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The analysis of 6-component measurements of a random electromagnetic wave field in a magnetoplasma – II. The integration kernels

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Summary. General expressions are derived for the kernels of the set of integral equations that relates the spectral matrix of the six components of a random electromagnetic wave field in a magnetoplasma to the wave distribution function for the field. The dependence of the kernels on wave-normal direction is examined, with particular reference to the propagation of very low-frequency waves in the whistler mode.

1 Introduction

This is the second paper in a series, the general subject of which is how to interpret a set of simultaneous measurements of the three electric and the three magnetic components of a random electromagnetic wave field in a magnetoplasma.

In two short review papers that preceded this series (Storey 1971; Storey & Lefeuvre 1974), we pointed out that a linear random field in a homogeneous magnetoplasma could be considered as the sum of fields due to a continuum of elementary plane waves, of different frequencies, propagated in different directions, without any mutual phase coherence. Such a field can only be described statistically. We chose to characterize it by a function that specifies the distribution of wave energy density with respect to frequency and to wave-normal direction; this we named the *wave distribution function* (WDF). The second of the two papers referenced above forms an introduction to the present series.

In the first paper of this series (Storey & Lefeuvre 1979), henceforth referred to as Paper I, we considered what we called the *direct problem.* Knowing the distribution function for the waves together with the values of the characteristic parameters of the plasma, we sought to determine the statistical properties of the six field components. These properties are described by the 36 auto-covariance and cross-covariance functions of the six components, or by their auto-spectra and cross-spectra. We argued that it is preferable to work with the 36 spectra, which we arranged for convenience in a 6×6 *spectral matrix* **S**. Then we showed that the elements of this matrix are related to the WDF by the following equation:

$$S_{ij}(\omega) = \frac{\pi}{2} \sum_{m} \oiint a_{ijm}(\omega, \varkappa) F_m(\omega, \varkappa)\, d\sigma. \qquad (1)$$

On the left-side, ω is the wave angular frequency, while the subscripts i and j refer to the six field components. $S_{ij}(\omega)$ is either the mean auto-power spectrum of a single field component (if $i = j$), or the mean cross-power spectrum of two components (if $i \neq j$).

On the right-side, the coefficients $a_{ijm}(\omega, \boldsymbol{\varkappa})$ are related to the corresponding spectra for an elementary plane wave in the magneto-ionic mode m (ordinary or extraordinary), propagated in the direction of the unit vector $\boldsymbol{\varkappa}$; we shall also specify this direction by a polar angle θ and an azimuthal angle ϕ, using spherical polar coordinates. $F_m(\omega, \boldsymbol{\varkappa})$ is the distribution function for waves in the mode m, while $d\sigma$ is an element of solid angle centred on the direction of $\boldsymbol{\varkappa}$.

Since each of the two subscripts ranges from 1 to 6, equation (1) actually stands for 36 different equations. This set of equations represents the solution of the direct problem, because it enables the spectral matrix to be predicted when the WDF is given.

For this purpose, however, it is necessary to know the functions $a_{ijm}(\omega, \boldsymbol{\varkappa})$, which are the *kernels* of this set of integral equations. There are 72 such functions, 36 for each of the two modes of propagation, though it will appear later that they are not all mutually independent. The purpose of the present paper is to derive algebraic expressions for these integration kernels, as explicit functions of the frequency ω, of the angles θ and ϕ, and of the characteristic parameters of the medium.

The basic formula for the kernels is equation (18) of Paper I, namely

$$a_{ijm}(\omega, \theta, \phi) = \frac{e_i e_j^*}{\rho} \tag{2}$$

where e_i and e_j are the complex amplitudes of the ith and jth components of the generalized electric field of an elementary plane wave in the mode m and characterized by the variables ω, θ and ϕ, and where ρ is the wave energy density. The SI unit for the kernels is the metre/farad. This formula applies only to real values of ω, θ and ϕ at which the elementary waves in the mode m are propagated freely, i.e. with purely real wavenumbers: elsewhere, the kernels are defined to be zero.

In this definition of the kernels $a_{ijm}(\omega, \theta, \phi)$, the field components in question are measured in a rectangular coordinate system $Oxyz$, with the axis Oz parallel to the steady magnetic field $\mathbf{B}_0$ (see Fig. 1 of Paper I). This type of system is quite convenient for practical data analysis. However, for the theoretical derivation of the kernels, it is simpler to work with the *complex principal axis coordinate system* (Westfold 1949), also known loosely as the *rotating* or *polarized* coordinate system. Vector components measured in this system will be called *circular components*. Use of this system simplifies the algebra, because it has gyrotropic symmetry, like the medium (*not* uniaxial symmetry, as was stated wrongly in Paper I). The kernels defined in terms of the circular field components will be denoted $b_{ijm}(\omega, \theta, \phi)$.

The six kernels a_{ii} (or b_{ii}), which are related to the auto-spectra of the six field components, will be referred to as *auto-kernels*. Similarly, the 30 kernels a_{ij} (or b_{ij}) with $i \neq j$, which are related to the cross-spectra of the field components, will be called *cross-kernels*.

In Section 2 of this paper, the definition of the a_{ij} is recapitulated, that of the b_{ij} is given in detail, and it is shown how these two sets of functions are related. Also various new symbols are defined for later use. So many symbols are needed that it has been impossible to avoid some duplication, nor some conflict with previous usage; warning is given of such cases.

The derivation of the general expressions for the kernels involves eliminating the complex amplitudes that appear in their definitions. For this purpose, it is helpful to have expressions for the ratios of the complex amplitudes of all possible pairs of field components. The ratios for pairs of components of the same type (electric or magnetic) are the *polarization ratios*, while those for mixed pairs are the *wave admittances* (or *wave impedances*). Therefore, as a preliminary step towards the derivation of the b_{ij}, expressions for these complex amplitude ratios in the complex principal axis coordinate system are derived in Section 3.

Expressions for the phase and group refractive indices are presented in Section 4, and for the wave energy density in Section 5. Then, in Section 6, these and the preceding results are combined to yield the general expressions for the integration kernels b_{ij}, from which the a_{ij} can be derived.

The property of the kernels that interests us the most is the way in which they vary as functions of θ and ϕ at fixed ω. The reason is that, when we analyse multi-component wave data to determine $F_m(\omega, \theta, \phi)$, we usually begin by dividing the frequency spectrum up into a number of narrow bands (see Section 2 of Paper I). Then, within any one of these bands (centred on the frequency ω_0, say), we try to estimate the functions $F_m(\omega_0, \theta, \phi)$ for the two wave modes. Only if the directional variations of the kernels at fixed frequency are substantial, and are different for the different kernels, can we hope to determine the directional variation of the WDF in detail, i.e. to obtain good directional resolution. In this respect, the b_{ij} behave more simply than the a_{ij} do, though the two sets of kernels are equivalent. Therefore our study of this question is limited to the b_{ij}. They vary with ϕ in simple ways which are described at the end of Section 6. However, their dependence on θ is more complicated, and is sensitive to the relative values of the wave frequency and of the characteristic frequencies of the plasma. As a result, it is too diverse to be discussed here systematically: only its symmetry is described in Section 6.

Nevertheless, this subject of the dependence of the kernels on the angle θ is pursued in Section 7, the discussion being limited to the whistler mode of propagation, and, within that mode, to the domain of validity of a simple approximate formula for the phase refractive index. We derive the corresponding approximations for the kernels. The whistler mode was chosen for this study because natural random wave fields, suitable for analysis by our methods, often occur in this mode, for instance in the Earth's magnetosphere.

Finally, the conclusions of the paper are presented in Section 8. Like those of Paper I, they rest on the assumptions that the plasma is cold and collisionless.

