
BY PERMISSION OF THE COUNCIL
EXCERPT FROM *GÉOTECHNIQUE*. VOL. 15, NO. 1

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GENERAL SLIP SURFACES

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PUBLISHED BY
THE INSTITUTION OF CIVIL ENGINEERS
GREAT GEORGE STREET · LONDON, S.W.1.
1965

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PRINTED IN GREAT BRITAIN BY WILLIAM CLOWES AND SONS, LIMITED
LONDON AND BECCLES

THE ANALYSIS OF THE STABILITY OF GENERAL SLIP SURFACES

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SYNOPSIS

Within the framework of limit equilibrium methods of stability analysis, no restriction need be placed at the outset upon the shape of the possible slip surface. In many cases, the critical surface may deviate significantly from a circle or a plane and therefore a method that facilitates the analysis of surfaces of arbitrary shape is of interest. A method for doing this is presented. The assumptions necessary to make the problem statically determinate are discussed. The solution of the governing equations ensures that all equilibrium and boundary conditions are satisfied. The method has been programmed for a digital computer and some examples of its application are given. Comparisons are also made with other methods of analysis.

Dans le cadre des méthodes d'équilibre limite d'analyse de stabilité, il n'y a pas besoin d'imposer de restrictions au départ sur la forme de la surface de glissement éventuelle. Dans bien des cas, la surface critique peut dévier d'une manière significative d'un cercle ou d'un plan et par conséquent une méthode qui facilite l'analyse des surfaces de formes arbitraires présente un intérêt. On présente une méthode pour accomplir cela. On discute des hypothèses nécessaires pour que le problème soit déterminé au point de vue de la statique. La solution des équations dominantes garantit que toutes les conditions d'équilibre et de limite soient satisfaites. La méthode a été programmée pour une calculatrice digitale et on donne quelques exemples de ses applications. D'autres méthodes d'analyse y sont comparées.

INTRODUCTION

The necessity of considering body forces, pore-water pressures, and a variety of soil types in the analysis of the stability of earth slopes vitiates the application of methods that are well-founded in the mechanics of continua and employ representative constitutive equations. As a result of this, limit equilibrium methods of analysis are commonly used. These methods invoke no kinematical considerations regarding soil behaviour and hence require that the shape of the potential slip surface be assumed. The selection of the shape of this surface is therefore, to a large degree, arbitrary. However, it is commonly assumed in the analysis of slope stability problems that the shape of the slip surface is circular. The choice of a circular slip surface is usually justified on the grounds that the computations are made simpler.

By appealing to field observations, it becomes apparent that slip surfaces that are approximately circular have been observed, although non-circular surfaces are more numerous. The best documented case of a circular slip is the slide at Lodalen described by Sevaldson (1956). Among the many examples of non-circular slips in natural slopes that could be cited are the Folkestone-Warren landslip (e.g. Legget, 1962) and the slides at Surte (Jakobson, 1952) and Furre (Hutchinson, 1961). Photographs of non-circular slip surfaces in natural slopes have been given by Varnes (1958). In the case of earth dam failures, the slip in the Chingford reservoir is an example of non-circular movement (Cooling and Golder, 1942) as is the more recent construction slip in the Tittesworth Dam (Twort, 1964) where a portion of the downstream slope of the old bank slid along a weak clay layer during the construction of the new bank.

The degree of surface disturbance of the sliding mass provides a useful visual method for distinguishing between circular and non-circular movement. When the soil mass displaces as a rigid body rotation, the disturbance of the surface is comparatively small. More severe

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surface disturbance is usually observed when failure occurs along a non-circular path. This type of movement induces intense shear stresses within the mass and a *graben* formation may result as illustrated in Fig. 1.

It is evident that because of departures from homogeneity the soil mass may slip along a non-circular surface, and thus the minimum factor of safety may be associated with such a surface. Current design practice often produces sections in which the strength parameters of the soil or the pore-pressures vary considerably within a dam or its foundation and, as noted by Bishop (1957), an analysis based solely upon circular slip surfaces may significantly overestimate the factor of safety. These conditions can arise when:

- (i) the presence of a soft layer in the foundation dictates the path of the slip surface with the lowest factor of safety,
- (ii) different types of soil or rock fill are used and the curvature of the potential failure surface varies with the different strengths or pore-pressures existing within the zones,
- (iii) drainage blankets are used to facilitate the dissipation of pore-pressures.

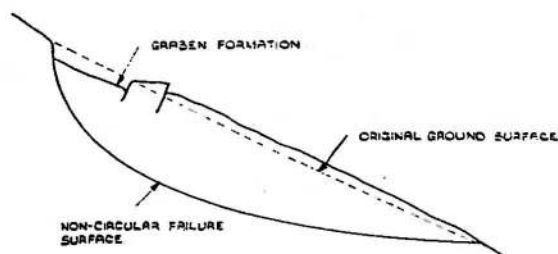


Fig. 1. Graben formation due to non-rotational slip

The influence of the foregoing conditions is illustrated diagrammatically in Fig. 2.

The analysis of the stability of rock slopes also requires the consideration of non-circular sliding surfaces. Here the configuration of the potentially unstable zone is usually dictated by the patterns of joints and fissures in the rock mass and a method of analysis is needed that allows one to determine the factor of safety of a slope with any set of observed planes of weakness.

It is also of interest to investigate whether in the case of a homogeneous soil profile, a surface can be found which gives a lower factor of safety than the conventional slip circle.

CONSIDERATION OF SOME PREVIOUS METHODS

Any method developed to calculate the factor of safety of non-circular slip surfaces must satisfy several requirements if it is to have other than only restricted usage. The method should be able to consider a wide variety of surfaces and any configuration composed of soils with differing shear strength properties. It must also be able to make allowance for complex pore-pressure distributions and hence treat the stability problem in terms of effective stress.

