THE ANALYSIS OF CATEGORICAL DATA
FROM MIXED MODELS

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## 1. Introduction

In considering two-way contingency tables, one often is concerned with two types of models: the "two response, no factor" situation in which neither margin is fixed and the "one response, one factor" situation in which one margin (say row totals) is fixed and the other is not. For the first case, the hypothesis of primary interest is one of independence or no association between the responses; while in the other case, the hypothesis of primary interest is one of homogeneity over the factor levels or equality of the distributions in the different rows.

This paper is concerned with the recognition of a third type of model which will be called a "mixed categorical data model of order 2" in the case of two-way tables. The experimental situation corresponding to such a model involves exposing each of n randomly chosen subjects from some homogeneous population to both levels of a binary factor (eg. control vs. treated) and classifying each of the two responses into one of r categories. The resulting data are then represented in the matrix of an r×r contingency table which will be assumed to follow a multinomial distribution. It is apparent that this is a categorical data version of the classical matched pairs design. As such, it has been studied in a somewhat different context recently by Miettinen [1968, 1969]

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This paper is concerned with the recognition of the underlying structure of contingency tables which are analogous to the well-known mixed model in analysis of variance. The experimental situation corresponding to such a model involves exposing each of n subjects to each of the d levels of a given factor and classifying each of the d responses into one of r categories. The resulting data are represented in an $r \times r \times \ldots \times r$ contingency table of d-dimensions. The hypothesis of principal interest is one of first order marginal symmetry or equality of the one-dimensional marginal distributions. Alternatively, if the r categories may be quantitatively scaled, then attention is directed at the hypothesis of equality of the mean scores over the d first order marginals. Test statistics for these hypotheses are developed in terms of minimum Neyman $\chi^2$ or equivalently weighted least squares analysis of underlying linear models. As such, they have a strong resemblance to the Hotelling $T^2$ procedures used with continuous data in mixed models. Several numerical examples are given to illustrate the use of the various methods discussed.

and others (Worcester [1964], Billewicz [1964, 1965], Bennett [1967], and Chase [1968]). In addition, the methods of Mantel and Haenszel [1959] are also of interest for this experimental situation. Although each of the previously cited papers deal with appropriate statistical procedures for the analysis of mixed categorical data models of order 2, they do not indicate the way in which such contingency table data can be analyzed as a special case of a unified general model.

One such model which will be the basis of inference in the remainder of this paper is that of Grizzle, Starmer, and Koch [1969]. As a result, test statistics are derived through weighted least squares analysis of certain appropriately formulated linear models and correspond identically with the minimum modified $\chi_1^2$-statistics due to Neyman [1949] or equivalently the generalized quadratic form criteria due to Wald [1943]. Alternative methods are also appropriate in certain contexts; eg. those of Lewis [1968] and Goodman [1970] which are based on maximum likelihood and that of Ku and Kullback [1968] based on minimum discrimination information.

To apply the methods of Grizzle, Starmer, and Koch (GSK), it is first necessary to recognize the underlying factor-response structure for the data (see Bhapkar and Koch [1968]). Since only the sample size n is fixed, this is a "two response, no factor" situation. However, the hypothesis of interest is one of homogeneity over the factor levels; i.e., there is no difference between the effects of control and treated. This hypothesis can be formulated specifically as an hypothesis of marginal symmetry in the two-way table.

The hypothesis of marginal symmetry in two-way contingency tables has been considered by a number of authors including Stuart [1955], Bhapkar [1966], and most recently Ireland, Ku, and Kullback [1969]. However, not a great deal of attention has been given to motivating the situations in which tests of marginal symmetry are of the most interest with respect to the interpretation of data from the biological, physical, or social sciences. Hence, one of the principal purposes of this paper is to demonstrate that tests of marginal symmetry are not only applicable to mixed categorical data models, but are the only tests directed at the question of primary interest associated with such experimental situations; namely homogeneity over factor levels. One should also note that the traditional hypothesis of independence which is usually applied to "two response, no factor" tables is of only casual interest here. This remark follows from the fact that responses from the same subject (or units in the same matched pair) to different factor levels are naturally expected to be associated since they have the same initial source in common.