2 Symbols and definitions

First we recall some symbols and definitions introduced in Paper I. From the axial components of the electric field **E** and of the magnetic field **H**, measured in the coordinate system of Fig. 1 of Paper I, we defined a six-component generalized electric field vector as follows:

$$\left.\begin{array}{lll}\mathscr{E}_1 = E_x & \mathscr{E}_2 = E_y & \mathscr{E}_3 = E_z \\ \mathscr{E}_4 = Z_0 H_x & \mathscr{E}_5 = Z_0 H_y & \mathscr{E}_6 = Z_0 H_z\end{array}\right\} \quad (3)$$

Z_0 is the wave impedance of free space. Subject to our assumptions, an arbitrary wave field $\mathscr{E}(t, \mathbf{r})$, varying with time t and with the position vector $\mathbf{r}$, can be regarded as the sum of a finite or infinite number of elementary monochromatic plane waves, each of which is a characteristic wave, i.e. a solution of the dispersion equation for the medium. For any one such elementary wave, the usual (three-component) electric and magnetic fields are of the form:

$$\left.\begin{array}{l}\mathbf{E}(t, \mathbf{r}) = \mathrm{Re}\,\{\mathbf{e} \exp\,[i(\omega t - \mathbf{k}\cdot\mathbf{r})]\} \\ \mathbf{H}(t, \mathbf{r}) = \mathrm{Re}\,\{\mathbf{h} \exp\,[i(\omega t - \mathbf{k}\cdot\mathbf{r})]\}\end{array}\right\} \quad (4)$$

$\mathbf{e} \equiv (e_x, e_y, e_z)$ and $\mathbf{h} \equiv (h_x, h_y, h_z)$ are the complex amplitude vectors, while $\mathbf{k}$ is the wave-number vector, and the symbol Re denotes the real part. In accordance with equation (10) of Paper I, the corresponding generalized electric field vector will be written

$$\boldsymbol{\epsilon}(t, \mathbf{r}) = \mathrm{Re}\,\{\mathbf{e} \exp [i(\omega t - \mathbf{k} \cdot \mathbf{r})]\} \tag{5}$$

where $\mathbf{e} \equiv (e_1, e_2, e_3, e_4, e_5, e_6)$ is now a six-component complex amplitude vector:

$$e_{1,2,3} = e_{x,y,z} \qquad e_{4,5,6} = Z_0 h_{x,y,z}. \tag{6}$$

It is hoped that, with this explanation, the ambiguous use of the symbol e in equations (4) and (5) will not cause confusion. In fact we shall use it infrequently, and only for the complex amplitude of the usual (three-component) electric field vector, as in equation (4). Greater use will be made of the axial components of the two vectors to which it refers. Numeric subscripts are used for the six components of the complex amplitude of the generalized electric field, and alphabetic subscripts for the three components of the complex amplitude of the usual electric field. Likewise the symbol h with an alphabetic subscript denotes one of the three components of the complex amplitude vector of the usual magnetic field.

In Sections 3–5 we shall need to manipulate these three-component complex amplitude vectors in the complex principal axis coordinate system mentioned in the Introduction. The definition of this system is rather long, so it has been relegated to the Appendix. First, we use it to define circular components for the electric and magnetic fields individually: $\mathbf{E} \equiv (E_\mathrm{R}, E_\mathrm{L}, E_\mathrm{P})$ and $\mathbf{H} \equiv (H_\mathrm{R}, H_\mathrm{L}, H_\mathrm{P})$. In the principal axis coordinates, the generalized electric field vector will be called $\mathscr{F}$. By analogy with equation (3), its six components are defined as follows:

$$\mathscr{F}_{1,2,3} = E_{\mathrm{R,L,P}} \qquad \mathscr{F}_{4,5,6} = Z_0 H_{\mathrm{R,L,P}}. \tag{7}$$

An arbitrarily varying field $\mathscr{F}(t, \mathbf{r})$ can be Fourier analysed into a sum of monochromatic elementary waves. Let the complex amplitudes of the electric and magnetic vectors for one such wave be $\mathbf{e} \equiv (e_\mathrm{R}, e_\mathrm{L}, e_\mathrm{P})$ and $\mathbf{h} \equiv (h_\mathrm{R}, h_\mathrm{L}, h_\mathrm{P})$. From equation (A11) of the Appendix, the relations between the components of each of these vectors in the rectangular and in the complex principal axis coordinate systems are

$$\left.\begin{array}{lll} e_\mathrm{R} = e_x + ie_y & e_\mathrm{L} = e_x - ie_y & e_\mathrm{P} = e_z \\ h_\mathrm{R} = h_x + ih_y & h_\mathrm{L} = h_x - ih_y & h_\mathrm{P} = h_z. \end{array}\right\} \tag{8}$$

Then, by analogy with equation (6), we define a six-component complex amplitude vector $\mathbf{f}$ for the generalized electric field of this elementary wave as follows.

$$f_{1,2,3} = e_{\mathrm{R,L,P}} \qquad f_{4,5,6} = Z_0 h_{\mathrm{R,L,P}}. \tag{9}$$

The symbols f_i and f_j, with the subscripts running from 1 to 6, will stand for any of these components. (Note: the symbol f was used in another sense in Section 2 of paper I.)

Next, by analogy with equation (2), we set

$$b_{ijm}(\omega, \theta, \phi) \equiv \frac{f_i f_j^*}{\rho}. \tag{10}$$

On the left-side, the variables θ and ϕ specify the direction of propagation of the elementary wave, i.e. the direction of $\mathbf{k}$. On the right-side, ρ is the wave energy density, which of course is independent of the choice of coordinates. The quantities defined by equation (10) will also be written as $b_{ijm}(\omega, \varkappa)$. Pursuing the analogy with the a_{ijm}, the b_{ijm} are defined to be non-zero and given by equation (10) only at values of the variables ω and $\varkappa$ corresponding to freely propagated waves: they are zero when the waves are evanescent.

By reasoning as in Section 4 of Paper I, we can show that the spectral matrix of the random field $\mathscr{F}(t, \mathbf{r})$ is related to the wave distribution function $F_m(\omega, \varkappa)$ by an equation similar to (1), but with the quantities $a_{ijm}(\omega, \varkappa)$ replaced by the $b_{ijm}(\omega, \varkappa)$. Thus the latter are the integration kernels in the complex principal axis coordinate system.

We shall need to know the relationship between the two sets of kernels, so that we can get the a_{ijm} once the b_{ijm} have been derived. First, by combining the equations (6), (8) and (9), we note that the following relations hold between the six-component complex amplitude vectors in the two coordinate systems:

$$e_1 = \frac{1}{2}(f_1 + f_2) \quad e_2 = -\frac{i}{2}(f_1 - f_2) \quad e_3 = f_3$$
$$e_4 = \frac{1}{2}(f_4 + f_5) \quad e_5 = -\frac{i}{2}(f_4 - f_5) \quad e_6 = f_6. \tag{11}$$

Substituting these expressions into equation (2), and using the definitions (9), we obtain the following results:

$$a_{11} = \frac{1}{4}(b_{11} + b_{12} + b_{21} + b_{22}) \tag{12a}$$

$$a_{12} = \frac{i}{4}(b_{11} - b_{12} + b_{21} - b_{22}) \tag{12b}$$

$$a_{13} = \frac{1}{2}(b_{13} + b_{23}) \tag{12c}$$

$$a_{22} = \frac{1}{4}(b_{11} - b_{12} - b_{21} + b_{22}) \tag{12d}$$

$$a_{23} = \frac{i}{2}(b_{23} - b_{13}) \tag{12e}$$

$$a_{33} = b_{33}. \tag{12f}$$

The three other kernels with i and $j \leq 3$ can be found by using the relation $a_{ji} = a_{ij}^*$. From this set of nine kernels, the remaining 27 with i and/or $j \geq 4$ can be obtained by adding 3 to the first and/or second subscript of every term in each expression. The results expressed by equations (12a)–(12f) are independent of the wave mode m.

3 Complex amplitude ratios

For the derivation of the kernels, we require the set of ratios of the complex amplitudes of the three electric and three magnetic components of an elementary plane wave. To get them, we start from the two Maxwell equations that relate the electric field $\mathbf{E}$ to the magnetic field $\mathbf{H}$. When applied to the monochromatic plane wave defined by equation (4), they yield the following relations between the complex amplitudes:

$$\mathbf{k} \times \mathbf{e} = \omega\mu_0\mathbf{h}, \tag{13}$$

$$\mathbf{k} \times \mathbf{h} = -\omega\epsilon_0\boldsymbol{\epsilon}\mathbf{e}. \tag{14}$$

μ_0 is the magnetic permeability and ϵ_0 the electric permittivity of free space, while $\boldsymbol{\epsilon}$ is the dielectric tensor of the plasma.

For a cold magnetoplasma, the dielectric tensor is independent of the wavenumber vector **k**: it depends only on the angular frequency ω, and on the plasma parameters which we now define. In this we follow Stix (1962), with some minor differences: in particular, we suppose that the plasma contains only electrons and positive ions, and that the latter, of which several different kinds may be present, are all singly charged. Denoting the various species of charged particle – including the electrons – by the subscript s, let n_s be the number density for each species, m_s the mass per particle, and q_s the charge per particle. For each species, we now define two characteristic frequencies: the angular plasma frequency Π_s, such that

$$\Pi_s^2 = \frac{n_s q_s^2}{\epsilon_0 m_s} \tag{15}$$

and the angular gyrofrequency

$$\Omega_s = -\frac{q_s B_0}{m_s} \tag{16}$$

where B_0 is the magnitude of the induction vector of the steady magnetic field. The minus sign has been put into this definition so as to give Ω_s the sign of the natural sense of gyration of the particles around the field (e.g. positive for the electrons, which gyrate in a right-handed sense). Lastly, we define three dimensionless quantities:

$$R = 1 - \sum_s \frac{\Pi_s^2}{\omega(\omega - \Omega_s)}, \tag{17a}$$

$$L = 1 - \sum_s \frac{\Pi_s^2}{\omega(\omega + \Omega_s)}, \tag{17b}$$

$$P = 1 - \sum_s \frac{\Pi_s^2}{\omega^2}. \tag{17c}$$

These quantities are identical with those that Stix denotes by the same symbols. In what follows, they represent the properties of the plasma.

