{h}_1$ is the equivalent channel vector such that

$$\mathbf{X}_1 = \begin{pmatrix} \mathbf{x}_{\rm I} & \mathbf{0} \\ \mathbf{x}_{\rm II} & \mathbf{x}_{\rm I} \end{pmatrix} \quad \text{and} \quad \mathbf{h}_1 = \begin{pmatrix} h_{\rm sd} \\ bh_{\rm sr}h_{\rm rd} \end{pmatrix}. \quad (8)$$

B. Multiple Relays

A half-duplex transmission system fitted with multiple relays is shown in Fig 2. There are N relay nodes which assist the transmission from the source node to the destination. Again, we assume that all nodes equipped with a single antenna. We denote the channel from the source to the destination by $h_{\rm sd}$ as in the case of single relay, the channel from the source to the nth relay node by $h_{{\rm sr}_n}$, and the channel from the nth relay to the destination by $h_{{\rm r}_n d}$, $n = 1, \cdots, N$. We also consider a symmetric system here, all channel gains are also assumed to be IID zero mean circularly Gaussian with unit variance, and remain unchanged during the period of observation. We focus on the multiple relay scheme that is also proposed in [6]. In this scheme, referred to as Scheme N in this paper, the relays take turns to assist the transmission from the source to the destination. At any instant, only one relay is active. The signal transmission between the source and the *active* relay at any instant assumes the mode of Scheme 1 as described in the Section II-A.

The original data symbols are assumed to be equally probable from the constellation set $\mathcal{S}$ and processed by a unitary precoder before being transmitted. Our analysis is in fact valid for any block length, however, for simplicity of consideration, we assume that the block length between the source and each active relay is of 2 (each frame length is one symbol, i.e., $P = 1$). For all the N relays, the total signal block length is therefore $2N$. We denote the original data vector by $\mathbf{s} = [s(1), \ \cdots, s(2N)]^T$, with covariance being $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathbf{I}_{2N}$, and $s(i)$, $i = 1, \cdots, 2N$, being the original data symbol to be transmitted at the ith instant. The precoded signal vector is $\mathbf{x}_{\rm N} = \mathbf{F}_{\rm N}\mathbf{s}$, where $\mathbf{F}_{\rm N}$ is a $2N \times 2N$ unitary matrix, and $\mathbf{x}_{\rm N} = [x(1), \ \cdots, x(2N)]^T$. The received signals at the destination are:

$$\begin{aligned} r(1) &= \sqrt{E_{\rm p}}h_{\rm sd}x(1) + v(1) \\ r(2) &= \sqrt{E_{\rm p}}h_{\rm sd}x(2) + bh_{{\rm r}_1{\rm d}}\big(\sqrt{E_{\rm p}}h_{{\rm sr}_1}x(1) + w(1)\big) + v(2) \\ r(3) &= \sqrt{E_{\rm p}}h_{\rm sd}x(3) + v(3) \\ r(4) &= \sqrt{E_{\rm p}}h_{\rm sd}x(4) + bh_{{\rm r}_2{\rm d}}\big(\sqrt{E_{\rm p}}h_{{\rm sr}_2}x(3) + w(3)\big) + v(4) \\ &\vdots \\ r(2N-1) &= \sqrt{E_{\rm p}}h_{\rm sd}x(2N-1) + v(2N-1) \\ r(2N) &= \sqrt{E_{\rm p}}h_{\rm sd}x(2N) \\ &\quad + bh_{{\rm r}_N{\rm d}}\big(\sqrt{E_{\rm p}}h_{{\rm sr}_N}x(2N-1) + w(2N-1)\big) + v(2N) \end{aligned} \quad (9)$$

where $x(\cdot)$ and $r(\cdot)$ denote respectively the transmitted precoded signal at the source and the received data at the destination, $v(\cdot)$ and $w(\cdot)$ denote respectively the IID zero-mean circularly Gaussian noise with variance σ^2 received at the destination and at the relay nodes. The relay nodes are assumed to have knowledge of the second order statistics of the channel from source to relays, therefore the amplifying coefficients b for all relay nodes are the same, i.e., $b = \sqrt{\frac{E_{\rm p}}{E_{\rm p}+\sigma^2}}$. Using similar arguments as in the previous subsection, the transmission model in (9) can be written in a compact matrix form,

$$\mathbf{r}_{\rm N} = \sqrt{E_{\rm p}}\mathbf{X}_{\rm N}\mathbf{h}_{\rm N} + \mathbf{n}_{\rm N} \quad (10)$$

where subscript $_{\rm N}$ denotes entities associated with the N-relay system, $\mathbf{r}_{\rm N}$ is the received vector given by $\mathbf{r}_{\rm N} = [r(1), \ \cdots, \ r(2N)]^T$, $\mathbf{X}_{\rm N}$ is the $2N \times (N+1)$ transmitted signal matrix, and $\mathbf{h}_{\rm N}$ is the $(N+1) \times 1$ equivalent channel vector respectively given by

$$\mathbf{X}_{\rm N} = \begin{pmatrix} x(1) & 0 & \cdots & \cdot & 0 \\ x(2) & x(1) & 0 & \cdots & 0 \\ x(3) & 0 & 0 & 0 & \cdots \\ x(4) & 0 & x(3) & 0 & \cdots \\ \vdots & & \ddots & & \ddots \\ x(2N-1) & 0 & \cdots & 0 & 0 \\ x(2N) & 0 & \cdots & 0 & x(2N-1) \end{pmatrix} \quad (11)$$

$$\text{and} \quad \mathbf{h}_{\rm N} = [h_{\rm sd}, \ bh_{{\rm sr}_1}h_{{\rm r}_1{\rm d}}, bh_{{\rm sr}_2}h_{{\rm r}_2{\rm d}}, \ \cdots, bh_{{\rm sr}_N}h_{{\rm r}_N{\rm d}}]^T \quad (12)$$

and $\mathbf{n}_{\rm N}$ is the $2N \times 1$ noise vector such that $\mathbf{n}_{\rm N} \sim \mathcal{N}(0, \sigma^2\mathbf{\Sigma}_{\rm N})$ with $\mathbf{\Sigma}_{\rm N}$ given by

$$\mathbf{\Sigma}_{\rm N} = \left(\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1+b^2|h_{\rm r_1 d}|^2 & \cdots & 0 \\ \ddots & & \ddots & \\ & & & \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1+b^2|h_{\rm r_N d}|^2 \end{array} \right). \quad (13)$$

III. Pair-wise Error Performance Analysis

We now present the results of the performance analysis for the half-duplex systems with a single relay as well as with multiple relays described in the previous section. The criterion on which the performance of the systems is analyzed is the pairwise error probability (PEP) which is explained as follows: Consider that a ML detector is employed at the receiver after the noise is pre-whitened. Then, in either case of single or multiple relay, for a given channel realization $\mathbf{h}_m, m=1,N$ ($m=1$ corresponds to the single relay system and $m=N$ to multiple relay systems), and an original symbol block $\mathbf{s} \in \mathcal{S}^{2P}$ or $\mathbf{s} \in \mathcal{S}^{2N}$, the PEP is defined as the probability of deciding in favor of $\mathbf{s}' \neq \mathbf{s}, \quad \mathbf{s}', \mathbf{s} \in \mathcal{S}^{2P}$ or $\mathcal{S}^{2N}$, and is given by

$$P_{em}(\mathbf{s} \rightarrow \mathbf{s}'|\mathbf{h}_m) = Q\Big(\frac{d_m(\mathbf{s},\mathbf{s}')}{2}\Big), \qquad m=1,N \quad (14)$$

where as shown in [16] the $Q(x)$ function is

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{z^2}{2}}\,dz = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \exp\big(-\frac{x^2}{2\sin^2\theta}\big)d\theta \quad (15)$$

and $d_m(\mathbf{s},\mathbf{s}')$ is the Euclidean distance between $\mathbf{s}$ and $\mathbf{s}'$ *at the ML detector*. This distance $d_m(\mathbf{s},\mathbf{s}')$ is a function of the error vector $\mathbf{e}$ representing the difference between $\mathbf{s}$ and $\mathbf{s}'$ *at the transmitter*, where

$$\mathbf{e} = \mathbf{s} - \mathbf{s}'. \quad (16)$$

We begin by examining the system with a single relay ($m=1$) and then move on to the case with multiple relays ($m=N$).