It is on the basis of these requirements that we shall exclude from discussion the method of analysis suggested by Nonveiller (1957). This method is directed towards the analysis of slip surfaces that are concave downwards in a dam composed of two materials and requires an assumption regarding the vertical shear force at the interface between the two materials. The factor of safety, as defined by Nonveiller, is strongly dependent upon this assumption. The method is not used widely.

The condition of limiting equilibrium together with the method of slices provides a satisfactory basis from which a method of sufficient generality may be developed. This has been recognized by Janbu (1954, 1957), Kenney (1956), Janbu, Bjerrum and Kjaernsli (1956), and Sherard (1960), who have developed analyses that consider the limiting equilibrium of a

potential sliding body composed of a series of slices. However, none of these methods satisfies all the equations of equilibrium and computations based upon them are in error by an unknown amount. More particularly, it is the moment equation of equilibrium that remains unsatisfied. It should be noted that this arises from the method of solution rather than from the equations which are in themselves correct.

The main theoretical justification for the method presented in this Paper is that all the equilibrium conditions are satisfied. It is also an application of the method of slices. The solution of the equations is carried out using a digital computer and comparisons will be made

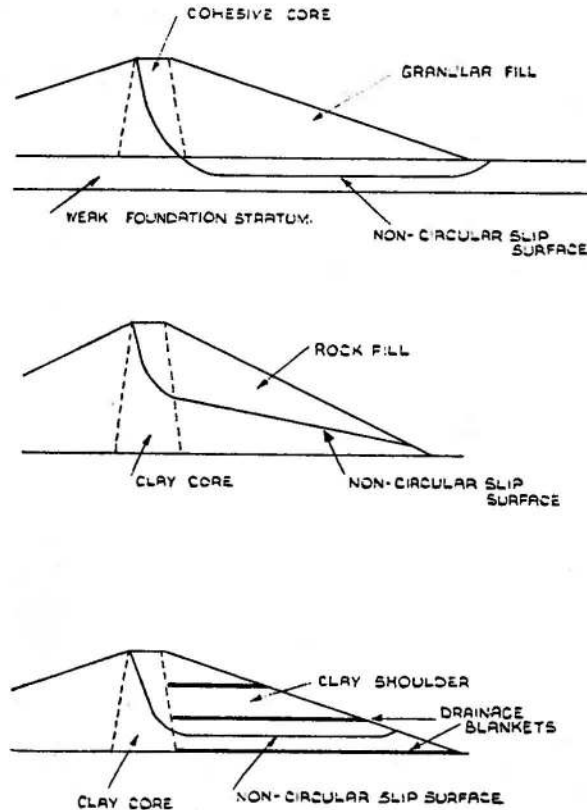


Fig. 2. Some conditions illustrating the occurrence of non-circular slip surfaces

between computations using this method and the methods of Janbu and Kenney. It is not possible to make direct comparisons with Sherard's method because of the difference in the definition of the factor of safety.

THE EQUATIONS OF EQUILIBRIUM

We investigate the equilibrium of the potential sliding mass shown in Fig. 3(a). In this Figure, the equation of the assumed slip surface is $y = y(x)$; the equation of the surface of the slope, which is taken as known, is $y = z(x)$; and the equation of the position of action of the effective horizontal thrust, the line of thrust, is $y = y'_i(x)$. This function is unknown. The line of thrust of the internal water pressure is $y = h(x)$.

The forces acting on an infinitesimal slice of width dx of the potential sliding mass are shown in Fig. 3(b). In this Figure:

- E' denotes the lateral thrust on the side of the slice in terms of effective stresses,
- X denotes the vertical shear force on the side of the slice,
- dW denotes the weight of the slice,
- P_w denotes the resultant water pressure acting on the side of the slice,

dP_b denotes the water pressure on the base of the slice,
 dN' denotes the effective normal pressure,
 dS denotes the shear force acting along the base of the slice, and
 α denotes the inclination of the base of the slice with respect to the horizontal.

The condition that there be no rotation of the slice is satisfied if the sum of the moments about the centre of the base of the slice is equal to zero. By taking moments about the mid-point of the base of the slice we find that:

$$E' \left[(y - y_i) - \left(-\frac{dy}{2} \right) \right] + P_w \left[(y - h) - \left(-\frac{dy}{2} \right) \right] - (E' + dE') \left[y + dy - y_i' - dy_i' + \left(-\frac{dy}{2} \right) \right] - X \frac{dx}{2} - (X + dX) \frac{dx}{2} - (P_w + dP_w) \left[(y + dy) - (h + dh) - \frac{dy}{2} \right] - dP_b \cdot g = 0 \quad (1)$$

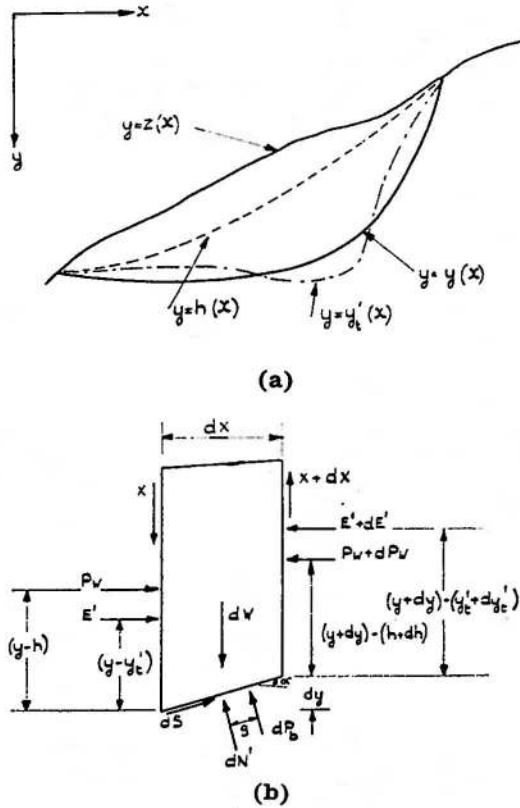


Fig. 3. (a) Potential sliding mass; (b) Forces acting on an infinitesimal slice

After simplifying and proceeding to the limit as $dx \rightarrow 0$ it can be readily shown that:

$$X = \frac{d}{dx} (E' \cdot y_i) - y \frac{dE'}{dx} + \frac{d}{dx} (P_w \cdot h) - y \frac{dP_w}{dx} \quad (2)$$