The expression for the dielectric tensor is simplest in the complex principal axis coordinate system defined in Section 2, where it is purely diagonal:

$$\boldsymbol{\epsilon} = \begin{vmatrix} R & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & P \end{vmatrix}. \tag{18}$$

Therefore we shall use this system in deriving the complex amplitude ratios.

First, let us derive the three ratios that involve only the electric field components. We begin by eliminating **h** between equations (13) and (14):

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{e}) + \omega^2 \mu_0 \epsilon_0 \boldsymbol{\epsilon} \mathbf{e} = 0. \tag{19}$$

We now introduce the refractive index vector

$$\mathbf{n} \equiv c\mathbf{k}/\omega \equiv n\boldsymbol{\varkappa}. \tag{20}$$

where $n = ck/\omega$ is the phase refractive index, with $c = (\mu_0 \epsilon_0)^{-1/2}$ the speed of light. In terms of **n**, equation (19) becomes

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{e}) + \boldsymbol{\epsilon} \mathbf{e} = 0. \tag{21}$$

The next step is to split this equation into its axial components in the rectangular coordinate system, using the fact that the components of the unit vector $\boldsymbol{\kappa}$, which appears in equation (20), are

$$\kappa_x = \sin\theta\cos\phi, \qquad \kappa_y = \sin\theta\sin\phi, \qquad \kappa_z = \cos\theta. \tag{22}$$

Then we go over to the complex principal axis coordinate system, using the following relations from Section 2:

$$e_x = \frac{1}{2}(e_R + e_L), \qquad e_y = -\frac{i}{2}(e_R - e_L), \qquad e_z = e_P. \tag{23}$$

The resulting set of three equations can be rearranged to yield the desired ratios:

$$\frac{e_R}{e_L} = \frac{L - n^2}{R - n^2}\exp(2i\phi), \tag{24a}$$

$$\frac{e_R}{e_P} = \frac{P - n^2}{R - n^2}\tan\theta\exp(i\phi), \tag{24b}$$

$$\frac{e_L}{e_P} = \frac{P - n^2}{L - n^2}\tan\theta\exp(-i\phi). \tag{24c}$$

These are the polarization ratios for the electric field.

Let us now derive some mixed ratios. Starting from the Maxwell equation (13) alone, we replace **k** by **n**, and split the equation into its components in the complex principal axis system as before. Remembering that $Z_0 = (\mu_0/\epsilon_0)^{1/2}$, we can write them as

$$Z_0 h_R = in[\cos\theta\, e_R - \sin\theta\, e_P\exp(i\phi)], \tag{25a}$$

$$Z_0 h_L = -in[\cos\theta\, e_L - \sin\theta\, e_P\exp(-i\phi)], \tag{25b}$$

$$Z_0 h_P = (i/2)n\sin\theta\,[e_L\exp(i\phi) - e_R\exp(-i\phi)]. \tag{25c}$$

Using equations (24a)–(24c), we obtain

$$Z_0\frac{h_R}{e_R} = in\frac{P - R}{P - n^2}\cos\theta, \tag{26a}$$

$$Z_0\frac{h_L}{e_L} = -in\frac{P - L}{P - n^2}\cos\theta. \tag{26b}$$

$$Z_0\frac{h_P}{e_P} = \frac{i}{2}n\frac{(P - n^2)(R - L)}{(R - n^2)(L - n^2)}\frac{\sin^2\theta}{\cos\theta}. \tag{26c}$$

The complex amplitude ratios on the left-sides of equations (26a)–(26c), which have been multiplied by Z_0 to make them dimensionless, are wave admittances. Note that if the medium is transparent (i.e. non-absorbing), in which case the refractive index n is real, then these ratios are purely imaginary; this fact will prove to be of importance later on.

All the remaining complex amplitude ratios may be found by combining equations (24a–c) and (26a–c). They consists of six more wave admittances, involving unlike components of the magnetic and electric fields, and the following three ratios involving the magnetic components alone:

$$\frac{h_R}{h_L} = -\frac{(P - R)(L - n^2)}{(P - L)(R - n^2)}\exp(2i\phi), \tag{27a}$$

$$\frac{h_R}{h_P} = 2 \frac{(P-R)(L-n^2)}{(P-n^2)(R-L)} \cot\theta \exp(i\phi), \tag{27b}$$

$$\frac{h_L}{h_P} = -2 \frac{(P-L)(R-n^2)}{(P-n^2)(R-L)} \cot\theta \exp(-i\phi). \tag{27c}$$

With these polarization ratios for the magnetic field, our list of complex amplitude ratios is complete. Using equation (9), they can be put into the form in which they will be required later, involving components of the generalized complex amplitude vector **f**.

4 Phase and group refractive indices

The expressions derived in the previous section apply equally to waves in either of the magneto-ionic modes. However, they involve the phase refractive index n, which is different for the two modes. If the vector equation (15) is split into its three axial components as described earlier, then the condition for their mutual consistency, namely that their determinant should vanish, leads to the following equation, the roots of which are the values of n^2 for the two modes (Stix 1962):

$$\tan^2\theta = -\frac{P(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)} \tag{28}$$

where S (which is not to be confused with the elements of the spectral matrix) is

$$S \equiv (R+L)/2. \tag{29}$$

Equation (28) is a quadratic with purely real roots. For a given θ and given plasma parameters, there may be two, one, or no roots with n^2 positive. Only such positive roots, which correspond to freely propagated modes, are to be considered here; negative roots correspond to evanescent modes. Moreover, for a given positive n^2, only the positive value of n will be considered, the negative value corresponding to a wave propagated in the opposite direction. On the questions of the nomenclature for the two wave modes, of their correspondence with the roots of the quadratic, and of the behaviour of these roots as functions of θ, the reader should consult the work of Stix (1962) or that of Allis, Buchsbaum & Bers (1963), and also the standard works on the magneto-ionic theory (Ratcliffe 1959; Budden 1961); additional information about ion effects at low frequencies has been given by Smith & Brice (1964).

When the roots of the quadratic equation for n^2 are inserted into the expressions of Section 3 for the complex amplitude ratios, and the latter are used for computing the kernels, difficulties may arise if n^2 goes to zero or to infinity. The former case is called a *cut-off*, and the latter a *resonance*. Computational difficulties arise at cut-offs and resonances because negative powers of n diverge in the first case and positive powers in the second. Either of these two difficulties can be avoided by writing the expressions for the kernels in forms containing only positive or only negative powers of n respectively, but clearly it is not possible to avoid both simultaneously. Therefore it is important to know which of the two is likely to be the more troublesome in practice.

For the reasons stated in Section 1, our prime concern is with the variations of the kernels as functions of wave-normal direction at a fixed frequency. Now, for a given plasma, the condition for a cut-off of a given type can be satisfied at only one frequency and is independent of direction, while the condition for a given type of resonance can be satisfied over a band of frequencies and depends on direction through the angle θ (Stix 1962). Hence the commonest difficulties are those associated with resonances, so priority should be given to forestalling them.

The condition for a plasma resonance ($n^2 = \infty$) is

$$\tan^2 \theta = -P/S. \tag{30}$$

There is a resonance whenever P and S have opposite signs. For a two-component plasma (i.e. one consisting only of electrons and of positive ions of a single type), this state of affairs exists in regions 3, 5, 7, 8, 10 and 13 of the CMA diagram (Stix 1962). In each of these regions, and for one or other of the magneto-ionic modes, a resonance occurs at the value of θ given by equation (30), which we shall call θ_r. It is said to be a *parallel resonance* if $\theta_r = 0$, or a *perpendicular resonance* if $\theta_r = \pi/2$, but evidently each of these *principal resonances* is the limit of an *oblique resonance* with $0 < \theta_r < \pi/2$. The family of all the wave normals having $\theta = \theta_r$ forms a cone around $\mathbf{B}_0$, which is known as the *resonance cone*; for a given plasma, the cone angle θ_r is a function of the wave frequency.

The resonance cone is the boundary between a range of θ in which $n^2 > 0$ and the kernels are non-zero, and a range where $n^2 < 0$ and the kernels are zero by definition. Fortunately, as θ tends to θ_r within the first of these two ranges, all the kernels tend either to zero or to finite limits; this result is proved in Section 6. However, care must be taken to write the general expressions for the kernels in forms involving only negative powers of n, so as not to have difficulties near the resonances.