Finally, it is important to recognize that if the r categories are describable in terms of some quantitative scale, then the question of principal interest is that of equality of mean scores over the two marginals. This hypothesis, which bears a definite resemblance to the relationship tested by the classical matched pairs t-test used with continuous data, is weaker than the hypothesis of marginal symmetry in the sense that it specifies a restriction on only one degree of freedom rather than (r-1) degrees of freedom. On the other hand, test statistics directed at this weaker hypothesis are more powerful than those directed at the hypothesis

of marginal symmetry when the inequality of the two mean scores is the
only alternative of real interest.

## 2. Mixed Categorical Data Models Of Order 2

First let us consider the following data (see Table 1) from case

TABLE 1

7477 Women Aged 30-39; Unaided Distance Vision

| Right Eye | Left Eye | | | | |
|---|---|---|---|---|---|
| | Highest Grade (1) | Second Grade (2) | Third Grade (3) | Lowest Grade (4) | Total |
| Highest Grade (1) | 1520 | 266 | 124 | 66 | 1976 |
| Second Grade (2) | 234 | 1512 | 432 | 78 | 2256 |
| Third Grade (3) | 117 | 362 | 1772 | 205 | 2456 |
| Lowest Grade (4) | 36 | 82 | 179 | 492 | 789 |
| Total | 1907 | 2222 | 2507 | 841 | 7477 |

records of the eye-testing of 7477 employees in Royal Ordnance factories
which have been used as an illustrative example for tests of marginal
symmetry by Stuart [1955], Bhapkar [1966], Grizzle, Starmer, and Koch
[1969], and Ireland, Ku, and Kullback [1969]. These data may be viewed
as arising from a mixed model in which each of n = 7477 subjects is
classified according to both levels of a factor which will be called eye
position (namely right eye and left eye) with the corresponding cell fre-
quencies being assumed to follow a multinomial distribution. As such,
the hypothesis of principal interest is that the distributions of grades
of vision are the same for both eyes; in other words marginal symmetry.
The values of test statistics, each approximately distributed as $\chi^2$ with
D.F. = 3 if the hypothesis is true, which have been obtained by the authors
cited above are as follows:

| Stuart  | $x^2 = 11.96$ |
| Bhapkar | $x^2 = 11.97$ |
| GSK     | $x^2 = 11.98$ |
| IKK     | $x^2 = 12.00$ |

These values are essentially the same and indicate that the two marginal distributions are significantly different.

Let us next look at the problem of marginal symmetry in a more formal way. If $\pi_{jj'}$ denotes the probability that a subject in the population is classified according to the j-th grade category on the right eye and the j'-th grade category on the left eye, then the hypothesis of marginal symmetry may be written as

$$\pi_{j.} = \pi_{.j'} \qquad j = j' = 1, 2, 3, 4 \qquad (2.1)$$

where $\pi_{j.} = \sum_{j'=1}^{4} \pi_{jj'}$ and $\pi_{.j'} = \sum_{j=1}^{4} \pi_{jj'}$. Following the GSK approach, one would first form the unrestricted maximum likelihood estimates $p_{jj'} = (n_{jj'}/n)$ of $\pi_{jj'}$ where $n_{jj'}$ is the number of subjects classified into the jj'-th cell. From these, the desired estimates of $\pi_{j.}$ and $\pi_{.j'}$ may be obtained by taking the appropriate marginal sums of $p_{jj'}$; namely

$p_{j.} = \sum_{j'=1}^{4} p_{jj'}$ and $p_{.j'} = \sum_{j=1}^{4} p_{jj'}$ respectively. If the following linear model is fitted to the $p_{j.}$ and $p_{.j'}$

$$E\begin{bmatrix} p_{1.} \\ p_{2.} \\ p_{3.} \\ p_{.1} \\ p_{.2} \\ p_{.3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \qquad (2.2)$$

by weighted least squares, then the residual sum of squares with D.F. $= 3$ is the GSK test statistic for (2.1) which is the same as that of Bhapkar [1966] (except for round-off error).