For equilibrium in the N direction,

$$dN' + dP_b = dW \cos \alpha - dX \cos \alpha - dE' \sin \alpha - dP_w \sin \alpha \quad (3)$$

From equilibrium in the S direction,

$$dS = dE' \cos \alpha + dP_w \cos \alpha - dX \sin \alpha + dW \sin \alpha \quad (4)$$

The Coulomb-Mohr failure criterion in terms of effective stresses may be expressed as:

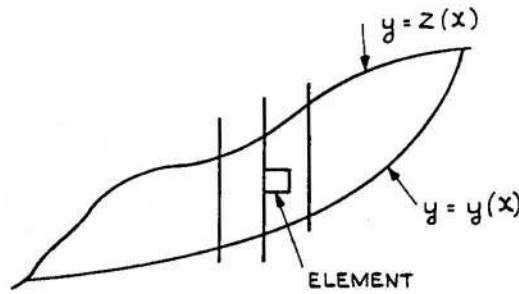
$$dS = \frac{1}{F} [c' dx \sec \alpha + (dN') \tan \phi'] \quad (5)$$

where c' is the cohesion intercept
 ϕ' is the angle of shearing resistance } in terms of effective stresses
 and F denotes the factor of safety

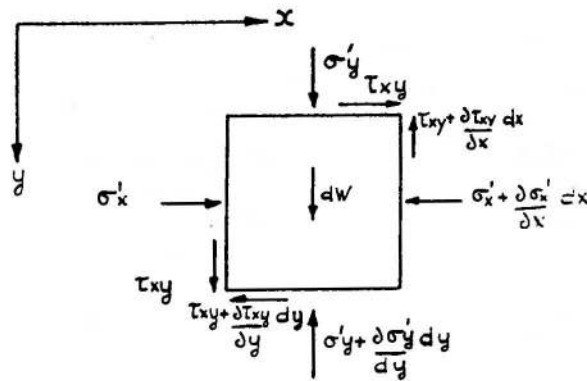
It should be noted that equation (5) also constitutes a definition of the factor of safety. The factor of safety with respect to shear strength has been adopted here. It is that value by which the shear strength parameters must be reduced in order to bring the potential sliding mass into a state of limiting equilibrium. An alternative definition of the factor of safety has been discussed elsewhere (Bishop and Morgenstern, 1960). It is clear that the factor of safety with respect to moment ratios cannot be utilized in non-circular analyses where the shape of the sliding surface is arbitrary.

Eliminating dS from equations (4) and (5) we obtain:

$$\frac{1}{F} [c' dx \sec \alpha + (dN') \tan \phi'] = dE' \cos \alpha + dP_w \cos \alpha - dX \sin \alpha + dW \sin \alpha \quad (6)$$



(a)



(b)

Fig. 4. (a) An element at an interface between two slices; (b) Effective stresses acting on an element

Eliminating dN' from equations (3) and (6), and dividing by $dx \cos \alpha$ it can be shown that:

$$\begin{aligned} \frac{c'}{F} \sec^2 \alpha + \frac{\tan \phi'}{F} \left[\frac{dW}{dx} - \frac{dX}{dx} - \frac{dE'}{dx} \tan \alpha - \frac{dP_w}{dx} \tan \alpha - \frac{dP_b}{dx} \sec \alpha \right] \\ = \frac{dE'}{dx} + \frac{dP_w}{dx} - \frac{dX}{dx} \tan \alpha + \frac{dW}{dx} \tan \alpha \quad (7) \end{aligned}$$

In the specified co-ordinate system, $\tan \alpha = -\frac{dy}{dx}$, and equation (7) becomes:

$$\frac{c'}{F} \left[1 + \left(\frac{dy}{dx} \right)^2 \right] + \frac{\tan \phi'}{F} \left\{ \frac{dW}{dx} - \frac{dX}{dx} + \frac{dE'}{dx} \cdot \frac{dy}{dx} + \frac{dP_w}{dx} \cdot \frac{dy}{dx} - r_u \frac{dW}{dx} \left[1 + \left(\frac{dy}{dx} \right)^2 \right] \right\} = \frac{dE'}{dx} + \frac{dP_w}{dx} + \frac{dX}{dx} \cdot \frac{dy}{dx} - \frac{dW}{dx} \cdot \frac{dy}{dx} \quad \dots \quad (8)$$

where $dP_b = r_u \cdot dW \cdot \sec \alpha \quad \dots \quad (9)$

and r_u is the pore-pressure ratio defined by Bishop and Morgenstern (1960).

Therefore, the two governing differential equations are:

$$X = \frac{d}{dx} (E' \cdot y_t) - y \frac{dE'}{dx} + \frac{d}{dx} (P_w \cdot h) - y \frac{dP_w}{dx} \quad \dots \quad (2)$$

$$\frac{dE'}{dx} \left[1 - \frac{\tan \phi'}{F} \frac{dy}{dx} \right] + \frac{dX}{dx} \left[\frac{\tan \phi'}{F} + \frac{dy}{dx} \right] = \frac{c'}{F} \left[1 + \left(\frac{dy}{dx} \right)^2 \right] + \frac{dP_w}{dx} \left[\frac{\tan \phi'}{F} \frac{dy}{dx} - 1 \right] + \frac{dW}{dx} \left\{ \frac{\tan \phi'}{F} + \frac{dy}{dx} - r_u \left[1 + \left(\frac{dy}{dx} \right)^2 \right] \frac{\tan \phi'}{F} \right\} \quad \dots \quad (10)$$

TREATMENT OF STATICAL INDETERMINACY

If y is specified as some function of x , we have, in general, a statically indeterminate problem involving the unknown functions E' , X and y_t and the two governing differential equations. The indeterminacy arises from our lack of knowledge of the stresses obtaining in the soil mass. If the stresses could be determined, the displacements could be predicted using a representative stress-strain relationship. This would obviate the need of doing limit equilibrium analyses. It is our inability to obtain an adequate stress analysis that justifies the application of limit equilibrium methods. At the same time, not knowing the stresses makes it necessary to invoke an assumption in order to render the problem statically determinate.