Similar considerations apply to the group refractive index n_g, which enters into the general expressions for the kernels via the wave energy density ρ; see equation (10). This index is

$$n_g \equiv c/V_g = \left| \frac{\partial}{\partial \omega} (\omega n) \right| \tag{31}$$

where V_g is the modulus of the component of the group velocity in the direction of the wave normal, as defined by equation (3) in Paper I. (Deferring to the custom in other branches of physics, we now use the term *group velocity* to mean the velocity with which the group travels in the ray direction: in Paper I, we called this the *group-ray velocity*.) The derivative in equation (31) is always positive in the present context, so henceforth we omit the modulus sign. An equivalent alternative expression for n_g is

$$n_g = n^3 \nu_g \tag{32}$$

where

$$\nu_g \equiv \frac{1}{2} \omega^3 \frac{\partial}{\partial \omega} [-(\omega n)^{-2}]. \tag{33}$$

As the angle θ approaches the value θ_r for a resonance, ν_g remains finite while n_g tends to infinity like n^3.

5 Wave energy density

In a transparent medium, the energy density of an electromagnetic wave is

$$\rho = P_n/V_g \tag{34}$$

where P_n (not to be confused with Stix's parameter P) is the component of the time-averaged Poynting vector parallel to the wave normal, and

$$V_g = \left| \left(\frac{\partial \omega}{\partial k}\right)_\theta \right| = c/n_g. \tag{35}$$

The time-averaged Poynting vector is

$$\mathbf{P} = \tfrac{1}{4}\,(\mathbf{e}\times\mathbf{h}^* + \mathbf{e}^*\times\mathbf{h}). \tag{36}$$

Its component parallel to the wave normal is

$$P_{\mathrm{n}} = \boldsymbol{\varkappa}\cdot\mathbf{P} = \tfrac{1}{4}\,(N + N^*) \tag{37}$$

where

$$N = \boldsymbol{\varkappa}\cdot(\mathbf{e}\times\mathbf{h}^*). \tag{38}$$

With the help of equations (22) and (23), this quantity may be expressed in terms of the components of $\mathbf{e}$ and of $\mathbf{h}$ in the complex principal axis coordinate system, and thence in terms of the corresponding complex amplitude vector $\mathbf{f}$ for the generalized electric field:

$$N = \frac{i}{2Z_0}\Big\{\cos\theta\,[f_1 f_4^* - f_2 f_5^*] + \sin\theta\, f_6^*\,[f_2\exp(i\phi) - f_1\exp(-i\phi)] - \sin\theta\, f_3\,[f_4^*\exp(i\phi) - f_5^*\exp(-i\phi)]\Big\}. \tag{39}$$

Now all the different terms on the right-side of equation (39) involve different types of product of one component of $\mathbf{f}$ with the complex conjugate of another. For the derivation, in Section 6, of each of the kernels b_{ij}, it will be necessary to have an expression for ρ involving only the product $f_i f_j^*$, for reasons already mentioned in Section 1. By extracting the product $f_i f_j^*$ from all the different terms above, we obtain an expression of this type for N:

$$N = \frac{i}{2Z_0} K_{ij} f_i f_j^* \tag{40}$$

with

$$K_{ij} = \psi\cos\theta\,\frac{f_1 f_4^*}{f_i f_j^*} + \sin\theta\exp(i\phi)\left[\eta\,\frac{f_2 f_6^*}{f_i f_j^*} - \zeta\,\frac{f_3 f_4^*}{f_i f_j^*}\right] \tag{41}$$

where the functions ψ, η and ζ are

$$\psi = 1 - f_2 f_5^*/f_1 f_4^*, \tag{42}$$

$$\eta = 1 - (f_1/f_2)\exp(-2i\phi), \tag{43}$$

$$\zeta = 1 - (f_5^*/f_4^*)\exp(-2i\phi). \tag{44}$$

These three functions are independent of the variable subscripts i and j.

Finally, by inserting the expression (40) into equation (37), and the result into equation (34), we arrive at a general expression for the energy density ρ as a multiple of any one of the 36 products $f_i f_j^*$, namely

$$\rho = \frac{i}{8Z_0 V_{\mathrm{g}}}\left(K_{ij} - K_{ij}^*\,\frac{f_i^* f_j}{f_i f_j^*}\right) f_i f_j^*. \tag{45}$$

This is the required result.

6 The integration kernels

6.1 GENERAL EXPRESSIONS

By combining the above result for ρ with their definition (10), we get the following general expression for the kernels in the complex principal axis coordinate system:

$$b_{ij} = 8iZ_0V_g\left(K_{ij}^*\frac{f_i^*f_j}{f_if_j^*} - K_{ij}\right). \tag{46}$$

It involves only the wave variables ω, θ and ϕ, and the plasma parameters R, L and P.

We remind the reader that equation (46) applies only at values of these quantities for which $n^2 > 0$: if $n^2 < 0$, then $b_{ij} = 0$ by definition. In regions of the CMA diagram where an oblique resonance exists, n^2 changes sign as θ varies from 0 to $\pi/2$, the change occurring by passage through $\pm\infty$. Before computing the set of kernels required for a given application, it is important to check whether there is a resonance, for instance by seeing whether the parameters P and S have opposite signs. If so, then the next step should be to calculate the resonance angle θ_r using equation (30), since it bounds the range of θ over which the computations need to be made.

The derivation of the explicit expressions for the individual kernels is lengthy; but does not involve any special difficulty. Before quoting them, however, it is helpful to define a new quantity which we shall call ξ. It involves ν_g, as defined by equation (33), and two other parameters:

$$\lambda \equiv 2(R-P)\left[1 + \frac{(L-P)^2(1-Rn^{-2})^2}{(R-P)^2(1-Ln^{-2})^2}\right]\cos^2\theta, \tag{47}$$

$$\mu \equiv \frac{(R-L)^2(1-Pn^{-2})^2}{(R-P)(1-Ln^{-2})^2}\sin^2\theta. \tag{48}$$

In these terms, the necessary quantity is

$$\xi \equiv 8\,[\epsilon_0\nu_g(\lambda+\mu)]^{-1}. \tag{49}$$

It has the dimensions of ϵ_0^{-1}, which are the same as those of the kernels. The above definitions of λ, μ and ξ differ in sign, and by multipliers that are integral powers of n, from those used by Lefeuvre (1977).

We are now in a position to list the 36 kernels b_{ij}. In point of fact, it suffices to list the six kernels with $i = j$, and 15 of the kernels with $i \neq j$, since the remaining 15 are given by the relation $b_{ji} = b_{ij}^*$, which follows from their definition (10). Here are the 21 expressions:

$$b_{11} = \frac{(1-Pn^{-2})^2}{R-P}\,\xi, \tag{50a}$$

$$b_{12} = \frac{(1-Rn^{-2})(1-Pn^{-2})^2}{(1-Ln^{-2})(R-P)}\,\xi\exp(2i\phi), \tag{50b}$$

$$b_{13} = \frac{(1-Rn^{-2})(1-Pn^{-2})}{R-P}\,\xi\cot\theta\exp(i\phi), \tag{50c}$$

$$b_{14} = -in^{-1}(1-Pn^{-2})\,\xi\cos\theta, \tag{50d}$$

$$b_{15} = i\,\frac{n^{-1}(1-Rn^{-2})(L-P)(1-Pn^{-2})}{(1-Ln^{-2})(R-P)}\,\xi\cos\theta\exp(2i\phi), \tag{50e}$$

$$b_{16}=i\frac{n^{-1}(R-L)(1-Pn^{-2})^2}{2(R-P)(1-Ln^{-2})}\xi\sin\theta\exp(i\phi), \tag{50f}$$

$$b_{22}=\frac{(1-Pn^{-2})^2(1-Rn^{-2})^2}{(R-P)(1-Ln^{-2})^2}\xi, \tag{50g}$$

$$b_{23}=\frac{(1-Pn^{-2})(1-Rn^{-2})^2}{(R-P)(1-Ln^{-2})}\xi\cot\theta\exp(-i\phi), \tag{50h}$$

$$b_{24}=-i\frac{n^{-1}(1-Pn^{-2})(1-Rn^{-2})}{1-Ln^{-2}}\xi\cos\theta\exp(-2i\phi), \tag{50i}$$

$$b_{25}=i\frac{n^{-1}(1-Pn^{-2})(L-P)(1-Rn^{-2})^2}{(R-P)(1-Ln^{-2})^2}\xi\cos\theta, \tag{50j}$$

$$b_{26}=i\frac{n^{-1}(R-L)(1-Rn^{-2})(1-Pn^{-2})^2}{2(R-P)(1-Ln^{-2})^2}\xi\sin\theta\exp(-i\phi), \tag{50k}$$

$$b_{33}=\frac{(1-Rn^{-2})^2}{R-P}\xi\cot^2\theta, \tag{50l}$$

$$b_{34}=-in^{-1}(1-Rn^{-2})\xi\cos\theta\cot\theta\exp(-i\phi), \tag{50m}$$

$$b_{35}=i\frac{n^{-1}(L-P)(1-Rn^{-2})^2}{(R-P)(1-Ln^{-2})}\xi\cos\theta\cot\theta\exp(i\phi), \tag{50n}$$

$$b_{36}=i\frac{n^{-1}(R-L)(1-Rn^{-2})(1-Pn^{-2})}{2(R-P)(1-Ln^{-2})}\xi\cos\theta, \tag{50o}$$

$$b_{44}=n^{-2}(R-P)\xi\cos^2\theta, \tag{50p}$$

$$b_{45}=-\frac{n^{-2}(L-P)(1-Rn^{-2})}{1-Ln^{-2}}\xi\cos^2\theta\exp(2i\phi), \tag{50q}$$

$$b_{46}=-\frac{n^{-2}(R-L)(1-Pn^{-2})}{2(1-Ln^{-2})}\xi\sin\theta\cos\theta\exp(i\phi), \tag{50r}$$

$$b_{55}=\frac{n^{-2}(L-P)^2(1-Pn^{-2})^2}{(R-P)(1-Ln^{-2})^2}\xi\cos^2\theta, \tag{50s}$$

$$b_{56}=\frac{n^{-2}(L-P)(R-L)(1-Pn^{-2})(1-Rn^{-2})}{2(R-P)(1-Ln^{-2})^2}\xi\sin\theta\cos\theta\exp(-i\phi), \tag{50t}$$

$$b_{66}=\frac{n^{-2}(R-L)^2(1-Pn^{-2})^2}{4(R-P)(1-Ln^{-2})^2}\xi\sin^2\theta. \tag{50u}$$