A. Single relay

In the case of a single relay, at the receiver, after the received signal has been processed by the noise whitener, the distance measure $d_1(\mathbf{s},\mathbf{s}')$ between the two signal vectors can be expressed as

$$\begin{aligned} d_1^2(\mathbf{s},\mathbf{s}') &= \frac{E_{\rm p}}{\sigma^2}\,(\mathbf{s}-\mathbf{s}')^H\mathbf{F}^H\mathbf{H}_1^H\mathbf{\Sigma}_1^{-1}\mathbf{H}_1\mathbf{F}(\mathbf{s}-\mathbf{s}') \\ &= \rho\,\mathbf{e}^H\mathbf{F}^H\mathbf{H}_1^H\mathbf{\Sigma}_1^{-1}\mathbf{H}_1\mathbf{F}\,\mathbf{e} \end{aligned}$$

where $\mathbf{e}$ is given by (16), and $\rho = E_{\rm p}/\sigma^2$ represents the signal-to-noise ratio (SNR). (Note that ρ is *not* the SNR defined by the equivalent noise at destination receiver). Denote $\mathbf{u} = [\mathbf{u}_{\rm I}^T \quad \mathbf{u}_{\rm II}^T]^T = \mathbf{x} - \mathbf{x}'$, then $\mathbf{u} = \mathbf{F}\mathbf{e} = \mathbf{F}(\mathbf{s}-\mathbf{s}')$. We can re-write the above measure for single relay as

$$d_1^2(\mathbf{s},\mathbf{s}') = \rho\;\mathbf{h}_1^H\mathbf{U}_1^H\mathbf{\Sigma}_1^{-1}\mathbf{U}_1\;\mathbf{h}_1 \quad (17)$$

where $\mathbf{U}_1$ is the error matrix after precoding such that

$$\mathbf{U}_1 = \left(\begin{array}{cc} \mathbf{u}_{\rm I} & \mathbf{0} \\ \mathbf{u}_{\rm II} & \mathbf{u}_{\rm I} \end{array}\right) = \mathbf{X}_1 - \mathbf{X}_1' \quad (18)$$

with $\mathbf{u}_{\rm i} = \mathbf{x}_{\rm i} - \mathbf{x}_{\rm i}', i = $ I, II, and $\mathbf{X}_1$ and $\mathbf{X}_1'$ as defined in (8). Using (14), (15), and (17), the *average* pairwise error probability is given by

$$\begin{aligned} P_{\rm e1}(\mathbf{s}\rightarrow\mathbf{s}') &= \mathbb{E}_{\mathbf{h}_1}\Big[Q\Big(\frac{d_1(\mathbf{s},\mathbf{s}')}{2}\Big)\Big] \\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\mathbb{E}_{\mathbf{h}_1}\Big[\exp\Big(-\rho\frac{\mathbf{h}_1^H\mathbf{U}_1^H\mathbf{\Sigma}_1^{-1}\mathbf{U}_1\mathbf{h}_1}{8\sin^2\theta}\Big)\Big]d\theta. \end{aligned} \quad (19)$$

We now present an asymptotic pairwise error expression of a single relay in the following theorem, the proof of which is given in Appendix I:

Theorem 1: Suppose $\det(\mathbf{U}_1^H\mathbf{U}_1) \neq 0$ where $\mathbf{U}_1$ is given by (18). Then at high SNR, the average PEP for the single relay system is given by

$$P_{\rm e1}(\mathbf{s}\rightarrow\mathbf{s}') = \frac{6}{\det(\mathbf{U}_1^H\mathbf{U}_1)}(\rho^{-2}\ln\rho) + \mathcal{O}\Big(\frac{|\ln\det(\mathbf{U}_1^H\mathbf{U}_1)|}{\rho^2}\Big). \quad (20)$$

The terms $\frac{1}{6}(\det(\mathbf{U}_1^H\mathbf{U}_1))$ and $(\rho^{-2}\ln\rho)$ are respectively designated the *coding gain* and the *diversity gain function* of the system. □

Comparing the result in Theorem 1 with the asymptotic PEP for a conventional MIMO system [17], the following observations are noted:

- An upper bound of the PEP for a conventional MIMO system is often characterized by the *diversity gain* which is defined as the slope at which the PEP decreases with the logarithm of SNR. However, as we observe from Theorem 1, the "diversity gain" for an AF single relay system is no longer simply a power function of SNR. It also involves a function of the logarithm of SNR. This is because the channel matrix contains a term of the product of two independent channel gains which is not IID Gaussian as in the case for a conventional MIMO system. We therefore have changed the designation here to *diversity gain function* to fully characterize the diversity behavior of the relay system. This log SNR factor in the diversity gain function had also be noted in [12]–[14] as mentioned in Section I.
- Theorem 1 indicates that the coding gain of the AF single relay system has a form similar to that of a conventional MIMO system. The condition to fully reach the diversity gain function is to have non-zero coding gain, i.e.,

$$\det(\mathbf{U}_1^H\mathbf{U}_1) = \|\mathbf{u}_{\rm I}\|^4 + \beta \neq 0 \quad (21)$$

where

$$\beta = \|\mathbf{u}_{\rm I}\|^2\|\mathbf{u}_{\rm II}\|^2 - \mathbf{u}_{\rm II}^H\mathbf{u}_{\rm I}\mathbf{u}_{\rm I}^H\mathbf{u}_{\rm II}. \quad (22)$$

We note that $\beta \geq 0$ by the Schwartz Inequality and $\beta = 0$ iff $\mathbf{u}_{\rm I} = \alpha\mathbf{u}_{\rm II}$ for some α. Similar to a MIMO system, this condition can be regarded as a *rank criterion* [17], i.e., the auto-correlation matrix of the error must be full rank to achieve the full diversity function. However the auto-correlation matrix of the error in the single relay system, unlike that in a conventional MIMO system, is lower triangular in its structure because the channel matrix in (6) is lower triangular in structure with equal diagonal elements.

B. Multiple relays

From the description in Section II-B, the input-output relation of the multiple relay system is given by $\mathbf{r}_{\text{N}} = \sqrt{E_{\text{p}}}\mathbf{X}_{\text{N}}\mathbf{h}_{\text{N}} + \mathbf{n}_{\text{N}}$ where $\mathbf{X}_{\text{N}}$ is given by (11). Thus, for this multi-relay system, the error matrix is given by,

$$\mathbf{U}_{\text{N}} = \mathbf{X}_{\text{N}} - \mathbf{X}'_{\text{N}} = \begin{pmatrix} u(1) & 0 & \cdots & & 0 \\ u(2) & u(1) & 0 & \cdots & \\ u(3) & 0 & 0 & 0 & \cdots \\ u(4) & 0 & u(3) & 0 & \cdots \\ \vdots & & \ddots & & \ddots \\ u(2N-1) & 0 & \cdots & 0 & 0 \\ u(2N) & 0 & \cdots & 0 & u(2N-1) \end{pmatrix} \tag{23}$$

with $u(i) = x(i) - x'(i),\ i = 1, 2, \cdots, 2N$. The analysis for such a system with N relays is similar to the system with a single relay, and its PEP is given by the following theorem, the proof of which is shown in Appendix II.

Theorem 2: Suppose that $\det(\mathbf{U}_{\text{N}}^H\mathbf{U}_{\text{N}}) = \xi\prod_{n=1}^{N}|u(2n-1)|^2 \neq 0$, where $\mathbf{U}_{\text{N}}$ is given in (23) and $\xi = \sum_{n=1}^{N}|u(2n-1)|^2$. Then, at high SNR, the average PEP of the multiple relay system is

$$P_{e\text{N}}(\mathbf{s} \to \mathbf{s}') = \frac{(2N+1)!!2^{(N+1)}}{(N+1)!\det(\mathbf{U}_{\text{N}}^H\mathbf{U}_{\text{N}})}\rho^{-(N+1)}\ln^N\rho + \mathcal{O}\left(\frac{|\ln\det(\mathbf{U}_{\text{N}}^H\mathbf{U}_{\text{N}})|}{\rho^{(N+1)}}\ln^{N-1}\rho\right) \tag{24}$$

where $(2N+1)!! = 1\cdot 3\cdots(2N+1)$. □

In this case, the diversity gain function is given by $\rho^{-(N+1)}\ln^N\rho$, while the coding gain is given by $\frac{(N+1)!}{(2N+1)!!2^{(N+1)}}\det(\mathbf{U}_{\text{N}}^H\mathbf{U}_{\text{N}})$. Parallel to Theorem 1, we can look upon the condition $\det(\mathbf{U}_{\text{N}}^H\mathbf{U}_{\text{N}}) \neq 0$ in Theorem 2 as a *rank criterion* under which *full diversity gain function* can be achieved.

IV. Precoder design and performance

The previous section addresses the performance of both the single relay and the multi-relay systems. In this section, we develop the design of the precoders at the transmitter of these relay communication systems so that their PEP can be improved. The results of our analysis in the previous section suggest that as long as the rank criterion is satisfied, the PEP of both the single-relay and the multi-relay systems will depend on the diversity gain function (which is a function of the SNR ρ) and the coding gain (which is a function of the error matrix). Therefore, our design of the precoder can be carried out by ensuring that: (i) the rank criterion is satisfied to achieve full diversity which implies that the determinant of the corresponding autocorrelation of the error matrix must be non-zero, and then (ii) the minimum of coding gain is maximized. Now, the PEP of a signal constellation depends on which pair of points is being addressed. For any signal constellation, the worse-case PEP dominates the performance. Thus, our design should minimize the worse case PEP of the relay communication system by maximizing the minimum of the determinant of the error correlation matrix, provided that the rank criterion is satisfied. (The principles of satisfying these two criteria can be viewed as a parallel to those in the design of space-time codes for a conventional MIMO channel [17]). In the following, we discuss the designs of the precoders under consideration of these two criteria.