There are three classes of assumptions that can be made:

1. The distribution of normal pressure along the sliding surface can be assumed. The friction circle analysis is an example of a method that adopts this type of assumption. We have chosen to eliminate the normal pressure from the equilibrium equations and thereby place the burden of the indeterminacy on the internal forces.
2. An assumption may be made regarding the position of the line of thrust. For example, if

$$y - y_t = a(y - z) \quad \dots \quad (11)$$

the moment equilibrium equation, neglecting the pore-pressure terms, becomes

$$X = E \frac{dy}{dx} - a \frac{d}{dx} [E(y - z)] \quad \dots \quad (12)$$

Equations (12) and (10) now define a statically determinate problem where the magnitudes of a and F must be found by satisfying the appropriate boundary conditions. An attempt has been made to obtain solutions using equations (10) and (12) but ill-conditioned functions, with ensuing numerical difficulties, arise in the solution of the differential equations and it was not found possible to obtain a satisfactory numerical procedure. It should be noticed that the magnitude of a in equation (12) must be determined as part of the solution. If it is specified at the outset as in Kenney's solution (1956) it is not possible to satisfy all equilibrium and boundary conditions.

3. Assumptions may be made regarding the relation between the E' and X forces. If we isolate an element at the interface between two slices, as shown in Fig. 4(a), the effective

stresses acting on this element will be as given in Fig. 4(b). For a specific geometry and slip surface the internal forces are determined by

$$E' = \int_z^y \sigma_x'(y) dy \quad \dots \dots \dots (13)$$

and

$$X = \int_z^y \tau_{xy}(y) dy \quad \dots \dots \dots (14)$$

We may therefore assume that

$$X = \lambda f(x) E' \quad \dots \dots \dots (15)$$

If $f(x)$ is specified the problem is statically determinate and λ and F may be found from a solution of the differential equations that satisfies the appropriate boundary conditions. The function $f(x)$ can take any prescribed form in principle. However, the behaviour of soil imposes certain limitations so that only a certain range of functions will be reasonable in practice. This will be discussed in more detail in a later paragraph. Estimates of the function can be obtained from elastic theory. More reliable field observations of internal stresses in dams will also be useful in estimating the distribution of the internal forces.

To simplify the equations it has been found convenient to define $\lambda f(x)$ by using the total horizontal stress E instead of the effective stress E' . Thus we define

$$E = E' + P_w \quad \dots \dots \dots (16)$$

and the point of application y_t of the total stress by

$$E y_t = E' y'_t + P_w h \quad \dots \dots \dots (17)$$

Then instead of (15) we assume

$$X = \lambda f(x) E \quad \dots \dots \dots (18)$$

THE SOLUTION OF THE EQUATIONS

In order to be able to investigate the stability of a soil mass with any slope and properties it has been assumed that the potential sliding body may be divided into a number of finite slices by vertical lines with co-ordinates $x_0, x_1 \dots x_n$. This division is carried out so that within each slice the portion of the slip surface is linear, the interface between different soil types and pore pressure zones are linear and the function f defined by equation (18) depends linearly on x . Hence within each slice we have

$$y = Ax + B \quad \dots \dots \dots (19)$$

$$\frac{dW}{dx} = px + q \quad \dots \dots \dots (20)$$

and

$$f = kx + m \quad \dots \dots \dots (21)$$

Equation (19) and (20) allow a section with any arbitrary shape to be approximated in the analysis. Equation (21) assumes that the ratio of the internal forces changes linearly over a segment of the sliding body. This assumption is not unduly restrictive because k and m may be chosen to vary from segment to segment and any continuous distribution of internal forces can be approximated in this way.

Using equations (16) to (21), equation (2) becomes

$$X = \frac{d}{dx} (E y_t) - y \frac{dE}{dx} \quad \dots \dots \dots (22)$$

and equation (10) becomes

$$(Kx + L) \frac{dE}{dx} + KE = Nx + P \quad \dots \dots \dots (23)$$

where
$$K = \lambda k \left(\frac{\tan \phi'}{F} + A \right) \quad \dots \dots \dots (23a)$$

$$L = \lambda m \left(\frac{\tan \phi'}{F} + A \right) + 1 - A \frac{\tan \phi'}{F} \quad \dots \dots \dots (23b)$$

$$N = p \left[\frac{\tan \phi'}{F} + A - r_u (1 + A^2) \frac{\tan \phi'}{F} \right] \quad \dots \dots \dots (23c)$$

and
$$P = \frac{c'}{F} (1 + A^2) + q \left[\frac{\tan \phi'}{F} + A - r_u (1 + A^2) \frac{\tan \phi'}{F} \right] \quad \dots \dots \dots (23d)$$

Equation (23) can be integrated across each slice in turn starting with the value $E=0$ at the beginning of the slip surface. If, for each slice, x is measured from the beginning of that slice, then the solution which satisfies

$$E = E_i \text{ when } x = 0 \quad \dots \dots \dots (24)$$

is
$$E = \frac{1}{L + Kx} \left[E_i L + \frac{Nx^2}{2} + Px \right] \quad \dots \dots \dots (25)$$

The value of E at the end of the slice is determined and this gives the starting value of E for the next slice unless the end of the slip surface has been reached. The boundary condition to be satisfied at the end of the slip surface is

$$E = E_n \text{ when } x = x_n \quad \dots \dots \dots (26)$$

where E_n is usually zero

Satisfying equation (25) and its boundary conditions is alone insufficient to ensure complete equilibrium since equation (22) must also be satisfied. This can be done by determining y_i from the values of E and found from equations (25) and (18) provided they satisfy the following necessary condition. By integrating equation (22) we have:

$$M = E(y_i - y) = \int_{z_0}^x \left(X - E \frac{dy}{dx} \right) dx \quad \dots \dots \dots (27)$$

Since in general $M=0$ when $x=x_n$, then:

$$M_n = \int_{z_0}^{x_n} \left(X - E \frac{dy}{dx} \right) dx = 0 \quad \dots \dots \dots (28)$$

If equation (28) is satisfied, values of y_i can be found from equation (27) thereby ensuring that each slice is in moment equilibrium.