Alternatively, if scores $a_1$, $a_2$, $a_3$, $a_4$ are assigned to the response categories, then the hypothesis of equality of mean scores over the two marginals may be written as

$$\alpha_{RE} \equiv \sum_{j=1}^{r} a_j \pi_{j\cdot} = \sum_{j'=1}^{r} a_{j'} \pi_{\cdot j'} \equiv \alpha_{LE} \tag{2.3}$$

One test statistic for this hypothesis is the residual sum of squares with D.F. $= 1$, which is obtained by forming $\hat{\alpha}_{RE} = \sum_{j=1}^{r} a_j p_{j\cdot}$ and

$\hat{\alpha}_{LE} = \sum_{j'=1}^{r} a_{j'} p_{\cdot j'}$ and fitting the linear model

$$E \begin{bmatrix} \hat{\alpha}_{RE} \\ \hat{\alpha}_{LE} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mu$$

by weighted least squares. This statistic may be more compactly written as

$$x^2 = n(\hat{\alpha}_{RE} - \hat{\alpha}_{LE})^2 / \{ \sum_{j=1}^{r} \sum_{j'=1}^{r} [(a_j - a_{j'}) - (\hat{\alpha}_{RE} - \hat{\alpha}_{LE})]^2 p_{jj'} \} \tag{2.4}$$

For the data in Table 1, the scores $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$ have been assigned to the grade categories. The appropriate value of $x^2$ may be computed as follows:

$$\hat{\alpha}_{RE} = \frac{17012}{7477} \qquad \hat{\alpha}_{LE} = \frac{17236}{7477} \qquad (\hat{\alpha}_{RE} - \hat{\alpha}_{LE})^2 = (\frac{-224}{7477})^2$$

$$\sum_{j=1}^{r} \sum_{j'=1}^{r} (a_j - a_{j'})^2 p_{jj'} = (1)(\frac{1678}{7477}) + 4(\frac{401}{7477}) + 9(\frac{102}{7477}) = \frac{4200}{7477}$$

$$X^2 = 7477 (\frac{-224}{7477})^2 / \{[(4200)(7477) - (224)^2] / (7477)^2\}$$

$$= 11.97$$

The similarity of the test given in (2.4) to the matched pairs t-test may be seen clearly by writing down the frequency distribution of the difference ($d_{jj'} = a_j - a_{j'}$) between the right eye grade and the left eye grade. These are shown in Table 2. The value of the average

TABLE 2

| Difference (d) | Frequency (f) |
|---|---|
| -3 | 66 |
| -2 | 202 |
| -1 | 903 |
| 0 | 5296 |
| 1 | 775 |
| 2 | 199 |
| 3 | 36 |

difference $\bar{d}$ is $(-224/7477) \approx -.03$ and its estimated standard error $s_{\bar{d}}$ is .0087. Hence

$$t^2 = \bar{d}^2 / s^2_{\bar{d}} = (-3.46)^2 = 11.97 \qquad (2.5)$$

which is identical to the value obtained for the $X^2$ test previously described.

As a second example, let us consider the following data (see

Table 3) from Miettinen [1969] which pertain to a study of the extent to

TABLE 3

Previous History of Induced Abortion in Propositi
with Ectopic Pregnancy and Matched Controls

|  |  | Number of Controls with Induced Abortion | | | | | |
|---|---|---|---|---|---|---|---|
|  |  | 0 | 1 | 2 | 3 | 4 | Total |
| Propositus with induced abortion | no 0 | 5 | 1 | 0 | 0 | 0 | 6 |
|  | yes 1 | 3 | 5 | 3 | 0 | 1 | 12 |
|  | total | 8 | 6 | 3 | 0 | 1 | 18 |

which induced abortions increase the risk of ectopic implantation in sub-

sequent pregnancies. To each of 18 propositi with ectopic pregnancy fol-

lowing at least one earlier pregnancy, four control subjects were matched

with respect to order of pregnancy, age, and husband's education. The

history of induced abortions terminating any of the preceding pregnancies

was then recorded for both the propositi and the controls. These data

may be viewed as arising from an asymmetric mixed model in which each of

the $n = 18$ sets of matched cases correspond to generalized subjects and

classification is with respect to each level of the factor propositus vs.

control. The complicating feature of these data is that the responses to

the different factor levels are measured on seemingly different scales;

i.e., propositus on a two-value scale and control on a five value scale.

