THE ANALYTIC-FUNCTIONAL CALCULUS FOR SEVERAL COMMUTING OPERATORS

BY

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In [15] we introduced a notion of spectrum for a commuting tuple of operators on a Banach space. The main objective of this paper is to develop a corresponding analytic functional calculus.

If X is a Banach space and $\alpha=(a_1, \ldots, a_n)$ a commuting tuple of operators on X, then associated with α is a certain chain complex (the Koszul complex). If this chain complex is exact then we say α is non-singular. The spectrum, $\mathrm{Sp}(\alpha, X)$, of α on X is defined to be the set of $z\in\mathbf{C}^n$ such that the tuple $z-\alpha=(z_1-a_1, \ldots, z_n-a_n)$ is singular (cf. [15], § 1). The set $\mathrm{Sp}(\alpha, X)$, defined in this manner, is compact and non-empty and has several other properties that cause it to deserve the name spectrum (cf. [15], § 3).

Classically, one would define the spectrum of α in terms of some commutative Banach algebra of operators containing $a_1, \ldots, a_n$. If A is such an algebra, then α is non-singular relative to A if the equation

$$a_1 b_1+\ldots+a_n b_n=\mathrm{id}$$

has a solution for $b_1, \ldots, b_n\in A$. A point $z\in\mathbf{C}^n$ is in the spectrum of α relative to A ($\mathrm{Sp}_A(\alpha)$) if $z-\alpha$ is singular relative to A.

One disadvantage of the classical notion of joint spectrum is that it depends intrinsically on the algebra A and it is not clear that there is an optimum choice for A. Furthermore, it may be very difficult to decide whether or not equation (1) has a solution in a given situation; hence, $\mathrm{Sp}_A(\alpha)$ may be very difficult to compute.

The spectrum, $\mathrm{Sp}(\alpha, X)$, that we have chosen does not involve questions of solvability

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of an operator equation like (1). It is based on a notion of non-singularity which is a direct generalization to several operators of the idea of a single operator being both injective and surjective. Furthermore, if A is any commutative Banach algebra containing $a_1, \ldots, a_n$, then $\mathrm{Sp}(\alpha, X) \subset \mathrm{Sp}_A(\alpha)$. There are examples where this containment is proper regardless of how A is chosen (cf. [15], § 4).

Of course, the functional calculus is the main reason behind any interest that exists in a notion of spectrum. If A is a commutative Banach algebra, $a_1, \ldots, a_n \in A$, and $\mathfrak{A}(\mathrm{Sp}_A(\alpha))$ denotes the algebra of functions analytic in a neighborhood of $\mathrm{Sp}_A(\alpha)$, then there is a homomorphism $f \to f(\alpha)$ of $\mathfrak{A}(\mathrm{Sp}_A(\alpha))$ into A such that $1(\alpha) = \mathrm{id}$ and $z(\alpha) = a_i$ for $i = 1, \ldots, n$. This very deep and useful result was proved by Shilov [14] for finitely generated algebras and by Arens–Calderon [4], Arens [3], and Waelbrock [16] for general algebras. Adaptations of the Cauchy–Weil integral formula provide the basis for [14] and [3].

Our main purpose here is to prove that a version of the Shilov–Arens–Calderon Theorem remains valid for tuples of operators with the spectrum chosen as $\mathrm{Sp}(\alpha, X)$. Since $\mathrm{Sp}(\alpha, X)$ is generally smaller than spectra defined in other ways, we obtain a richer functional calculus. The theorem is as follows: there is a homomorphism $f \to f(\alpha)$ of $\mathfrak{A}(\mathrm{Sp}(\alpha, X))$ into $(\alpha)''$, the algebra of all operators on X that commute with all operators commuting with each a_i, such that $1(\alpha) = \mathrm{id}$ and $z_i(\alpha) = a_i$. (Theorem 4.3 and Corollary 4.4.)

As in [3], we shall obtain the map $f \to f(\alpha)$ via an abstract form of the Cauchy–Weil integral. However, the version of the Cauchy–Weil integral we use here, and the methods we use to obtain it, appear to be quite new and possibly of independent interest. In fact, a second major objective of this paper is to give a detailed development of a quite general form of the Cauchy–Weil integral. We pause to outline some of the features of this development.

Let X be a Banach space, U a domain in $\mathbf{C}^n$, and $\alpha = (a_1, \ldots, a_n)$ a commuting tuple of analytic operator valued functions on U with values in the space $L(X)$ of bounded linear operators on X. If the tuple $\alpha(z) = (a_1(z), \ldots, a_n(z))$ is non-singular for $z \in U \setminus K$, where K is some compact subset of U, then in § 3 we define a continuous linear map

$$f \to \int_U R_{\alpha(z)} f(z) \wedge dz_1 \wedge \ldots \wedge dz_n \tag{2}$$

from the space of analytic X-valued functions on U to X. This map satisfies several transformation laws which justify calling it a Cauchy–Weil integral. Furthermore, the expression $\int_U R_{\alpha(z)} f(z) \wedge dz_1 \wedge \ldots \wedge dz_n$ depends analytically (continuously) on any parameter on which α and f depend analytically (continuously). If α is the scalar valued tuple $\alpha(z) = z - w = (z_1 - w_1, \ldots, z_n - w_n)$ for some $w \in U$, then we obtain the Cauchy–Weil integral formula

$$f(w)=\frac{1}{(2\pi i)^n}\int_U R_{z-w}f(z)\wedge dz_1\wedge\ldots\wedge dz_n.$$

Versions of the Cauchy–Weil integral have been published by many authors, notably Weil [17], Arens [3], and Gleason [8]. In Arens' version $a_1, \ldots, a_n$ and f are allowed to have values in a Banach algebra A. He assumes that $a_1(z)b_1+\ldots+a_n(z)b_n=\mathrm{id}$ has a solution in A for each $z\in U\setminus K$. Our version appears considerably stronger since we make no assumptions regarding the solvability of such an equation.

Our development of the Cauchy–Weil integral is almost entirely algebraic. In § 1 we use the formalism of exterior algebra to construct a map $f\to R_\alpha f$ which supplies the integrand of the Cauchy–Weil integral. We construct the map R_α in a quite general algebraic context. In order to apply the results of § 1 to the situation which interests us, it is necessary to construct in § 2 a special space of vector valued functions which we call $\mathfrak{B}(U, X)$. If we were using a classical notion of spectrum based on solvability of an operator equation like (1), then it would suffice to use for $\mathfrak{B}(U, X)$ the space of C^∞ X-valued functions on U. Section 2 would not be necessary in this situation.

In § 3 we apply the results of § 1 to the space constructed in § 2 and obtain our Cauchy–Weil integral. An interesting feature of our construction is this: we do not use geometric integration theory; the only integration that appears is ordinary Lebesgue integration in $\mathbf{C}^n$. All of the combinatorial considerations involved in the Cauchy-Weil integral are taken care of by the algebraic construction of R_α in § 1. This approach makes it very easy to obtain a variety of transformation laws for the Cauchy–Weil integral. We also obtain an analogue of Fubini's Theorem which relates an iterated Cauchy–Weil integral to a double integral.

In § 4 obtain the analytic functional calculus and investigate some of its properties. The functional calculus makes it possible, in § 5, to obtain geometric relationships between $\mathrm{Sp}(\alpha, X)$ and $\mathrm{Sp}_A(\alpha)$ for various choices of a Banach algebra A.

1. Algebraic machinery

In this section we distill the algebraic portion of our development of the Cauchy–Weil integral. Our basic tool is elementary exterior algebra. We refer the reader to MacLane and Birkoff [12], Chapter XVI, for background in this area.

We shall work in the context of modules over a commutative ring. Although eventually we shall be concerned with modules which arise as function spaces with considerable additional structure, the discussion is much simpler if we ignore the additional structure at this stage.

We shall freely use terminology and elementary results from the theory of cochain complexes. We shall attempt to use terminology consistent with [11], Chapter 2, with one exception: a cochain map will be a homomorphism of graded modules which commutes with the coboundary map—we do not insist that it be of degree zero.

Throughout this section K will be a fixed commutative ring with identity. Modules will be K-modules unless otherwise specified. Tensor product will mean tensor product over K.

Notation 1.1. If $\sigma=(s_1, ..., s_n)$ is an n-tuple of indeterminates, then $\Lambda[\sigma]$ will denote the exterior algebra (over K) with generators $s_1, ..., s_n$, while $\Lambda^p[\sigma]$ will denote the module consisting of elements of degree p in $\Lambda[\sigma]$ (cf. [12], XVI, § 6).

If X is any module, then $X\otimes\Lambda[\sigma]$ and $X\otimes\Lambda^p[\sigma]$ will be denoted by $\Lambda[\sigma, X]$ and $\Lambda^p[\sigma, X]$ respectively. An element $x\otimes s_{j_1}\wedge\ldots\wedge s_{j_p}\in\Lambda^p[\sigma, X]$ will be written simply as $xs_{j_1}\wedge\ldots\wedge s_{j_p}$. Note that $\Lambda[\sigma, X]=\{\Lambda^p[\sigma, X]\}_p$ is a graded module (cf. [11] or [12]).

If A is an algebra (over K) then $\Lambda[\sigma, A]$ has the structure of a graded algebra under the operation $(\alpha, \beta)\to\alpha\wedge\beta$, where if

$$\alpha=\sum_{j_1\ldots j_p} a_{j_1\ldots j_p} s_{j_1}\wedge\ldots\wedge s_{j_p}\in\Lambda^p[\sigma, A]$$

and

$$\beta=\sum_{k_1\ldots k_q} b_{k_1\ldots k_q} s_{k_1}\wedge\ldots\wedge s_{k_q}\in\Lambda^q[\sigma, A]$$

then

$$\alpha\wedge\beta=\sum_{j_1\ldots k_q}(a_{j_1\ldots j_p} b_{k_1\ldots k_q}) s_{j_1}\wedge\ldots\wedge s_{j_p}\wedge s_{k_1}\wedge\ldots\wedge s_{k_q}\in\Lambda^{p+q}[\sigma, A].$$

Of course, if A fails to be commutative then $\Lambda[\sigma, A]$ will not be an exterior algebra, i.e.: it will not be true that $\alpha\wedge\alpha=0$ for every α. In fact, if $\alpha=a_1s_1+\ldots+a_ns_n\in\Lambda^1[\sigma, A]$, then $\alpha\wedge\alpha=\sum_{i<j}(a_i a_j-a_j a_i) s_i\wedge s_j$.

If X is a module and A an algebra of endomorphisms of X, then the graded algebra $\Lambda[\sigma, A]$ acts on the graded module $\Lambda[\sigma, X]$ via the operation $(\alpha, \psi)\to\alpha\psi=\alpha\wedge\psi$, where if

$$\alpha=\sum a_{j_1\ldots j_p} s_{j_1}\wedge\ldots\wedge s_{j_p}\in\Lambda^p[\sigma, A]$$

and

$$\psi=\sum x_{k_1\ldots k_q} s_{k_1}\wedge\ldots\wedge s_{k_q}\in\Lambda^q[\sigma, X]$$

then

$$\alpha\wedge\psi=\sum (a_{j_1\ldots j_p} x_{k_1\ldots k_q}) s_{j_1}\wedge\ldots\wedge s_{j_p}\wedge s_{k_1}\wedge\ldots\wedge s_{k_q}\in\Lambda^{p+q}[\sigma, X].$$

If $\alpha=a_1s_1+\ldots+a_ns_n\in\Lambda^1[\sigma, A]$ then $\psi\to\alpha\wedge\psi$ is a graded module homomorphism of degree 1 (i.e.: α maps $\Lambda^p[\sigma, X]$ to $\Lambda^{p+1}[\sigma, X]$ for each p). If $\alpha\wedge\alpha=0$ then α acts as a coboundary operator on $\Lambda[\sigma, X]$. This happens if and only if $a_i a_j=a_j a_i$ for $i, j=1, ..., n$.

Definition 1.2. If X is a module then $\mathcal{E}(X)$ will denote its algebra of endomorphisms. An element $\alpha=a_1s_1+\ldots+a_ns_n\in\Lambda^1[\sigma, \mathcal{E}(X)]$ will be called commutative if $\alpha\wedge\alpha=0$.

If $\alpha \in \Lambda^1[\sigma, \mathcal{E}(X)]$ is commutative, then we denote by $F(X, \alpha)$ the cochain complex whose graded module is $\Lambda[\sigma, X]$ and whose coboundary operator is α. The cohomology of $F(X, \alpha)$ is the graded module $H(X, \alpha) = \{H^p(X, \alpha)\}$, where $H^p(X, \alpha) = \ker\{\alpha: \Lambda^p[\sigma, X] \to \Lambda^{p+1}[\sigma, X]\}/\operatorname{Im}\{\alpha: \Lambda^{p-1}[\sigma, X] \to \Lambda^p[\sigma, X]\}$.

We shall say that $\alpha = a_1 s_1 + \ldots + a_n s_n$ is non-singular on X if the complex $F(X, \alpha)$ is exact, i.e., if $H^p(X, \alpha) = 0$ for each p.

The reader who is familiar with [15] will note that we seem to be changing horses in mid-stream. In [15] we declared a tuple $\alpha = (a_1, \ldots, a_n)$ of endomorphisms of X to be non-singular if a certain chain complex $E(X, \alpha)$ (the Koszul complex) was exact. In the above definition we have said α is non-singular if the cochain complex $F(X, \alpha)$ is exact. However, it is trivial to check that if the grading in $F(X, \alpha)$ is reversed by replacing p by $n - p$, then one obtains a chain complex isomorphic to $E(X, \alpha)$. Hence, which of these we use is purely a matter of convenience. We have chosen to use $F(X, \alpha)$ here because it not only formally resembles a complex of differential forms, but in some situations in § 3 it will actually be a complex of differential forms. In these situations it would be needlessly confusing to depart from the standard symbolism for differential forms.

For convenience, we now restate three lemmas from [15] that we shall have several occasions to use.

LEMMA 1.3. (Lemma 1.3 of [15].) *Let $a_1, \ldots a_n, b_1, \ldots, b_m$ be mutually commuting endomorphisms of X and set $\alpha = a_1 s_1 + \ldots + a_n s_n$, $\beta = b_1 t_1 + \ldots + b_m t_m$, $\alpha \oplus \beta = a_1 s_1 + \ldots + a_n s_n + b_1 t_1 + \ldots + b_m t_m$ where $\sigma = (s_1, \ldots, s_n)$, $\tau = (t_1, \ldots, t_m)$, and $\sigma \cup \tau = (s_1, \ldots, t_m)$ are tuples of indeterminates. If α is non-singular on X, then so is $\alpha \oplus \beta$.*

LEMMA 1.4. (Lemma 1.1 of [15]). *If $\alpha = a_1 s_1 + \ldots + a_n s_n$ where $(a_1, \ldots, a_n)$ is a commuting n-tuple of endomorphisms of X, then $H(X, \alpha)$ may be considered a graded left module over the algebra A of elements of $\mathcal{E}(X)$ that commute with each a_i (since* $\ker\{\alpha: \Lambda^p[\sigma, X] \to \Lambda^{p+1}[\sigma, X]\}$ *and* $\operatorname{Im}\{\alpha: \Lambda^{p-1}[\sigma, X] \to \Lambda^p[\sigma, X]\}$ *are invariant under A for each p). Under this action of A on $H(X, \alpha)$ we have $a_i H(X, \alpha) = 0$ for $i = 1, \ldots, n$. Hence, if the ideal generated by $a_1, \ldots, a_n$ in A is A, then $H(X, \alpha) = 0$ and α is non-singular on X.*

LEMMA 1.5. (Lemma 1.2 of [15]). *Let X, Y, and Z be modules over a fixed algebra A, $a_1, \ldots, a_n$ be mutually commuting elements of A, and $\alpha = a_1 s_1 + \ldots + a_n s_n$. If $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is a short exact sequence of A-modules, then there is a corresponding short exact sequence $0 \to F(X, \alpha) \xrightarrow{u^\wedge} F(Y, \alpha) \xrightarrow{v^\wedge} F(Z, \alpha) \to 0$* of cochain maps ($u^\wedge = u \otimes \mathrm{id}: \Lambda[\sigma, X] \to \Lambda[\sigma, Y]$ and $v^\wedge = v \otimes \mathrm{id}: \Lambda[\sigma, Y] \to \Lambda[\sigma, Z]$), *which induces a long exact sequence*

$$\ldots \to H^p(X, \alpha) \to H^p(Y, \alpha) \to H^p(Z, \alpha) \to H^{p+1}(X, \alpha) \to \ldots$$

of cohomology. Hence, if α is non-singular on any two of X, Y, and Z then it is non-singular on the third as well.

Our development of the Cauchy–Weil integral will be based on properties of certain transformations of $\Lambda[\sigma, X]$. Most of these transformations fall into a special category which we now describe.

Definition 1.6. Let X and Y be modules and $\sigma=(s_1, \ldots, s_n)$, $\tau=(t_1, \ldots, t_m)$ tuples of indeterminates. By a special transformation $u:\Lambda[\sigma, X]\to\Lambda[\tau, Y]$ we shall mean a graded module homomorphism (of degree zero) determined by a homomorphism $u^0: X\to Y$ and an $m\times n$-matrix (u_{ij}) of commuting elements of $\mathcal{E}(Y)$ in the following way:

$$u(\sum x_{j_1\ldots j_p} s_{j_1}\wedge\ldots\wedge s_{j_p})=\sum u^0(x_{j_1\ldots j_p})\,u(s_{j_1})\wedge\ldots\wedge u(s_{j_p}),$$

where
$$u(s_j)=u_{1j}t_1+\ldots+u_{mj}t_m \quad \text{for } j=1,\ldots,n,$$

(Here we agree that $yu_{ij}=u_{ij}(y)$ for $y\in Y$).

Note that if $1\leqslant j_1<j_2<\ldots<j_p\leqslant n$ and $x\in X$

then
$$u(xs_{j_1}\wedge\ldots\wedge s_{j_p})=u^0(x)\,u(s_{j_1})\wedge\ldots\wedge u(s_{j_p})$$

$$=u^0(x)\,(u_{1j_1}t_1+\ldots+u_{mj_1}t_m)\wedge\ldots\wedge(u_{1j_n}t_1+\ldots+u_{mj_n}t_m)$$

$$=\sum_{j_1\ldots i_p}(u_{i_1j_1}\ldots u_{i_pj_p}u^0(x))\,t_{i_1}\wedge\ldots\wedge t_{i_p}$$

$$=\sum_{i_1<i_2\ldots<i_p}(\det(u_{i_kj_l})_{k,l}\,u^0(x))\,t_{i_1}\wedge\ldots\wedge t_{i_p},$$

if u is a special transformation determined by $(u^0, \{u_{ij}\})$.

LEMMA 1.7. *Let u: $\Lambda[\sigma, X]\to\Lambda[\tau, Y]$ be a special transformation. Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots b_m)$ be commuting tuples of elements of $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ respectively and set $\alpha=a_1s_1+\ldots+a_ns_n$ and $\beta=b_1t_1+\ldots+b_mt_m$. Then u is a cochain map from $F(X,\alpha)$ to $F(X,\beta)$ if and only if the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \Lambda^1[\sigma,X] \\ \downarrow{\scriptstyle u^0} & & \downarrow{\scriptstyle u^1} \\ Y & \xrightarrow{\beta} & \Lambda^1[\tau,Y] \end{array}$$

is commutative, where u^1 is u restricted to $\Lambda^1[\sigma, X]$. Of course, in this case u induces a homomorphism u^*: $H(X,\alpha)\to H(Y,\beta)$ of cohomology (cf. [11], Chapter 2).