With use of the appropriate values of n and of ξ, these results apply to both magneto-ionic modes. Their rather awkward algebraic forms, involving negative powers of n, have been adopted in order to avoid computational difficulties near the plasma resonances (see Section 4).

At a resonance, the kernels involving only electric field components tend to finite limits, those involving both electric and magnetic components tend to zero like n^{-1}, while those involving only magnetic components tend to zero like n^{-2}. This behaviour reflects the fact that an electromagnetic wave acquires an increasingly electrostatic character as a resonance is approached: its electric field becomes relatively strong, and is mainly directed parallel to the wave normal (Allis *et al.* 1963).

For waves having their normal directions extremely close to the resonance cone, the phase velocities are not large compared to the random thermal velocities of the plasma electrons. Consequently these waves do not conform to the predictions of cold-plasma theory: in particular, they suffer collisionless damping. The analysis developed in the present series of papers is founded on the assumption that the waves are undamped (see Paper I), so it is not altogether valid if an appreciable fraction of the energy of the random field is carried by resonant waves. Various natural plasma instabilities tend to produce such waves. If there is reason to suspect that this is happening, then our analysis must be applied with caution.

Returning now to the general expressions for the kernels, we might expect that the 36 kernels defined by equations (50a)–(50u) would involve altogether 36 different functions, namely the six real auto-kernels b_{ii} together with the 15 real and 15 imaginary parts of the cross-kernels b_{ij}, for instance with $i<j$ as above. However, a close inspection of these equations reveals that the cross-kernels b_{14}, b_{25} and b_{36} are purely imaginary. These are the kernels in which one component of **e** is associated with the same component of **h**; they are purely imaginary because the wave admittances given by equations (26a)–(26c) have this property, as we pointed out in Section 3. Thus the equations (50a)–(50u) define only 33 independent functions.

From these various expressions for the kernels in the complex principal axis system, the expressions appropriate to any other set of coordinates may be derived readily. For instance, the kernels a_{ij}, which belong to the rectangular system $Oxyz$, are obtained from the b_{ij} by means of equations (12a)–(12f). Since the a_{ij} are tensors of rank 2, the kernels in any other rectangular system can be got from them by a straightforward tensor transformation.

It remains for us to discuss the general properties of the kernels, and especially their variations with wave-normal direction at fixed frequency, which is of the greatest interest to us for the reasons given in Section 1. In this respect the b_{ij} behave more simply than the a_{ij} do, because every one of the b_{ij} is the product of a function of θ and a function of ϕ, which is not the case for the a_{ij}. Therefore our discussion of the properties of the kernels will concern only the b_{ij}; the ways in which they depend on θ and on ϕ will be discussed separately, beginning with their dependence on ϕ.

6.2 DEPENDENCE ON ϕ

This dependence is very simple, involving no other variables. An inspection of equations (50a)–(50u) reveals that the kernels fall into three categories, as follows:

– all the b_{ij} involving field components along the same axis (that is, with $i=j$ or $|i-j|=3$) are independent of ϕ;
– all the b_{ij} involving one component along the axis parallel to $\mathbf{B}_0$ (that is, with i or j equal to 3 or 6), and with the other component not along this axis, vary as $\exp(i\phi)$ or as $\exp(-i\phi)$;
– all the b_{ij} with $i \neq j$ but with neither i nor j equal to 3 or 6, vary as $\exp(2i\phi)$ or as $\exp(-2i\phi)$.

The first category comprises the six auto-kernels, which are purely real, together with the cross-kernels b_{14}, b_{25} and b_{36} which – as we noted in Section 6.1 above – are purely imaginary.

In the general expressions for the other cross-kernels, belonging to the second or third categories, the factors that multiply the terms exp $(\pm i\phi)$ or exp $(\pm 2i\phi)$ are either purely real or purely imaginary. They are purely real for the kernels involving either two electric components $(1 \leqslant i, j \leqslant 3)$ or two magnetic components $(4 \leqslant i, j \leqslant 6)$. They are purely imaginary for the kernels involving one electric and one magnetic component $(|i - j| \geqslant 3)$. Thus it is only in virtue of their ϕ-dependence that these kernels take complex values.

6.3 DEPENDENCE ON θ

This dependence is relatively complicated, since it involves the quantities R, L, P, n and ξ, all of which are functions of θ. Only one aspect of it is amenable to general discussion, namely its symmetry. The θ-dependence either has reflection symmetry about the angle $\pi/2$, meaning that $b_{ij}(\pi - \theta) = b_{ij}(\theta)$, or skew symmetry, in which case $b_{ij}(\pi - \theta) = -b_{ij}(\theta)$. The rules that govern the symmetry can be inferred from equations (50a)–(50u), or more readily from equations (56a)–(56u) in Section 7, where the presence of the factor s indicates that the kernel concerned is skew-symmetric. Knowing them, the dependence of the kernels on θ is specified fully when it is given in the range $0 \leqslant \theta \leqslant \pi/2$.

The details of the θ-dependence are governed by the relative values of the wave frequency and of the various characteristic frequencies of the plasma. They are discussed in Section 7, in the important special case of the whistler mode.

7 The whistler mode

The whistler mode is the right-polarized (ordinary) mode of propagation that exists at wave frequencies less than both the plasma frequency and the electron gyro-frequency (i.e. for a two-component plasma, in regions 7, 8, 11 and 13 of the CMA diagram). In the upper part of this frequency range only the electrons influence the properties of the waves, but in the lower part the ions are equally important; the division occurs at the lower hybrid resonance (LHR) frequency. In the band below the LHR frequency, whistler-mode waves are propagated freely in all directions in a two-component plasma, but in a plasma containing more than one type of positive ion their behaviour is relatively complicated (Smith & Brice 1964). Above the LHR frequency there is an oblique resonance, and propagation is possible only with $\theta < \theta_r$. In this upper band of frequencies, the whistler mode is the only one in which electromagnetic waves can be propagated: waves in the left-polarized (extraordinary) mode are evanescent.

We shall study the θ-dependence of the integration kernels for waves in the whistler mode, at frequencies in the upper band. For this purpose, we shall replace the exact expressions (50a)–(50u) for the kernels by simpler approximate expressions based on the quasi-longitudinal (QL) approximation (Stix 1962), the condition for which is

$$\Omega_e^2 \sin^4 \theta \ll 4\omega^2 (1 - \Pi_e^2/\omega^2)^2 \cos^2 \theta \tag{51}$$

where Ω_e is the angular electron gyrofrequency and Π_e is the angular electron plasma frequency. Moreover we shall assume that $\Pi_e \gg \Omega_e$, in which case the QL approximation for the phase refractive index becomes

$$n^2 \simeq \frac{B^2}{\Lambda(\chi - \Lambda)} \tag{52}$$

where, in the notation of Helliwell (1965),

$$\Lambda = \omega/\Omega_e, \qquad B = \Pi_e/\Omega_e. \tag{53}$$

Λ is a normalized wave frequency and B a normalized plasma frequency (*not* a magnetic induction). Additionally we have written

$$\chi = |\cos\theta|. \tag{54}$$

There is an oblique resonance at the angle

$$\theta_r = \cos^{-1}\Lambda. \tag{55}$$

Thus θ_r decreases with increasing frequency. With $B \gg 1$ as assumed, the inequality (51) is satisfied for all $\theta \leqslant \theta_r$, which is the range of θ where wave propagation is possible ($n^2 > 0$). In point of fact, for whistler-mode waves above the LHR frequency, these various assumptions and approximations hold reasonably well throughout much of the magnetosphere.