This difficulty can be resolved by formulating the hypothesis of marginal

symmetry in an appropriate fashion. In particular, let $\pi_{jj'}$ denote the

probability that a matched set is classified according to the j-th level

with respect to propositus and the j'-th level with respect to controls.

If scores $a_0 = 0$, $a_1 = 1$ are attached to the responses for propositus

and scores $b_0 = 0$, $b_1 = 1/4$, $b_2 = 2/4$, $b_3 = 3/4$, and $b_4 = 4/4$ are attached

to the responses for controls, then

$$\alpha_p \equiv \sum_{j=0}^{1} a_j \pi_{j.} = \sum_{j=0}^{1} \sum_{j'=0}^{4} a_j \pi_{jj'}$$

$$(2.6)$$

$$\alpha_c \equiv \sum_{j'=0}^{4} b_{j'} \pi_{.j'} = \sum_{j=0}^{1} \sum_{j'=0}^{4} b_{j'} \pi_{jj'}$$

represent the probability of observing an induced abortion in a propositus

and the expected relative frequency of observing an induced abortion in a

set of four controls respectively. Hence, the hypothesis of interest is

$H_0: \alpha_p = \alpha_c$. Essentially, the same test indicated in (2.4) is applicable

here.

Following the GSK approach, one would first apply the A-matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ .00 & .25 & .50 & .75 & 1.00 & .00 & .25 & .50 & .75 & 1.00 \end{bmatrix}$$

to the data in Table 3 arranged in a single (10×1) column vector with

the transpose of row 1 on top and that of row 2 underneath. This would

generate $\hat{\alpha}_p$ and $\hat{\alpha}_c$ to which the linear model

$$E \begin{bmatrix} \hat{\alpha}_p \\ \hat{\alpha}_c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mu \quad \text{where} \quad \begin{bmatrix} \hat{\alpha}_p \\ \hat{\alpha}_c \end{bmatrix} = \begin{bmatrix} 0.667 \\ 0.222 \end{bmatrix}$$

is fitted by weighted least squares. The test statistic (with D.F. = 1)

again is the residual sum of squares and for these data is given by

$X^2 = 20.85$. This result is consistent with that of Miettinen [1969] who

observed essentially $X^2 = 16.00$ without continuity correction and $X^2 = 13.69$

with continuity correction. This is not unexpected since both the test

procedure given here and that of Miettinen are in some sense asymptotically

equivalent. The advantage of the one proposed here is that it is an easily

produced special case of a very general program.

### 3. Mixed Categorical Data Models of Higher Orders

In general a "mixed categorical data model of order d" involves ex-

posing each of n randomly chosen subjects from some homogeneous population

to each level of a factor with d levels and classifying each response into

one of r categories. The resulting data are then represented in an $r\times r\times...\times r$

contingency table of d dimensions which will be assumed to follow a multi-

nomial distribution. This represents a categorical version of the classical

mixed model with n subjects and d treatments (see Eisenhart [1947] and Koch

and Sen [1968]). The hypothesis of interest is equality of treatment effects

or homogeneity over factor levels. For the case $r = 2$, Cochran [1950] has

proposed a test (sometimes called the Q-test) by a permutation argument.