Proof. We must prove that the diagram

$$\begin{array}{ccc} \Lambda^p[\sigma, X] & \xrightarrow{\alpha} & \Lambda^{p+1}[\sigma, X] \\ u \downarrow & & u \downarrow \\ \Lambda^p[\tau, Y] & \xrightarrow{\beta} & \Lambda^{p+1}[\tau, Y] \end{array} \tag{1.1}$$

is commutative for every p if it is commutative for $p=0$.

Hence, let $\psi = xs_{j_1} \wedge \ldots \wedge s_{j_p} \in \Lambda^p[\sigma, X]$. Then $u(\alpha\psi) = u(\alpha x \wedge s_{j_1} \wedge \ldots \wedge s_{j_p}) = u(\alpha x) \wedge u(s_{j_1}) \wedge \ldots \wedge u(s_{j_p})$. However, commutativity of (1.1) for $p=0$ implies that $u(\alpha x) = \beta(u^0(x))$. Hence, $u(\alpha\psi) = \beta(u^0(x)) \wedge u(s_{j_1}) \wedge \ldots \wedge u(s_{j_p}) = \beta(u^0(x) \wedge u(s_{j_1}) \wedge \ldots \wedge u(s_{j_p})) = \beta(u\psi)$, and (1.1) is commutative for every p.

We are now prepared to construct what will eventually be the integrand of our Cauchy–Weil integral.

Definition 1.8. Let $\mathfrak{B}_0 \subset \mathfrak{B}$ be K-modules and let $\mathcal{E}(\mathfrak{B}|\mathfrak{B}_0)$ be the algebra of endomorphisms in $\mathcal{E}(\mathfrak{B})$ which leave $\mathfrak{B}_0$ invariant. Let $(a_1, \ldots, a_n, d_1, \ldots, d_m)$ be a commuting tuple of elements of $\mathcal{E}(\mathfrak{B}|\mathfrak{B}_0)$ and set $\alpha = a_1 s_1 + \ldots + a_n s_n$, $\delta = d_1 t_1 + \ldots + d_m t_m$. We shall then call $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ a Cauchy–Weil system if α is non-singular on $\mathfrak{B}/\mathfrak{B}_0$ (which is an $\mathcal{E}(\mathfrak{B}|\mathfrak{B}_0)$-module since $\mathfrak{B}_0$ is invariant under $\mathcal{E}(\mathfrak{B}|\mathfrak{B}_0)$).

If $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ is a Cauchy–Weil system, then we shall construct a homomorphism R_α: $H^p(\mathfrak{B}, \delta) \to H^{n+p}(\mathfrak{B}_0, \delta)$ for each p. The map R_α will be the composition of the three homomorphisms constructed below.

If $F \in \Lambda^p[\tau, \mathfrak{B}]$ we set $sF = F \wedge s_1 \wedge \ldots \wedge s_n \in \Lambda^{n+p}[\sigma \cup \tau, \mathfrak{B}]$. Note that $s\delta F = \delta F \wedge s_1 \wedge \ldots \wedge s_n = \alpha F \wedge s_1 \wedge \ldots \wedge s_n + \delta F \wedge s_1 \wedge \ldots \wedge s_n$, since $\alpha \wedge s_1 \wedge \ldots \wedge s_n = 0$. Hence, $s\delta F = (\alpha \oplus \delta) F \wedge s_1 \wedge \ldots \wedge s_n = (\alpha \oplus \delta) sF$. Thus, s: $F(\mathfrak{B}, \delta) \to F(\mathfrak{B}, \alpha \oplus \delta)$ is a cochain map of degree n. It follows that s induces a homomorphism s^*: $H(\mathfrak{B}, \delta) \to H(\mathfrak{B}, \alpha \oplus \delta)$ of degree n.

Let i: $\mathfrak{B}_0 \to \mathfrak{B}$ be the inclusion map. We have that $0 \to \mathfrak{B}_0 \xrightarrow{i} \mathfrak{B} \to \mathfrak{B}/\mathfrak{B}_0 \to 0$ is an exact sequence which may be considered an exact sequence of $\mathcal{E}(\mathfrak{B}|\mathfrak{B}_0)$-modules. By hypothesis α is non-singular on $\mathfrak{B}/\mathfrak{B}_0$. By Lemma 1.3, $\alpha \oplus \delta$ is also non-singular on $\mathfrak{B}/\mathfrak{B}_0$. It follows from Lemma 1.5 that i^*: $H(\mathfrak{B}_0, \alpha \oplus \delta) \to H(\mathfrak{B}, \alpha \oplus \delta)$ is an isomorphism.

Finally, let π: $\Lambda[\sigma \cup \tau, \mathfrak{B}_0] \to \Lambda[\tau, \mathfrak{B}_0]$ be the special transformation (cf. 1.6) such that π^0: $\mathfrak{B}_0 \to \mathfrak{B}_0$ is the identity and $\pi(s_i) = 0$ $(i = 1, \ldots, n)$, $\pi(t_j) = t_j$ $(j = 1, \ldots, n)$. Note that $\pi(\alpha \oplus \delta) f = \delta f$ for $f \in \mathfrak{B}_0$. It follows from 1.7 that π: $F(\mathfrak{B}_0, \alpha \oplus \delta) \to F(\mathfrak{B}_0, \delta)$ is a cochain map of degree zero and induces a map π^*: $H(\mathfrak{B}_0, \alpha \oplus \delta) \to H(\mathfrak{B}_0, \delta)$.

Definition 1.9. If $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ is a Cauchy–Weil system, then for each p we define R_α: $H^p(\mathfrak{B}, \delta) \to H^{n+p}(\mathfrak{B}_0, \delta)$ to be the composition of the maps

$$H^p(\mathfrak{B}, \delta) \xrightarrow{s^*} H^{n+p}(\mathfrak{B}, \alpha \oplus \delta) \xrightarrow{i^{*-1}} H^{n+p}(\mathfrak{B}_0, \alpha \oplus \delta) \xrightarrow{\pi^*} H^{n+p}(\mathfrak{B}_0, \delta)$$

defined above, followed by multiplication by $(-1)^n$.

The properties we require of the Cauchy–Weil integral will follow from certain invariance properties of the map R_α. Developing these invariance properties occupies the remainder of the section.

PROPOSITION 1.10. *Let $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ and $(\mathfrak{B}', \mathfrak{B}'_0, \alpha', \delta')$ be Cauchy–Weil systems, where $\alpha = a_1 s_1 + \ldots + a_n s_n$, $\alpha' = a'_1 s_1 + \ldots + a'_n s_n$, $\delta = d_1 t_1 + \ldots + d_m t_m$, and $\delta' = d'_1 t'_1 + \ldots + d'_n t'_{m'}$. Let $u: \Lambda[\tau, \mathfrak{B}] \to \Lambda[\tau', \mathfrak{B}']$ be a special transformation which satisfies the following conditions:*

(1) *$u^0: \mathfrak{B} \to \mathfrak{B}'$ maps $\mathfrak{B}_0$ into $\mathfrak{B}'_0$;*
(2) *$\mathfrak{B}'_0$ is invariant under each u_{ij};*
(3) *$u^0 a_i f = a'_i u^0 f$ for $i = 1, \ldots, n$ and $f \in \mathfrak{B}$;*
(4) *the diagram*

$$\begin{tikzcd} \mathfrak{B} \arrow[r, "\delta"] \arrow[d, "u^0"] & \Lambda^1[\tau, \mathfrak{B}] \arrow[d, "u^1"] \\ \mathfrak{B}' \arrow[r, "\delta'"] & \Lambda^1[\tau', \mathfrak{B}'] \end{tikzcd}$$

is commutative.

Then, if $u^*: H(\mathfrak{B}, \delta) \to H(\mathfrak{B}', \delta')$ is the map guaranteed by 1.7, the diagram

$$\begin{tikzcd} H^p(\mathfrak{B}, \delta) \arrow[r, "R_\alpha"] \arrow[d, "u^*"] & H^{n+p}(\mathfrak{B}_0, \delta) \arrow[d, "u^*"] \\ H^p(\mathfrak{B}', \delta') \arrow[r, "R_{\alpha'}"] & H^{n+p}(\mathfrak{B}'_0, \delta') \end{tikzcd}$$

is commutative.

Proof. Conditions (1) and (2) on u simply guarantee that $u: \Lambda[\tau, \mathfrak{B}] \to \Lambda[\tau', \mathfrak{B}']$ maps $\Lambda[\tau, \mathfrak{B}_0]$ into $\Lambda[\tau', \mathfrak{B}'_0]$.

We define a special transformation $u^\wedge: \Lambda[\sigma \cup \tau, \mathfrak{B}] \to \Lambda[\sigma \cup \tau', \mathfrak{B}']$ by setting $\hat{u}^0 = u^0$: $\mathfrak{B} \to \mathfrak{B}'$, $\hat{u}(t_j) = u(t_j)$ $(j = 1, \ldots, m)$, and $\hat{u}(s_i) = s_i$ $(i = 1, \ldots, m)$. It then follows from 1.7 and conditions (3) and (4) on u, that each of $u: F(\mathfrak{B}, \delta) \to F(\mathfrak{B}', \delta')$, $u: F(\mathfrak{B}_0, \delta) \to F(\mathfrak{B}'_0, \delta')$, $\hat{u}: F(\mathfrak{B}, \alpha \oplus \delta) \to F(\mathfrak{B}', \alpha' \oplus \delta')$, and $\hat{u}: F(\mathfrak{B}_0, \alpha \oplus \delta) \to F(\mathfrak{B}'_0, \alpha' \oplus \delta')$ is a cochain map. Furthermore, the following diagram is commutative:

$$\begin{tikzcd} F(\mathfrak{B}, \delta) \arrow[r, "s"] \arrow[d, "u"] & F(\mathfrak{B}, \alpha \oplus \delta) \arrow[d, "\hat{u}"] & F(\mathfrak{B}_0, \alpha \oplus \delta) \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[d, "\hat{u}"] & F(\mathfrak{B}_0, \delta) \arrow[d, "u"] \\ F(\mathfrak{B}', \delta') \arrow[r, "s"] & F(\mathfrak{B}', \alpha' \oplus \delta') & F(\mathfrak{B}'_0, \alpha' \oplus \delta') \arrow[l, "i"'] \arrow[r, "\pi"] & F(\mathfrak{B}'_0 \delta') \end{tikzcd}$$

It follows that $u^* R_\alpha = R_{\alpha'} u^*$.

In the above proposition, the transformation u did not effect α in any essential way. In the next proposition we derive the affect on R_α of a transformation of α.

Proposition 1.11. *Let $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ and $(\mathfrak{B}, \mathfrak{B}_0, \beta, \delta)$ be Cauchy–Weil systems, where $\alpha = a_1 s_1 + \ldots + a_n s_n$, $\beta = b_1 s_1' + \ldots + b_n s_n'$ and $\delta = d_1 t_1 + \ldots + d_m t_m$. If α and β are related by $b_i = \sum u_{ij} a_j$ $(i = 1, \ldots, n)$, where $\{u_{ij}\}$ is an $n \times n$-matrix of elements of $\mathcal{E}(\mathfrak{B} \mid \mathfrak{B}_0)$ which commute with each other and with $a_1, \ldots, a_n, d_1, \ldots, d_m$, then $R_\beta \circ \det(u_{ij}) = R_\alpha$.*

Proof. Consider the special transformation u: $\Lambda[\sigma \cup \tau, \mathfrak{B}] \to \Lambda[\sigma' \cup \tau, \mathfrak{B}]$, where u^0: $\mathfrak{B} \to \mathfrak{B}$ is the identity and $u(s_j) = \sum u_{ij} s_i'$, $u(t_j) = t_j$. This carries $\Lambda[\sigma \cup \tau, \mathfrak{B}_0]$ into $\Lambda[\sigma' \cup \tau, \mathfrak{B}_0]$ and u: $F(\mathfrak{B}, \alpha \oplus \delta) \to F(\mathfrak{B}, \beta \oplus \delta)$ and u: $F(\mathfrak{B}_0, \alpha \oplus \delta) \to F(\mathfrak{B}_0, \beta \oplus \delta)$ are cochain maps. Furthermore, the following diagram is commutative:

$$\begin{tikzcd}
F(\mathfrak{B}, \delta) \arrow[r, "s"] \arrow[d, "det\,(u_{ij})"] & F(\mathfrak{B}, \alpha \oplus \delta) \arrow[d, "u"] & F(\mathfrak{B}_0, \alpha \oplus \delta) \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[d, "u"] & F(\mathfrak{B}_0, \delta) \arrow[d, "id"] \\
F(\mathfrak{B}, \delta) \arrow[r, "s'"] & F(\mathfrak{B}, \beta \oplus \delta) & F(\mathfrak{B}_0, \beta \oplus \delta) \arrow[l, "i"'] \arrow[r, "\pi"] & F(\mathfrak{B}_0, \delta),
\end{tikzcd}$$

where $\det(u_{ij}) \in \mathcal{E}(\mathfrak{B} \mid \mathfrak{B}_0)$ determines a cochain map $\det(u_{ij})$: $F(\mathfrak{B}, \delta) \to F(\mathfrak{B}, \delta)$, since each u_{ij} commutes with $d_1, \ldots, d_m$. Upon passing to cohomology, the proposition follows from the commutativity of this diagram.

If $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ is a Cauchy–Weil system, then each a_i commutes with each d_j. Hence, each a_i acts as an endomorphism of $H^p(\mathfrak{B}, \delta)$. We then have:

Proposition 1.12. *If $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ is a Cauchy–Weil system and $a = a_1 s_1 + \ldots + a_n s_n$, then $R_\alpha \circ a_i$: $H^p(\mathfrak{B}, \delta) \to H^{n+p}(\mathfrak{B}_0, \delta)$ is zero for each i.*

Proof. It follows from 1.4, that $a_i H^{n+p}(\mathfrak{B}, \alpha \oplus \delta) = 0$ for each i. The proposition follows from this and the definition of R_α.

Our next result is a factorization lemma which will yield a formula relating iterated and double Cauchy–Weil integrals.

Proposition 1.13. *Let $\mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \mathfrak{B}$ be modules and $a_1, \ldots, a_n$, $b_1, \ldots, b_{n'}$, $d_1, \ldots, d_m$ commuting elements of $\mathcal{E}(\mathfrak{B})$ which leave $\mathfrak{B}_0$ and $\mathfrak{B}_1$ invariant. Let $\alpha = a_1 s_1 + \ldots + a_n s_n$, $\beta = b_1 s_1' + \ldots + b_n s_{n'}'$, and $\delta = d_1 t_1 + \ldots + d_m t_m$. If $(\mathfrak{B}, \mathfrak{B}_1, \alpha, \delta)$ and $(\mathfrak{B}_1, \mathfrak{B}_0, \beta, \delta)$ are Cauchy–Weil systems, then so is $(\mathfrak{B}, \mathfrak{B}_0, \alpha \oplus \beta, \delta)$ and the following diagram is commutative:*

$$\begin{tikzcd}
H^p(\mathfrak{B}, \delta) \arrow[rr, "R_{\alpha \oplus \beta}"] \arrow[dr, "R_\alpha"'] & & H^{n+n'+p}(\mathfrak{B}_0, \delta) \\
& H^{n+p}(\mathfrak{B}_1, \delta) \arrow[ur, "R_\beta"'] &
\end{tikzcd}$$

Proof. Since $0 \to \mathfrak{B}_1/\mathfrak{B}_0 \to \mathfrak{B}/\mathfrak{B}_0 \to \mathfrak{B}/\mathfrak{B}_1 \to 0$ is a short exact sequence and α is non-singular on $\mathfrak{B}/\mathfrak{B}_1$ and β is non-singular on $\mathfrak{B}_1/\mathfrak{B}_0$, 1.3 and 1.5 imply that $\alpha \oplus \beta$ is non-singular on $\mathfrak{B}/\mathfrak{B}_0$ and $(\mathfrak{B}, \mathfrak{B}_0, \alpha\oplus\beta, \delta)$ is a Cauchy–Weil system. Now consider the diagram

$$\begin{tikzcd}
F(\mathfrak{B},\delta) \arrow[r, "s\wedge s'"] \arrow[d, "s"] & F(\mathfrak{B},\alpha\oplus\beta\oplus\delta) & F(\mathfrak{B}_1,\alpha\oplus\beta\oplus\delta) \arrow[l, "i_1"'] \arrow[dr, "\pi_1"] & F(\mathfrak{B}_0,\alpha\oplus\beta\oplus\delta) \arrow[l, "i_0"'] \arrow[r, "\pi"] \arrow[dr, "\pi_1"] & F(\mathfrak{B}_0,\delta) \\
F(\mathfrak{B},\alpha\oplus\delta) \arrow[ur, "s'"] & F(\mathfrak{B}_1,\alpha\oplus\delta) \arrow[l, "i_1"'] \arrow[r, "\pi_1"] \arrow[ur, "s'"] & F(\mathfrak{B}_1,\delta) \arrow[r, "s'"] & F(\mathfrak{B}_1,\beta\oplus\delta) & F(\mathfrak{B}_0,\beta\oplus\delta) \arrow[l, "i_0"'] \arrow[u, "\pi_2"']
\end{tikzcd}$$

where $sf = f\wedge s_1 \wedge \ldots \wedge s_n$, $s'f = f \wedge s_1' \wedge \ldots \wedge s_{n'}'$, $(s\wedge s')f = f\wedge s_1 \wedge \ldots \wedge s_n \wedge s_1' \wedge \ldots \wedge s_{n'}'$, $\pi_1(s_i) = 0$ for $i = 1, \ldots, n$, $\pi_1(s_i') = s_i'$ for $i = 1, \ldots, n'$, $\pi_1(t_i) = t_i$ for $i = 1, \ldots, m$, $\pi_2(s_i') = 0$ for $i = 1, \ldots, n'$, $\pi_2(t_i) = t_i$ for $i = 1, \ldots, m$, and $\pi(s_i) = \pi(s_j') = 0$ for $i = 1, \ldots, n$, $j = 1, \ldots, n'$, $\pi(t_i) = t_i$ for $i = 1, \ldots, m$. The maps i_1 and i_0 are induced by the inclusions $i_1\colon \mathfrak{B}_1 \to \mathfrak{B}$ and $i_0\colon \mathfrak{B}_0 \to \mathfrak{B}_1$.

The diagram is clearly commutative and each of its maps is a cochain map. On passing to cohomology, we have $R_\alpha = (-1)^n \pi_1^* i_1^{*-1} s^*$, $R_\beta = (-1)^{n'} \pi_2^* i_0^{*-1} s'^*$, and $R_{\alpha\oplus\beta} = (-1)^{n+n'} \pi^* i_0^{*-1} i_1^{*-1} (s\wedge s')^*$, since $i\colon \mathfrak{B}_0 \to \mathfrak{B}$ is $i_1 \circ i_0$. The commutativity of the above diagram now implies that $R_{\alpha\oplus\beta} = R_\beta \circ R_\alpha$.