Hence in order to find values of λ and F such that all the equations of equilibrium are satisfied, we start with guessed values of λ and F and then integrate across all the slices to obtain the values of E_n and M_n which in general will not both be zero. Then, by a systematic iterative method of modifying λ and F , values are finally obtained for which E_n and M_n are zero. This has been programmed for an electronic computer and the numerical techniques used will be reported elsewhere.

There is one complication, however, which will be described here. In some cases there is more than one pair of values of λ and F which satisfy the equations for a given surface and one has to decide which is the physically meaningful solution. The rule which has been adopted is that the values of λ and F should make the values of $(L + Kx)$ positive for the complete range of x of the slip surface. With this rule, in all the cases examined, unique values of λ and F have been found. From equation (25) it can be seen that if $(L + Kx)$ is zero for any value of x , E becomes infinite at that point. This is not physically possible. Although the values of $(L + Kx)$ are discontinuous across any of the boundaries of the slices x_1, x_2, \dots, x_{n-1} , one would not expect $(L + Kx)$ to change sign across these boundaries. If we consider a very thin

slice round each x_i in which some soil properties are very quickly varying but continuous, then if $(L + Kx)$ changes sign across a discontinuity, it would be zero at some point in the corresponding thin hypothetical region. This indicates that $(L + Kx)$ should be of the same sign for all x , and from experience it appears that it should be positive due to the dominance of the contribution of unity to L in equation (23b).

Some special assumptions relating the X and E forces are of interest. If it is assumed that

$$X = \frac{E dy}{dx} \dots \dots \dots (29)$$

it can be shown that

$$y_i = y \dots \dots \dots (30)$$

and the factor of safety can be found from a modified form of equation (26) alone. The assumption stated by equation (29) is similar to that adopted in the conventional circular analysis which has been discussed by Bishop (1955). Although this assumption simplifies the numerical calculations it is not acceptable because the implied internal forces are likely to be physically inadmissible. There is also no reason for the relation between the internal forces to depend solely upon the inclination of an arbitrarily chosen slip surface.

Another assumption has been suggested by Janbu (1954) and by Sherard (1960) in the form:

$$X = \psi E \dots \dots \dots (31)$$

where ψ is some *specified* constant.

It is clear that overall moment equilibrium cannot be ensured if ψ is specified. It is, however, possible to make the problem statically determinate by assuming:

$$X = \lambda E \dots \dots \dots (32)$$

where λ must be computed together with F from equations (26) and (28).

Before describing some examples of the application of the preceding analysis, it is important to discuss several aspects of soil behaviour that have not been considered in the method and how this omission influences the interpretation of the results. For the analysis to be physically acceptable, not only must the equilibrium and boundary conditions and failure criterion along the slip surface be satisfied but the implied state of stress within the soil mass must also be possible. In particular, the failure criterion within the soil mass above the slip surface must not be violated and, since it is commonly accepted that soils do not take tension, no state of tension must be implied to exist above the slip surface. For an arbitrarily chosen slip surface it is not possible to ensure that all these conditions are satisfied. It is therefore necessary in each particular case to calculate the internal forces and the position of the line of thrust in order to inspect whether the failure criterion is exceeded internally and whether a state of tension is implied. In the former case this is easily done since the allowable ratio between the shear force and the lateral thrust may be readily computed from the failure criterion. To investigate whether tension is implied, the position of the line of thrust may be calculated from equation (22) and if it falls outside of the potential sliding mass tension must exist within. In the discussion of the examples that follows, it will be seen that the factor of safety and the internal forces are relatively insensitive to variation in the assumed function relating the internal forces provided that the function is reasonable. Hence, for an arbitrarily chosen slip surface it may not be possible to obtain physically admissible internal forces and in such a case it must be concluded that the surface itself is unlikely.

The selection of a reasonable function also requires some comment. As mentioned earlier, functions can be estimated from available elasticity solutions using equations (13) and (14). Alternatively, they may be specified on the basis of the intuitive assumption that for most cross-sections the higher the rate of curvature of the slip surface, the greater the ratio between

the shear and horizontal forces at the slice interface. The pore pressures in the soil mass constitute another factor to be considered when estimating the function. In particular, if the slice interface is in a zone of high pore pressure the amount of shear that could be mobilized would be reduced accordingly and the function should therefore take a lower value in this region. Ultimately, reliable field measurements of internal stresses will provide the best guide to estimating this function. In the following examples, comparisons are made for particular cross-sections using functions that are considered reasonable and some that have been chosen arbitrarily.