An alternative test due to Bhapkar [1965] has been formulated in terms of

d-th order marginal symmetry and can be produced from the weighted least

squares approach. This procedure is illustrated in Grizzle, Starmer, and Koch

[1969] for data indicating the responses of 46 subjects to each of the drugs

A, B, and C as either being favorable or not (see Table 4). This example

TABLE 4

Tabulation of Responses to Drugs A, B, and C

| Response to A | favorable | | | not favorable | | | total |
|---|---|---|---|---|---|---|---|
| Response to B | fav. | not fav. | total | fav. | not fav. | total | |
| Response to C — fav. | 6 | 2 | 8 | 2 | 6 | 8 | 16 |
| Response to C — not fav. | 16 | 4 | 20 | 4 | 6 | 10 | 30 |
| total | 22 | 6 | 28 | 6 | 12 | 18 | 46 |

is a mixed categorical data model of order 3 with drug type being the factor and having levels A, B, and C. The hypothesis of interest is equality of the probability of a favorable response for each of the three drugs; i.e., equality of the corresponding marginal distributions or marginal symmetry of the three one-way margins.

In the continuous case, the hypothesis of equality of treatment effects in mixed models is most appropriately tested by a Hotelling $T^2$-procedure (see Scheffé [1959]) unless certain symmetry relationships hold in which case an analysis of variance test applies. Both of these tests together with certain non-parametric counterparts are described in Koch and Sen [1968]. Hence, it is apparent that in the general categorical data setting of interest here, the test statistics applied to the hypothesis of marginal symmetry bear the same analogy to the Hotelling $T^2$-test as the test statistics used in the previous section bore to the classical matched pairs t-test. For the data in Table 4, the GSK approach led to a $X^2$ = 6.58 with D.F. = 2 indicating a significant difference ($\alpha$ = .05) between drugs. The reader is referred to that paper for the details leading to this result and the Appendix for the pertinent theory.

A more complex example can be seen in terms of the following data (see Table 5) of Lessler [1962] which have been described in Bhapkar and Koch [1968]. In this case, there are two groups of male subjects in a study of the nature of sexual symbolism in objects: Group A members were not told the purpose of the experiment while Group C subjects were. Each subject was asked to classify as masculine or feminine (M or F) certain objects which had been previously characterized as being culturally

TABLE 5

Responses of Males to Culturally Masculine and Anatomically
Feminine Objects with Weak Intensity

| Resp. at 1/5 Sec. | | M | | | F | | | Total |
|---|---|---|---|---|---|---|---|---|
| Resp. at 1/100 Sec. | | M | F | Total | M | F | Total | |
| A | Resp. at M 1/1000 Sec. | 171 | 6 | 177 | 7 | 7 | 14 | 191 |
| | F | 18 | 12 | 30 | 7 | 56 | 63 | 93 |
| | Total | 189 | 18 | 207 | 14 | 63 | 77 | 284 |
| C | Resp. at M 1/1000 Sec. | 184 | 10 | 194 | 7 | 20 | 27 | 221 |
| | F | 38 | 14 | 52 | 7 | 114 | 121 | 173 |
| | Total | 222 | 24 | 246 | 14 | 134 | 148 | 394 |

masculine and anatomically feminine but with weak intensity at three dif-

ferent exposure rates:  1/5 second, 1/100 second, and 1/1000 second.

Hence, this is a "three response, one factor" situation with Group being

the factor and the M or F classification at different exposure rates being

the responses.  However, since it is of interest to compare the responses

at different exposure rates, it is apparent that the data from either one

of the groups A or C by themselves is a mixed categorical data model of

order 3.  Thus, one may call the data for the combined groups a "split plot

categorical data" model.  Hence, the hypotheses of interest are equality

of group (whole-plot) effects, equality of exposure rate (split-plot) effects,

and no interaction between exposure rates and groups.  All of these may be

readily formulated by using the GSK approach.  First let us define the

following relative frequency vectors for the two groups

$$p_A' = (171, 18, 6, 12, 7, 7, 7, 56)(1/284)$$

$$p_C' = (184, 38, 10, 14, 7, 7, 20, 114)(1/394)$$

(3.1)

$$p' = (p_A', p_C')$$

The overall vector $\underset{\sim}{p}$ is to be pre-multiplied by the following A-matrix to determine functions $\underset{\sim}{F} = \underset{\sim}{A}\ \underset{\sim}{p}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad (3.2)$$