Proposition 1.10 shows that $R_\alpha\colon H^p(\mathfrak{B}, \delta) \to H^{n+p}(\mathfrak{B}_0, \delta)$ is "natural" relative to a certain class of transformations of δ. The final proposition of this section is another result concerning the "naturality" of R_α relative to δ. This result is not needed in our development of the Cauchy–Weil integral or the functional calculus. However, we will need it to obtain a relation between the spectrum of a tuple $(f_1(\alpha), \ldots, f_m(\alpha))$ and the spectrum of $\alpha = (a_1, \ldots, a_n)$ for $f_1, \ldots, f_m \in \mathfrak{A}(\mathrm{Sp}\,(a, X))$ (cf. Theorem 4.8).

Let $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta\oplus\beta)$ be a Cauchy–Weil system, where $\alpha = a_1 s_1 + \ldots + a_n s_n$, $\delta = d_1 t_1 + \ldots + d_n t_n$, and $\beta = b_1 t_1' + \ldots + b_m t_m'$. Without loss of generality we may assume that $b_1, \ldots, b_m$ are elements of the ground ring K so that all of the objects in the discussion are modules over a ring containing these elements. In particular we may consider $(b_1, \ldots, b_m)$ as a tuple of operators on $H^p(\mathfrak{B}, \delta)$ and $H^p(\mathfrak{B}_0, \delta)$ for each p.

There are two ways of constructing a map $R_\alpha^*\colon H^p(H^0(\mathfrak{B}, \delta), \beta) \to H^p(H^n(\mathfrak{B}_0, \delta), \beta)$ using the tuple α. On the one hand, we could compute $R_\alpha\colon H^0(\mathfrak{B}, \delta) \to H^n(\mathfrak{B}_0, \delta)$ for the Cauchy–Weil system $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ and then let $R_\alpha^*\colon H^p(H^0(\mathfrak{B}, \delta), \beta) \to H^p(H^n(\mathfrak{B}_0, \delta), \beta)$ be the map induced on cohomology by the corresponding cochain map $R_\alpha\colon F(H^0(\mathfrak{B}, \delta), \beta) \to F(H^n(\mathfrak{B}_0, \delta), \beta)$. On the other hand, there are natural maps $j^*\colon H^p(H^0(\mathfrak{B}, \delta), \beta) \to H^p(\mathfrak{B}, \delta\oplus\beta)$ and $k^*\colon H^{n+p}(\mathfrak{B}_0, \delta\oplus\beta) \to H^p(H^n(\mathfrak{B}_0, \delta), \beta)$ which we shall describe below. Hence, we could define R_α^* as $k^* R_\alpha j^*\colon H^p(H^0(\mathfrak{B}, \delta), \beta) \to H^p(H^n(\mathfrak{B}, \delta), \beta)$, where $R_\alpha\colon H^p(\mathfrak{B}, \delta\oplus\beta) \to H^{n+p}(\mathfrak{B}_0, \delta\oplus\beta)$ is computed for the Cauchy–Weil system $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta\oplus\beta)$. We shall show that these two definitions of R^*_α agree. There is a very short proof based on the theory of spectral sequences and the fact that j^* and k^* are just edge homomorphisms (cf. [11], Chapter XI). However, we shall attempt to give a more elementary proof.

Let j: $H^0(\mathfrak{B}, \delta) = \ker\{\delta: \mathfrak{B} \to \Lambda^1[\tau, \mathfrak{B}]\} \to \mathfrak{B}$ be the inclusion map and consider the corresponding inclusion $j^\wedge: \Lambda[\tau', H^0(\mathfrak{B}, \delta)] \to \Lambda[\tau \cup \tau', \mathfrak{B}]$. Note that since $\delta = 0$ on $H^0(\mathfrak{B}, \delta)$ we have that $j^\wedge: F(H^0(\mathfrak{B}, \delta), \beta) \to F(\mathfrak{B}, \delta \oplus \beta)$ is a cochain map and, hence, it induces a map j^*: $H(H^0(\mathfrak{B}, \delta), \beta) \to H(\mathfrak{B}, \delta \oplus \beta)$.

Let k: $\Lambda^n[\tau, \mathfrak{B}_0] \to H^n(\mathfrak{B}_0, \delta) = \Lambda^n[\tau, \mathfrak{B}_0]/\mathrm{Im}\{\delta: \Lambda^{n-1}[\tau, \delta] \to \Lambda^n[\tau, \delta]\}$ be the factor map. There is a corresponding factor map $k^\wedge: \Lambda[\tau \cup \tau', \mathfrak{B}_0] \to \Lambda[\tau', H^n(\mathfrak{B}_0, \delta)]$ defined by $k^\wedge \psi = 0$ if ψ has degree less than n in $t_1, \ldots, t_n$ and $k^\wedge \psi = k\psi$ if ψ has degree n in $t_1, \ldots, t_n$. Since $\ker k^\wedge$ contains $\mathrm{Im}\{\delta: \Lambda[\tau \cup \tau', \mathfrak{B}_0] \to \Lambda[\tau \cup \tau', \mathfrak{B}_0]\}$, the map $k^\wedge$: $F(\mathfrak{B}_0, \delta \oplus \beta) \to F(H^n(\mathfrak{B}_0, \delta), \beta)$ is a cochain map of degree $-n$ and induces a map k^*: $H^{n+p}(\mathfrak{B}_0, \delta \oplus \beta) \to H^p(H^n(\mathfrak{B}_0, \delta), \beta)$ for each p.

PROPOSITION 1.14. *Let* α, β, δ *and* j^* *and* k^* *be as above. Let* R^*_α: $H^p(H^0(\mathfrak{B}, \delta), \beta) \to H^p(H^n(\mathfrak{B}_0, \delta), \beta)$ *be induced by* R_α: $H^0(\mathfrak{B}, \delta) \to H^n(\mathfrak{B}_0, \delta)$ *and let* R_α: $H^p(\mathfrak{B}, \delta \oplus \beta) \to H^{n+p}(\mathfrak{B}, \delta \oplus \beta)$ *be the map* R_α *relative to the Cauchy–Weil system* $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta \oplus \beta)$. *Then the following diagram is commutative:*

$$\begin{array}{ccc}
H^p(H^0(\mathfrak{B}, \delta), \beta) & \xrightarrow{R^*_\alpha} & H^p(H^n(\mathfrak{B}_0, \delta), \beta) \\
\downarrow j^* & & \uparrow k^* \\
H^p(\mathfrak{B}, \delta \oplus \beta). & \xrightarrow{R_\alpha} & H^{n+p}(\mathfrak{B}_0, \delta \oplus \beta)
\end{array}$$

Proof. Let $\mathfrak{C}$ denote either $\mathfrak{B}$ or $\mathfrak{B}_0$ and γ either δ or $\alpha \oplus \delta$. We introduce a filtration in $F(\mathfrak{C}, \gamma \oplus \beta)$ by letting $D^p F(\mathfrak{C}, \gamma \oplus \beta)$ be the subcomplex of $F(\mathfrak{C}, \gamma \oplus \beta)$ consisting of elements of degree at least p in $t'_1, \ldots, t'_m$. Note that for each p there is a cochain map v: $D^p F(\mathfrak{C}, \gamma \oplus \beta) \to F(\Lambda^p[\tau', \mathfrak{C}], \gamma)$ of degree $-p$ defined by $v\psi = 0$ for $\psi \in D^{p+1} F(\mathfrak{C}, \gamma \oplus \beta)$ and $v\psi = \psi$ considered as an element of $\Lambda[\tau, \Lambda^p[\tau', \mathfrak{B}]]$ if ψ has degree p in $t'_1, \ldots, t'_m$.

The maps s, i, and π of Definition 1.9 for the Cauchy–Weil system $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta \oplus \beta)$ respect the filtration $\{D^p\}$. Hence, we have the following commutative diagram of cochain maps:

$$\begin{array}{ccccccc}
F(\mathfrak{B}, \delta \oplus \beta) & \xrightarrow{s} & F(\mathfrak{B}, \alpha \oplus \delta \oplus \beta) & \xleftarrow{i} & F(\mathfrak{B}_0, \alpha \oplus \delta \oplus \beta) & \xrightarrow{\pi} & F(\mathfrak{B}_0, \delta \oplus \beta) \\
\uparrow u & & \uparrow u & & \uparrow u & & \uparrow u \\
D^p F(\mathfrak{B}, \delta \oplus \beta) & \xrightarrow{s} & D^p F(\mathfrak{B}, \alpha \oplus \delta \oplus \beta) & \xleftarrow{i} & D^p F(\mathfrak{B}_0, \alpha \oplus \delta \oplus \beta) & \xrightarrow{\pi} & D^p F(\mathfrak{B}_0, \delta \oplus \beta) \\
\downarrow v & & \downarrow v & & \downarrow v & & \downarrow v \\
F(\Lambda^p[\tau', \mathfrak{B}], \delta) & \xleftarrow{s} & F(\Lambda^p[\tau', \mathfrak{B}], \alpha \oplus \delta) & \xleftarrow{i} & F(\Lambda^p[\tau', \mathfrak{B}_0], \alpha \oplus \delta) & \xrightarrow{\pi} & F(\Lambda^p[\tau', \mathfrak{B}_0], \delta)
\end{array}$$

where u is the inclusion map. Note that since $\Lambda^p[\tau', \mathfrak{B}]$ is just a direct sum of copies of $\mathfrak{B}$ the map i: $F(\Lambda^p[\tau', \mathfrak{B}_0], \alpha\oplus\delta) \to F(\Lambda^p[\tau', \mathfrak{B}], \alpha\oplus\delta)$ induces an isomorphism of cohomology. Using this fact, the map v, and induction on $n-p$, one can prove that i: $D^pF(\mathfrak{B}_0, \alpha\oplus\delta\oplus\beta) \to D^pF(\mathfrak{B}, \alpha\oplus\delta\oplus\beta)$ also induces an isomorphism of cohomology. Hence, with $R_\alpha = (-1)^n \pi^* i^{*-1} s^*$, we have a commutative diagram

$$\begin{array}{ccc}
H^p(\mathfrak{B}, \delta\oplus\beta) & \xrightarrow{R_\alpha} & H^{n+p}(\mathfrak{B}_0, \delta\oplus\beta) \\
\uparrow u^* & & \uparrow u^* \\
H^p(D^pF(\mathfrak{B}, \delta\oplus\beta)) & \xrightarrow{R_\alpha} & H^{n+p}(D^pF(\mathfrak{B}_0, \delta\oplus\beta)) \\
\downarrow v^* & & \downarrow v^* \\
H^0(\Lambda^p[\tau', \mathfrak{B}], \delta) = \Lambda^p[\tau', H^0(\mathfrak{B}, \delta)] & \xrightarrow{R_\alpha} & \Lambda^p[\tau', H^n(\mathfrak{B}_0, \delta)] = H^n(\Lambda^p[\tau', \mathfrak{B}_0], \delta).
\end{array}$$

Furthermore, for each p, v^* maps $H^p(D^pF(\mathfrak{B}, \delta\oplus\beta))$ and $H^{n+p}(D^pF(\mathfrak{B}_0, \delta\oplus\beta))$ into $\ker\{\beta: \Lambda^p[\tau', H^0(\mathfrak{B}, \delta)] \to \Lambda^{p+1}[\tau', H^0(\mathfrak{B}, \delta)]\}$ and $\ker\{\beta: \Lambda^p[\tau', H^n(\mathfrak{B}_0, \delta)] \to \Lambda^{p+1}[\tau', H^n(\mathfrak{B}_0, \delta)]\}$ respectively. Hence, v^* induces maps v^{**} so the following diagram is commutative:

$$\begin{array}{ccc}
H^p(\mathfrak{B}, \delta\oplus\beta) & \xrightarrow{R_\alpha} & H^{n+p}(\mathfrak{B}_0, \delta\oplus\beta) \\
\uparrow u^* & & \uparrow u^* \\
H^p(D^pF(\mathfrak{B}, \delta\oplus\beta)) & \xrightarrow{R_\alpha} & H^{n+p}(D^pF(\mathfrak{B}_0, \delta\oplus\beta)) \\
\downarrow v^{**} & & \downarrow v^{**} \\
H^p(H^0(\mathfrak{B}, \delta), \beta) & \xrightarrow{R_\alpha^*} & H^p(H^n(\mathfrak{B}, \delta), \beta).
\end{array}$$

To complete the proof, we simply note that v^{**}: $H^p(D^pF(\mathfrak{B}, \delta\oplus\beta)) \to H^p(H^0(\mathfrak{B}, \delta), \beta)$ is an isomorphism and $j^* = u^*(v^{**})^{-1}$, and that u^*: $H^{n+p}(D^pF(\mathfrak{B}_0, \delta\oplus\beta)) \to H^{n+p}(\mathfrak{B}_0, \delta\oplus\beta)$ is an isomorphism and $k^* = v^{**}(u^*)^{-1}$.

COROLLARY 1.15. *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements of the ground ring K and $d_1, \ldots, d_n$ a commuting tuple in $\mathcal{E}(\mathfrak{B}|\mathfrak{B}_0)$. Let $\alpha = a_1 s_1 + \ldots + a_n s_n$, $\beta = b_1 s_1' + \ldots + b_n s_n'$, $\delta = d_1 t_1 + \ldots + d_n t_n$, $(\alpha-\beta) = (a_1 - b_1) s_1'' + \ldots + (a_n - b_n) s_n''$, and suppose $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ and $(\mathfrak{B}, \mathfrak{B}_0, \beta, \delta)$ are Cauchy–Weil systems. Then the maps R_α: $H^0(\mathfrak{B}, \delta) \to H^n(\mathfrak{B}_0, \delta)$ and R_β: $H^0(\mathfrak{B}, \delta) \to H^n(\mathfrak{B}_0, \delta)$ induce identical homomorphisms of $H^p(H^0(\mathfrak{B}, \delta), (\alpha-\beta))$ into $H^p(H^n(\mathfrak{B}, \delta), (\alpha-\beta))$ for each p.*

Proof. By Proposition 1.14 it suffices to show that R_α: $H^p(\mathfrak{B}, \delta\oplus(\alpha-\beta)) \to H^{n+p}(\mathfrak{B}_0, \delta\oplus(\alpha-\beta))$ and R_β: $H^p(\mathfrak{B}, \delta\oplus(\alpha-\beta)) \to H^{n+p}(\mathfrak{B}_0, \delta\oplus(\alpha-\beta))$ are the same homomorphism. To do this, we define a special transformation u: $F(\mathfrak{B}, \alpha\oplus\delta\oplus(\alpha-\beta))\to F(\mathfrak{B}, \beta\oplus\delta\oplus(\alpha-\beta))$ by defining $u^0=\mathrm{id}$: $\mathfrak{B}\to\mathfrak{B}$ and $u(t_i)=t_i$, $u(s_i'')=s_i''-s_i'$, and $u(s_i)=s_i'$. Then the following is a commutative diagram of cochain maps:

$$\begin{array}{ccccccc}
F(\mathfrak{B}, \delta\oplus(\alpha-\beta)) & \xrightarrow{s} & F(\mathfrak{B}, \alpha\oplus\delta\oplus(\alpha-\beta)) & \xleftarrow{i} & F(\mathfrak{B}_0, \alpha\oplus\delta\oplus(\alpha-\beta)) & \xrightarrow{\pi} & F(\mathfrak{B}_0, \delta\oplus(\alpha-\beta)) \\
\updownarrow id & & \downarrow u & & \downarrow u & & \updownarrow id \\
F(\mathfrak{B}, \delta\oplus(\alpha-\beta)) & \xrightarrow{s'} & F(\mathfrak{B}, \beta\oplus\delta\oplus(\alpha-\beta)) & \xleftarrow{i} & F(\mathfrak{B}_0, \beta\oplus\delta\oplus(\alpha-\beta)) & \xrightarrow{\pi} & F(\mathfrak{B}_0, \delta\oplus(\alpha-\beta))
\end{array}$$

On passing to cohomology, we conclude that

$$R_\alpha = R_\beta\colon H^p(\mathfrak{B}, \delta\oplus(\alpha-\beta)) \to H^{n+p}(\mathfrak{B}_0, \delta\oplus(\alpha-\beta)).$$

2. A special function space

In § 3 we shall develop the Cauchy–Weil integral by applying the results of § 1 to a Cauchy–Weil system $(\mathfrak{B}, \mathfrak{B}_0, \alpha, \delta)$ in which $\mathfrak{B}$ is a space of vector valued functions on a domain $U\subset\mathbf{C}^n$ and $\mathfrak{B}_0$ is the subspace of $\mathfrak{B}$ consisting of functions with compact support. In this section we introduce the space $\mathfrak{B}$ and develop its relevant properties.

Notation 2.1. If X is a Frechet space and U a locally compact Hausdorff space, then $C(U, X)$ will denote the space of continuous X-valued functions on U with the compact-open topology. If $U\subset\mathbf{C}^n$ is a domain, then $\mathfrak{A}(U, X)$ will denote the space of analytic X-valued functions on U, also with the compact-open topology. The theory of analytic functions with values in a Frechet space does not differ in any essential way from the theory of numerical valued analytic functions. We refer the reader to [5] and [7] for discussions of this matter.

If X and Y are Frechet spaces, then $L(X, Y)$ will denote the space of continuous linear maps from X to Y with the topology of uniform convergence on bounded subsets of X.

In [15] we discussed parameterized chain complexes. We shall be using cochain complexes in this paper. Hence, we restate Definition 2.1 of [15] as follows:

Definition 2.2. Let $\{Y^p\}$ be a family of Banach spaces indexed by the integers and let U be a locally compact Hausdorff space. Let $\{\alpha^p\}$ be an indexed family of maps with $\alpha^p\in C(U, L(Y^p, Y^{p+1}))$ for each p. If $\alpha^p(z)\circ\alpha^{p-1}(z)=0$ for each p and each $z\in U$, then we shall call $Y=\{Y^p, \alpha^p\}$ a parameterized cochain complex on U. If, in addition, U is a domain in $\mathbf{C}^n$ and $\alpha^p\in\mathfrak{A}(U, L(Y^p, Y^{p-1}))$ for each p, then we shall call Y an analytically parameterized cochain complex.

If $Y=\{Y^p, \alpha^p\}$ is a parameterized (analytically parameterized) cochain complex on U and V is an open subset of U, then $C(V, Y)$ ($\mathfrak{A}(V, Y)$) will denote the cochain complex $\{C(V, Y^p), \alpha^p\}$ ($\{\mathfrak{A}(V, Y^p), \alpha^p\}$), where $(\alpha^p f)(z)=\alpha^p(z)f(z)$ for $z\in V$, $f\in C(V, Y^p)$ ($\mathfrak{A}(V, Y^p)$).