Here are the corresponding approximations for the kernels:

$$b_{11} = (2\epsilon_0^{-1})B^{-2}[\Lambda(1-\Lambda)^2(1+\chi)^2\chi^{-1}], \tag{56a}$$

$$b_{12} = (2\epsilon_0^{-1})B^{-2}[\Lambda(1-\Lambda^2)(1-\chi^2)\chi^{-1}]\exp(2i\phi), \tag{56b}$$

$$b_{13} = s(2\epsilon_0^{-1})B^{-2}[\Lambda^2(1-\Lambda)(1-\chi^2)^{1/2}(1+\chi)\chi^{-1}]\exp(i\phi), \tag{56c}$$

$$b_{14} = -is(2\epsilon_0^{-1})B^{-1}[\Lambda^{1/2}(1-\Lambda)(1+\chi)^2(\chi-\Lambda)^{1/2}\chi^{-1}], \tag{56d}$$

$$b_{15} = is(2\epsilon_0^{-1})B^{-1}[\Lambda^{1/2}(1-\Lambda)(1-\chi^2)(\chi-\Lambda)^{1/2}\chi^{-1}]\exp(2i\phi), \tag{56e}$$

$$b_{16} = i(2\epsilon_0^{-1})B^{-1}[\Lambda^{1/2}(1-\Lambda)(1-\chi^2)^{1/2}(1+\chi)(\chi-\Lambda)^{1/2}\chi^{-1}]\exp(i\phi), \tag{56f}$$

$$b_{22} = (2\epsilon_0^{-1})B^{-2}[\Lambda(1+\Lambda)^2(1-\chi)^2\chi^{-1}], \tag{56g}$$

$$b_{23} = s(2\epsilon_0^{-1})B^{-2}[\Lambda^2(1+\Lambda)(1-\chi^2)^{1/2}(1-\chi)\chi^{-1}]\exp(-i\phi), \tag{56h}$$

$$b_{24} = -is(2\epsilon_0^{-1})B^{-1}[\Lambda^{1/2}(1+\Lambda)(1-\chi^2)(\chi-\Lambda)^{1/2}\chi^{-1}]\exp(-2i\phi), \tag{56i}$$

$$b_{25} = is(2\epsilon_0^{-1})B^{-1}[\Lambda^{1/2}(1+\Lambda)(1-\chi)^2(\chi-\Lambda)^{1/2}\chi^{-1}], \tag{56j}$$

$$b_{26} = i(2\epsilon_0^{-1})B^{-1}[\Lambda^{1/2}(1+\Lambda)(1-\chi^2)^{1/2}(1-\chi)(\chi-\Lambda)^{1/2}\chi^{-1}]\exp(-i\phi), \tag{56k}$$

$$b_{33} = (2\epsilon_0^{-1})B^{-2}[\Lambda^3(1-\chi^2)\chi^{-1}], \tag{56l}$$

$$b_{34} = -i(2\epsilon_0^{-1})B^{-1}[\Lambda^{3/2}(1-\chi^2)^{1/2}(1+\chi)(\chi-\Lambda)^{1/2}\chi^{-1}]\exp(-i\phi), \tag{56m}$$

$$b_{35} = i(2\epsilon_0^{-1})B^{-1}[\Lambda^{3/2}(1-\chi^2)^{1/2}(1-\chi)(\chi-\Lambda)^{1/2}\chi^{-1}]\exp(i\phi), \tag{56n}$$

$$b_{36} = is(2\epsilon_0^{-1})B^{-1}[\Lambda^{3/2}(1-\chi^2)(\chi-\Lambda)^{1/2}\chi^{-1}], \tag{56o}$$

$$b_{44} = (2\epsilon_0^{-1})[(1+\chi)^2(\chi-\Lambda)\chi^{-1}], \tag{56p}$$

$$b_{45} = -(2\epsilon_0^{-1})[(1-\chi^2)(\chi-\Lambda)\chi^{-1}]\exp(2i\phi), \tag{56q}$$

$$b_{46} = -s(2\epsilon_0^{-1})[(1-\chi^2)^{1/2}(1+\chi)(\chi-\Lambda)\chi^{-1}]\exp(i\phi), \tag{56r}$$

$$b_{55} = (2\epsilon_0^{-1})[(1-\chi)^2(\chi-\Lambda)\chi^{-1}], \tag{56s}$$

$$b_{56} = s(2\epsilon_0^{-1})[(1-\chi^2)^{1/2}(1-\chi)(\chi-\Lambda)\chi^{-1}]\exp(-i\phi), \tag{56t}$$

$$b_{66} = (2\epsilon_0^{-1})[(1-\chi^2)(\chi-\Lambda)\chi^{-1}]. \tag{56u}$$

In the expressions above,

$$s = \mathrm{sn}(\cos\theta) = (\cos\theta)/|\cos\theta| \tag{57}$$

where sn is the signum function.

Before studying how these kernels depend on the angle θ, let us first examine briefly their dependence on the other variables.

Their dependence on the plasma frequency is represented by the factor B, which is absent from the expressions for the kernels that involve magnetic field components only. It appears as B^{-1} in the kernels involving one magnetic and one electric component, and as B^{-2} in kernels involving two electric components. Consequently, as the plasma frequency increases (relative to the electron gyro-frequency), the electric field components become weaker in comparison with the magnetic components; this is due to the increase in the phase refractive index n, which is proportional to B and which enters as a multiplier into the equations (26a)–(26c) for the wave admittances. This influence of n on the different types of kernel is also apparent in equations (50a)–(50u).

The dependence of the kernels on the wave frequency is represented by the factor Λ. We note that the expressions for all the kernels that involve the P-component of the electric field (i.e. those with a subscript 3) contain Λ to a power of 3/2 or more as a multiplier. Hence a decrease in frequency is accompanied by reductions of these kernels in comparison with the others. The reduction is most marked for b_{33}, which is proportional to Λ^3. It is due to the fact that, at the lowest frequencies, the wave electric field is almost exactly perpendicular to $\mathbf{B}_0$; this can be shown from the equations (24a)–(24c) for the electric polarization ratios.

We come now to the dependence of the kernels on θ, which is represented by the factors s and χ. The expressions (56a)–(56u) have been written in terms of these factors because, when evaluating the integral of equation (1), it is helpful to take $\cos\theta$ as a variable of integration (see Paper I).

The presence of s in an expression indicates that the θ-dependence of the kernel concerned is skew-symmetric with respect to the point $\theta = \pi/2$. Another symmetry is exhibited by the kernels that involve either the electric field components only, or the magnetic field components only: whenever the expressions for them contain the factor s, it is always accompanied by a θ-dependence of the form $\exp(i\phi)$ or $\exp(-i\phi)$. This implies that $b_{ij}(\theta, \phi) = b_{ij}(\pi - \theta, \pi + \phi)$, or – in words – that the kernel is unchanged if the wave-normal direction is reversed. On the other hand, all the kernels that involve both electric and magnetic components change sign in these circumstances. Of course, these various symmetries are independent of the wave mode, but they are more easily perceived in the approximate expressions (56a)–(56u) for the whistler-mode kernels than in the general expressions (50a)–(50u).

The functional dependence of the kernels on θ in the range $0 \leqslant \theta \leqslant \pi/2$ is given by the dimensionless terms in the rectangular brackets, which contain the factor χ and which we shall call β_{ij}. They are plotted in Figs 1–3, at values of Λ ranging from 0.1 to 0.8, for the 21 kernels listed above. These graphs exhibit a number of noteworthy features.

At the origin ($\theta = 0$), only the kernels b_{11}, b_{14} and b_{44} remain finite; each of them involves only the R-components of the field. All the others are zero at the origin, and for most of them the slope is zero also. Exceptions are the kernels that involve one R-component and one P-component, namely b_{13}, b_{16}, b_{34} and b_{46}; consistent with their finite slopes at the origin, their θ-dependence is of the form $\exp(i\theta)$ or $\exp(-i\theta)$.

At the resonance angle ($\theta = \theta_r$), the six kernels that involve only electric field components remain finite; the lower the frequency, the higher their values (Figs 1 and 2). All the other kernels tend to zero. The kernels involving only magnetic components do so with finite slopes, while the cross-kernels involving one electric and one magnetic component have infinite slopes at θ_r.

Between the origin and this upper limit of θ, the behaviour of the kernels is quite diverse, in fulfilment of the hopes that we expressed in Section 1.