A. Precoder design for full diversity

Single Relay System From the discussion on Theorem 1, the condition to achieve the full diversity gain function (the rank criterion of (21)) is re-written here as

$$\|\mathbf{u}_{\text{I}}\|^4 + \beta \neq 0 \tag{25}$$

where $\beta = \|\mathbf{u}_{\text{I}}\|^2\|\mathbf{u}_{\text{II}}\|^2 - \mathbf{u}_{\text{II}}^H\mathbf{u}_{\text{I}}\mathbf{u}_{\text{I}}^H\mathbf{u}_{\text{II}}$. As noted before, β is non-negative but can be equal to zero (when $\mathbf{u}_{\text{I}} \propto \mathbf{u}_{\text{II}}$). Thus, a necessary and sufficient condition for the rank criterion for Scheme 1 can be written as

$$\|\mathbf{u}_{\text{I}}\|^4 \neq 0. \tag{26}$$

A sufficient condition to guarantee (26) is given by: $|u(k)|^2 = 0,\ \forall k \in [1, 2P]$ if and only if $\mathbf{s} = \mathbf{s}',\ \mathbf{s},\ \mathbf{s}' \in \mathcal{S}^{2P}$. A precoder which ensures this condition can be readily obtained by applying the design scheme in [18]–[23] as stated in the following Lemma:

Lemma 1: Define the Cyclotomic ring [24] as $\mathbb{Z}[\zeta_r] = \{\sum_{i=1}^{r} c_i\zeta_r^i:\ c_i \in \mathbb{Z},\ i = 1, \cdots, r-1\}$, where $\mathbb{Z}$ denotes the integer ring $\{\cdots, 0, \pm 1, \pm 2, \cdots\}$ and $\zeta_r = \exp(\frac{j2\pi}{r})$. Let $L = \prod_{k=1}^{K}\ell_k^{n_k}$, where ℓ_k is prime, n_k is a positive integer for $k = 1, \cdots, K$. Form the integer $L_1 = L_2\prod_{k=1}^{K}\ell_k^{m_k}$, where $m_k \geq 1$ with L_2 being prime to L and let $Q = LL_1$. Define the $L \times L$ precoder matrix

$$\mathbf{F}_{\text{od}} = \mathbf{W}_L^H\ \ \text{diag}(1,\ \zeta_Q,\ \cdots,\ \zeta_Q^{L-1}) \tag{27}$$

where $\mathbf{W}_L$ is a normalized discrete Fourier transform matrix of size L. If $\mathbf{q} = \mathbf{F}_{\text{od}}\mathbf{p}$, $\mathbf{p} \in \mathbb{Z}^L[\zeta_{L_1}]$ and $\mathbf{p} \neq \mathbf{0}$, then all the entries in vector $\mathbf{q}$ are nonzero. □

The proof of Lemma 1 can be found in [23]. It tells us that matrix $\mathbf{F}_{\text{od}}$ precodes vector $\mathbf{p}$ in such a way that all the components of the precoded vector $\mathbf{q}$ are non-zero unless $\mathbf{p}$ is a zero vector. Obviously if our precoder $\mathbf{F}$ for Scheme 1 is chosen as $\mathbf{F}_{\text{od}}$ in (27) with $L = 2P$, then the condition in (26) is guaranteed by carefully choosing $L_1,\ L_2$ and Q in Lemma 1. Let us look at a simple example:

Example 1: Consider $P = 1$. It is obvious that for this value of P, we have $L = 2,\ K = 1$ and $n_1 = 1$. Then, any odd number will be prime to L and thus, $L_2 = 1, 3, 5, \cdots$. If we choose $L_2 = 1,\ m_1 = 2$, then we can form $L_1 = L_2 2^{m_1} = 4$ and hence $Q = LL_1 = 8$. Now, for $L = 2$, $\mathbf{W}_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then, for any $\mathbf{e} = [e(1)\ \ e(2)]^T = (\mathbf{s} - \mathbf{s}')^T$,

$$\mathbf{u} = \mathbf{F}_{\text{od}}\mathbf{e} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{j\frac{\pi}{4}} \\ 1 & -e^{j\frac{\pi}{4}} \end{pmatrix}\mathbf{e}. \tag{28}$$

Thus, $u(1) = e(1) + e^{j\frac{\pi}{4}}e(2)$ and $u(2) = e(2) - e^{j\frac{\pi}{4}}e(2)$. For any square QAM constellation, $e(1),\ e(2)$ are rational complex numbers. Since $e^{j\frac{\pi}{4}}$ is irrational, hence, $u(1) = 0 \Leftrightarrow e(1) = e(2) = 0$. ■

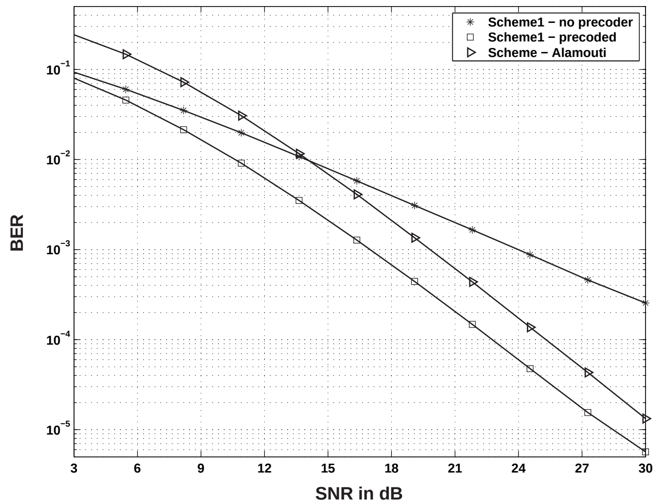

Fig. 3. BER performance for single relay Scheme 1 (with and without precoder) and Scheme with Alamouti code

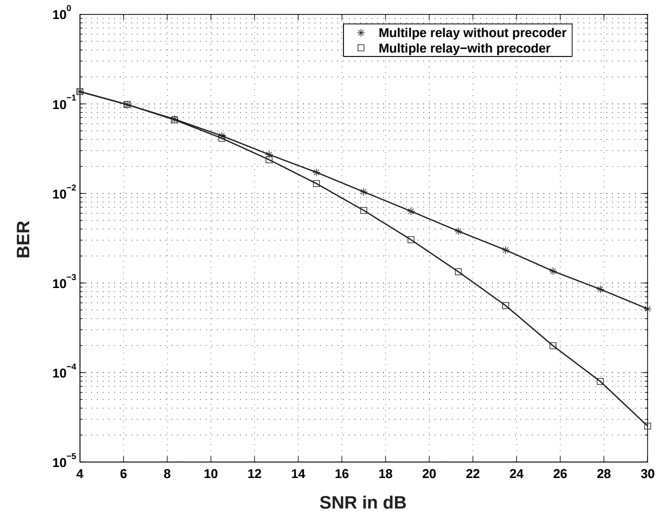

Fig. 4. BER performance for Scheme N(2-relays) with and without precoder

Let us examine the effect of such a precoder on the performance of the single relay communication system.

Example 2: For the single relay system operating in the mode of Scheme 1, we compare the bit error rate (BER) of the system without a precoder to that equipped with the proposed precoder in (27). We employ a data block length of 2, and BPSK is used for transmitting the signal. We also evaluate the BER performance of the scheme proposed in [5] where the Alamouti code [25] is used. This scheme requires 4 time slots to transmit 2 symbols. In the 1st and 3rd time slots, the source transmits two symbols to the relay node and there is no transmission to the destination. In the 2nd and 4th time slots, both the source and the relay transmit their symbols to the destination (this scheme can actually be considered as a special case of Scheme 1). In this case, the transmission procedure is similar to that of a MIMO system with 2 transmitter antennas and one receiver antenna thus Alamouti's code can be applied. For this Alamouti coded relay scheme, we employ 4-QAM signalling so that the transmission is carried out at the same bit rate as Scheme 1. The BER performance of these different single-relay schemes are shown in Fig. 3. It can be seen that in comparison to the system without percoding, the designed precoder provides approximately a 6dB gain at moderate-to-high SNR. It can also be observed that the proposed scheme outperforms the Alamouti coded relay scheme at moderate-to-high SNRs. ■

Multiple Relay System From Theorem 2, the rank criterion for Scheme N is given by

$$\det(\mathbf{U}_{\text{N}}^H\mathbf{U}_{\text{N}}) = \xi \prod_{n=1}^{N} |u(2n-1)|^2 \neq 0. \tag{29}$$

Similar to Scheme 1, a sufficient condition for (29) is: $|u(k)|^2 = 0,\ \forall k \in [1, 2N]$ if and only if $\mathbf{s} = \mathbf{s}',\ \mathbf{s},\ \mathbf{s}' \in \mathcal{S}^{2N}$. Therefore, a precoder of the form given by (27) with $L = 2N$ will also satisfy the rank criterion for Scheme N, i.e., a precoder of size $2N$ achieving full diversity for the multiple relay system can take on the form,

$$\mathbf{F}_{\text{od}} = \mathbf{W}_{2N}^H\ \text{diag}(1,\ \zeta_Q,\ \cdots,\ \zeta_Q^{L-1}) \tag{30}$$

where $\mathbf{W}_{2N},\ L,$ and Q are defined in Lemma 1. It can be seen that the precoder proposed above for the multiple relay system has the same form as that for the single-relay system. Both satisfy the respective rank criteria so that full diversity can be achieved. To illustrate how the precoder for the multiple-relay system performs, we now present some computer simulation results.