The following is a restatement of Theorems 2.1 and 2.2 of [15]:

LEMMA 2.3. *If $Y=\{Y^p, \alpha^p\}$ is a parameterized cochain complex on U, then for each p the set of $z\in U$ for which $Y(z)=\{Y^p, \alpha^p(z)\}$ is exact at the pth stage is an open set. If $Y(z)$ is exact for all $z\in V$, where V is an open set in U, then $C(V, Y)$ is exact.*

If $U=U_1\times\ldots\times U_n$ is a polydisc in $\mathbf{C}^n$, Y is analytically parameterized on U, and $Y(z)$ is exact for each $z\in U$, then $\mathfrak{A}(U, Y)$ is also exact.

Given a domain $U\subset\mathbf{C}^n$ and a Banach space X, we seek a space $\mathfrak{B}(U, X)$ of X-valued functions on U which has certain special properties. Specifically, we need that $\mathfrak{B}(U, X)$ is closed under multiplication by functions in $C^\infty(U)$, the differential operators $\partial/\partial\bar{z}_1, \ldots, \partial/\partial\bar{z}_n$ act on $\mathfrak{B}(U, X)$, and $\mathfrak{B}(U, Y)$ is exact whenever Y is an analytically parameterized cochain complex such that $Y(z)$ is exact for each $z\in U$. The obvious choice is $C^\infty(U, X)$. Unfortunately, we have been unable to prove that $C^\infty(U, Y)$ is exact when Y is a pointwise exact analytically parameterized complex. Hence, we are forced to seek another choice for $\mathfrak{B}(U, X)$.

Our choice is the following: $\mathfrak{B}(U, X)$ will be the subspace of $C(U, X)$ consisting of functions f for which the derivatives $(\partial/\partial\bar{z}_1)^{p_1}\ldots(\partial/\partial\bar{z}_n)^{p_n}f$ (in the distribution sense) are elements of $C(U, X)$ for all multi-indices $(p_1, \ldots, p_n)$. We make this more precise below.

Definition 2.4. Let X be a Frechet space and U a domain in $\mathbf{C}$. If $f, g\in C(U, X)$ we shall say $f\in\mathfrak{B}^1(U, X)$ and $(\partial/\partial\bar{z})f=g$ provided

$$\iint_U\left(\frac{\partial}{\partial\bar{z}}\varphi\right)f\,dz\wedge d\bar{z}=-\iint_U\varphi g\,dz\wedge d\bar{z} \tag{2.1}$$

for every $\varphi\in\mathcal{D}(U)$ — the space of C^∞ functions with compact support in U.

We define $\mathfrak{B}^k(U,X)$ for $k>1$ by saying $f\in\mathfrak{B}^k(U,X)$ if $f\in\mathfrak{B}^1(U,X)$ and $\partial f/\partial\bar{z}\in\mathfrak{B}^{k-1}(U,X)$. We set $\mathfrak{B}^0(U, X)=C(U, X)$.

Note that the fact that X is complete insures that both integrals in (2.1) exist. Clearly, equation (2.1) uniquely defines $g=\partial f/\partial\bar{z}$ in terms of f.

We topologize $\mathfrak{B}^k(U, X)$ in the following manner: we give $\mathfrak{B}^k(U, X)$ the Frechet space topology in which a sequence $\{f_n\}$ converges to zero if and only if $(\partial/\partial\bar{z})^p f_n\to 0$ uniformly on compact subsets of U for $p=0, 1, \ldots, k$.

One might suspect that it is always the case that $\mathfrak{B}^k(U, X)=C^k(U, X)$. It is true that a real valued function in $\mathfrak{B}^k(U, \mathbf{C})$ is also in $C^k(U)$. However, the function $z \to z \ln|\ln|z||$ is in $B^1(D, \mathbf{C})$ but not in $C^1(D)$, where D is the unit disc.

The next few lemmas develop elementary properties of the class $\mathfrak{B}^k$. For convenience, we set $\partial/\partial\bar{z} = \bar{\partial}$ and $(\partial/\partial\bar{z})^k = \bar{\partial}^k$.

LEMMA 2.5. *If* $f \in \mathfrak{B}^k(U, X)$ *and* $\varphi \in C^\infty(U)$ *then* $\varphi f \in \mathfrak{B}^k(U, X)$ *and* $\bar{\partial}(\varphi f) = (\bar{\partial}\varphi) f + \varphi(\bar{\partial} f)$.

Proof. This is an elementary computation.

LEMMA 2.6. *If* $f \in \mathfrak{B}^k(U, X)$ *then there is a sequence* $\{f_n\}$ *of functions in* $C^\infty(U, X)$ *such that* $f_n \to f$ *in the topology of* $\mathfrak{B}^k(U, X)$.

Proof. If $f \in \mathfrak{B}^k(U, X)$ has compact support, we may consider f an element of $\mathfrak{B}^k(\mathbf{C}, X)$. We let $\{u_n\} \subset \mathcal{D}(\mathbf{C})$ be a convolution approximate identity. Then $u_n * f \in C^\infty(\mathbf{C}, X)$, $\bar{\partial}^p(u_n * f) = u_n * (\bar{\partial}^p f)$ for $p = 0, 1, \ldots, k$, and $\bar{\partial}^p(u_n * f) \to \bar{\partial}^p f$ uniformly for $p = 0, 1, \ldots, k$. Hence, f is the limit of a sequence in $C^\infty(U, X)$ if f has compact support. By 2.5 every element of $\mathfrak{B}^k(U, X)$ is the limit of a sequence of elements with compact support. The lemma follows.

The next Lemma is obviously true for $f \in C^\infty(U, X)$ and in view of 2.6, it is true for all $f \in \mathfrak{B}^k(U, X)$:

LEMMA 2.7. *If* X *and* Y *are Frechet spaces and* $\alpha \in \mathfrak{A}(U, L(X, Y))$, $f \in \mathfrak{B}^k(U, X)$ *then* $\alpha f \in \mathfrak{B}^k(U, Y)$ *and* $\bar{\partial}(\alpha f) = \alpha(\bar{\partial} f)$, *where* $\alpha f(z) = \alpha(z) f(z)$.

LEMMA 2.8. *If* $f \in C(U, X)$ *and* $k \geq 0$, *then* $f \in \mathfrak{B}^{k+1}(U, X)$ *if and only if there is a* $g \in \mathfrak{B}^k(U, X)$ *such that every compact set* $K \subset U$ *is contained in a compact set* $D \subset U$ *with piecewise smooth boundary such that*

$$f(z) = \frac{1}{2\pi i}\left[\int_{\partial D} (\zeta - z)^{-1} f(\zeta)\, d\zeta + \iint_D (\zeta - z)^{-1} g(\zeta)\, d\zeta \wedge d\bar{\zeta}\right] \tag{2.2}$$

for every $z \in \operatorname{int} D$.

Proof. If $f \in C^\infty(U, X)$ and $\bar{\partial} f = g$, then (2.2) is just the generalized Cauchy integral formula (cf. [10], 1.2.1). It follows from 2.6 that (2.2) holds for all $f \in \mathfrak{B}^{k+1}(U, X)$.

To prove the converse, we let $\varphi \in \mathcal{D}(U)$ and choose a compact set D containing the support of φ. If we assume (2.2) holds for f, g, D with $f \in C(U, X)$ and $g \in \mathfrak{B}^k(U, X)$, then

$$\begin{aligned} 2\pi i \iint \bar{\partial}\varphi(z) f(z)\, dz \wedge d\bar{z} &= \iint \left\{\int_{\partial D} \bar{\partial}\varphi(z)\, (\zeta - z)^{-1} f(\zeta)\, d\zeta\right\} dz \wedge d\bar{z} \\ &\quad + \iint \left\{\iint_D \bar{\partial}\varphi(z)\, (\zeta - z)^{-1} g(\zeta)\, d\zeta \wedge d\bar{\zeta}\right\} dz \wedge d\bar{z}. \end{aligned} \tag{2.3}$$

If we reverse the order of integration and replace z by $w+\zeta$, then (2.3) becomes

$$-\int_{\partial D} f(\zeta)\left\{\iint \bar{\partial}\varphi(w+\zeta)\, w^{-1}dw\wedge d\bar{w}\right\} d\zeta - \iint_D g(\zeta)\left\{\iint \bar{\partial}\varphi(w+\zeta)\, w^{-1}dw\wedge d\bar{w}\right\} dz\wedge d\bar{z}. \quad (2.4)$$

If we use the fact that (2.2) holds for C^∞-functions and that φ has compact support in D, then (2.4) becomes

$$-2\pi i\left\{\int_{\partial D}\varphi(\zeta) f(\zeta)\, d\zeta + \iint_D g(\zeta)\,\varphi(\zeta)\, d\zeta\wedge d\bar{\zeta}\right\} = -2\pi i\iint \varphi(\zeta)\, g(\zeta)\, d\zeta\wedge d\bar{\zeta}. \quad (2.5)$$

It follows that equation (2.1) holds for f and g and that $f\in\mathfrak{B}^{k+1}(U, X)$ and $\bar{\partial} f = g$.

LEMMA 2.9. *Let $U\subset \mathbf{C}$ be a domain and V a domain with compact closure in U. If $k\geqslant 0$ and $g\in\mathfrak{B}^k(U, X)$ then there exists $f\in\mathfrak{B}^{k+1}(U, X)$ such that $\bar{\partial} f = g$ on V.*

Proof. By multiplying g by a C^∞ function which is one on V and has compact support in U, we obtain $g_0\in\mathfrak{B}^k(U, X)$ such that g_0 has compact support and $g_0 = g$ on V. We set

$$f(z) = \frac{1}{2\pi i}\iint (\zeta - z)^{-1} g_0(\zeta)\, d\zeta\wedge d\bar{\zeta}.$$

It follows from Lemma 2.8 that $f\in\mathfrak{B}^{k+1}(U, X)$ and $\bar{\partial} f = g_0$.

We now proceed to the case of several variables. Let U be a domain in $\mathbf{C}^n$ and X a Frechet space. For each i, we denote the operator $\partial/\partial\bar{z}_i = \frac{1}{2}[\partial/\partial x_i + i(\partial/\partial y_i)]$ by $\bar{\partial}_i$. As in Definition 2.4, if $f\in C(U, X)$ we shall say $\bar{\partial}_i f\in C(U, X)$ if there is a $g\in C(U, X)$ $(g = \bar{\partial}_i f)$ such that

$$\int_U (\bar{\partial}_i\varphi)\, f\, dm = -\int \varphi g\, dm \quad (2.6)$$

for every $\varphi\in\mathcal{D}(U)$, where m is Lebesgue measure in $\mathbf{C}^n$.

LEMMA 2.10. *If $f\in C(U, X)$, $\bar{\partial}_i f\in C(U, X)$, and $\bar{\partial}_j(\bar{\partial}_i f)\in C(U, X)$ for some pair i, j, then $\bar{\partial}_j f\in C(U, X)$ and $\bar{\partial}_i(\bar{\partial}_j f) = \bar{\partial}_j(\bar{\partial}_i f)$.*

Proof. Since the hypothesis and conclusion are clearly local statements about functions in $C(U, X)$, we may assume without loss of generality that $U = U_1\times\dots\times U_n$ is a polydisc and f has compact support in U. Also, we may assume $i = n$.

We set
$$h(z) = \frac{1}{2\pi i}\iint_{U_n}\bar{\partial}_j\bar{\partial}_n f(z_1, \dots, z_{n-1}, \zeta)\,(\zeta - z_n)^{-1} d\zeta\wedge d\bar{\zeta},$$

and
$$g(z) = \frac{1}{2\pi i}\iint_{U_n}\bar{\partial}_n f(z_1, \dots, z_{n-1}, \zeta)\,(\zeta - z_n)^{-1} d\zeta\wedge d\bar{\zeta}.$$

Note that $h, g \in C(U, X)$ and $\bar{\partial}_j g = h$. Furthermore, since f has compact support, it follows from Lemma 2.8 that $g = f$. Hence, $\bar{\partial}_j f = h \in C(U, X)$. That $\bar{\partial}_n(\bar{\partial}_j f) = \bar{\partial}_j(\bar{\partial}_n f)$ follows from the fact that $\bar{\partial}_n \bar{\partial}_j \varphi = \bar{\partial}_j \bar{\partial}_n \varphi$ for $\varphi \in \mathcal{D}(U)$.

The above lemma makes it possible to define, in a non-ambiguous manner, a space $\mathfrak{B}^j(U, X)$ for each multi-index $j = (j_1, ..., j_n)$.

Definition 2.11. If $j = (0, ..., 0)$ is the zero index, we set $\mathfrak{B}^j(U, X) = C(U, X)$. For each i let $1_i = (0, ..., 0, 1, 0, ..., 0)$ (1 in the ith position). If $j = (j_1, ..., j_n)$ $(0 \leqslant j_1 < \infty)$ and $\mathfrak{B}^j(U, X)$ has been defined, we define $\mathfrak{B}^{j+1_i}(U, X) = \{f \in C(U, X) \colon \bar{\partial}_i f \in \mathfrak{B}^j(U, X)\}$.

If j and k are multi-indices with $j \leqslant k$ $(j_i \leqslant k_i$ for each $i)$ then we define $\bar{\partial}^j \colon \mathfrak{B}^k(U, X) \to \mathfrak{B}^{k-j}(U, K)$ by $\bar{\partial}^j = \bar{\partial}^{j_1} \dots \bar{\partial}^{j_n}$.

Without Lemma 2.10, the above definition would be ambiguous since there are many ways to form a multi-index j by beginning with 0 and successively adding indices 1_i.

As before, we give $\mathfrak{B}^k(U, X)$ the Frechet space topology in which $f_n \to 0$ if $\bar{\partial}^j f_n \to 0$ uniformly on compact sets for each $j \leqslant k$.

Let U and V be domains in $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively. We may identify $C(U \times V, X)$ and $C(U, C(V, X))$ by identifying the function $(z, w) \to f(z, w)$ in $C(U \times V, X)$ with the function $z \to f(z, \cdot)$ in $C(U, C(V, X))$. If $j = (j_1, ..., j_n)$, $k = (k_1, ..., k_m)$, and $j \cup k = (j_1, ..., j_n, k_1, ..., k_m)$, then it is evident that this identification also identifies $\mathfrak{B}^{j \cup k}(U \times V, X)$ with $\mathfrak{B}^j(U, \mathfrak{B}^k(V, X))$.

We now introduce a special notation for use in the next two lemmas. If $j = (j_1, ..., j_n)$ where $j_i = 0, 1, 2, ...$ or $j_i = -\infty$ for each i, then $\mathfrak{B}^j(U, X)$ will represent the subspace of $C(U, X)$ consisting of functions f such that f is analytic in z_i if $j_i = -\infty$ and $f \in \mathfrak{B}^{j'}(U, X)$ for $j' = (j'_1, ..., j'_n)$, where $j'_1 = 0$ if $j_i = -\infty$ and $j'_i = j_i$ if $j_i \neq -\infty$.

By $\mathfrak{B}^j_U[X]$ we will mean the sheaf of germs of the presheaf $V \to \mathfrak{B}^j(V, X)$ for $V \subset U$ (cf. [6], or [9], IV).

LEMMA 2.12. *For each $j = (j_1, ..., j_n)$ $(j_k = -\infty$, or $0, 1, 2, ...)$ and for each $i = 1, ..., n$ with $j_i \neq -\infty$, the sequence*

$$0 \to \mathfrak{B}^{j-\infty_i}_U[X] \to \mathfrak{B}^{j+1_i}_U[X] \xrightarrow{\bar{\partial}_i} \mathfrak{B}^j_U[X] \to 0$$

is exact, where $\infty_i = (0, ..., \infty, 0, ..., 0)$ and $1_i = (0, ..., 1, ..., 0)$ (∞ resp. 1 in the ith *position).*

Proof. If $f \in \mathfrak{B}^{j+1_i}(V, X)$ for some domain $V \subset X$, then $\bar{\partial}_i f = 0$ implies that f is analytic in the variable z_i on V. Hence, $f \in \mathfrak{B}^{j-\infty_i}(V, X)$.

If $z_0 \in U$ and $f \in \mathfrak{B}^j(V, X)$ for some neighborhood V of z_0, then it follows from Lemma 2.9 that there is a function $g \in \mathfrak{B}^{j+1_i}(V^1, X)$ for some possibly smaller neighborhood V^1

such that $\bar{\partial}_i g = f$ on V^1. To see this, note that if $V = V_1 \times \ldots \times V_n$ is a polydisc, we may write $\mathfrak{B}^j(V, X) = \mathfrak{B}^{j_i}(V_i, X')$ where $X' = \mathfrak{B}^{j'}(V_1 \times \ldots \times V_{i-1} \times V_{i+1} \times \ldots \times V_n, X)$ and $j' = (j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n)$. This reduces the problem to the one dimensional situation described in Lemma 2.9.

On passing to germs, the above considerations show that $\bar{\partial}_i : \mathfrak{B}_U^{j+1_i}[X] \to \mathfrak{B}_U^j[X]$ is a surjective map with kernel $\mathfrak{B}_U^{j-\infty_i}[X]$.

LEMMA 2.13. *Let $Y = \{Y^p, \alpha^p\}$ be an analytically parameterized cochain complex on a domain $U \subset \mathbf{C}^n$. If $Y(z)$ is exact for each point z of U, then the complex $\mathfrak{B}_U^j[Y] = \{\mathfrak{B}_U^j[Y^p], \alpha^p\}$ is an exact sequence of sheaves on U for each multi-index $j = (j_1, \ldots, j_n)$ $(j_i = -\infty, 0, 1, \ldots)$.*

Proof. It follows from Proposition 2.3 that if each j_i is $-\infty$ or 0 then $\mathfrak{B}_U^j[Y]$ is exact. The proof for general j proceeds by induction.

Suppose $\mathfrak{B}_U^k[Y]$ is exact for all $k < j$, where $j = (j_1, \ldots, j_n)$ and $k < j$ means $k_i \leqslant j_i$ for all i and inequality holds for some i. We may assume that $j_i > 0$ for at least one i. Then, by Lemma 2.12 the sequence

$$0 \to \mathfrak{B}_U^{j-\infty_i}[X] \to \mathfrak{B}_U^j[X] \xrightarrow{\bar{\partial}_i} \mathfrak{B}_U^{j-1_i}[X] \to 0$$

is exact. Hence,

$$0 \to \mathfrak{B}_U^{j-\infty_i}[Y] \to \mathfrak{B}_U^j[Y] \xrightarrow{\bar{\partial}_i} \mathfrak{B}_U^{j-1_i}[Y] \to 0$$

is an exact sequence of cochain complexes (of sheaves). By the induction hypothesis, $\mathfrak{B}_U^{j-\infty_i}[Y]$ and $\mathfrak{B}_U^{j-1_i}[Y]$ are both exact. It follows that $\mathfrak{B}_U^j[Y]$ is also exact (cf. [11], Chapter 2). This completes the induction.

LEMMA 2.14. *If Y is an analytically parameterized cochain complex on $U \subset \mathbf{C}^n$ and if $Y(z)$ is exact for each $z \in U$, then $\mathfrak{B}^k(U, Y)$ is exact for $k = (k_1, \ldots, k_n)$ $(k_i \geqslant 0)$.*

Proof. Each of the spaces $\mathfrak{B}^k(V, Y^p)$ $(V \subset U)$ is closed under multiplication by functions in $C^\infty(U)$ (cf. Lemma 2.5). It follows that $\mathfrak{B}_U^k[Y^p]$ is a fine sheaf for each p (cf. [6] or [9]). Hence, by Lemma 2.13, $\mathfrak{B}_U^k[Y]$ is an exact sequence of fine sheaves. It follows that the corresponding sequence of global sections $\mathfrak{B}^k(U, Y)$ is also exact (cf. [9], VI or [6], II, 4).