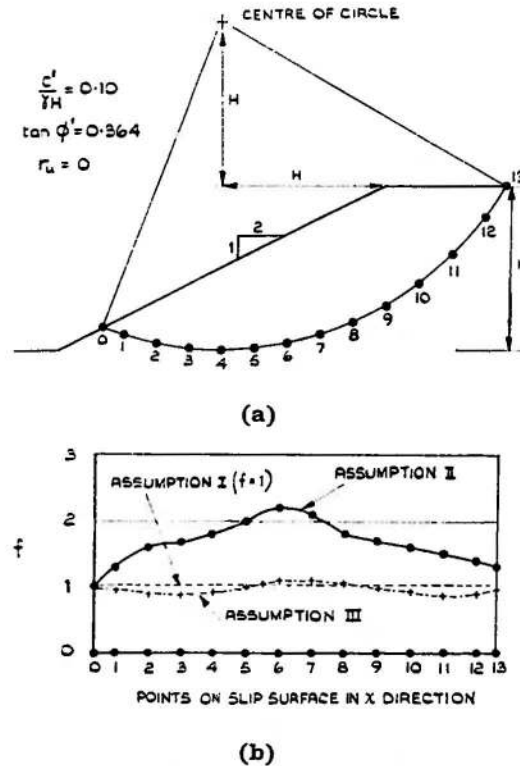


Fig. 5. (a) Example of a circular slip surface; (b) Three assumed functions

SOME APPLICATIONS OF THE ANALYSIS OF GENERAL SLIP SURFACES

1. A circular slip surface

It has been shown, that to all practical purposes, the friction circle method of analysis and slip circle analysis, as developed by Bishop (1955), give the same factors of safety for a given circle when the problem is considered in terms of effective stresses (Bishop and Morgenstern, 1960). However the routine analysis presented by Bishop being the first stage of a more complete iterative process does not satisfy statical equilibrium. In particular, the influence of the X forces is neglected. Nevertheless, the fact that the two methods give the same factor of safety while implying different internal force distributions suggests that for circular slip surfaces, the factor of safety is relatively insensitive to the distribution of the internal forces, as discussed in detail by Bishop (1955).

By assuming different relationships between the X and E forces and computing the factor of safety for a potential circular slip surface it is possible to demonstrate this insensitivity. The calculations have been carried out using the general slip surface analysis and hence the property that the slip surface is circular has not been used explicitly.

A slip-circle analysis has first been carried out on an electronic computer for the circle shown in Fig. 5(a) using the programme described by Little and Price (1958). The factor of

safety was found to be 2.098. Three different assumptions regarding the relationship between the internal forces are shown in Fig. 5(b). All three assumptions have been selected arbitrarily. The values of F and λ found from the three non-circular analyses calculated using these relationships are summarized in Table 1 together with the factor of safety obtained from the slip-circle analysis.

Table 1. Analysis of a circular slip surface

Case	F	λ
Slip-circle analysis . .	2.098	—
Generalized method (Assumption I) . .	2.045	-0.256
Generalized method (Assumption II) . .	2.136	-0.208
Generalized method (Assumption III) . .	2.134	-0.393

Although the fact that the values of F with assumptions II and III are so similar is a coincidence the results illustrate that, at least for a circular surface in a homogeneous section, the factor of safety is not strongly dependent upon the internal distribution and that the assumptions in Bishop's circular arc analysis provide reasonable results.

2. A non-circular surface in a homogeneous section

Fig. 6(a) illustrates an assumed non-circular slip surface in a section composed of a single soil type having only one pore-pressure ratio. This section has also been studied by Kenney (1956) who found that the factor of safety was approximately 1.7 using his method.* Using Kenney's figures for his first iteration, the factor of safety given by Janbu's analysis is readily shown to be 1.73 (Janbu, Bjerrum and Kjaernsli, 1956). Two distributive assumptions have been considered in the computer analysis and they are given in Fig. 6(b). The second assumption is considered reasonable. The factor of safety for both cases was found to be almost the same. The results are tabulated in Table 2 and the computed internal forces for both cases are plotted in Fig. 6(c).

Table 2. Analysis of a non-circular surface in a homogeneous section

Case	F	
Kenney's analysis . .	1.66 - 1.72	
Janbu's analysis . .	1.73	
Generalized method (Assumption I) . .	1.592	-0.328
Generalized method (Assumption II) . .	1.609	-1.190

It is clear that both Kenney's and Janbu's methods overestimate the factor of safety on the unsafe side, but by less than 8%.

* In his analysis Kenney assumed several positions for the line of thrust and found that the factor of safety varied between 1.66 and 1.72.

As in the case of the slip-circle analysis, the factor of safety for this surface is not sensitive to the assumption of the relationship between the internal forces. This will not always be the case. The influence of this assumption depends to a large degree upon the shape of the slip surface and the variation of strength parameters and pore-pressures along it.

The line of thrust compatible with the calculations based upon the second assumption has also been computed. It is shown by the dotted line in Fig. 6(a). The numbers in parentheses give the ratio of the distance between the slip surface and the line of thrust to the height of the slice at each interface. It is apparent that no internal tensional stresses are implied and