Example 3: Fig. 4 shows the BER performance comparison for the systems with multiple relays. We compare the performance of the 2-relay systems with and without the precoder. Here, the precoder for the system is given by (30) with $N = 2$ and $Q = 16$. The signals are transmitted using QPSK. It can be observed that the precoded scheme again provides substantial gain over the unprecoded scheme. ■

B. Precoder design for maximizing coding gain in single relay — the 2×2 case

In the previous section, we have outlined how the precoder of a relay system can be designed to reach full diversity gain function. Here, we examine how the precoder can further be designed to obtain a high coding gain. To maximize the coding gain one has to take into account all the possible pairs of symbol in a signal constellation. In general, it is very complicated [17] to consider this criterion for arbitrary block length and size of constellation. Hence, we consider a simple case of the single relay system in which the data block length is of 2, i.e., $P = 1$ and the transmitting signals are from 4-QAM constellation. In this case, the coding gain is proportional to the factor $\det(\mathbf{U}_1^H\mathbf{U}_1) = |u(1)|^4$. We now seek an optimal 2 by 2 unitary precoder to maximize the minimum of the coding gain.

A general 2 by 2 unitary matrix group can be expressed as

[26],

$$\mathbf{F}_{\rm c}=\begin{pmatrix} e^{j\alpha_1} & 0 \\ 0 & e^{j\alpha_2}\end{pmatrix}\begin{pmatrix} \cos\theta & e^{j\phi}\sin\theta \\ -e^{-j\phi}\sin\theta & \cos\theta\end{pmatrix} \quad (31)$$

where $\alpha_1,\ \alpha_2,\ \theta\in[0,2\pi]$, and $\phi\in[0,\pi]$. Now, the value of $|u(1)|^4=|e(1)\cos\theta+e(2)e^{j\phi}\sin\theta|^4$ does not depend on the choice of α_1 and α_2. Therefore, θ and ϕ are the variables for the optimization problem. The optimal values of θ and ϕ are given by the following theorem, the proof of which is provided in Appendix III.

Theorem 3: For the single relay transmission system having a data block length of 2 and transmitting 4-QAM signals, the optimal precoder $\mathbf{F}_{\rm oc}$ as given in (31) that maximizes the minimum coding gain has values of θ and ϕ given respectively by

$$\theta_0=\sin^{-1}\left(\sqrt{\frac{3-\sqrt{3}}{6}}\right)$$
$$\phi_0=\pi/12 \quad (32)$$

and the corresponding maximum value of the minimum coding gain is given by $C_0=\frac{2}{3}(1-\frac{1}{\sqrt{3}})^2$. □

Remarks on the 2×2 *optimal precoder:*

- While Theorem 3 clearly indicates that the chosen values of θ_0 and ϕ_0 in (32) maximize the minimum coding gain, these values also enable the AF system to achieve full diversity gain function even though the matrix $\mathbf{F}_{\rm oc}$ does not conform to the structure of $\mathbf{F}_{\rm od}$ in (27). This is because the structure of $F_{\rm od}$ is only sufficient (but not necessary) to achieve full diversity gain function.
- In a MIMO transmission system with 2 transmission antennas and one receiver antenna, a standard diagonal space-time code [18], [22], [27], [28] uses unitary matrix $\mathbf{F}_{\rm st}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{j\phi} \\ 1 & -e^{j\phi}\end{pmatrix}$ generating the code words. To maximize the minimum of coding gain for this MIMO system, the value of $\phi=\pi/4$ has been suggested [18], [22], [27], [28]. This happens to be the design in (28) to achieve full diversity for a single relay system with data block length of 2. Apparently the design is not optimized for coding gain for the relay system. A simple evaluation shows that the corresponding coding gain is equal to 0.02, a value far inferior to that of $C_0=\frac{2}{3}(1-\frac{1}{\sqrt{3}})^2=0.357$ given by Theorem 3 using the optimum precoder $\mathbf{F}_{\rm oc}$ for maximizing the coding gain. In fact, the form of $\mathbf{F}_{\rm st}$ corresponds to the choice of $\theta=\frac{\pi}{4}$, and $\alpha_1=0,\alpha_2=\phi+\pi$ in $\mathbf{F}_{\rm c}$ of (31). In such a case, maximizing the minimum coding gain, we arrive at a precoder $\mathbf{F}_{\rm ost}$ having the optimum value of $\phi=\phi_{\rm ost}=\pi/6$ and the corresponding coding gain equal to 0.048 (calculations of these coding gains are given in Appendix III). This precoder is clearly better than the choice of $\phi=\pi/4$ in $\mathbf{F}_{\rm st}$, but is still much inferior to the value given in Theorem 3. The above discussion illustrates that *the optimum designs for space-time block coding in a MIMO system may, in general, not be directly applicable to the optimum design in a relay transmission system.*

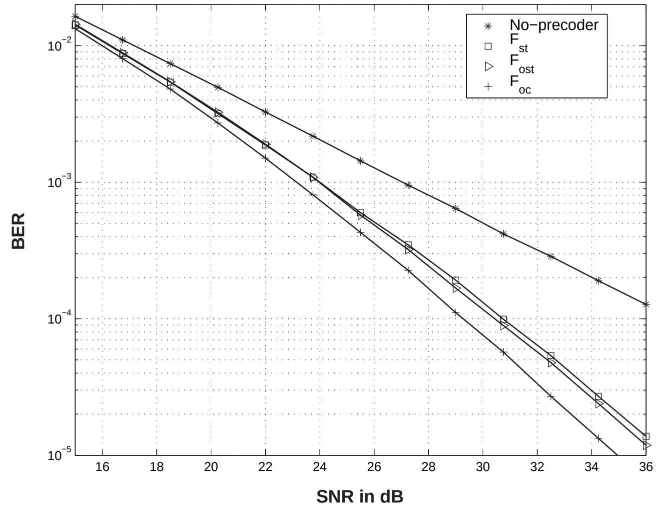

Fig. 5. BER performance of Scheme 1 with precoders of different coding gains

We now illustrate the effect of the optimized coding gain precoder on the performance of the single-relay system which employs a data block length 2 and compare it with that of the various precoders.

Example 4: In this example, we test by simulations the performance of the following single-relay systems operating in the mode of Scheme 1: i) without precoder (the precoder is actually an identity matrix), ii) with optimum coding gain precoder $\mathbf{F}_{\rm oc}$, the values of θ_0 and ϕ_0 as given in Theorem 3 are used. iii) with precoder $\mathbf{F}_{\rm st}$ which provides optimum coding gain for the MIMO system and full diversity for the relay system, the value of $\phi=\pi/4$ is used, and iv) with precoder $\mathbf{F}_{\rm ost}$, the value of $\phi=\pi/6$ is used. In all the tests, the signals are transmitted in 4-QAM through the relay system with added white Gaussian noise. Fig. 5 shows the BER performance. Among the precoded relay systems, the one employing $\mathbf{F}_{\rm ost}$ is slightly better than that using $\mathbf{F}_{\rm st}$, while the system employing $\mathbf{F}_{\rm oc}$ provides the best performance, being superior to both the other precoded systems by a margin of nearly 2dB in the high SNR region. On the other hand, the unprecoded transmission suffers badly in performance, needing more than 8dB SNR compensation for a BER of 10^{-4}. ■

Example 5: In this example, we compare the performance of Scheme 1 equipped with the optimal coding gain precoder $\mathbf{F}_{\rm oc}$ with two other schemes: the scheme described in Example 2 which uses the Alamouti's code, and the scheme proposed in [7] called *an orthogonal AF protocol*. The orthogonal scheme can also be considered as a special case of Scheme 1, it differs from Scheme 1 in that there is no transmission from the source node at all in the 2nd P-time-slots. It has the same symbol rate as the Alamouti coded scheme. Therefore, to compare their performance under the same bit rate, we transmit 4-QAM signals for Scheme 1 and 16-QAM signals for the other two schemes. Fig. 6 shows the comparison of the performance. The performance of Scheme 1 *without a precoder* is also plotted for completeness of comparison. It

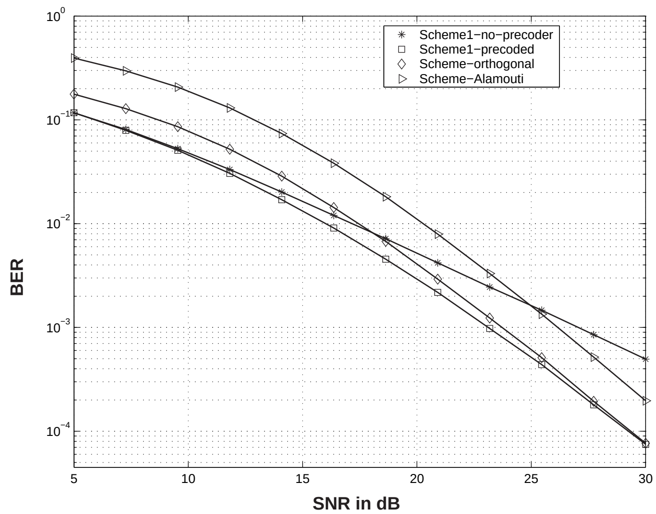

Fig. 6. BER performance of Scheme 1 (with and without optimal precoder), Alamouti's coded scheme, and the orthogonal scheme

can be seen that Scheme 1 equipped with the optimal coding gain precoder yields the best performance among all schemes. It should be noted that all the schemes in this example achieve full diversity as defined in Section IV-A. The advantage of Scheme 1 with the optimal 2×2 precoder over the other schemes comes from achieving the optimum coding gain. ■

V. CONCLUSION

In this paper, we have analyzed the performance of an AF half-duplex relay transmission system equipped with a single antenna. We have focused on a non-orthogonal AF protocol in which the source transmits the first half data block to both the destination and the relay in the first P time-slot, and during the second P time-slots, sends the second half data block to the destination while the relay simply amplifies and forwards what it receives in the first P time-slots. This is referred to as Scheme 1 in the paper. We have also analyzed the multiple relay system, Scheme N, which is the multiple relay version of Scheme 1. The exact asymptotic expressions of the pairwise error probabilities of both schemes have been derived. The diversity gains and the coding gains of the systems have been identified. While the diversity gains have been shown to be the product of a power function and log function of SNR, the coding gains have been shown to be proportional to the determinant of the autocorrelation of the coding error matrices. These results are obtained over a symmetric relay system where all the channel gains are IID Gaussian with zero mean and unit variance. However our analysis can be extended to cases in which channel gains are independent yet have difference variances.