We are now ready to define $\mathfrak{B}(U, X)$ and prove that it possesses the properties that we require.

Definition. 2.15. If X is a Frechet space and $U \subset \mathbf{C}^n$ a domain, then we define $\mathfrak{B}(U, X) = \bigcap \{\mathfrak{B}^j(U, X) : j = (j_1, \ldots, j_n), j_i \geqslant 0\}$. We give $\mathfrak{B}(U, X)$ the Frechet space topology in which $f_n \to 0$ if and only if $\bar{\partial}^j f_n \to 0$ uniformly on compact sets for each j.

THEOREM 2.16. *The space $\mathfrak{B}(U, X)$ has the following properties:*

(1) *$\mathfrak{B}(U, X)$ is closed under multiplication by functions in $C^\infty(U)$;*

(2) *if $f \in \mathfrak{B}(U, X)$ then $\bar{\partial}^j f \in \mathfrak{B}(U, X)$ for each multi-index $j = (j_1, ..., j_n)$ and $\mathfrak{A}(U, X) = \{f \in \mathfrak{B}(U, X): \bar{\partial}_i f = 0$ for $i = 1, ..., n\}$;*

(3) *if $\varphi: V \to U$ is an analytic map, then $\varphi^*: C(U, X) \to C(V, X)$ maps $\mathfrak{B}(U, X)$ into $\mathfrak{B}(V, X)$;*

(4) *if X_1 and X_2 are Banach spaces and $\alpha \in \mathfrak{A}(U, L(X_1, X_2))$, then $\alpha f \in \mathfrak{B}(U, X_2)$ for each $f \in \mathfrak{B}(U, X_1)$;*

(5) *if Y is an analytically parameterized cochain complex of Banach spaces on U and $Y(z)$ is exact for each $z \in U$, then $\mathfrak{B}(U, Y)$ is also exact.*

Proof. Properties (1), (2), (3), and (4) are routine in view of the preceding lemmas. Hence, we concentrate on part (5).

Suppose each Y^p is a Banach space and $f \in \mathfrak{B}(U, Y^p)$ with $\alpha^p f = 0$. If $j(i) = (i, i, ..., i)$ for $i = 0, 1, ...,$ then 2.14 implies that there exists $g_i \in \mathfrak{B}^{j(i)}(U, Y^{p-1})$ for each i such that $\alpha^{p-1} g_i = f$. Also, since $\alpha^{p-1}(g_{i+1} - g_i) = 0$, we may choose $h_i \in \mathfrak{B}^{j(i)}(U, Y^{p-2})$ such that $\alpha^{p-2} h_i = g_{i+1} - g_i$. We then have

$$g_m = g_0 + \sum_{i=0}^{m-1} \alpha^{p-2} h_i \quad \text{for each } m.$$

Clearly, $\mathfrak{B}(U, Y^{p-2})$ is dense in $\mathfrak{B}^{j(i)}(U, Y^{p-2})$ for each i. Hence, if V is a domain with compact closure in U, we may choose $k_i \in \mathfrak{B}(U, Y^{p-2})$ such that $\|\bar{\partial}^j h_i(z) - \bar{\partial}^j k_i(z)\| < 1/2^i$ for all $z \in \bar{V}$ and $j \leqslant j(i)$. If $h = \sum_{i=0}^\infty (h_i - k_i)$ and $g = g_0 + \alpha^{p-2} h$, then $\alpha^{p-1} g = f$ and for each m

$$g = g_m - \alpha^{p-1} \sum_{i=0}^{m-1} k_i + \alpha^{p-2} \sum_{i=m}^{\infty} (h_i - k_i).$$

Since $\sum_{i=m}^\infty \bar{\partial}^j (h_i - k_i)$ converges uniformly on V for each $j \leqslant j(m)$, we conclude that $g \in \mathfrak{B}(V, Y^{p-1})$.

Since the above argument works for any V with compact closure in U and since $\mathfrak{B}(U, Y^{p-1})$ is closed under multiplication by functions in $C^\infty(U)$, a standard partition of unity argument yields that $\mathfrak{B}(U, Y)$ is exact.

We should point out that the construction of the space $\mathfrak{B}(U, X)$ can be carried out if X is any quasi-complete locally convex topological vector space. However, the results on exactness of $\mathfrak{B}(U, Y)$, when Y is a pointwise exact parameterized cochain complex, use the fact that each Y^p is a Banach space in an essential way. This is the only barrier to extending all of our results to some very general category of topological vector spaces.

Since Lemma 2.3 is patently false if it is not true that each Y^p is a Banach space, an extension of these results to a larger class of topological vector spaces will require a strengthened notion of exactness at a point for a parameterized cochain complex. A similar problem occurs in the spectral theory of an operator on a topological vector space which is not a Banach space. The approach to this problem used in [1] and [13] may suggest ways of extending the results of this paper.

3. The Cauchy–Weil integral

It is now a simple matter to combine the results of the preceding two sections to obtain a very general form of the Cauchy–Weil integral.

Notation 3.1. Throughout this section X will be a Banach space and $L(X)$ will denote the space of bounded linear operators on X.

Let $U \subset \mathbf{C}^{n+m}$ be a domain. We shall write points of U in the form $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_m)$. Corresponding to the coordinates in $\mathbf{C}^{n+m}$ we choose tuples of indeterminates $d\bar{z} = (d\bar{z}_1, \dots, d\bar{z}_n)$, $d\bar{w} = (d\bar{w}_1, \dots, d\bar{w}_n)$. We consider $\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}, \frac{\partial}{\partial \bar{w}_1}, \dots, \frac{\partial}{\partial \bar{w}_m}$ to be operators on $\mathfrak{B}(U, X)$ and set $\bar{\partial}_z = \frac{\partial}{\partial \bar{z}_1} d\bar{z}_1 + \dots + \frac{\partial}{\partial \bar{z}_n} d\bar{z}_n \in \Lambda^1[d\bar{z}, \mathcal{E}(\mathfrak{B}(U, X))]$ and $\bar{\partial}_w = \frac{\partial}{\partial \bar{w}_1} d\bar{w}_1 + \dots + \frac{\partial}{\partial \bar{w}_m} d\bar{w}_m \in \Lambda^1[d\bar{w}, \mathcal{E}(\mathfrak{B}(U, X))]$ (cf. § 1).

Let $a_1, \dots, a_n$ be operator valued functions in $\mathfrak{A}(U, L(X))$ such that $a_i(z, w) a_j(z, w) = a_j(z, w) a_i(z, w)$ for each i, j and each $(z, w) \in U$. We choose a tuple of indeterminates $\sigma = (s_1, \dots, s_n)$ and set $\alpha = a_1 s_1 + \dots + a_n s_n$ and $\alpha(z, w) = a_1(z, w) s_1 + \dots + a_n(z, w) s_n$. Each a_i is considered an operator on $\mathfrak{B}(U, X)$, where $a_i f(z, w) = a_i(z, w) f(z, w)$.

For our ground ring K we choose an arbitrary commutative subring of $\mathfrak{A}(U, L(X))$ whose elements commute with $a_1, \dots, a_n$. We may then consider $\mathfrak{B}(U, X)$ to be a K-module and $a_1, \dots, a_n$, $\partial/\partial\bar{z}_1, \dots, \partial/\partial\bar{z}_n, \partial/\partial\bar{w}_1, \dots, \partial/\partial\bar{w}_m$ to be module endomorphisms.

Definition 3.2. Let $\pi_w\colon \mathbf{C}^{n+m} \to \mathbf{C}^m$ be the projection of $\mathbf{C}^{n+m}$ on its last m coordinates and set $V = \pi_w U$.

A closed subset $K \subset U$ will be called z-compact if $K \cap \{\mathbf{C}^n \times L\}$ is compact in U for each compact set $L \subset V$.

The submodule of $\mathfrak{B}(U, X)$ consisting of functions with compact support (z-compact support) will be called $\mathfrak{B}_0(U, X)$ ($\mathfrak{B}_1(U, X)$).

3.3 Lemma. *The space $\mathfrak{B}(U, X)/\mathfrak{B}_0(U, X)$ is the inductive limit of the system $\{\mathfrak{B}(V, X)$: $V \subset U$, $U \setminus V$ compact$\}$, where $\{V\colon V \subset U, U \setminus V$ compact$\}$ is directed downward by inclusion* *and for $V_1 \subset V_2$ we map $\mathfrak{B}(V_2, X)$ into $\mathfrak{B}(V_1, X)$ by restriction. The same statement holds for $\mathfrak{B}(U, X)/\mathfrak{B}_1(U, X)$ with compact replaced by z-compact.*

Proof. The restriction map ν: $\mathfrak{B}(U, X) \to \mathfrak{B}(V, X)$ obviously induces a homomorphism $\nu^\wedge$ of $\mathfrak{B}(U, X)$ into $\varinjlim \{\mathfrak{B}(V, X);\ V \subset U,\ U \setminus V \text{ compact}\}$ whose kernel is $\mathfrak{B}_0(U, X)$. Hence, we need only show that $\nu^\wedge$ is surjective.

If $V \subset U$, $U \setminus V$ is compact, and $f \in \mathfrak{B}(V, X)$ then there exist compact sets K_1 and K_2 with $U \setminus V \subset \operatorname{int} K_1 \subset K_1 \subset \operatorname{int} K_2 \subset K_2 \subset U$. We may choose $\varphi \in C^\infty(U)$ such that $\varphi = 0$ on K_1 and $\varphi = 1$ on $U \setminus K_2$. If $g = \varphi f$ on V and $g = 0$ on $U \setminus V$, then $g \in \mathfrak{B}(U, X)$ and g agrees with f on $U \setminus K_2$. Hence, $\nu^\wedge g$ is the element of $\varinjlim \{\mathfrak{B}(V, X)\}$ determined by f.

3.4 LEMMA. *If the set $S(\alpha)$, of points $(z, w) \in U$ for which $\alpha(z, w) = a_1(z, w)s_1 + \dots + a_n(z, w)s_n$ is singular on X, is z-compact in U, then $(\mathfrak{B}(U, X),\ \mathfrak{B}_1(U, X),\ \alpha,\ \bar\partial_z \oplus \bar\partial_w)$ is a Cauchy-Weil system (cf. Def. 1.8).*

Proof. Clearly each of

$$a_1, \dots, a_n, \frac{\partial}{\partial \bar z_1}, \dots, \frac{\partial}{\partial \bar z_n}, \frac{\partial}{\partial \bar w_1}, \dots, \frac{\partial}{\partial \bar w_m}$$

leaves $\mathfrak{B}_1(U, X)$ invariant. Furthermore, as we mentioned earlier, any pair of these operators commute. Hence, it suffices to prove that α is non-singular on $\mathfrak{B}/\mathfrak{B}_1$.

By hypothesis, $S(\alpha)$ is z-compact. If $V = U \setminus S(\alpha)$ then for $(z, w) \in V$ the complex $F(X, \alpha(z, w))$ is exact (cf. Def. 1.2). It follows from Theorem 2.16 that the complex

$$F(\mathfrak{B}(V', X), \alpha) = \mathfrak{B}(V', F(X, \alpha))$$

is exact for any open set $V' \subset V$. Since inductive limits preserve exactness it follows that $F(\mathfrak{B}/\mathfrak{B}_1, \alpha)$ is exact, i.e., α is non-singular on $\mathfrak{B}/\mathfrak{B}_1$.

We now have that if $S(\alpha)$ is z-compact in U, then the homomorphism

$$R_\alpha: H^p(\mathfrak{B}(U, X), \bar\partial_z \oplus \bar\partial_w) \to H^{n+p}(\mathfrak{B}_1(U, X), \bar\partial_z \oplus \bar\partial_w)$$

of Def. 1.9 is defined. To complete our Cauchy–Weil integral, we will use the Lebesgue integral in $\mathbf{C}^n$ to define a homomorphism of $H^{n+p}(\mathfrak{B}_1(U, X), \bar\partial_z \oplus \bar\partial_w)$ into $H^p(\mathfrak{B}(V, X), \bar\partial_w)$.

If $$d_z = \frac{\partial}{\partial z_1} dz_1 + \dots + \frac{\partial}{\partial z_n} dz_n + \frac{\partial}{d\bar z_1} d\bar z_1 + \dots + \frac{\partial}{\partial \bar z_n} d\bar z_n = \partial_z + \bar\partial_z,$$

then the map $f \to f \wedge dz_1 \wedge \dots \wedge dz_n$ $(f \in F(\mathfrak{B}_1(U, X), \bar\partial_z \oplus \bar\partial_w)$ defines a cochain map of degree n from $F(\mathfrak{B}_1(U, X), \bar\partial_z \oplus \bar\partial_w)$ into $F(\mathfrak{B}_1(U, X), \partial_z \oplus \bar\partial_z \oplus \bar\partial_w)$.

We define a cochain map $f \to \int f$ of $F(\mathfrak{B}_1(U, X), d_z \oplus \bar{\partial}_w)$ into $F(\mathfrak{B}_1(V, X), \bar{\partial}_w)$ in the following way: we write $f = g d\bar{z}_1 \wedge dz_1 \wedge \ldots \wedge d\bar{z}_n \wedge dz_n + h$, where h contains only terms of degree less than $2n$ in $dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_n$; then

$$\int f = \int g(z, w)\, d\bar{z}_1 \wedge dz_1 \wedge \ldots \wedge d\bar{z}_n \wedge dz_n$$

is $(2i)^n$ times the ordinary Lebesgue integral of g, where for each $w \in V$, $z \to g(z, w)$ is considered a function with compact support in $\mathbf{C}^n$ with values in $\Lambda[d\bar{w}, X]$.

Note that

$$\begin{aligned}\int (d_z \oplus \bar{\partial}_w) f &= \int (\bar{\partial}_w g)\, d\bar{z}_1 \wedge dz_1 \wedge \ldots \wedge d\bar{z}_n \wedge dz_n + \int d_z h + \int \bar{\partial}_w h \\ &= \int (\bar{\partial}_w g)\, d\bar{z}_1 \wedge dz_1 \wedge \ldots \wedge d\bar{z}_n \wedge dz_n + \int \bar{\partial}_w h = \int \bar{\partial}_w f.\end{aligned}$$

This is due to the fact that $\int d_z h = 0$ since $h(z, w)$ has compact support in z for each w (Stokes Theorem). Hence $f \to \int f$ is a cochain map of degree $-2n$.

If we combine the maps $f \to f \wedge dz_1 \wedge \ldots \wedge dz_n$ and $g \to \int g$ we obtain a cochain map $f \to \int f \wedge dz_1 \wedge \ldots \wedge dz_n$ of $F(\mathfrak{B}_1(U, X), \bar{\partial}_z \oplus \bar{\partial}_w) \to F(\mathfrak{B}(V, X), \bar{\partial}_w)$ of degree $-n$. We shall occasionally denote this map by ϱ.

It follows that ϱ defines a homomorphism ϱ^*: $H^{n+p}(\mathfrak{B}_1(U, X), \bar{\partial}) \to H^p(\mathfrak{B}(V, X), \bar{\partial}_w)$ for each ϱ. We shall also denote this homomorphism by $f \to \int f \wedge dz_1 \wedge \ldots \wedge dz_n$.

Definition 3.5. If $U \subset \mathbf{C}^{n+m}$ is a domain, X a Banach space, $a_1, \ldots, a_n \in \mathfrak{A}(U, L(X))$ a commuting n-tuple such that $\alpha(z, w) = a_1(z, w) s_1 + \ldots + a_n(z, w) s_n$ is non-singular except on a z-compact subset of U, then for each $f \in H^p(\mathfrak{B}(U, X), \bar{\partial}_z \oplus \bar{\partial}_w)$ the Cauchy–Weil integral of f with respect to α is the element

$$\varrho^* R_\alpha f = \int (R_\alpha f) \wedge dz_1 \wedge \ldots \wedge dz_n$$

of $H^p(\mathfrak{B}(V, X), \bar{\partial}_w)$, where $V = \pi_w U$.

Note that if $p = 0$ then $H^0(\mathfrak{B}(U, X), \bar{\partial}_z \oplus \bar{\partial}_w) = \mathfrak{A}(U, X)$. $H^0(\mathfrak{B}(V, X), \bar{\partial}_w) = \mathfrak{A}(V, X)$, and so the Cauchy–Weil integral maps $\mathfrak{A}(U, X)$ into $\mathfrak{A}(V, X)$. It is the case $p = 0$ that really interests us; however, for technical reasons we must consider the integral in each degree until we complete some additional machinery.

Also note that if $m = 0$, then we consider $\mathbf{C}^m$ to be the singleton point $\{0\}$ and set $V = \{0\}$, $\mathfrak{B}(V, X) = X$, $F^p(\mathfrak{B}(V, X), \bar{\partial}_w) = H^p(\mathfrak{B}(V, X), \bar{\partial}_w) = X$ for $p = 0$ and 0 otherwise.

In this case, $\int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n$ is zero if $f \in H^p(\mathfrak{B}(U, X), \bar\partial)$ with $p \neq 0$ and is an element of X if $p=0$.

The key to computing the Cauchy–Weil integral for specific $a_1, \ldots, a_n$ is an analogue of Fubini's Theorem which we present below.