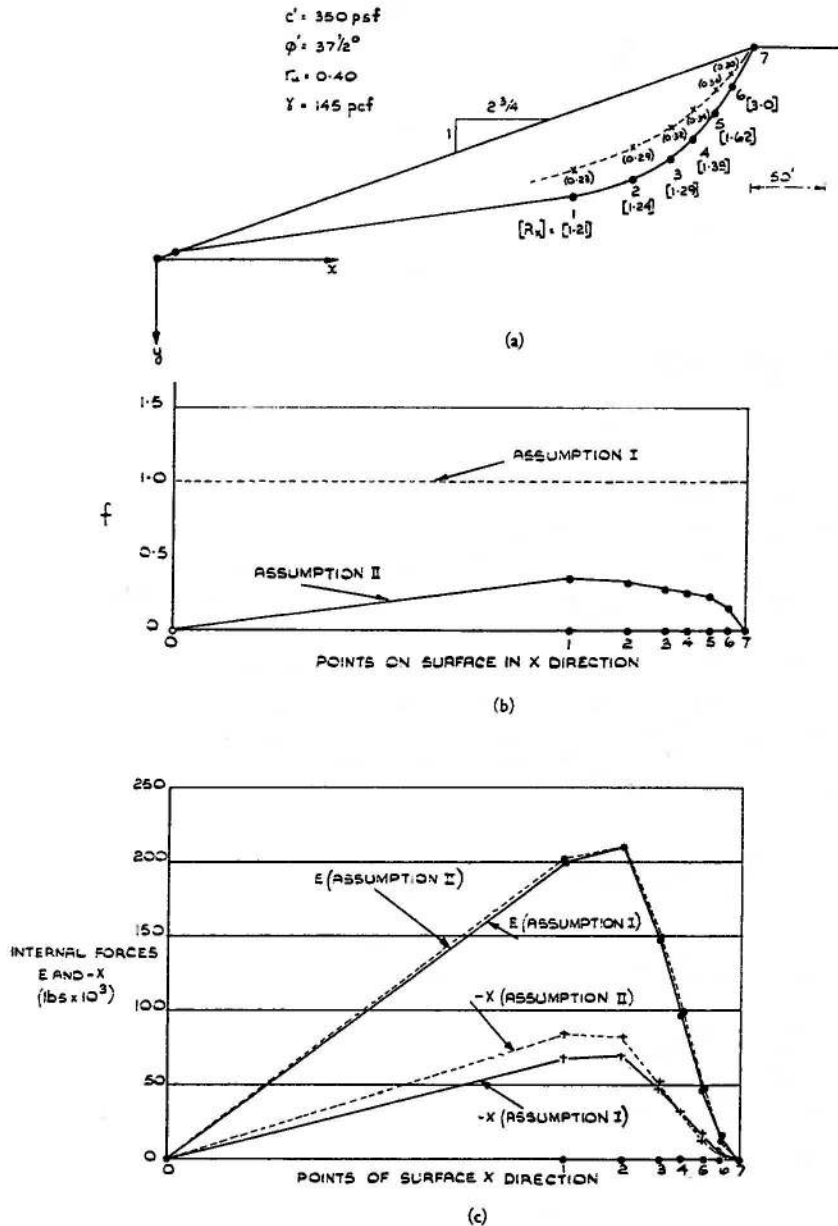


Fig. 6. (a) Non-circular slip surface in a homogeneous section; (b) Two assumed functions; (c) Internal force distributions

that the computed internal forces are physically acceptable on the basis of this criterion. They are not, of course, the only admissible set of internal forces.

The yield criterion within the sliding soil mass would be violated if the calculated shear force at a slice interface necessary for equilibrium exceeded the shearing resistance that could be mobilized along the interface. Since the total normal force acting on the interface has been calculated and since the pore pressures, and strength parameters obtaining along the interface are also known the available shearing resistance for the second assumption may also be computed. They are compared with the calculated X forces in Fig. 6(a) where the ratio R_x of the available shearing resistance to the shear force required for equilibrium is given. All values lie above unity and therefore the yield criterion is satisfied within the sliding mass.

It should be noted that the shear and normal stresses that must be acting on a critical surface are not the same as those actually in the soil mass. However, it is not possible in general to make comparisons because of the lack of knowledge of stress distributions computed using representative stress-strain relations.

3. A typical earth dam section

The non-circular surface specified in Fig. 7(a) is typical of the type of surface that should be analysed during the design of an earth dam on a foundation with low shear strength properties.

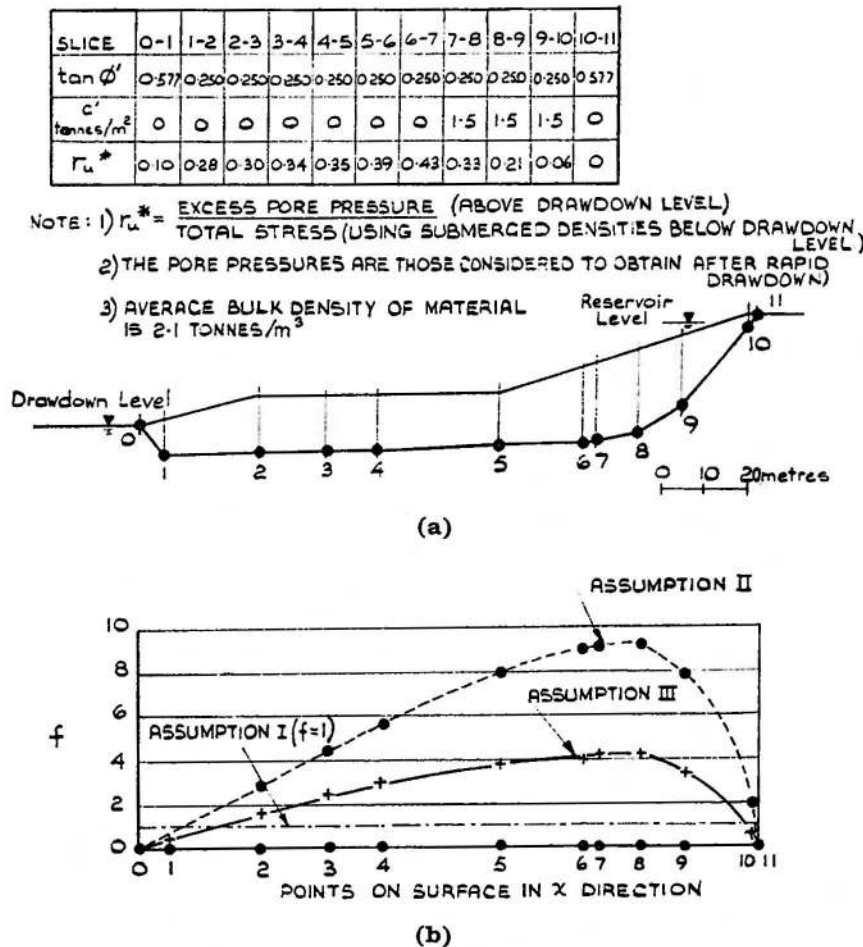


Fig. 7. (a) Typical non-circular slip surface in a dam with a weak foundation; (b) Three assumed functions

In this example the pore-pressures are those that might obtain after a rapid drawdown of the reservoir. An analysis of this surface using Janbu's method gave a factor of safety of 1.28 (Janbu, Bjerrum and Kjaernsli, 1956). Three distributions have been assumed for the

internal forces for analysis on the computer. These distributions are shown in Fig. 7(b). The results of the calculations are presented in Table 3.