The resulting statistics are estimates of the probability of masculine classification at the exposure rates 1/5 second, 1/100 second, 1/1000 second respectively in Groups A and C respectively. The next step in the analysis is to fit the following linear model by weighted least squares

$$E\{\underset{\sim}{F}\} = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \mu_A \\ \lambda_{A_1} \\ \lambda_{A_2} \\ \mu_C \\ \lambda_{C_1} \\ \lambda_{C_2} \end{bmatrix} \qquad \text{where } \underset{\sim}{F} = \begin{bmatrix} 0.729 \\ 0.715 \\ 0.673 \\ 0.624 \\ 0.599 \\ 0.561 \end{bmatrix} \qquad (3.3)$$

and $\mu_A$ is a mean effect in Group A, $\lambda_{A_1}$ is the linear effect of exposure in Group A, $\lambda_{A_2}$ is the non-linear effect of exposure in Group A, and $\mu_C$, $\lambda_{C_1}$, $\lambda_{C_2}$ are similarly defined quantities in Group C. The hypothesis of no group $\times$ exposure rate interaction may be formulated as

$$H_0: \lambda_{A_1} = \lambda_{C_1}, \ \lambda_{A_2} = \lambda_{C_2}$$

which may be tested by using the following hypothesis matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

in accordance with the principles of weighted regression analysis. The resulting $X^2 = 0.20$ with D.F. $= 2$ indicates that the data are consistent with this hypothesis of no interaction. Hence, the model is revised as

$$E\{\underset{\sim}{F}\} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \mu \\ \gamma \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \qquad (3.4)$$

in which $\mu$ is an overall mean, $\gamma$ is the differential group effect, $\lambda_1$ is the linear effect of exposure, and $\lambda_2$ is the non-linear effect of exposure. The hypothesis of no group main effects may be formulated as

$$H_0: \gamma = 0$$

and tested by using the hypothesis matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

while the hypothesis of no exposure rate main effects may be formulated as

$$H_0: \lambda_1 = \lambda_2 = 0$$

and tested by using the hypothesis matrix

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad .$$

The resulting test statistics respectively are $X^2 = 11.57$ with D.F. $= 1$ and $X^2 = 14.55$ with D.F. $= 2$. Hence, it is apparent that there are both significant differences between the two groups and among the three exposure rates. In fact, the differences among the three exposure rates are linear as can be seen by testing separately for the linear and for the non-linear effect of exposure. Thus, we see that split-plot contingency tables can be analyzed by the linear model approach as readily as the more standard types of multi-response, multi-factor experiments treated in Grizzle, Starmer, and Koch [1969].

## 4. Some Other Related Problems

The mixed models considered in this paper may be viewed as belonging
to Case III of Koch and Sen [1968] for which the Hotelling $T^2$-test is ap-
plicable. However, since it is stated here that a single multinomial dis-
tribution applies to the entire table for the underlying homogeneous popu-
lation, it is apparent that the discussion given here not only requires
additive subject effects but also no subject effects at all. On the other
hand if subject effects are additive and are selected at random, then
this same analysis is applicable in an approximate sense.

The categorical situation, analogous to Case IV of Koch and Sen in
which the analysis of variance F-test is appropriate, occurs when the re-
sponses of the same subject over repeated trials to the different levels of
the factor are independent. This means, in the case of a mixed model of
order d, that the r×r×...×r contingency table with n observations in it
can be replaced by a d×r table with nd observations in which the d rows
correspond to the d different treatments or factor levels and the r columns
correspond to the levels of the response. The frequencies in each row are
the appropriate first-order marginal distributions corresponding to each of
the d dimensions of the r×r×...×r table. Hence, the hypothesis of marginal
symmetry in the complex table may be translated here into the traditional
hypothesis of homogeneity in a "one factor, one response" two-way table.
Thus, it may be tested by applying the classical $\chi^2$-test to this two-way
table. One important difference between this test procedure and the one in
the complex table is that it is based on an effective sample size of nd
rather than n. Thus, if Case IV applies, then this type of test is natur-
ally more powerful than those given for Case III. Finally, as pointed out

in Bhapkar [1968], if the response is quantitatively scaled, then the hypothesis of equality of mean scores in the two-way table may be tested instead of the usual homogeneity test. Here, this test would be a Case IV alternative to the type of test described in (2.4).