THEOREM 3.6. *Let U be a domain in $\mathbf{C}^{n+m}$ and $V = \pi_w U \subset \mathbf{C}^m$. Let $a_1, \ldots, a_n \in \mathfrak{A}(U, L(X))$ and $b_1, \ldots, b_m \in \mathfrak{A}(V, L(X))$ be commuting tuples with $a_i(z, w) b_j(w) = b_j(w) a_i(z, w)$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $(z, w) \in U$. Let $\alpha = a_1 s_1 + \ldots + a_n s_n$ and $\beta = b_1 s_1' + \ldots + b_m s_m'$ and suppose $S(\alpha)$ is z-compact in U and $S(\beta)$ is compact in V. Then for each $f \in \mathfrak{A}(U, X)$,*

$$\int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n \in \mathfrak{A}(V, X) = H^0(\mathfrak{B}(V, X), \bar\partial_w)$$

and

$$\int R_\beta \left\{ \int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n \right\} \wedge dw_1 \wedge \ldots \wedge dw_n$$

$$= \int R_{\alpha \oplus \beta} f \wedge dz_1 \wedge \ldots \wedge dz_n \wedge dw_1 \wedge \ldots \wedge dw_m \in X.$$

Proof. Recall that $\mathfrak{B}_1(U, X)$ is the space of functions in $\mathfrak{B}(U, X)$ with z-compact support, and $\mathfrak{B}_0(U, X)$ is the space of functions in $\mathfrak{B}(U, X)$ with compact support. The hypothesis on α and β guarantee that $(\mathfrak{B}(U, X), \mathfrak{B}_1(U, X), \alpha, \bar\partial_z \oplus \bar\partial_w)$ and $(\mathfrak{B}_1(U, X), \mathfrak{B}_0(U, X), \beta, \bar\partial_z \oplus \bar\partial_w)$ are both Cauchy–Weil systems. By Proposition 1.13, $(\mathfrak{B}(U, X), \mathfrak{B}_0(U, X), \alpha \oplus \beta, \bar\partial_z \oplus \bar\partial_w)$ is also a Cauchy–Weil system and $R_{\alpha \oplus \beta} = R_\beta \circ R_\alpha$. Hence, we may write

$$\int_{\mathbf{C}^{n+m}} R_{\alpha \oplus \beta} f \wedge dz_1 \wedge \ldots \wedge dz_n \wedge dw_1 \wedge \ldots \wedge dw_n$$

$$= \int_{\mathbf{C}^m} \left\{ \int_{\mathbf{C}^n} R_\beta \circ R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n \right\} \wedge dw_1 \wedge \ldots \wedge dw_n.$$

To complete the proof, we need only show that the diagram

$$\begin{array}{ccc} H^n(\mathfrak{B}_1(U, X), \bar\partial) & \xrightarrow{R_\beta} & H^{n+m}(\mathfrak{B}_0(U, X), \bar\partial) \\ \big\downarrow \varrho_z^* & & \big\downarrow \varrho_z^* \\ H^0(\mathfrak{B}(V, X), \bar\partial_w) & \xrightarrow{R_\beta} & H^m(\mathfrak{B}_0(V, X), \bar\partial_w) \end{array}$$

is commutative, where ϱ_z: $F(\mathfrak{B}_1(U, X), \bar\partial) \to F(\mathfrak{B}_0(V, X), \bar\partial_w)$ is defined by $\varrho_z(f) = \int_{\mathbf{C}^n} f \wedge dz_1 \wedge \ldots \wedge dz_n$ and ϱ_z^* is the corresponding map of cohomology. However, this follows readily, since the fact that $b_1, \ldots, b_m$ are independent of z forces the following diagram to be commutative:

$$\begin{array}{ccccccc}
F(\mathfrak{B}_1(U,X),\bar\partial_z\oplus\bar\partial_w) & \xrightarrow{s'} & F(\mathfrak{B}_1(U,X),\beta\oplus\bar\partial_z\oplus\bar\partial_w) & \xleftarrow{i} & F(\mathfrak{B}_0(U,X),\beta\oplus\bar\partial_z\oplus\bar\partial_w) & \xrightarrow{\pi} & F(\mathfrak{B}_0(U,X),\bar\partial_z\oplus\bar\partial_w) \\
\downarrow \varrho_z & & \downarrow \varrho_z & & \downarrow \varrho_z & & \downarrow \varrho_z \\
F(\mathfrak{B}(V,X),\bar\partial_w) & \xrightarrow{s'} & F(\mathfrak{B}(V,X),\beta\oplus\bar\partial_w) & \xleftarrow{i} & F(\mathfrak{B}_0(V,X),\beta\oplus\bar\partial_w) & \xrightarrow{\pi} & F(\mathfrak{B}_0(V,X),\bar\partial_w)
\end{array}$$

(cf. Definition 1.9). This completes the proof.

If U and $\alpha=a_1s_1+\ldots+a_ns_n$ satisfy the conditions of Definition 3.5, then for each $w\in V=\pi_w U$ and $f\in\mathfrak{A}(U,X)$ there are two ways of obtaining an element of X. One could compute $g=\int R_\alpha f\wedge dz_1\wedge\ldots\wedge dz_n\in\mathfrak{A}(V,X)$ and then evaluate at w, or one could set $\alpha^w(z)=\alpha(z,w)$ and $f^w(z)=f(z,w)$ and compute $\int_{\mathbf{C}^n} R_{\alpha w} f^w\wedge dz_1\wedge\ldots\wedge dz_n\in X$. We shall show that these yield the same result. This makes Theorem 3.6 much more useful than it would be otherwise.

LEMMA 3.7. *If U, X, and α satisfy the conditions of Definition 3.5, $w\in V=\pi_w U$, and $\alpha^w(z)=\alpha(z,w)$ then for each $f\in\mathfrak{A}(U,X)$ we have*

$$\left(\int_{\mathbf{C}^n} R_\alpha f\wedge dz_1\wedge\ldots\wedge dz_n\right)(w)=\int_{\mathbf{C}^n} R_{\alpha w} f^w\wedge dz_1\wedge\ldots\wedge dz_n,$$

where $f^w(z)=f(z,w)$.

Proof. Let $U_w=\{z\in\mathbf{C}^n:(z,w)\in U\}$ and consider the special transformation $u: F(\mathfrak{B}(U,X),\bar\partial_z\oplus\bar\partial_w)\to F(\mathfrak{B}(U_w,X),\bar\partial_z)$ determined by $(u^0f)(z)=f(z,w)=f^w(z)$ for $f\in\mathfrak{B}(U,X)$ and $u(d\bar z_i)=d\bar z_i$, $u(d\bar w_j)=0$ for $i=1,\ldots,n$, $j=1,\ldots,m$ (cf. Def. 1.6). Clearly $u^0(a_if)=a_i^w u^0(f)$ for $f\in\mathfrak{B}(U,X)$. Hence, it follows from Proposition 1.10 that the diagram

$$\begin{array}{ccc}
H^0(\mathfrak{B}(U,X),\bar\partial_z\oplus\bar\partial_w) & \xrightarrow{R_\alpha} & H^n(\mathfrak{B}(U,X),\bar\partial_z\oplus\bar\partial_w) \\
\downarrow u^* & & \downarrow u^* \\
H^0(\mathfrak{B}(U_w,X),\bar\partial_z) & \xrightarrow{R_{\alpha w}} & H^n(\mathfrak{B}(U_w,X),\bar\partial_z)
\end{array}$$

is commutative.

Also, u^* clearly commutes with the map $g\to\int g\wedge dz_1\wedge\ldots\wedge dz_n$. Hence,

$$u^*\int R_\alpha f\wedge dz_1\wedge\ldots\wedge dz_n=\int R_{\alpha w}u^*f\wedge dz_1\wedge\ldots\wedge dz_n,$$

which is precisely the conclusion of the Lemma.

The above lemma makes the following notation unambiguous: for $f\in\mathfrak{A}(U,X)$ and $w\in V$, we set

$$\int (R_{\alpha(z,w)} f(z,w)) \wedge dz_1 \wedge \ldots \wedge dz_n = \left(\int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n\right)(w)$$
$$= \int R_{\alpha w} f^w \wedge dz_1 \wedge \ldots \wedge dz_n.$$

This is analogous to writing $\int f(x, y)\,dx$ for the integral of a function of two variables with respect to one of these variables.

With Theorem 3.5 and Lemma 3.6 out of the way, we may restrict attention to the case where $m = p = 0$, which is our real interest. That is, U will be a domain in $\mathbf{C}^n$, $(a_1, \ldots, a_n)$ a commuting n-tuple in $\mathfrak{A}(U, L(X))$ such that $\alpha = a_1 s_1 + \ldots + a_n s_n$ is non-singular except on a compact subset of U, and we consider $f \to \int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n$ a map from $\mathfrak{A}(U, X)$ to X. This will be the setting for the next four Theorems.

PROPOSITION 3.7. *If A is the algebra of all operators on X which commute with $a_i(z)$ for each i and each $z \in U$, then $f \to \int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n$ is an A-module homomorphism from $\mathfrak{A}(U, X)$ to X. Furthermore, if $f = a_1 g_1 + \ldots + a_n g_n$ for some $g_1, \ldots, g_n \in \mathfrak{A}(U, X)$, then $\int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n = 0$.*

Proof. By construction, R_α is a K-module homomorphism for any commutative ring $K \subset \mathfrak{A}(U, X)$ consisting of elements which commute with each a_i. In particular, K could be chosen to be any commutative subring of A (where elements of A are considered to be constant functions on U). Hence, $a \in A$ implies that

$$\int R_\alpha a f \wedge dz_1 \wedge \ldots \wedge dz_n = \int a R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n = a \int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n.$$

If $f = a_1 g_1 + \ldots + a_n g_n$, then $R_\alpha f = 0$ by Proposition 1.12.

PROPOSITION 3.8. *Let $X \xrightarrow{u} Y$ be a bounded linear map from the Banach space X to the Banach space Y and $a_1, \ldots, a_n \in \mathfrak{A}(U, L(X))$, $b_1, \ldots, b_n \in \mathfrak{A}(U, L(Y))$ be commuting n-tuples with $u(a_i(z)x) = b_i(z)u(x)$ for all i, z. If $\alpha = a_1 s_1 + \ldots + a_n s_n$, $\beta = b_1 s_1 + \ldots + b_n s_n$, and $S(\alpha)$ and $S(\beta)$ are both compact in U, then*

$$u \int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n = \int R_\beta u f \wedge dz_1 \wedge \ldots \wedge dz_n$$

for every $f \in \mathfrak{A}(U, X)$

Proof. This follows directly from Proposition 1.10.

As an application of the above theorem, we prove the continuity of the map $f \to \int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n$. Suppose Λ is a compact Hausdorff space and $a_1, \ldots a_n \in C(U \times \Lambda, L(X))$

is a commuting tuple of operator valued functions such that $z \to a_i(z, \lambda)$ is analytic on U for each $\lambda \in \Lambda$. If we set $Y = C(\Lambda, X)$ then each a_i determines an element $a'_i \in \mathfrak{A}(U, L(Y))$, where $(a'_i(z)f)(\lambda) = a_i(z, \lambda)f(\lambda)$ for $f \in C(\Lambda, X) = Y$. If $\alpha = a_1 s_1 + \ldots + a_n s_n$ is non-singular except on a compact subset of $U \times \Lambda$, then it follows from Lemma 2.3 that $\alpha' = a'_1 s_1 + \ldots + a'_n s_n$ is non-singular on Y, except for z in a compact subset of U.

If $f \in C(U \times \Lambda, X)$ is analytic in $z \in U$ for each $\lambda \in \Lambda$, then $f' \in \mathfrak{A}(U, Y)$, where $f'(z) = f(z, \cdot)$. Hence, the Cauchy–Weil integral $\int R_{\alpha'} f' \wedge dz_1 \wedge \ldots \wedge dz_n$ exists and is an element of $Y = C(\Lambda, X)$.

Now fix $\lambda \in \Lambda$ and consider the Cauchy–Weil integral $\int R_{\alpha(z,\lambda)} f(z, \lambda) \wedge dz_1 \wedge \ldots \wedge dz_n = \int R_{\alpha(\cdot,\lambda)} f(\cdot, \lambda) dz_1 \wedge \ldots \wedge dz_n \in X$. Since the evaluation map u_λ: $C(\Lambda, X) \to X$ and the tuples α' and $\alpha(\cdot, \lambda)$ satisfy the conditions of Theorem 3.8 we have that

$$\left(\int R_{\alpha'} f' \wedge dz_1 \wedge \ldots \wedge dz_n\right)(\lambda) = \int R_{\alpha(z,\lambda)} f(z, \lambda) \wedge dz_1 \wedge \ldots \wedge dz_n$$

for each $\lambda \in \Lambda$. Hence, we have the following corollary to Theorem 3.8:

COROLLARY 3.9. *With Λ, α, and f as above we have*

$$\lambda \to \int R_{\alpha(z,\lambda)} f(z, \lambda) \wedge dz_1 \wedge \ldots \wedge dz_n$$

is a continuous function of $\lambda \in \Lambda$.

If $\{f_i\}$ is a sequence in $\mathfrak{A}(U, X)$ which converges uniformly on compact sets to $f \in \mathfrak{A}(U, X)$, then we let Λ be the one point compactification of the positive integers and set $f(z, i) = f_i(z)$ for $i = 1, 2, \ldots$ and $f(z, \infty) = f(z)$. We set $\alpha(z, i) = \alpha(z)$ for $i = 1, 2, \ldots, \infty$. The above corollary then implies that

$$\int R_\alpha f_i \wedge dz_1 \wedge \ldots \wedge dz_n \to \int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n.$$

Hence, we have

COROLLARY 3.10. *The map $f \to \int R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n$ is a continuous map from $\mathfrak{A}(U, X)$ to X, where $\mathfrak{A}(U, X)$ is given the topology of uniform convergence on compact sets.*

The next theorem shows that the Cauchy–Weil integral is, in a sense, independent of U. Strictly speaking, it is the map R_α which depends on U and not the map $\varrho^* = (f \to \int f \wedge dz_1 \wedge \ldots \wedge dz_n)$. However, we shall still use the symbol $\int_U R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n$ to indicate the Cauchy–Weil integral of $f \in \mathfrak{A}(U, X)$ computed relative to the Cauchy–Weil system $(\mathfrak{B}(U, X), \mathfrak{B}_0(U, X), \alpha, \bar{\partial})$.

PROPOSITION 3.11. *Let U_1 and U_2 be domains in $\mathbf{C}^n$ and $a_1, \ldots, a_n \in \mathfrak{A}(U_1 \cup U_2, X)$ a commuting tuple. If $\alpha = a_1 s_1 + \ldots + a_n s_n$ and $S(\alpha) \cap U_1 = S(\alpha) \cap U_2$ is compact, then*

$$\int_{U_1} R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n = \int_{U_2} R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n$$

for every $f \in \mathfrak{A}(U_1 \cup U_2, X)$.

Proof. Without loss of generality we may assume that $U_1 \subset U_2$ and $S(\alpha) \cap U_2$ is a compact subset of U_1. This ensures that if we consider $\mathfrak{B}_0(U_1, X)$ to be a subspace of $\mathfrak{B}_0(U_2, X)$, then $(\mathfrak{B}(U_2, X), \mathfrak{B}_0(U_1, X), \alpha, \bar{\partial})$ is a Cauchy–Weil system. Likewise, $(\mathfrak{B}(U_2, X), \mathfrak{B}_0(U_2, X), \alpha, \bar{\partial})$ and $(\mathfrak{B}(U_1, X), \mathfrak{B}_0(U_1, X), \alpha, \bar{\partial})$ are Cauchy-Weil systems.

If u: $\mathfrak{B}(U_2, X) \to \mathfrak{B}(U_1, X)$ is restriction and v: $\mathfrak{B}_0(U_1, X) \to \mathfrak{B}_0(U_2, X)$ is inclusion, then the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{B}(U_2, X) & \mathfrak{B}_0(U_2, X) \arrow[l, "i"'] \\
\mathfrak{B}(U_2, X) \arrow[u, leftrightarrow, "\mathrm{se}", "\mathrm{id}"'] & \mathfrak{B}_0(U_1, X) \arrow[l, "i"'] \arrow[u, "v"'] \\
\mathfrak{B}(U_1, X) & \mathfrak{B}_0(U_1, X). \arrow[l, "i"'] \arrow[u, leftrightarrow, "\mathrm{id}"']
\arrow[from=2-1, to=3-1, "u"]
\end{tikzcd}$$

The proposition now follows from a double application of Proposition 1.10.

The following proposition is an immediate consequence of Proposition 1.11:

PROPOSITION 3.12. *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be commuting tuples of elements of $\mathfrak{A}(U, L(X))$ which are related by $b_i = \sum u_{ij} a_j$, where (u_{ij}) is an $n \times n$-matrix of commuting elements of $\mathfrak{A}(U, L(X))$ which commute with each a_j and each b_i. If $\alpha = a_1 s_1 + \ldots + a_n s_n$ and $\beta = b_1 s_1 + \ldots + b_n s_n$ are both non-singular off some compact subset of U, then*

$$\int R_\alpha(\det (u_{ij}) f) \wedge dz_1 \wedge \ldots \wedge dz_n = \int R_\beta f \wedge dz_1 \wedge \ldots \wedge dz_n,$$

for every $f \in \mathfrak{A}(U, X)$.

If U and V are connected domains in $\mathbf{C}^n$ and φ: $U \to V$ is a proper analytic map, then except on a set of measure zero φ is a k-sheeted covering map for some integer k (cf. [9], III. B. 21).

THEOREM 3.13. *Let U and V be connected domains in $\mathbf{C}^n$, where we use coordinate systems $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ in U and V respectively. Let φ: $U \to V$ be a proper analytic map which* *is a k-sheeted analytic cover (cf.* [9], *III. B.). If* $a_1, \ldots, a_n \in \mathfrak{A}(V, X)$ *is a commuting tuple and* $\alpha = a_1 s_1 + \ldots + a_n s_n$ *is non-singular except on a compact subset* $K \subset V$*, then* $\alpha \circ \varphi = a_1 \circ \varphi s_1 + \ldots + a_n \circ \varphi s_n$ *is non-singular except on the compact subset* $\varphi^{-1}(K)$ *of* U*, and*

$$k \int R_\alpha f \wedge dw_1 \wedge \ldots \wedge dw_n = \int \det(\varphi')\,(R_{\alpha\circ\varphi} f \circ \varphi) \wedge dz_1 \wedge \ldots \wedge dz_n,$$

where φ' *is the matrix* $\left(\frac{\partial \varphi_i}{\partial z_j}\right)$.

Proof. Consider the special transformation $u\colon F(\mathfrak{B}(V, X), \bar\partial_w) \to F(\mathfrak{B}(U, X), \bar\partial_z)$ defined by $u_0(f) = f \circ \varphi$ for $f \in \mathfrak{B}(V, X)$ and

$$u(d\bar w_i) = \sum \frac{\partial \bar\varphi_i}{\partial \bar z_j}\, d\bar z_j = \sum \overline{\left(\frac{\partial \varphi_i}{\partial z_j}\right)}\, d\bar z_j.$$

It follows from the chain rule and Proposition 1.7 that u is a cochain map. Hence, by Proposition 1.10 we have

$$R_{\alpha\circ\varphi}(f \circ \varphi) = R_{\alpha\circ\varphi}(uf) = u(R_\alpha f) = (\overline{\det \varphi'})\,(R_\alpha f) \circ \varphi.$$

Hence, $(\det \varphi')\, R_{\alpha\circ\varphi}(f\circ\varphi) = |\det \varphi'|^2 (R_\alpha f) \circ \varphi$. However, $|\det \alpha'|^2$ is the Jacobian of the transformation φ as a map from $U \subset R^{2n}$ to $V \subset R^{2n}$. Hence, it follows from the change of variables formula for the Lebesgue integral in R^{2n} that

$$\int (\det \varphi')\, R_{\alpha\circ\varphi}(f\circ\varphi) \wedge dz_1 \wedge \ldots \wedge dz_n = \int |\det \varphi'|^2 (R_\alpha f) \circ \varphi \wedge dz_1 \wedge \ldots \wedge dz_n$$
$$= k \int R_\alpha f \wedge dw_1 \wedge \ldots \wedge dw_n.$$

We conclude this section by showing that our version of the Cauchy–Weil integral behaves as it should in the usual special cases.

LEMMA 3.14. *Let* U *be a domain in* $\mathbf{C}$, $a \in \mathfrak{A}(U, L(X))$ *an operator valued function such that* $a^{-1}(z)$ *exists except on a compact subset* $K \subset U$*, and* Γ *a Jordan curve in* U *with* K *contained in the union of the bounded complementary components of* Γ*. Then if* $\alpha = a s_1$ *we have:*

$$\int R_\alpha f \wedge dz = \int_\Gamma a^{-1}(z)\, f(z)\, dz$$

for every $f \in \mathfrak{A}(U, X)$.