Our derived PEP analysis results suggest that two criteria can be used for the precoder design: One being the rank criterion to achieve the maximum diversity gain, and the second being the coding gain criterion to obtain an optimum gain. We have shown that a precoder to satisfy the first criterion can be obtained by employing a currently available design scheme in MIMO systems. To obtain a general precoder with arbitary size (greater than 2×2) that satisfies the second criterion proves to be a hard problem. In this paper, we have focused on the case of a single relay, data block length of 2 and the transmitted signal being in 4-QAM. For this case, we derived a closed form optimum precoder over the 2×2 unitary matrix class. Simulations show that the proposed designs significantly improve the BER performance of the relay systems.

APPENDIX I
PROOF OF THEOREM 1

From (19), the average PEP for the single relay system can be written as,

$$P_{e1}(\mathbf{s} \to \mathbf{s}') = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \mathbb{E}_{\mathbf{h}_1}\left[\exp\left(-\rho\frac{\mathbf{h}_1^H \mathbf{U}_1^H \mathbf{\Sigma}_1^{-1}\mathbf{U}_1\mathbf{h}_1}{8\sin^2\theta}\right)\right]d\theta \tag{33}$$

where $\mathbf{h}_1 = \mathbf{T}\begin{pmatrix} h_{\rm sd} \\ h_{\rm sr}\end{pmatrix}$, and $\mathbf{T} = \mathrm{diag}(1,\ bh_{\rm rd})$. Taking the expected value in (33) with respect to $h_{\rm sd}$ and $h_{\rm sr}$ first, we obtain

$$P_{e1}(\mathbf{s} \to \mathbf{s}') = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \mathbb{E}_{h_{\rm rd}}\left[\frac{1}{\det(\mathbf{I}_2 + \frac{\rho}{8\sin^2\theta}\mathbf{T}^H\mathbf{U}_1^H\mathbf{\Sigma}_1^{-1}\mathbf{U}_1\mathbf{T})}\right]d\theta \tag{34}$$

where we have used the fact that given a complex circularly distributed Gaussian random column vector $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$, and a Hermitian matrix $\mathbf{A}$, then $\mathbb{E}[\exp(-\mathbf{z}^H\mathbf{A}\mathbf{z})] = \frac{1}{\det(\mathbf{I}+\mathbf{\Sigma}\mathbf{A})}$. Let $y = |h_{\rm rd}|^2$. The probability distribution function of y is $\frac{e^{-\frac{y}{2}}}{2}$, since $h_{\rm rd}$ is zero-mean Gaussian. Then (34) becomes

$$P_{e1}(\mathbf{s} \to \mathbf{s}') = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{e^{-\frac{y}{2}}}{\det(\mathbf{I}_2 + \frac{\rho}{8\sin^2\theta}\mathbf{U}_1^H\mathbf{\Sigma}_1^{-1}\mathbf{U}_1\mathbf{T}\mathbf{T}^H)}dyd\theta \tag{35}$$

where we have used the fact that $\det(\mathbf{I}+\mathbf{CD}) = \det(\mathbf{I}+\mathbf{DC})$. The determinant in (35) can be evaluated such that,

$$\left(\det(\mathbf{I}_2 + \frac{\rho}{8\sin^2\theta}\mathbf{U}_1^H\mathbf{\Sigma}_1^{-1}\mathbf{U}_1\mathbf{T}\mathbf{T}^H)\right)^{-1} = \frac{1}{\mu^2}(1 + \frac{A_1}{y+\lambda_1} + \frac{A_2}{y+\lambda_2}) \tag{36}$$

where

$$\mu = 1 + \frac{\rho||\mathbf{u}_{\rm I}||^2}{8\sin^2\theta},$$
$$\lambda_1 = \frac{1}{2b^2}\left(a_1 + \sqrt{a_1^2 - 4a_2}\right),$$
$$\lambda_2 = \frac{1}{2b^2}\left(a_1 - \sqrt{a_1^2 - 4a_2}\right),$$
$$A_1 = \frac{\frac{1}{b^4}(1-a_2) - \frac{1}{b^2}(2-a_1)\lambda_1}{\lambda_2 - \lambda_1},$$
$$A_2 = \frac{-\frac{1}{b^4}(1-a_2) + \frac{1}{b^2}(2-a_1)\lambda_2}{\lambda_2 - \lambda_1},$$
$$a_1 = 1 + \frac{1}{\mu^2}\left(1 + \frac{\rho||\mathbf{u}||^2}{8\sin^2\theta} + \frac{\rho^2\beta}{64\sin^4\theta}\right),$$
$$a_2 = \frac{1}{\mu^2}\left(1 + \frac{\rho||\mathbf{u}||^2}{8\sin^2\theta}\right),$$
$$\text{with}\quad ||\mathbf{u}||^2 = ||\mathbf{u}_{\rm I}||^2 + ||\mathbf{u}_{\rm II}||^2,$$

$$\beta = ||\mathbf{u}_{\rm I}||^2||\mathbf{u}_{\rm II}||^2 - \mathbf{u}_{\rm II}^H\mathbf{u}_{\rm I}\mathbf{u}_{\rm I}^H\mathbf{u}_{\rm II}.$$

The pairwise error probability can thus be re-written as,

$$\begin{aligned}
&P_{e1}(\mathbf{s}\to\mathbf{s}')\\
&= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\frac{1}{\mu^2}\int_0^{\infty}(1+\frac{A_1}{y+\lambda_1}+\frac{A_2}{y+\lambda_2})e^{-\frac{y}{2}}dyd\theta\\
&= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\frac{1}{\mu^2}\big(2+A_1J(\lambda_1)+A_2J(\lambda_2)\big)d\theta \qquad (37)
\end{aligned}$$

where $J(\nu)=\int_0^\infty \frac{e^{-\frac{u}{2}}}{u+\nu}du$. Given the assumption $\|\mathbf{u}_{\rm I}\|^4+\beta\neq 0$, then at high SNR, λ_1,λ_2,A_1, and A_2 can be asymptotically expressed as,

$$\lambda_1 = (1+\frac{\beta}{||\mathbf{u}_{\rm I}||^4}) + O(\frac{1}{||\mathbf{u}_{\rm I}||^4}\rho^{-1}) \qquad (38)$$

$$\lambda_2 = \frac{8\|\mathbf{u}\|^2\sin^2\theta}{(||\mathbf{u}_{\rm I}||^4+\beta)}\rho^{-1} + O\big(\frac{1}{||\mathbf{u}_{\rm I}||^4+\beta}\rho^{-2}\big) \qquad (39)$$

$$A_1 = \frac{-\beta^2}{||\mathbf{u}_{\rm I}||^4(||\mathbf{u}_{\rm I}||^4+\beta)} + O\big(\frac{1}{||\mathbf{u}_{\rm I}||^4+\beta}\rho^{-1}\big) \qquad (40)$$

$$A_2 = \frac{||\mathbf{u}_{\rm I}||^4}{(||\mathbf{u}_{\rm I}||^4+\beta)} + O\big(\frac{1}{||\mathbf{u}_{\rm I}||^4+\beta}\rho^{-1}\big) \qquad (41)$$

where we have used the fact $b^{-2}=1+O(\rho^{-1})$. From Section IV-A, we know $\|\mathbf{u}_{\rm I}\|^4=0$ if and only if $\|\mathbf{u}_{\rm I}\|^4+\beta=0$. Therefore all the quantities in (38) - (41) exist. The asymptotic behavior of $J(\nu)$ when $\rho\to\infty$ depends on ν. From [29], $J(\nu)$ can be expressed as

$$J(\nu) = \int_0^\infty \frac{e^{-u}}{u+\nu/2}du = e^{\frac{\nu}{2}}\int_{\frac{\nu}{2}}^{\infty}\frac{e^{-u}}{u}du = e^{\frac{\nu}{2}}\mathrm{E_i}(\frac{\nu}{2})$$

where $\mathrm{E_i}(x)=\int_x^\infty\frac{e^{-u}}{u}du = -\big(\gamma+\ln x+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!n}\big)$, and γ is the Euler constant. If $\nu=c_1\rho^{-1}+O(c_2\rho^{-2})$, $0<c_1,c_2<\infty$, then when ρ is large,

$$\begin{aligned}
J(\nu) =& -\Big(1+O(c_1\rho^{-1})\Big)\Big(\gamma+\ln(\frac{c_1+O(c_2\rho^{-1})}{2})\\
&-\ln\rho+\sum_{n=1}^{\infty}\frac{(-1)^{-n}\big(c_1\rho^{-1}+O(c_2\rho^{-2})\big)^n}{2^n n! n}\Big)\\
=& \ln\rho + O(|\ln c_1|). \qquad (42)
\end{aligned}$$