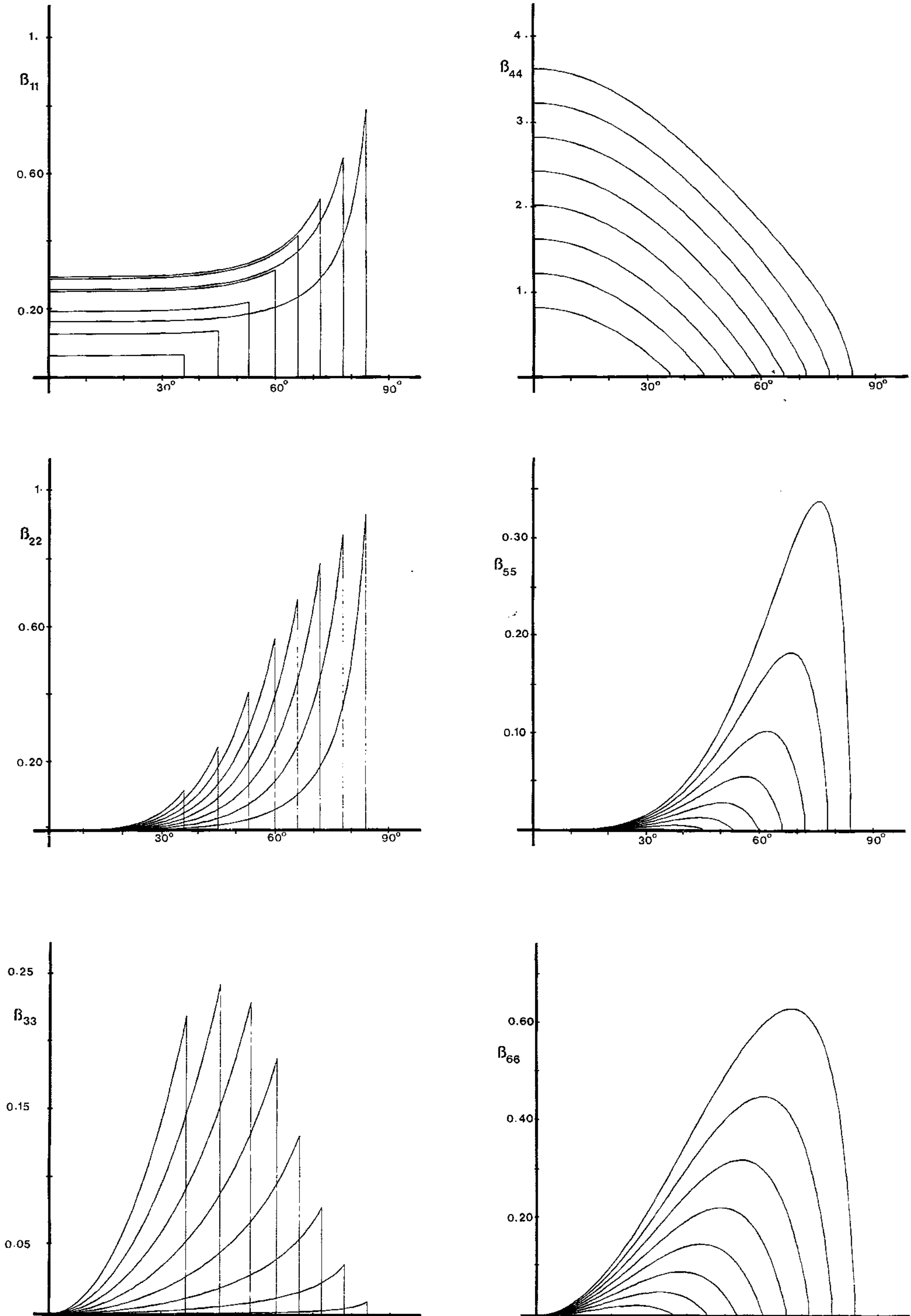

Figure 1. θ-dependence of the six auto-kernels for the whistler mode of propagation. Those in the column on the left involve electric field components only, while those on the right involve magnetic components only. The eight curves for each kernel correspond to $\Lambda = 0.1, 0.2 \ldots 0.7, 0.8$, in descending order of their intercepts with the horizontal axis at $\theta_r = \cos^{-1} \Lambda$.

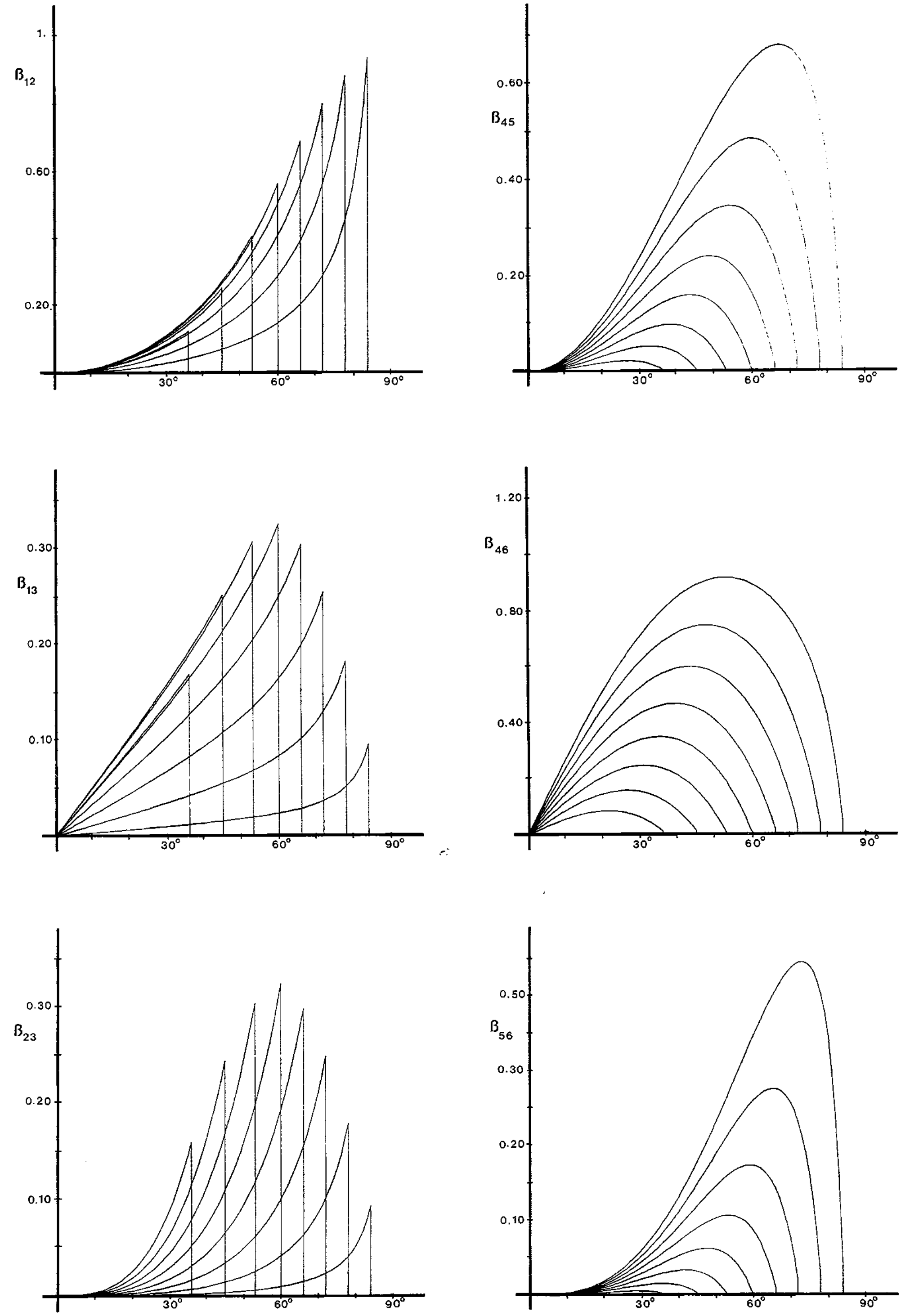

Figure 2. θ-dependence of the six whistler-mode cross-kernels involving field components of the same type. Those in the column on the left involve electric field components only, while those on the right involve magnetic components only. The values of the parameter Λ are the same as in Fig. 1.