Table 3. Analysis of a non-circular surface in typical earth dam

Case	F	λ
Janbu's analysis — —	1.28	—
Generalized method (Assumption I) — —	1.346	-0.132
Generalized method (Assumption II) — —	1.282	-0.022
Generalized Method (Assumption III) — —	1.279	-0.046

For this case we see that Janbu's method gives a factor of safety that is the same as that obtained using a sensible distribution. The factor of safety from the computer analysis is also reasonably insensitive to the distributive assumption. Not enough computations have been carried out so far to enable one to make general statements about the influence of this assumption on the factor of safety. However, it is significant that in this example which is representative of actual design considerations the influence appears to be slight for reasonable assumptions.

CONCLUSIONS

A method has been developed for the determination of the factor of safety of a sliding body of any shape containing materials with varying shear strength parameters and pore-pressures. It is based solely upon the principles of limiting equilibrium. Not only must the shape of the potential slip surfaces be chosen but an assumption must also be made regarding the distribution of internal forces. The factor of safety does not appear to be very sensitive to this assumption. Reasonable assumptions can be inferred either from a knowledge of the approximate internal stress distribution or from field observations of internal stresses. The method is useful in the design of earth dams or the analysis of the stability of natural slopes and cuttings. The solution ensures that all equilibrium and boundary conditions are satisfied. Comparison in two cases with other existing solutions reveals that they can be in error by as much as 8% on the unsafe side. Further computations are required to investigate the occurrence of these differences in more detail.

The use of internal stress measurements to aid in the determination of the relationship between the internal forces plays a role analogous to carrying out effective stress calculations where the influence of the pore-pressure is considered in the analysis. One of the main advantages of an effective stress analysis is that the factor of safety depends upon the magnitude of the pore-pressures. If these have been predicted, it is possible to observe them and hence corroborate, in part, the design calculations. The use of internal stress measurements to help define the possible relationships between the internal forces means that computations are based upon another measurable quantity and, to that degree, may be further substantiated by field observations.

In the case of a circular slip surface treated as a case of a non-circular analysis, it was shown that both the method presented here and that suggested by Bishop (1955) gave approximately the same result. Furthermore, the factor of safety in this case, as well as in the two other examples discussed, is insensitive to varying the relationship between the internal forces.

Both earth pressure and bearing capacity problems can also be considered in terms of limit equilibrium analyses. It would be possible to extend the solution developed here to the computation of earth pressure coefficients and bearing capacity factors in a manner similar to that described by Janbu (1957).

The influence of earthquakes could also be introduced into the analysis. This is commonly done in limit equilibrium methods by considering an inclined body force, some fraction of gravity, acting on each slice. This force can be incorporated into the equations of equilibrium. However the shear stress distribution within a soil mass subject to an earthquake will also change and it would be necessary to use an assumption for the internal forces in this case which differs from that used in the static case.

ACKNOWLEDGEMENT

The Authors gratefully acknowledge the assistance of Mrs J. Skinner and other members of the staff of the English Electric-Leo London Computing Service in programming the solution and carrying out the computations. The Authors are also grateful for many helpful discussions with Dr A. W. Bishop.

This study was supported in its early stages by a research grant awarded by the Department of Scientific and Industrial Research.

REFERENCES

- BISHOP, A. W., 1955. "The use of the slip circle in the stability analysis of earth slopes." *Géotechnique*, 5:1:7-17.
- BISHOP, A. W., 1957. Contribution to discussion on "The Usk scheme for the water supply of Swansea". G. A. R. Sheppard and L. B. Ayles, *Proc. Instn civ. Engrs*, 7:281.
- BISHOP, A. W., and N. MORGENSTERN, 1960. "Stability coefficients for earth slopes." *Géotechnique*, 10:4:129-150.
- COOLING, L. F., and H. O. GOLDBER, 1942. "The analysis of the failure of an earth dam during construction." *J. Instn civ. Engrs*, 19:38-55.
- HUTCHINSON, J. N., 1961. "A landslide on a thin layer of quick clay at Furre, Central Norway." *Géotechnique*, 11:2:69-94.
- JAKOBSON, B., 1952. "The landslide at Surte on the Gota River." *Proc. Roy. Swedish Geotech. Inst. No. 5*, 120 pp.
- JANBU, N., 1954. "Application of composite slip surfaces for stability analysis." *Proc. European Conf. on Stability of Earth Slopes, Stockholm*, 3:43-49.
- JANBU, N., 1957. "Earth pressure and bearing capacity by generalized procedure of slices." *Proc. Fourth Int. Conf. Soil Mech.*, 2:207-212.
- JANBU, N., L. BJERRUM, and B. KJAERNSLI, 1956. "Soil mechanics applied to some engineering problems." *Norwegian Geotech. Inst., Pub. No. 16*.
- KENNEY, T. C., 1956. "An examination of the methods of calculating the stability of slopes." *M.Sc. Thesis, University of London*.
- LEGGETT, R. F., 1962. "Geology and engineering." (2nd Ed.) p. 426, *McGraw Hill, New York*.
- LITTLE, A. L., and V. E. PRICE, 1958. "The use of an electronic computer for stability analysis." *Géotechnique*, 8:3:113-120.
- NONVEILLER, E., 1957. "Stabilnost nehomogenih nasipa." *Hidroteknički Institut, Belgrade*.
- SEVALDSON, R. A., 1956. "The slide in Lodalen, October 6, 1954." *Géotechnique*, 6:4:167.
- SHERARD, J. L., 1960. "An investigation of the influence of side forces in 'slice' and 'wedge' methods of stability analysis for earth dams." *Unpublished report*.
- TWORT, A. C., 1964. "The new Tittesworth dam." *J. Instn Water Eng.*, 18:125-179.
- VARNES, D. J., 1958. "Landslide types and processes" in "Landslides and engineering practice". Ed. by E. B. Eckel, *Highway Research Board, Special Report 29, Washington, D.C.*