Proof. We compute $R_\alpha\colon H^0(\mathfrak{B}(U, X), \bar\partial) \to H^1(\mathfrak{B}(U, X), \bar\partial)$. We let $f \in \mathfrak{A}(U, X)$ and refer to Definition 1.9. We have $sf = f s_1 \in \Lambda^1[\sigma \cup d\bar z, \mathfrak{B}(U, X)]$.

Let V be a domain with compact closure such that $K \subset V \subset \bar{V} \subset \operatorname{int} \Gamma$. If we multiply $a^{-1}(z) f(z)$ by an appropriate function in $C^\infty(U)$, we obtain $g \in \mathfrak{B}(U, X)$ such that $g(z) = a^{-1}(z) f(z)$ on $U \setminus V$. It follows that $\bar{\partial} g = (\partial g / \partial \bar{z})\, d\bar{z}$ has compact support in U. We set $h = fs_1 - \alpha \wedge g - \bar{\partial} g = (f - ag) s_1 - (\partial g / \partial \bar{z})\, d\bar{z}$.

Since $ag = f$ on $U \setminus V$ we have that h has compact support and $h \in \Lambda^1[\sigma \cup d\bar{z}, \mathfrak{B}_0(U, X)]$. Note that $ih \in \Lambda^1[\sigma \cup d\bar{z}, \mathfrak{B}(U, X)]$ is equal to $sf - (\alpha \oplus \bar{\partial}) g$. It follows that $s^*[f] = i^*[h]$, where [ ] represents cohomology class.

Note that $\pi h = \pi(fs_1 - ags_1 - (\partial g / \partial \bar{z})\, d\bar{z}) = -(\partial g / \partial \bar{z})\, d\bar{z} = -\bar{\partial} g$. Hence, $R_\alpha f = -\pi^* i^{*-1} s^* f = [\bar{\partial} g]$ and

$$\int_U R_\alpha f \wedge dz = \int_U \bar{\partial} g \wedge dz = \int_U d(g \wedge dz) = \int_\Gamma g dz = \int_\Gamma a^{-1}(z) f(z)\, dz.$$

This completes the proof.

If we combine the above Lemma with Theorem 3.5 and use induction, we obtain:

COROLLARY 3.15. *Let $U = U_1 \times \ldots \times U_n$ be a polydomain in $\mathbf{C}^n$ and let $a_i \in \mathfrak{A}(U_i, L(X))$ for $i = 1, \ldots, n$. If $a_1, \ldots, a_n$ commute and a_i^{-1} exists off a compact set K_i of U_i for each i, then*

$$\int_U R_\alpha f \wedge dz_1 \wedge \ldots \wedge dz_n = \int_{\Gamma_1} \ldots \int_{\Gamma_n} a_1^{-1}(z_1) \ldots a_n^{-1}(z_n) f(z)\, dz_1\, dz_2 \ldots dz_n,$$

where for each i, Γ_i is a Jordan curve in U_i with $K_i \subset \operatorname{int} \Gamma_i$.

Finally, we have:

THEOREM 3.16. *Let U be a domain in $\mathbf{C}^n$, $w \in U$, and set $z - w = (z_1 - w_1) s_1 + \ldots + (z_n - w_n) s_n$. Then for any $f \in \mathfrak{A}(U, X)$,*

$$f(w) = \frac{1}{(2\pi i)^n} \int_U (R_{z-w} f(z)) \wedge dz_1 \wedge \ldots \wedge dz_n.$$

Proof. By Theorem 3.11, we may assume U is a polydisc containing w. The theorem then follows from Corollary 3.15 and the ordinary Cauchy integral formula.

Note that, now that we have Theorem 3.16, we can use Theorem 3.13 to derive a formula for the number of sheets k in an analytic cover $\varphi: U \to V$. In fact, if $\varphi = (\varphi_1, \ldots, \varphi_n)$,

$$w^0 \in \varphi(U), \quad \alpha(w) = (w_1 - w_1^0) s_1 + \ldots + (w_n - w_n^0) s_n \quad \text{and} \quad \varphi' = \left(\frac{\partial \varphi_i}{\partial z_j}\right),$$

then

$$\frac{1}{(2\pi i)^n} \int_U \det(\varphi')\, (R_{\alpha \circ \varphi} 1) \wedge dz_1 \wedge \ldots \wedge dz_n = \frac{k}{(2\pi i)^n} \int_V R_\alpha 1 \wedge dw_1 \wedge \ldots \wedge dw_n = k.$$

4. The functional calculus

We now develop an analytic functional calculus for a commuting n-tuple of operators on a Banach space. The form of the Cauchy–Weil integral that we have developed makes this task almost as easy for n-operators as it is for a single operator.

Notation 4.1. In this section, X will be a Banach space and $(a_1, \ldots, a_n)$ will be a commuting tuple of operators on X. We set $\alpha = a_1 s_1 + \ldots + a_n s_n \in \Lambda^1[\sigma, L(X)]$.

In [15] we defined $\mathrm{Sp}(\alpha, X)$ to be the set of all $z \in \mathbf{C}^n$ such that $z - \alpha = (z_1 - a_1)s_1 + \ldots + (z_n - a_n)s_n$ is singular on X. The set $\mathrm{Sp}(\alpha, X)$ is always a compact non-empty subset of the closed polydisc D_ν of multiradius $\nu = (\nu_1, \ldots, \nu_n)$, where $\nu_i = \lim \|a_i^n\|^{1/n}$ (cf. [15], § 3).

Definition 4.2. If $f \in \mathfrak{A}(U)$ for some open set $U \supset \mathrm{Sp}(\alpha, X)$, then for each $x \in X$ we define

$$f(\alpha)\, x = \frac{1}{(2\pi i)^n} \int_U (R_{z-\alpha} f(z)\, x) \wedge dz_1 \wedge \ldots \wedge dz_n.$$

For the convenience of the reader, we recall briefly the steps involved in computing the above expression. Since $z - \alpha$ is non-singular except on $\mathrm{Sp}(\alpha, X) \subset U$, it follows that $(\mathfrak{B}(U, X), \mathfrak{B}_0(U, X), z - \alpha, \bar{\partial}_z)$ is a Cauchy–Weil system (cf. Def. 1.8 and Lemma 3.4). Here, we set $m = 0$ in Lemma 3.4 so that the variables $w_1, \ldots, w_m$ do not appear and we have $\mathfrak{B}_0(U, X) = \mathfrak{B}_1(U, X)$. It follows that the map $R_{z-\alpha}$: $H^p(\mathfrak{B}(U, X), \bar{\partial}_z) \to H^{n+p}(\mathfrak{B}_0(U, X), \bar{\partial}_z)$ of Definition 1.9 is defined. With $p = 0$ this yields a map $R_{z-\alpha}$: $\mathfrak{A}(U, X) \to H^n(\mathfrak{B}_0(U, X), \bar{\partial}_z)$. Hence, if $f \in \mathfrak{A}(U)$ then $(z \to f(z)x) \in \mathfrak{A}(U, X)$ and $R_{z-\alpha} f(z) x$ has a representative g which is a differential form of degree n in $d\bar{z}_1, \ldots, d\bar{z}_n$ with coefficients in $\mathfrak{B}_0(U, X)$. Thus, $g \wedge dz_1 \wedge \ldots \wedge dz_n$ is a differential form of degree $2n$ with coefficients in $\mathfrak{B}_0(U, X)$. We integrate this over U to obtain

$$(2\pi i)^n f(\alpha) x = \int (R_{z-\alpha} f(z) x) \wedge dz_1 \wedge \ldots \wedge dz_n \in X.$$

Note that if $\alpha = (a)$ $(a \in L(X))$ is a singleton, then $\mathrm{Sp}(\alpha, X) = \{z \in \mathbf{C}\colon z - a \text{ is not invertible}\}$ since the complex $F(X, z-\alpha)$ is simply the sequence $0 \to X \xrightarrow{z-a} X \to 0$. Furthermore, by Lemma 3.14 we have

$$\int R_{z-\alpha} f(z)\, x \wedge dz = \int_\Gamma (z-a)^{-1} f(z)\, x dz$$

for a Jordan curve $\Gamma \subset U$ which encloses $\mathrm{Sp}(\alpha, X)$. Hence, in the case of a single operator, the expression $f(\alpha)x$ of Definition 4.2 agrees with that determined by the classical operational calculus.

THEOREM 4.3. *If U is an open set containing $\mathrm{Sp}(\alpha, X)$, then $f \to f(\alpha)$ is a continuous homomorphism of the algebra $\mathfrak{A}(U)$ into the Banach algebra $(\alpha)''$ of all operators on X which* *commute with all operators commuting with each* a_i. *Furthermore*, $1(\alpha)=\mathrm{id}$ *and* $z_i(\alpha)=a_i$ *for* $i=1, \ldots, n$.

Proof. It follows from Proposition 3.7 and Corollary 3.10 that $(f, x)\to f(\alpha)x$ is a continuous linear map of $\mathfrak{A}(U)\times X$ into X and that $f(\alpha)$ commutes with each operator that commutes with each a_i. Hence, $f\to f(\alpha)$ is a continuous map into $(\alpha)''$ with the norm topology.

To prove that $1(\alpha)=\mathrm{id}$ and $z_i(\alpha)=a_i$ for each i, note that for any polynomial P and $x\in X$, the funtion $z\to P(z)x$ is in $\mathfrak{A}(\mathbf{C}^n, X)$. Since, $z_i-a_i\in\mathfrak{A}(\mathbf{C}, X)$ for each i, it follows from Proposition 3.11 that we may replace U with $\mathbf{C}^n$ without loss of generality. It then follows from Corollary 3.15 that

$$P(\alpha)\,x=\frac{1}{(2\pi i)^n}\int_{\mathbf{C}^n}(R_{z-\alpha}P(z)\,x)\wedge dz_1\wedge\ldots\wedge dz_n$$

$$=\frac{1}{(2\pi i)^n}\int_{\Gamma_1}\cdots\int_{\Gamma_n}(z_1-a_1)^{-1}\ldots(z_n-a_n)^{-1}P(z)\,x dz_1\wedge\ldots\wedge dz_n=P(a_1,\ldots,a_n)\,x$$

for $\Gamma_1,\ldots,\Gamma_n$ sufficiently large circles.

To prove that $f\to f(\alpha)$ is multiplicative, note that

$$f(\alpha)\,g(\alpha)x=\frac{1}{(2\pi i)^{2n}}\int_U R_{z-\alpha}f(z)\left\{\int_U(R_{w-\alpha}g(w)\,x)\wedge dw_1\wedge\ldots\wedge dw_n\right\}\wedge dz_1\wedge\ldots\wedge dz_n$$

$$=\frac{1}{(2\pi i)^{2n}}\int_{U\times U}\{R_{(w-\alpha)\oplus(z-\alpha)}f(z)\,g(w)\,x\}\wedge dw_1\wedge\ldots\wedge dw_n\wedge dz_1\wedge\ldots\wedge dz_n,$$

by Theorem 3.6. If we transform the tuple $(w_1-a_1, \ldots, w_n-a_n, z_1-a_1, \ldots, z_n-a_n)$ by the matrix (u_{ij}), where $u_{ij}=1$ if $i=j$, $u_{ij}=-1$ if $j=n+i$, and $u_{ij}=0$ otherwise, then we obtain the tuple $(w_1-z_1, \ldots, w_n-z_n, z_1-a_1, \ldots, z_n-a_n)$. Also, since $\det(u_{ij})=1$ it follows from Proposition 3.12 that

$$\frac{1}{(2\pi i)^{2n}}\int_{U\times U}\{R_{(w-\alpha)\oplus(z-\alpha)}f(z)\,g(w)\,x\}\wedge dw_1\wedge\ldots\wedge dw_n\wedge dz_1\wedge\ldots\wedge dz_n$$

$$=\frac{1}{(2\pi i)^{2n}}\int_{U\times U}\{R_{(w-z)\oplus(z-\alpha)}f(z)\,g(w)\,x\}\wedge dw_1\wedge\ldots\wedge dw_n\wedge dz_1\wedge\ldots\wedge dz_n$$

$$=\frac{1}{(2\pi i)^{2n}}\int_U R_{(z-\alpha)}f(z)\left\{\int_U(R_{(w-z)}g(w)\,x)\wedge dw_1\wedge\ldots\wedge dw_n\right\}\wedge dz_1\wedge\ldots\wedge dz_n$$

$$=\frac{1}{(2\pi i)^n}\int_U(R_{(z-\alpha)}f(z)\,g(z)\,x)\wedge dz_1\wedge\ldots\wedge dz_n=(fg)\dot{}\,(\alpha)\cdot x.$$

Hence, $f(\alpha)g(\alpha)=(fg)(\alpha)$ and the proof is complete.

If $\mathfrak{A}(\mathrm{Sp}(\alpha, X))$ denotes the algebra of functions defined and analytic in some neighborhood of $\mathrm{Sp}(\alpha, X)$ – that is, the inductive limit of the spaces $\mathfrak{A}(U)$ over neighborhoods U descending on $\mathrm{Sp}(a, X)$ – then since Proposition 3.11 implies that the map $f \rightarrow f(\alpha)$ commutes with restriction, we have the following corollary to Theorem 4.3:

COROLLARY 4.4. *The map $f \rightarrow f(\alpha)$ of Theorem 4.3 defines a homomorphism of $\mathfrak{A}(\mathrm{Sp}(a, X))$ into $(a)''$.*

The following invariance law for the functional calculus follows directly from Proposition 3.8:

PROPOSITION 4.5. *Let X and Y be Banach spaces and $u: X \rightarrow Y$ a bounded linear map. Let $a_1, \ldots, a_n \in L(X)$ and $b_1, \ldots, b_n \in L(Y)$ be commuting tuples related by $ua_i = b_i u$ for $i = 1, \ldots, n$. If f is analytic in a neighborhood of $\mathrm{Sp}(\alpha, X) \cup \mathrm{Sp}(\beta, Y)$, then $uf(\alpha) = f(\beta)u$.*

Note that at this point we have all of the conclusions of the usual Shilov–Arens–Calderon Theorem. In fact, if A is a commutative Banach algebra with identity and $a_1, \ldots, a_n \in A$, then we may consider $a_1, \ldots, a_n$ to be operators on A via the regular representation. It turns out that $\mathrm{Sp}(\alpha, A)$ is then just the usual spectrum of an n-tuple in a Banach algebra (cf. [15]). In this case, the functional calculus of Theorem 4.3 reduces to the usual functional calculus in a commutative Banach algebra. If h is a complex homomorphism of A, then Proposition 4.5—with $u = h$, $Y = \mathbf{C}$, and $b_i = h(a_i)$—implies that $h(f(\alpha)) = f(h(a_1), \ldots, h(a_n))$.

Our next Proposition gives a powerful relationship between the functional calculi for two different tuples of operators.

PROPOSITION 4.6. *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be a commuting tuple of operators on X and set $\alpha = (a_1, \ldots, a_n)$, $\beta = (b_1, \ldots, b_n)$, and $\alpha - \beta = (a_1 - b_1, \ldots, a_n - b_n)$. If f is analytic in a neighborhood of $\mathrm{Sp}(\alpha, X) \cup \mathrm{Sp}(\beta, X)$ then $f(\alpha) - f(\beta)$ acts as the zero operator on $H^p(X, \alpha - \beta)$ for each p.*

Proof. Let $U \supset \mathrm{Sp}(\alpha, X) \cup \mathrm{Sp}(\beta, X)$. If $f \in \mathfrak{A}(U)$ and $k \in F^p(X, \alpha - \beta)$, then

$$(f(\alpha) - f(\beta))\, k = \frac{1}{(2\pi i)^n} \int (R_{z-\alpha} - R_{z-\beta}) f(z)\, k \wedge dz_1 \wedge \ldots \wedge dz_n$$

(recall $F^p(X, \alpha - \beta)$ is a direct sum of $\binom{n}{p}$ copies of X). Now, by Corollary 1.15, $(R_{z-\alpha} - R_{z-\beta}) f(z) k$ is cohomologous to zero as an element of $F^p(H^n(\mathfrak{B}_0(U, X), \bar{\partial}_z), \alpha - \beta)$. Also, the integral commutes with the coboundary operator determined by $\alpha - \beta$. Hence, $(f(\alpha) - f(\beta))\, k$

is cohomologous to zero in $F^p(X, \alpha-\beta)$. It follows that $f(\alpha)-f(\beta)$ is the zero operator on $H^p(X, \alpha-\beta)$.

COROLLARY 4.7. *If* $\alpha=(a_1, ..., a_n)$ *is a commuting tuple of operators on* X, U *a domain containing* $\operatorname{Sp}(\alpha, X)$, *and* $f \in \mathfrak{A}(U)$, *then for each* $z \in U$ *the operator* $f(\alpha)$ *acts as the scalar operator* $f(z)$ *on* $H^p(X, z-\alpha)$ *for each* p.

This serves as an effective tool in pinpointing the action of the operator $f(\alpha)$ on X—as we shall see in the next theorem.

THEOREM 4.8. *Let* $\alpha=(a_1, ..., a_n)$ *be a commuting tuple in* $L(X)$, U *a domain containing* $\operatorname{Sp}(a, X)$, *and* $f_1, ..., f_m \in \mathfrak{A}(U)$. *Let* $f\colon U \to \mathbf{C}^m$ *be defined by* $f(z)=(f_1(z), ..., f_m(z))$ *and let* $f(\alpha)$ *be the tuple of operators* $(f_1(\alpha), ..., f_m(\alpha))$. *Then* $\operatorname{Sp}(f(\alpha), X)=f(\operatorname{Sp}(\alpha, X))$.

Proof. It follows from Theorem 3.2 of [15] that the spectrum of $f(\alpha)=(f_1(\alpha), ..., f_m(\alpha))$ is the projection on the last m-coordinates of the spectrum of $(a_1, ..., a_n, f_1(\alpha), ..., f_m(\alpha))= \alpha \oplus f(\alpha)$. Hence, it suffices to prove that the spectrum of $\alpha \oplus f(\alpha)$ is $\{(z, w) \in \mathbf{C}^{n+m}\colon z \in \operatorname{Sp}(\alpha, X),\ w=f(z)\}$. To prove this it is sufficient to prove that if $z \in \operatorname{Sp}(\alpha, X)$ then $(z-\alpha)\oplus f(\alpha)$ is non-singular if and only if $f(z) \neq 0$. We prove this by induction on m.

We assume that $m \geqslant 0$ is an integer such that the following two statements are true of any m-tuple $f_1, ..., f_m \in \mathfrak{A}(U)$ and any $z \in U$:

(1) if $g \in \mathfrak{A}(U)$ then $g(\alpha)$ acts as the scalar operator $g(z)$ on $H(X, (z-\alpha)\oplus f(\alpha))$; and

(2) $(z-\alpha)\oplus f(\alpha)$ is non-singular on X if and only if $f(z) \neq 0$.