The more general Cases I and II of Koch and Sen allow for complex subject effects, and hence in the case of categorical data, involve product multinomial distributions. As a result, the methods given here are not directly applicable. However, such experimental situations can be handled to some extent by fitting complex linear models with both treatment effects and subject effects. The details of this type of analysis of product multinomial models are given in Grizzle, Starmer and Koch [1969].

## 5. Appendix of Theoretical Results

Let $\pi_{j_1 j_2 \cdots j_d}$ denote the probability of cell $(j_1, j_2, \ldots, j_d)$ where $j_\xi = 1, 2, \ldots, r$ with $\xi = 1, 2, \ldots, d$ in a d-dimensional table. Define $\phi_{\xi j}$ to be the marginal probability of level j for the $\xi$-th dimention. Hence

$$\phi_{\xi j} = \sum_{\substack{\nu=1 \\ \nu \neq \xi}}^{d} \sum_{\substack{j_\nu=1 \\ j_\xi=j}}^{r} \pi_{j_1 j_2 \cdots j_d} \tag{5.1}$$

The hypothesis of d-th order marginal symmetry on the one-way margins may be written

$$\phi_{1j} = \phi_{2j} = \cdots = \phi_{dj} \qquad j = 1, 2, \ldots, r \tag{5.2}$$

This hypothesis may be readily tested by letting $n_{j_1 j_2 \cdots j_d}$ denote the observed frequency of cell $(j_1, j_2, \ldots, j_d)$. An A-matrix is then produced which generates the estimates $p_{\xi j}$ of $\phi_{\xi j}$ obtained as

$$p_{\xi j} = \frac{1}{n} \sum_{\substack{\nu=1 \\ \nu\neq\xi}}^{d} \sum_{\substack{j_\nu=1 \\ j_\xi=j}}^{r} n_{j_1 j_2 \cdots j_d}$$ where n is the total sample size. For

example, if d = 3 and r = 3, we have

$$
A = \begin{bmatrix}
1\ 1\ 1 & 1\ 1\ 1 & 1\ 1\ 1 & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 \\
0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & 1\ 1\ 1 & 1\ 1\ 1 & 1\ 1\ 1 & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 \\
0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & 0\ 0\ 0 & 1\ 1\ 1 & 1\ 1\ 1 & 1\ 1\ 1 \\
1\ 1\ 1 & 0\ 0\ 0 & 0\ 0\ 0 & 1\ 1\ 1 & 0\ 0\ 0 & 0\ 0\ 0 & 1\ 1\ 1 & 0\ 0\ 0 & 0\ 0\ 0 \\
0\ 0\ 0 & 1\ 1\ 1 & 0\ 0\ 0 & 0\ 0\ 0 & 1\ 1\ 1 & 0\ 0\ 0 & 0\ 0\ 0 & 1\ 1\ 1 & 0\ 0\ 0 \\
0\ 0\ 0 & 0\ 0\ 0 & 1\ 1\ 1 & 0\ 0\ 0 & 0\ 0\ 0 & 1\ 1\ 1 & 0\ 0\ 0 & 0\ 0\ 0 & 1\ 1\ 1 \\
1\ 0\ 0 & 1\ 0\ 0 & 1\ 0\ 0 & 1\ 0\ 0 & 1\ 0\ 0 & 1\ 0\ 0 & 1\ 0\ 0 & 1\ 0\ 0 & 1\ 0\ 0 \\
0\ 1\ 0 & 0\ 1\ 0 & 0\ 1\ 0 & 0\ 1\ 0 & 0\ 1\ 0 & 0\ 1\ 0 & 0\ 1\ 0 & 0\ 1\ 0 & 0\ 1\ 0 \\
0\ 0\ 1 & 0\ 0\ 1 & 0\ 0\ 1 & 0\ 0\ 1 & 0\ 0\ 1 & 0\ 0\ 1 & 0\ 0\ 1 & 0\ 0\ 1 & 0\ 0\ 1
\end{bmatrix}
$$