Combining (38)-(41), and (42) with (37) results in

$$\begin{aligned}
&P_{e1}(\mathbf{s}\to\mathbf{s}')\\
&=\frac{32}{\pi(||\mathbf{u}_{\rm I}||^4+\beta)}\rho^{-2}\ln\rho\int_0^{\frac{\pi}{2}}\sin^4\theta d\theta + O\Big(\frac{|\ln(||\mathbf{u}_{\rm I}||^4+\beta)|}{\rho^2}\Big)\\
&= \frac{6}{(||\mathbf{u}_{\rm I}||^4+\beta)}\rho^{-2}\ln\rho + O\Big(\frac{|\ln(||\mathbf{u}_{\rm I}||^4+\beta)|}{\rho^2}\Big) \qquad (43)\\
&= \frac{6}{\det(\mathbf{U}_1^H\mathbf{U}_1)}\rho^{-2}\ln\rho + O\Big(\frac{|\ln\det(\mathbf{U}_1^H\mathbf{U}_1)|}{\rho^2}\Big) \qquad (44)
\end{aligned}$$

□

APPENDIX II
PROOF OF THEOREM 2

The pairwise error probability for the multiple relays can be expressed as

$$P_{e\rm N}(\mathbf{s}\to\mathbf{s}') = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\mathbb{E}_{\mathbf{h}_{\rm N}}\Big[\exp(-\rho\frac{\mathbf{h}_{\rm N}^H\mathbf{U}_{\rm N}^H\mathbf{\Sigma}_{\rm N}^{-1}\mathbf{U}_{\rm N}\mathbf{h}_{\rm N}}{8\sin^2\theta}\Big]d\theta \qquad (45)$$

where $\mathbf{U}_{\rm N}$ is given in (23), and $\mathbf{h}_{\rm N}$ is given by $\mathbf{h}_{\rm N}=\mathbf{T}_{\rm N}\,[h_{\rm sd},\ h_{{\rm sr}_1},\ \cdots,\ h_{{\rm sr}_N}]^T$ with $\mathbf{T}_{\rm N}$ being an $(N+1)\times(N+1)$ matrix such that $\mathbf{T}_{\rm N}=\mathrm{diag}(1,\ bh_{{\rm r}_1{\rm d}},\ bh_{{\rm r}_2{\rm d}},\ \cdots,\ bh_{{\rm r}_N{\rm d}})$. Taking expectation with respect to $h_{\rm sd},h_{{\rm sr}_1},\cdots,h_{{\rm sr}_N}$ in (45), we have

$$\begin{aligned}
&P_{e\rm N}(\mathbf{s}\to\mathbf{s}')\\
&=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\mathbb{E}_{h_{{\rm r}_1{\rm d}},\cdots,h_{{\rm r}_N{\rm d}}}\Big[\frac{1}{\det(\mathbf{I}_{N+1}+\frac{\rho}{8\sin^2\theta}\mathbf{T}_{\rm N}^H\mathbf{U}_{\rm N}^H\mathbf{\Sigma}_{\rm N}^{-1}\mathbf{U}_{\rm N}\mathbf{T}_{\rm N})}\Big]d\theta. \qquad (46)
\end{aligned}$$

Similar to the proof of Theorem 1, we let $y_n=|h_{{\rm rd}_n}|^2, n=1,2,\cdots,N$. Then, the PEP in (46) can be accordingly expressed as,

$$\begin{aligned}
&P_{e\rm N}(\mathbf{s}\to\mathbf{s}')\\
&=\frac{1}{2^N\pi}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\cdots\int_0^{\infty}f_1e^{-\frac{y_1}{2}}dy_1\prod_{k=2}^{N}g_ke^{-\frac{y_k}{2}}dy_2\cdots dy_Nd\theta \qquad (47)
\end{aligned}$$

where

$$\begin{aligned}
f_1 &= \frac{1+b^2y_1}{b^2y_1\mu_1\big(1+\frac{\rho}{8\sin^2\theta}(\xi+\varepsilon)\big)+1+\frac{\rho}{8\sin^2\theta}(\xi+\varepsilon+|u(2)|^2)}\\
g_k &= \frac{1+b^2y_k}{1+b^2y_k\mu_k},\quad k=2,\cdots,N\\
\xi &= \sum_{n=1}^{N}|u(2n-1)|^2,\\
\mu_n &= 1+\frac{|u(2n-1)|^2\rho}{8\sin^2\theta},\quad n=1,\cdots,N \qquad (48)\\
\varepsilon &= \sum_{i=2}^{N}\frac{|u(2i)|^2}{1+b^2y_i\mu_i}. \qquad (49)
\end{aligned}$$

Rewriting f_1 as $f_1=\frac{1}{\mu_1v_1}(1+\frac{\frac{1}{b^2}-\frac{q_1}{b^2}}{y_1+\frac{q_1}{b^2}})$, where $v_1=1+\frac{\rho}{8\sin^2\theta}(\xi+\varepsilon)$ and $q_1=\frac{1}{\mu_1v_1}\big(v_1+\frac{|u(2)|^2\rho}{8\sin^2\theta}\big)$, then, the integral with respect to y_1 in (47) becomes,

$$I_1 = \int_0^\infty f_1e^{-\frac{y_1}{2}}dy_1 = \frac{1}{\mu_1v_1}\Big(2+(\frac{1}{b^2}-\frac{q_1}{b^2})J(\frac{q_1}{b^2})\Big). \qquad (50)$$

Since $y_n\geq 0$ and $\mu_n>0$, $n=1,2,\cdots,N$, then from (49), we have $0\leq\varepsilon\leq\sum_{i=2}^{N}|u(2i)|^2$. Given the assumption $\prod_{n=1}^{N}|u(2n-1)|^2\neq 0$ (this implies $\xi\neq 0$), then at high SNR, we have

$$\begin{aligned}
\frac{1}{\mu_1} &= \frac{8\sin^2\theta}{|u(1)|^2}\rho^{-1}+O\Big(\frac{1}{|u(1)|^2}\rho^{-2}\Big)\\
\frac{1}{v_1} &= \frac{8\sin^2\theta}{\xi}\rho^{-1}+O\Big(\frac{1}{\xi}\rho^{-2}\Big)\\
q_1 &= \frac{8\big(\xi+|u(2)|^2\big)\sin^2\theta}{|u(1)|^2\xi}\rho^{-1}+O\Big(\frac{1}{\xi|u(1)|^2}\rho^{-2}\Big).
\end{aligned}$$

Putting these quantities into (50), then, at high SNR, we have

$$I_1 = \frac{(8\sin^2\theta)^2\rho^{-2}}{|u(1)|^2\xi}\Big(\ln\rho+O\big(|\ln(\xi|u(1)|^2)|\big)\Big). \qquad (51)$$

Next we calculate the integral of y_k, $k=2,3,\cdots,N-1$:

$$I_k = \int_0^\infty g_ke^{-\frac{y_k}{2}}dy_k$$

$$= \int_0^\infty \frac{1}{\mu_k}\big(1 + \frac{\frac{1}{b^2} - \frac{1}{b^2\mu_k}}{y_k + \frac{1}{b^2\mu_k}}\big)e^{-\frac{y_k}{2}}dy_k$$
$$= \frac{1}{\mu_k}\Big(2 + (\frac{1}{b^2} - \frac{1}{b^2\mu_k})J(\frac{1}{b^2\mu_k})\Big). \quad (52)$$

Under the assumption $\prod_{n=1}^N |u(2n-1)|^2 \neq 0$, then at high SNR we have

$$I_k = \frac{8\sin^2\theta}{|u(2k-1)|^2}\rho^{-1}\Big(\ln\rho + O\big(|\ln|u(2k-1)|^2|\big)\Big). \quad (53)$$

From (51) and (52), the PEP in (47) can be written as,

$$P_{e_N}(\mathbf{s}\to\mathbf{s}') = \frac{1}{2^N\pi}\int_0^{\frac{\pi}{2}}\frac{(8\sin^2\theta)^{(N+1)}}{\xi\prod_{n=1}^N|u(2n-1)|^2}\rho^{-(N+1)}\ln^N\rho d\theta + O\Big(\frac{|\ln\big(\xi\prod_{n=1}^N|u(2n-1)|^2\big)|}{\rho^{N+1}}\ln^{N-1}\rho\Big).$$