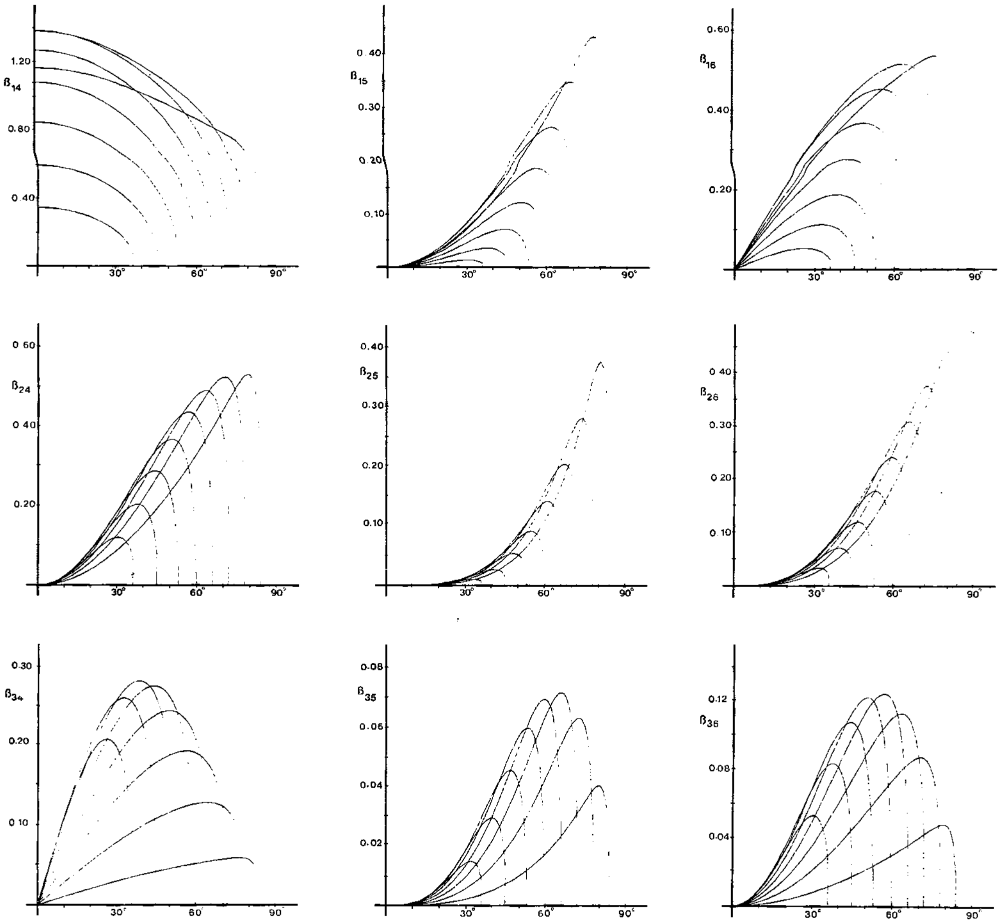

Figure 3. θ-dependence of the nine whistler-mode cross-kernels involving field components of different types, one electric and one magnetic. The values of the parameter Λ are the same as in Fig. 1.

8 Conclusion

On the assumptions that the plasma is cold and collisionless, general expressions have been derived for the kernels in the set of integral equations (1) that relates the spectral matrix to the wave distribution function.

These expressions have been given in algebraic forms that are usable for computation at all values of the angle θ, including the resonance angle θ_r. At θ_r, the kernels either vanish or remain finite. Very close to the resonance, however, cold-plasma theory breaks down because the phase velocities of the waves cease to be much larger than the electron thermal velocities; the consequences of this for our analysis have yet to be examined.

Besides these general expressions, simpler approximate expressions have been given that apply, under well-defined conditions, to waves propagated in the whistler mode. In the remaining papers of the present series, these approximate kernels will be used in worked examples demonstrating our various methods of analysis. The significance of the ways in which they depend on ϕ and on θ, as discussed in Sections 6 and 7 and illustrated in Figs 1–3, will then become apparent.

The information in the present paper, added to the findings of Paper I, completes the solution of the *direct problem* in the analysis of six-component measurements of a random electromagnetic wave field. We have now to deal with the corresponding *inverse problem*: namely, knowing the kernels and given measurements from which we can estimate the 36 elements of the spectral matrix, to what extent can the wave distribution function be determined and how should this be done? The solution of the inverse problem is the subject of the remaining papers.

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Appendix: the circular components of a time-varying vector

The concept of circular components has been explained most clearly by Kodera, Gendrin & de Villedary (1977), whom we shall refer to as KGV for short. However, we shall differ from them slightly in our definition of these components, for the sake of consistency with the work of Buneman (1961), of Allis *et al.* (1963) and of Lefeuvre (1977).

First, let us define the circular components for some real vector **U** that varies with time in an arbitrary way. In the rectangular coordinate system *Oxyz*, its components are U_x, U_y and U_z. With notation similar to that of KGV, we define the complex *analytic signals* $U_{xa}(t)$ and $U_{ya}(t)$ corresponding to the real signals $U_x(t)$ and $U_y(t)$:

$$\left.\begin{aligned} U_{xa}(t) &= U_x(t) + i\mathscr{H}[U_x(t)], \\ U_{ya}(t) &= U_y(t) + i\mathscr{H}[U_y(t)], \end{aligned}\right\} \qquad \text{(A1)}$$

where $\mathscr{H}$ is the Hilbert transform:

$$\mathscr{H}[x(t)] \equiv PP\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}\,d\tau\right\}. \tag{A2}$$

PP signifies the principal part. From these analytic signals, KGV define the two circular components of **U** as follows:

$$\left.\begin{aligned} U_+(t) &\equiv [U_{xa}(t) + iU_{ya}(t)]/2, \\ U_-(t) &\equiv [U_{xa}^*(t) + iU_{ya}^*(t)]/2. \end{aligned}\right\} \tag{A3}$$

However, we prefer to work with two equivalent but slightly different quantities, namely

$$\left.\begin{aligned} U_R(t) &\equiv 2U_+(t) = U_{xa}(t) + iU_{ya}(t), \\ U_L(t) &\equiv 2U_-^*(t) = U_{xa}(t) - iU_{ya}(t). \end{aligned}\right\} \tag{A4}$$

In the present paper, the quantities defined by equation (A4) are called the *circular components* of **U**, or alternatively, its *components in the complex principal axis coordinate system.* The subscripts R and L signify *right-handed* and *left-handed,* referring to the sense of rotation around the z-axis; this point is explained below. Finally, for the third component of **U**, directed *parallel* to the z-axis, we define $U_p \equiv U_z$. Thus, in the complex principal axis coordinates, we have $\mathbf{U} \equiv (U_R, U_L, U_P)$; we shall refer to these circular components as the *R-component*, the *L-component* and the *P-component.*

The significance of these definitions may be appreciated by applying them to a vector of which all three components vary sinusoidally, for instance

$$\mathbf{V}(t) = \text{Re}\ [\mathbf{v} \exp (i\omega t)]. \tag{A5}$$

The constant vector **v** is the complex amplitude of $\mathbf{V}(t)$. The components of $\mathbf{V}(t)$ in the *Oxy* plane are

$$\left.\begin{aligned} V_x(t) &= \text{Re}\ [v_x \exp (i\omega t)], \\ V_y(t) &= \text{Re}\ [v_y \exp (i\omega t)]. \end{aligned}\right\} \tag{A6}$$

The corresponding analytic signals are simply

$$\left.\begin{aligned} V_{xa}(t) &= v_x \exp (i\omega t), \\ V_{ya}(t) &= v_y \exp (i\omega t). \end{aligned}\right\} \tag{A7}$$

Therefore, from equation (A3),

$$\left.\begin{aligned} V_+(t) &= \tfrac{1}{2}\ (v_x + iv_y) \exp (i\omega t), \\ V_-(t) &= \tfrac{1}{2}\ (v_x^* + iv_y^*) \exp (-i\omega t). \end{aligned}\right\} \tag{A8}$$

These quantities are complex numbers with constant – but generally different – moduli, and with arguments that vary linearly with time, positively for $V_+(t)$ and negatively for $V_-(t)$. If *Oxy* is regarded as the complex plane, with *Ox* the real axis and *Oy* the imaginary axis, then the vectors representing $V_+(t)$ and $V_-(t)$ rotate around *Oz* at the angular frequency ω, in a right-handed and in a left-handed sense respectively. The loci of the end-points of the two vectors are circles, which is why KGV called these quantities circular components.

We, however, reserve this name for the related quantities

$$\left.\begin{aligned} V_R(t) &\equiv 2V_+(t) = (v_x + iv_y) \exp (i\omega t), \\ V_L(t) &\equiv 2V_-^*(t) = (v_x - iv_y) \exp (i\omega t). \end{aligned}\right\} \tag{A9}$$

The principal advantage of our definition is that, since the expressions for $V_R(t)$ and $V_L(t)$ contain the same exponent, the phase difference between these two complex components is constant, so it is meaningful to speak of their phase relationship. We also have

$$V_P(t) \equiv V_z(t) = \mathrm{Re}\,[v_z \exp(i\omega t)]. \tag{A10}$$

This third component of $\mathbf{V}(t)$ is purely real. Hence, in the complex principal axis coordinates, the complex amplitude of $\mathbf{V}(t)$ is the constant vector $\mathbf{v}$ with components

$$v_R = v_x + iv_y, \qquad v_L = v_x - iv_y, \qquad v_P = v_z \tag{A11}$$

but we note that the relation between each component of $\mathbf{V}(t)$ and the corresponding component of $\mathbf{v}$ is different for v_R and v_L from what it is for v_P. Equations analogous to equation (A11) were used by Buneman (1961) and by Allis *et al.* (1963) for defining the circular components of a harmonically varying vector, but in order to make the coordinate transformation unitary they included in their definitions a factor of $\sqrt{2}$ which we prefer to omit, following Quemada (1968).

The definitions that our equations (A1)–(A4) provide, for the circular components of an arbitrarily varying vector $\mathbf{U}(t)$, amount to applying the reasoning behind equations (A5)–(A9) to each of the elementary harmonic vectors into which $\mathbf{U}(t)$ can be decomposed by Fourier analysis.

These various definitions and concepts are applied to an electromagnetic field in Section 2 of the present paper.