Since the $\phi_{\xi j}$ necessarily satisfy restrictions of the type $\sum_{j=1}^{r} \phi_{\xi j} = 1$,

the A-matrix is modified to generate $p_{\xi j}$ for only the combinations

$\xi = 1, 2, \ldots, d$, $j = 1, 2, \ldots, (r-1)$. In the example, this means rows
3, 6, 9 would be deleted from A. To the resulting estimates $p_{\xi j}$, the

following linear model would be fitted by weighted least squares

$$E\{p_{\xi j}\} = \mu_j \qquad \begin{aligned} \xi &= 1, 2, \ldots, d \\ j &= 1, 2, \ldots, (r-1) \end{aligned} \tag{5.3}$$

This can be written more fully as

$$
E \begin{bmatrix}
p_{11} \\ p_{12} \\ \cdots \\ p_{1,r-1} \\ p_{21} \\ p_{22} \\ \cdots \\ p_{2,r-1} \\ \cdots \\ \cdots \\ \cdots \\ p_{d1} \\ p_{d2} \\ \cdots \\ p_{d,r-1}
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
  &   & \cdots &   \\
0 & 0 & \cdots & 1 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
  &   & \cdots &   \\
0 & 0 & \cdots & 1 \\
  &   & \cdots &   \\
  &   & \cdots &   \\
  &   & \cdots &   \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
  &   & \cdots &   \\
0 & 0 &        & 1
\end{bmatrix}
\begin{bmatrix}
\mu_1 \\ \mu_2 \\ \cdots \\ \mu_{r-1}
\end{bmatrix}
\tag{5.4}
$$

The residual sum of squares with D.F. = $(d-1)(r-1)$ is the test statistic

for the hypothesis of marginal symmetry in (5.2). When (5.2) holds, this

statistic is approximately distributed as $\chi^2$ with D.F. = $(d-1)(r-1)$.

Alternatively, if scores $a_1$, $a_2$,..., $a_r$ are assigned to the response

categories, then the hypothesis of equality of mean scores over the d mar-

ginals may be written as

$$\alpha_1 = \alpha_2 = \ldots = \alpha_d \qquad (5.5)$$

where $\alpha_\xi = \sum_{j=1}^{r} a_j \phi_{\xi j}$. The hypothesis (5.5) may be tested by obtaining the

estimates $\hat{\alpha}_\xi = \sum_{j=1}^{r} a_j p_{\xi j}$ where $\xi = 1, 2,\ldots,$ d and fitting the linear

model

$$E \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \ldots \\ \hat{\alpha}_d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{bmatrix} \mu \qquad (5.6)$$

by weighted least squares; again the test statistic is the residual sum of

squares with D.F. = $(d-1)$.

The tests given here for (5.2) and (5.5) appear similar to the classi-

cal ones of homogeneity and equality of mean scores in "one response, one

factor" tables. However, here the underlying table is d-dimensional and the

$p_{\xi j}$ are correlated with each other while in the "one response, one factor"

tables the underlying table is two-dimensional and the $p_{\xi j}$ are independent

of $p_{\xi' j}$ for all $\xi' \neq \xi$. Hence, in the underlying estimates of variance and

covariance, the two types of procedures are quite different.

Finally, it is obvious that other types of hypotheses of marginal symmetry (eg. equality of all two-way margins) may be easily tested by this approach. All that is required is choosing an appropriate A-matrix, and then fitting a linear model specific to the hypothesis of interest. Since such tests are usually not readily interpreted experimentally in the  context of mixed models, they will not be discussed in any more detail here.

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