Since

$$\int_0^{\frac{\pi}{2}}\sin^{2(N+1)}\theta d\theta = \frac{1\cdot3\cdot5\cdots(2N+1)}{2\cdot4\cdot6\cdots2(N+1)}\frac{\pi}{2} \quad (54)$$

then, we can write

$$\begin{aligned}P_{e_N}(\mathbf{s}\to\mathbf{s}') =& \frac{1\cdot3\cdot5\cdots(2N+1)2^{(N+1)}}{(N+1)!\xi\prod_{n=1}^N|u(2n-1)|^2}\rho^{-(N+1)}\ln^N\rho\\ &+ O\Big(\frac{|\ln\big(\xi\prod_{n=1}^N|u(2n-1)|^2\big)|}{\rho^{N+1}}\ln^{N-1}\rho\Big)\\ =& \frac{(2N+1)!!2^{(N+1)}}{(N+1)!\det(\mathbf{U}_N^H\mathbf{U}_N)}\rho^{-(N+1)}\ln^N\rho\\ &+ O\Big(\frac{|\ln\det(\mathbf{U}_N^H\mathbf{U}_N)|}{\rho^{N+1}}\ln^{N-1}\rho\Big).\end{aligned}$$

□

APPENDIX III
PROOF FOR THE OPTIMAL PRECODER $\mathbf{F}_{OC}$

Let the set of 4-QAM signals be $\mathcal{S} = \{c + dj : c, d = \pm1\}$, and let $\mathcal{S}_d = \{a + bj : a, b = 0, \pm2\}$ be the possible set of difference of elements in $\mathcal{S}$. The error vector after precoded by $\mathbf{F}_c$ in (31) is

$$\begin{pmatrix} u(1) \\ u(2)\end{pmatrix} = \mathbf{F}_c\begin{pmatrix} e(1) \\ e(2)\end{pmatrix} = \mathbf{F}_c\mathbf{e}. \quad (55)$$

Then, the coding gain is given by $C = \frac{1}{6}|u(1)|^4 = \frac{1}{6}|e(1)\cos\theta + e(2)e^{j\phi}\sin\theta|^4,\ \theta\in[0,2\pi],\ \phi\in[0,\pi]$. Now, let $G(\mathbf{e},\theta,\phi) = |e(1)\cos\theta + e(2)e^{j\phi}\sin\theta|^2$. Then, the coding gain criterion can be fulfilled by seeking the optimal values of θ and ϕ to maximize the minimum of $G(\mathbf{e},\theta,\phi)$ over all non-zero error vectors. First we show that it is sufficient to consider $\phi,\theta\in[0,\frac{\pi}{4}]$. Let

$$G_o = \max_{\substack{\theta\in[0,2\pi]\\ \phi\in[0,\pi]}}\min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi).$$

Then, we have

$$G_o = \max\{\max_{\substack{\theta\in[0,\pi]\\ \phi\in[0,\pi]}}\min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi),\ \max_{\substack{\theta\in[\pi,2\pi]\\ \phi\in[0,\pi]}}\min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi)\} \quad (56)$$

For $\mathbf{e}\in\mathcal{S}_d^2$, and $\mathbf{e}\neq\mathbf{0}$, we have

$$\begin{aligned}&|e(1)\cos\tilde\theta + e(2)e^{j\phi}\sin\tilde\theta|^2_{\substack{\tilde\theta\in[\pi,2\pi]\\ \phi\in[0,\pi]}}\\ &= |e(1)\cos(2\pi-\theta) + e(2)e^{j\phi}\sin(2\pi-\theta)|^2_{\substack{\theta\in[0,\pi]\\ \phi\in[0,\pi]}}\\ &= |e(1)\cos\theta - e(2)e^{j\phi}\sin\theta|^2_{\substack{\theta\in[0,\pi]\\ \phi\in[0,\pi]}}.\end{aligned}$$

We note that when $[e(1)\quad e(2)]^T$ covers all possible values in $\mathcal{S}_d^2$, $[e(1)\quad -e(2)]^T$ also covers the same values. Therefore we have,

$$\max_{\substack{\theta\in[0,\pi]\\ \phi\in[0,\pi]}}\min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi) = \max_{\substack{\theta\in[\pi,2\pi]\\ \phi\in[0,\pi]}}\min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi). \quad (57)$$

From (56), the feasible set $\theta\in[0,2\pi]$ can be reduced to $\theta\in[0,\pi]$; i.e.;

$$G_o = \max_{\substack{\theta\in[0,\pi]\\ \phi\in[0,\pi]}}\min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi).$$

Similarly, the feasible set $\theta,\phi\in[0,\pi]$ can be further reduced to $\theta,\phi\in[0,\frac{\pi}{4}]$. The reason is as follows:

$$\begin{aligned}G_o &= \max\{\max_{\substack{\theta\in[0,\frac{\pi}{2}]\\ \phi\in[0,\pi]}}\min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi)\} && (58)\\ &= \max\{\max_{\substack{\theta\in[0,\frac{\pi}{2}]\\ \phi\in[0,\frac{\pi}{2}]}}\min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi)\} && (59)\\ &= \max\{\max_{\substack{\theta\in[0,\frac{\pi}{4}]\\ \phi\in[0,\frac{\pi}{4}]}}\min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi)\}. && (60)\end{aligned}$$

Eq. (58) holds because for $\mathbf{e}\in\mathcal{S}_d^2,\ \mathbf{e}\neq\mathbf{0}$,

$$\begin{aligned}&|e(1)\cos\tilde\theta + e(2)e^{j\phi}\sin\tilde\theta|^2_{\substack{\tilde\theta\in[\frac{\pi}{2},\pi]\\ \phi\in[0,\pi]}}\\ &= |-e(1)\cos\theta + e(2)e^{j\phi}\sin\theta|^2_{\substack{\theta\in[0,\frac{\pi}{2}]\\ \phi\in[0,\pi]}} \quad (61)\end{aligned}$$

and (59) holds because

$$\begin{aligned}&|e(1)\cos\theta + e(2)e^{j\tilde\phi}\sin\theta|^2_{\substack{\theta\in[0,\frac{\pi}{2}]\\ \tilde\phi\in[\frac{\pi}{2},\pi]}}\\ &= |e(1)\cos\theta + je(2)e^{j\phi}\sin\theta|^2_{\substack{\theta\in[0,\frac{\pi}{2}]\\ \phi\in[0,\frac{\pi}{2}]}} \quad (62)\end{aligned}$$

Eq. (60) holds because of the following equalities,

$$\begin{aligned}&|e(1)\cos\theta + e(2)e^{j\tilde\phi}\sin\theta|^2_{\substack{\theta\in[0,\frac{\pi}{2}]\\ \tilde\phi\in[\frac{\pi}{4},\frac{\pi}{2}]}}\\ &= |e(1)\cos\theta + e(2)e^{j(\pi/2-\phi)}\sin\theta|^2_{\substack{\theta\in[0,\frac{\pi}{2}]\\ \phi\in[0,\frac{\pi}{4}]}}\\ &= |e^*(1)\cos\theta - je^*(2)e^{j\phi}\sin\theta|^2_{\substack{\theta\in[0,\frac{\pi}{2}]\\ \phi\in[0,\frac{\pi}{4}]}} \quad (63)\\ &|e(1)\cos\tilde\theta + e(2)e^{j\phi}\sin\tilde\theta|^2_{\substack{\tilde\theta\in[\frac{\pi}{4},\frac{\pi}{2}]\\ \phi\in[0,\frac{\pi}{4}]}}\\ &= |e(1)\cos(\pi/2-\theta) + e(2)e^{j(\phi)}\sin(\pi/2-\theta)|^2_{\theta,\phi\in[0,\frac{\pi}{4}]}\\ &= |e^*(2)\cos\theta + e^*(1)e^{j\phi}\sin\theta|^2_{\theta,\phi\in[0,\frac{\pi}{4}]} \quad (64)\end{aligned}$$

where * denotes the conjugate of a complex quantity. Therefore, with the regions of search for θ and ϕ both reduced to

$[0, \frac{\pi}{4}]$, the optimization problem becomes

$$G_o = \max_{\substack{\theta\in[0,\frac{\pi}{4}]\\ \phi\in[0,\frac{\pi}{4}]}} \min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi). \tag{65}$$

Let us first examine $\min_{\mathbf{e}\in\mathcal{S}_d^2,\mathbf{e}\neq\mathbf{0}} G(\mathbf{e},\theta,\phi)$ for $\theta,\phi \in [0,\frac{\pi}{4}]$. We note that for $e(1)$ and $e(2)$ being from the set $\mathcal{S}_d$ of difference of 4-QAM signals, we have the following three cases:

Case 1: $|e(1)| = |e(2)| = r$, where, depending the magnitude of $e(1)$ and $e(2)$, r is either equal to 2 or $2\sqrt{2}$. The function $G(\mathbf{e},\theta,\phi)$, in this case, can be written as

$$G_1(\mathbf{e},\theta,\phi) = \begin{cases} r^2(1 \pm \sin 2\theta\cos\phi) & \text{if} \quad e(1) = \pm e(2) \\ r^2(1 \mp \sin 2\theta \sin\phi) & \text{if} \quad e(1) = \pm j e(2) \end{cases} \tag{66}$$

Case 2: $e(1)e(2) = 0$, then $G(\mathbf{e},\theta,\phi)$ can be written as

$$G_2(\mathbf{e},\theta,\phi) = \begin{cases} r^2\cos^2\theta & \text{if} \quad e(1) = 0 \\ r^2\sin^2\theta & \text{if} \quad e(2) = 0 \end{cases} \tag{67}$$

Case 3: $|e(1)| \neq |e(2)|,\ e(1)e(2) \neq 0$, then $G(\mathbf{e},\theta,\phi)$ can be written as

$$G_3(\mathbf{e},\theta,\phi) = \begin{cases} 4\big(1 \pm \sin^2\theta \pm \sin 2\theta(\cos\phi \pm \sin\phi)\big) \\ \quad \text{if} \quad |e(1)| = 2,\ |e(2)| = 2\sqrt{2} \\ 4\big(1 \pm \cos^2\theta \pm \sin 2\theta(\cos\phi \pm \sin\phi)\big) \\ \quad \text{if} \quad |e(1)| = 2\sqrt{2},\ |e(2)| = 2 \end{cases} \tag{68}$$

From (66), (67) and (68), the corresponding minimum coding gains over non-zero error vectors in Cases 1, 2 and 3, i.e., $\min_{\mathbf{e}\in\mathcal{S}_d^2,\mathbf{e}\neq\mathbf{0}} G_i(\mathbf{e},\theta,\phi),\ i = 1,2,3$ are respectively

$$\begin{aligned} g_1 &= \min_{\mathbf{e}\in\mathcal{S}_d^2,\mathbf{e}\neq\mathbf{0}} G_1(\mathbf{e},\theta,\phi) = 4(1 - \sin 2\theta\cos\phi) \\ g_2 &= \min_{\mathbf{e}\in\mathcal{S}_d^2,\mathbf{e}\neq\mathbf{0}} G_2(\mathbf{e},\theta,\phi) = 4\sin^2\theta \\ g_3 &= \min_{\mathbf{e}\in\mathcal{S}_d^2,\mathbf{e}\neq\mathbf{0}} G_3(\mathbf{e},\theta,\phi) = 4\big(1+\sin^2\theta - \sin 2\theta(\cos\phi+\sin\phi)\big). \end{aligned}$$

Then (65) becomes $G_o = \max_{\theta,\phi\in[0,\frac{\pi}{4}]} \min\{g_1,g_2,g_3\}$. Since g_2 is not a function of ϕ, we then have

$$G_o = \max_{\theta\in[0,\frac{\pi}{4}]} \min\{g_2, \max_{\phi\in[0,\frac{\pi}{4}]} \min\{g_1,g_3\}\}. \tag{69}$$

Furthermore, since g_1 is an increasing function of ϕ and g_3 is a decreasing function of ϕ for $\phi \in [0,\frac{\pi}{4}]$, then the second term in (69), $\max_{\phi\in[0,\frac{\pi}{4}]} \min\{g_1,g_3\}$ is achieved at the point where $g_1 = g_3$. The condition for $g_1 = g_3$ is given by

$$\sin\theta = 2\cos\theta\sin\phi. \tag{70}$$

Therefore, we have

$$\begin{aligned} \max_{\phi\in[0,\frac{\pi}{4}]} \min\{g_1,g_3\} &= g_1|_{\sin\theta=2\cos\theta\sin\phi} && (71) \\ &= 4(1 - \sin\theta\sqrt{4\cos^2\theta - \sin^2\theta}) \\ &= \tilde{g}_1. && (72) \end{aligned}$$

Now (69) becomes,

$$G_o = \max_{\theta\in[0,\frac{\pi}{4}]} \min\{g_2,\tilde{g}_1\}. \tag{73}$$

We observe that g_2 is an increasing function of $\theta \in [0,\frac{\pi}{4}]$ and that $\tilde{g}_1$ is decreasing if $\theta \in [0,\theta_1]$ and increasing if $\theta \in (\theta_1,\frac{\pi}{4}]$, where $\theta_1 = \sin^{-1}\sqrt{\frac{2}{5}}$. Therefore for $\theta \in [0,\theta_1]$ the value of $\max_{\theta\in[0,\theta_1]} \min\{g_2,\tilde{g}_1\}$ is obtained when $g_2 = \tilde{g}_1$. The solution to make $g_2 = \tilde{g}_1,\ \theta \in [0,\theta_1]$ hold is

$$\theta_o = \sin^{-1}\sqrt{\frac{3-\sqrt{3}}{6}}. \tag{74}$$

The corresponding value of g_2 at $\theta = \theta_o$ is

$$g_2|_{\theta=\theta_o} = \tilde{g}_1|_{\theta=\theta_o} = 2(1 - \frac{1}{\sqrt{3}}). \tag{75}$$

Furthermore, since $\tilde{g}_1|_{\theta=\theta_o}|_{\theta=\theta_o} > \tilde{g}_1|_{\theta=\frac{\pi}{4}}$, then the optimal value of G_o in (73) for $\theta \in [0,\frac{\pi}{4}]$ is $G_o = \tilde{g}_1|_{\theta=\theta_o} = 2(1 - \frac{1}{\sqrt{3}})$. Inserting (74) into (70), we have the optimal value of ϕ given by

$$\sin^2\phi_o = \frac{2-\sqrt{3}}{4}, \ \text{or} \ \ \phi_o = \frac{\pi}{12}, \tag{76}$$

and the maximum of minimum coding gain is $C_o = \frac{1}{6}G_o^2 = \frac{2}{3}(1 - \frac{1}{\sqrt{3}})^2 = 0.3573$.

Next let us look at the case that $\mathbf{F}_{\text{c}}$ takes the form $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{j\phi} \\ 1 & -e^{j\phi} \end{pmatrix} = \mathbf{F}_{\text{st}}$ as described in the remark after Theorem 3. Then

$$G_{\text{ost}} = \max_{\phi\in[0,\frac{\pi}{4}]} \min_{\substack{\mathbf{e}\in\mathcal{S}_d^2\\ \mathbf{e}\neq\mathbf{0}}} G(\mathbf{e},\theta,\phi)|_{\theta=\frac{\pi}{4}}. \tag{77}$$

This is a special case of $\mathbf{F}_{\text{c}}$ when θ is fixed at $\frac{\pi}{4}$. The optimal ϕ in this case is given by the condition in (70) with $\theta = \frac{\pi}{4}$, i.e., $\sin\theta = 2\sin\phi\cos\theta|_{\theta=\frac{\pi}{4}}$. Solving this equation and inserting the solution to (77), we obtain the optimal ϕ and G_{ost} such that,

$$\phi_{\text{ost}} = \frac{\pi}{6},\ G_{\text{ost}} = (4 - 2\sqrt{3}).$$

The maximum of the worst coding gain over non-zero error vectors in this case is given by $C_{\text{ost}} = G_{\text{ost}}^2/6 = \frac{2}{3}(2-\sqrt{3})^2 = 0.0479$ (the worst case occurs when $\mathbf{e} = [2+2j \quad -2]^T$). Furthermore, if ϕ is assigned the value of $\frac{\pi}{4}$ as in the MIMO space-time code design, then minimum of $|u(1)|^2$ happens when $\mathbf{e} = [2+2j \quad -2]^T$, in this case $G = (6 - 4\sqrt{2})$, the worst coding gain is then $\frac{1}{6}(6-4\sqrt{2})^2 = 0.0196$. □

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He is now with the Department of Electrical and Computer Engineering at McMaster University, Hamilton, Ont., Canada. His research interests include multirate filterbanks, wavelet and multiwavelet transforms and their applications, number theory transform and fast algorithm. His current research focuses on multiscale wavelet transform, random matrices, precoding and space-time coding for block transmissions and multicarrier and wideband wireless communication systems.

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Kon Max Wong was born in Macau. He received his BSc(Eng), DIC, PhD, and DSc(Eng) degrees, all in electrical engineering, from the University of London, England, in 1969, 1972, 1974 and 1995, respectively. He started working at the Transmission Division of Plessey Telecommunications Research Ltd., England, in 1969. In October 1970 he was on leave from Plessey pursuing postgraduate studies and research at Imperial College of Science & Technology, London. In 1972, he rejoined Plessey as a research engineer and worked on digital signal processing and signal transmission. In 1976, he joined the Department of Electrical Engineering at the Technical University of Nova Scotia, Canada, and in 1981, moved to McMaster University, Hamilton, Canada, where he has been a Professor since 1985 and served as Chairman of the Department of Electrical and Computer Engineering in 1986-87, and 1988-94. Professor Wong was on leave as Visiting Professor at the Department of Electronic Engineering of the Chinese University of Hong Kong from 1997 to1999. At present, he holds the NSERC-Mitel Professorship of Signal Processing and is the Director of the Communication Technology Research Centre at McMaster University. He is also serving another five-year term as Chair of the Department. His research interest is in signal processing and communication theory and has published over 200 papers in the area.

Professor Wong was the recipient of the IEE Overseas Premium for the best paper in 1989, and is a Fellow of IEEE, a Fellow of the Institution of Electrical Engineers, a Fellow of the Royal Statistical Society, and a Fellow of the Institute of Physics. He was an Associate Editor of the IEEE Transaction on Signal Processing, 1996-98 and served as Chair of the Sensor Array and Multi-channel Signal Processing Technical Committee of the Signal Processing Society in 2002-04. Professor Wong was the recipient of a medal presented by the International Biographical Centre, Cambridge, England, for his“outstanding contributions to the research and education in signal processing” in May 2000, and was honoured with the inclusion of his biography in the two books: Outstanding People of the 20th Century and 2000 Outstanding Intellectuals of the 20th Century published by IBC to celebrate the arrival of the new millennium. He is also the co-author of the paper that received the IEEE Signal Processing Society “Best Young Author” award of 2006.