Let $f_1, ..., f_{m+1}$ be an $(m+1)$-tuple in $\mathfrak{A}(U)$. By Lemma 1.3 of [15], there is an exact sequence

$$\ldots \to H^p(X, (z-\alpha)\oplus f(\alpha)) \to H^p(X, (z-a)\oplus f'(\alpha)) \to H^{p+1}(X, (z-\alpha)\oplus f(\alpha)) \xrightarrow{f_{m+1}(\alpha)} H^{p+1}(X, (z-\alpha)\oplus f(\alpha)) \to \ldots$$

where $f'=(f_1, ..., f_{m+1})$ and $f=(f_1, ..., f_m)$. If $g \in \mathfrak{A}(U)$ then $g(\alpha)$ acts as $g(z)$ on $H^p(X, (z-\alpha)\oplus f(\alpha))$ for each p. It follows from the above sequence that $g(\alpha)$ acts as $g(z)$ on $H^p(X, (z-\alpha)\oplus f'(\alpha))$ as well. Also, since $f_{m+1}(\alpha)$ acts as $f_{m+1}(z)$ on $H^p(X, (z-\alpha)\oplus f(\alpha))$ for each p it follows that $H^p(X, (z-\alpha)\oplus f'(\alpha))=0$ if $f_{m+1}(z) \neq 0$ and $H^p(X, (z-\alpha)\oplus f'(\alpha)) \approx H^p(X, (z-\alpha)\oplus f(\alpha))$ if $f_{m+1}(z)=0$. Hence, since statement (2) above holds for $f_1, ..., f_m$, it continues to hold for $f_1, ..., f_{m+1}$.

Since (1) and (2) clearly hold if $m=0$, they hold for all m by induction. This completes the proof.

We close this section with an extension of the Shilov idempotent theorem (cf. [14]). That it is a true extension stems from two facts. First, our spectrum $\operatorname{Sp}(\alpha, X)$ is in general smaller—hence more likely to be disconnected—than the spectrum of $\alpha=(a_1, \dots, a_n)$ as computed in terms of some enveloping commutative Banach algebra of operators. Second, $\operatorname{Sp}(\alpha, X)$ is likely to be computable in situations where it is virtually impossible to tell when an operator equation $(z_1-a_1)b_1+\dots+(z_n-a_n)b_n=\mathrm{id}$ can or cannot be solved.

THEOREM 4.9. *If $\alpha=(a_1, \dots, a_n)$ is a commuting tuple of operators on X, and if $\operatorname{Sp}(\alpha, X)=K_1\cup K_2$ where K_1 and K_1 are disjoint compact sets, then there are closed subspaces X_1 and X_2 of X such that:*

(1) $X=X_1\oplus X_2$;

(2) *X_1 and X_2 are invariant under any operator which commutes with each a_i; and*

(3) $\operatorname{Sp}(\alpha, X_1)=K_1$ *and* $\operatorname{Sp}(\alpha, X_2)=K_2$.

Proof. Let U_1 and U_2 be disjoint open sets in $\mathbf{C}^n$ containing K_1 and K_2 respectively. If χ_{U_1} is the characteristic function of U_1, then $\chi_{U_1}\in\mathfrak{A}(U_1\cup U_2)$. Hence, there exists an idempotent $p\in(\alpha)''$ such that $\chi_{U_1}(\alpha)=p$. If $X_1=\operatorname{Im} p$ and $X_2=\operatorname{Ker} p$, then (1) and (2) above clearly hold for X_1, X_2. Condition (3) follows from Theorem 4.8 applied to the tuples $(z_1\chi_{U_1}, \dots, z_n\chi_{U_1})$ and $(z_1\chi_{U_2}, \dots, z_n\chi_{U_2})$.

COROLLARY 4.10. *If z is an isolated point of $\operatorname{Sp}(\alpha, X)$, then $X=X_1\oplus X_2$ where each z_i-a_i is quasi-nilpotent on X_1 and $z\notin\operatorname{Sp}(\alpha, X_2)$.*

5. Spectral hull

The functional calculus allows us to derive relationships between our notion of spectrum and notions based on Banach algebra theory.

Notation 5.1. Let X be a Banach space and A a Banach algebra of operators on X. If $\alpha=(a_1, \dots, a_n)$ is a tuple of operators in the center of A and $w\in\mathbf{C}^n$, then we shall say $w\in\operatorname{Sp}_A(\alpha)$ if the equation

$$(w_1-a_1)b_1+\dots+(w_n-a_n)b_n=\mathrm{id} \tag{5.1}$$

fails to have a solution for $b_1, \dots, b_n\in A$. We pointed out in [15] that $\operatorname{Sp}_A(\alpha)=\operatorname{Sp}(\alpha, A)$, if $a_1, \dots, a_n$ are considered operators on A via multiplication.

If A is an algebra of operators, then A' will denote the algebra of all operators that commute with each element of A. If (α) denotes the Banach algebra generated by $a_1, \dots, a_n$ in $L(X)$, then for any Banach algebra A with $a_1, \dots, a_n\in$ center (A) we have $(\alpha)\subset A\subset(\alpha)'$ and $\operatorname{Sp}(\alpha, X)\subset\operatorname{Sp}_{(\alpha)'}(\alpha)\subset\operatorname{Sp}_A(\alpha)\subset\operatorname{Sp}_{(\alpha)}(\alpha)$ (cf. [15], § 4).

Let $\mathfrak{A}(\alpha)$ denote the norm closure of the algebra of operators of the form $f(\alpha)$ for f analytic in a neighborhood of $\mathrm{Sp}(\alpha, X)$. Note that (α), $(\alpha)''$, and $\mathfrak{A}(\alpha)$ are commutative algebras and $(\alpha)\subset\mathfrak{A}(\alpha)\subset(\alpha)''\subset(\alpha)'$. Hence, we have

$$\mathrm{Sp}(\alpha, X)\subset \mathrm{Sp}_{(\alpha)'}(\alpha)\subset \mathrm{Sp}_{(\alpha)''}(\alpha)\subset \mathrm{Sp}_{\mathfrak{A}(\alpha)}(\alpha)\subset \mathrm{Sp}_{(\alpha)}(\alpha). \tag{5.2}$$

There are examples where each of the containments in (5.2) are proper. Examples where $\mathrm{Sp}_{\mathfrak{A}(\alpha)}(\alpha)\neq\mathrm{Sp}_{(\alpha)}(\alpha)$ abound; one such example is the single operator $f\to zf$ on $C(\Gamma)$, where Γ is the unit circle. In § 4 of [15] we gave an example where $\mathrm{Sp}(\alpha, X)\neq \mathrm{Sp}_{(\alpha)'}(\alpha)$. We shall reproduce this example here and show how it can be modified to obtain examples in the other two cases.

Let D be a compact polydisc and U an open polydisc with compact closure such that $0\in \operatorname{int} D\subset D\subset U\subset \mathbf{C}^2$. We set $V=U\setminus D$. Let $C(\overline{V})$ be the space of continuous functions on the closure $\overline{V}$ of V and $C^1(\overline{V})$ be the subspace of $C(\overline{V})$ consisting of functions with uniformly continuous first partial derivatives on V. We give $C(\overline{V})$ the sup norm. For $f\in C^1(\overline{V})$ we define $\|f\|$ to be the sum of the sup norms of f and its first partial derivatives. We set $X=C^1(\overline{V})\oplus C(\overline{V})$.

We define five operators on X as follows: $a_1(f,g)=(z_1 f, z_1 g)$, $a_2(f,g)=(z_2 f, z_2 g)$, $a_3(f,g)=(0, \partial f/\partial \bar z_1)$, $a_4(f,g)=(0,\partial f/\partial \bar z_2)$, and $a_5(f,g)=(0,f)$. Note that $(a_1, \ldots, a_5)$ is a commuting tuple of bounded linear operators on X. Note also that $(a_1, a_2)'$ contains all operators of the form $(f,g)\to(hf, hg)$ $(h\in C^1(\bar U))$ as well as a_3, a_4, and a_5.

Since $0\notin\overline{V}$ the equation $z_1h_1+z_2h_2=1$ can be solved for $h_1, h_2\in C^1(\overline{V})$. It follows that $0\notin \mathrm{Sp}_{(a_1,a_2)'}(a_1,a_2)$. However, $0\in \mathrm{Sp}_{(a_1,a_2)''}(a_1,a_2)$, for if we could solve $a_1b_1+a_2b_2=1$ with $b_1, b_2\in(a_1,a_2)''$ then this equation would remain valid on X_1/X_0, where $X_1=\ker a_3\cap\ker a_4=\{(f,g)\in X\colon f \text{ is analytic on } V\}$ and $X_0=\ker a_5=\{(0,g)\in X\}$: however, X_1/X_0 is isomorphic to the space of continuous functions on $\bar U$ which are analytic on U (cf. [15], § 4). Since $0\in U$ and $a_1 f=z_1 f$, $a_2 f=z_2 f$ we have a contradiction. Hence,

$$\mathrm{Sp}_{(a_1,a_2)''}(a_1,a_2)\neq \mathrm{Sp}_{(a_1,a_2)'}(a_1,a_2).$$

A similar argument (which appears in [15]) shows that $\mathrm{Sp}(\alpha, X)\neq\mathrm{Sp}_{(\alpha)'}(\alpha)$ for $\alpha=(a_1,\ldots,a_5)$.

It is simpler to obtain an example where $\mathrm{Sp}_{\mathfrak{A}(\alpha)}(\alpha)\neq \mathrm{Sp}_{(\alpha)''}(\alpha)$. We let V and $C(\overline{V})$ be as above. Let $X=C(\overline{V})$ and define $a_i\in L(X)$ $(i=1,2)$ by $a_i f=z_i f$. It can be shown that $(a_1,a_2)''$ consists of the operators $f\to hf$, where $h\in C(\overline{V})$. It follows that $0\notin\mathrm{Sp}_{(a_1,a_2)''}(a_1,a_2)$. However, $\mathfrak{A}(a_1,a_2)$ is the algebra of operators of the form $f\to gf$, where g is continuous on $\overline{V}$ and analytic on V. Any such g can be uniquely extended to be analytic on U (cf. [9], I). Since $0\in U$ we have $0\in \mathrm{Sp}_{\mathfrak{A}(a_1,a_2)}(a_1,a_2)$.

It turns out that $\mathrm{Sp}_{(\alpha)}(\alpha)$ and $\mathrm{Sp}_{\mathfrak{A}(\alpha)}(\alpha)$ are determined by the geometry of $\mathrm{Sp}(\alpha, X)$ as a subset of $\mathbf{C}^n$.

If $K \subset \mathbf{C}^n$ is compact, then the polynomial hull of K is $\{z \in \mathbf{C}^n: |p(z)| \leqslant \sup_{w \in K} |p(w)|\}$ for all polynomials $p\}$ (cf. [9]).

THEOREM 5.2. *For any commuting tuple* $\alpha = (a_1, \ldots, a_n)$, $\mathrm{Sp}_{(\alpha)}(\alpha)$ *is the polynomial hull of* $\mathrm{Sp}(\alpha, X)$.

Proof. Since (α) is the closure of the image of the map $p \to p(\alpha)$ from the algebra P of polynomials into (α), the spectrum of α is just the set of $z \in \mathbf{C}^n$ for which the complex homomorphism $p \to p(z)$ of P extends to a complex homomorphism of (α). It is easily seen that this set is exactly the polynomial hull of $\mathrm{Sp}(\alpha, X)$.

Definition 5.3. If $K \subset \mathbf{C}^n$ is compact then the spectral hull of K is the set of all $w \in \mathbf{C}^n$ such that the equation

$$(z_1 - w_1) f_1(z) + \ldots + (z_n - w_n) f_n(z) = 1 \tag{5.3}$$

fails to have a solution for $f_1, \ldots, f_n$ analytic in a neighborhood of K.

THEOREM 5.4. *For any commuting n-tuple* $\alpha = (a_1, \ldots, a_n)$ *of operators,* $\mathrm{Sp}_{\mathfrak{A}(\alpha)}(\alpha)$ *is the spectral hull of* $\mathrm{Sp}(\alpha, X)$.

Proof. It suffices to show that for $w \in \mathbf{C}^n$ equation (5.3) can be solved for $f_1, \ldots, f_n \in \mathfrak{A}(\mathrm{Sp}(\alpha, X))$ if and only if equation (5.1) can be solved for $b_1, \ldots, b_n \in \mathfrak{A}(\alpha)$.

If $f_1, \ldots, f_n \in \mathfrak{A}(\mathrm{Sp}(\alpha, X))$ satisfy equation (5.3) then clearly the operators $b_1 = f_1(\alpha)$, $\ldots, b_n = f_n(\alpha)$, given by the functional calculus, satisfy equation (5.1).

Conversely, if $b_1, \ldots, b_n \in \mathfrak{A}(\alpha)$ satisfy (5.1), then there exist functions $g_1, \ldots, g_n \in \mathfrak{A}(\mathrm{Sp}(\alpha, X))$ such that $\|(w_1 - a_1)(g_1(\alpha) - b_1) + \ldots + (w_n - a_n)(g_n(\alpha) - b_n)\| < 1$. It follows that $(w_1 - a_1) g_1(\alpha) + \ldots + (w_n - a_n) g_n(\alpha) = h(\alpha)$ is invertible in $\mathfrak{A}(\alpha)$, where $h(z) = (w_1 - z_1) g_1(z) + \ldots + (w_n - z_n) g_n(z)$. However, it follows from Theorem 4.8 that h cannot vanish on $\mathrm{Sp}(\alpha, X)$ if $h(\alpha)$ is invertible. Hence, $h^{-1} \in \mathfrak{A}(\mathrm{Sp}(\alpha, X))$ and $f_1 = h^{-1} g_1, \ldots, f_n = h^{-1} g_n$ is a solution of (5.3).

A compact set $K \subset \mathbf{C}^n$ is polynomially convex if it is equal to its polynomial hull. Similarly, we call K spectrally convex if it is equal to its spectral hull.

THEOREM 5.5. *If* $\mathrm{Sp}(\alpha, X)$ *is polynomially convex, then* $\mathrm{Sp}_A(\alpha) = \mathrm{Sp}(\alpha, X)$ *for any closed subalgebra* $A \subset L(X)$ *with* $a_1, \ldots, a_n \in \mathrm{center}(A)$.

If $\mathrm{Sp}(\alpha, X)$ *is spectrally convex, then* $\mathrm{Sp}_A(\alpha) = \mathrm{Sp}(\alpha, X)$ *for any closed subalgebra* $A \subset L(X)$ *such that* $\mathfrak{A}(\alpha) \subset \mathrm{center}(A)$.

Proof. If $a_1, \ldots, a_n \in \mathrm{center}(A)$ then $(\alpha) \subset A$ and $\mathrm{Sp}(\alpha, X) \subset \mathrm{Sp}_A(\alpha) \subset \mathrm{Sp}_\alpha(\alpha)$. By Theorem 5.2, if $\mathrm{Sp}(\alpha, X)$ is polynomially convex then $\mathrm{Sp}(\alpha, X) = \mathrm{Sp}_A(\alpha) = \mathrm{Sp}_\alpha(\alpha)$.

If $\mathfrak{A}(\alpha) \subset$ center (A) then $\operatorname{Sp}(\alpha, X) \subset \operatorname{Sp}_A(\alpha) \subset \operatorname{Sp}_{\mathfrak{A}(\alpha)}(\alpha)$ and Theorem 5.3 implies the three are equal if $\operatorname{Sp}(\alpha, X)$ is spectrally convex.

There are several conditions that ensure that a set $K \subset \mathbf{C}^n$ is spectrally convex. For example, K is spectrally convex if K has trivial cohomology relative to the sheaf of germs of analytic functions. Hence, K is spectrally convex if it is an $\mathfrak{A}(U)$-convex subset of a domain of holomorphy U (cf. [9]).

We close with a few comments concerning the algebra $\mathfrak{A}(\alpha)$. It follows from Theorem 4.8 that $\mathfrak{A}(\alpha)$ is closed under the application of analytic functions. Hence, $\mathfrak{A}(\alpha)$ may be viewed as an analytic functional completion of the algebra (α). Warning: although it is true that $\operatorname{Sp}_{\mathfrak{A}(\alpha)}(\alpha)$ is the spectral hull of $\operatorname{Sp}(\alpha, X)$, this may not be the maximal ideal space of $\mathfrak{A}(\alpha)$. If Δ is the maximal ideal space of $\mathfrak{A}(\alpha)$ and $\hat{a_1}, ..., \hat{a_n}$ are the Gelfand transforms of the elements $a_1, ..., a_n$, then the map $\hat{\alpha}: \Delta \to \mathbf{C}^n$ $(\hat{\alpha} = (\hat{a_1}, ..., \hat{a_n}))$ maps Δ onto the spectral hull of $\operatorname{Sp}(\alpha, X)$. However, $\hat{a_1}, ..., \hat{a_n}$ may fail to separate points of Δ. We give an example to show what can happen.

Example 5.6. Let $r_1 > r_2 > ...$ be a sequence of positive numbers converging to zero and, with $n > 1$, set $S_k = \{z \in \mathbf{C}^n: |z| = (|z_1|^2 + ... + |z_n|^2)^{\frac{1}{2}} = r_k\}$. We set $K = \{0\} \cup (\bigcup_{k=1}^{\infty} S_k)$ and $X = C(K)$. The operators $a_1, ..., a_n$ are defined by $(a_i f)(z) = z_i f(z)$.

If f is a function defined and analytic in a neighborhood of a set of the form $\{z \in \mathbf{C}^n: r - \varepsilon < |z| < r + \varepsilon\}$ then f has a unique extension to a function analytic on $\{z \in \mathbf{C}^n: |z| < r + \varepsilon\}$ provided $n > 1$ (cf. [9]). It follows that the spectral hull of the set K above is $\{z \in \mathbf{C}^n: |z| \leqslant r_1\}$.

Since the equation $(z_1 - w_1) f_1(z) + ... + (z_n - w_n) f_n(z) = 1$ can be solved for $f_1, ..., f_n \in C(K)$ if $w = (w_1, ..., w_n) \notin K$, we have $\operatorname{Sp}(\alpha, X) \subset \operatorname{Sp}_{(\alpha)'}(\alpha) \subset K$. On the other hand, if $w \in K$ then the above equation cannot be solved for $f_1, ..., f_n \in C(K) = X$; hence, the map $(w - \alpha)^{n-1}$: $F^{n-1}(X, w - \alpha) = \oplus^n X \to X = F^n(X, w - \alpha)$ fails to be onto, and the complex $F(X, w - \alpha)$ is not exact (cf. Def. 1.2). It follows that $\operatorname{Sp}(\alpha, X) = K$ and $\operatorname{Sp}_{\mathfrak{A}(\alpha)}(\alpha)$ is the spectral hull of K which is $\{z \in \mathbf{C}^n: |z| \leqslant r_1\}$. Note, however, that the maximal ideal space Δ of $\mathfrak{A}(\alpha)$ is the one point compactification of the disjoint union of the sets $\{z \in \mathbf{C}^n: |z| \leqslant r_i\}$. This follows from the fact that $\mathfrak{A}(K) \approx \oplus_{i=1}^{\infty} \mathfrak{A}(\{z: |z| \leqslant r_i\})$. The map $\hat{\alpha}: \Delta \to \operatorname{Sp}_{\mathfrak{A}(\alpha)}(\alpha) = \{z: |z| \leqslant r_1\}$ is just the map induced on Δ by the inclusions $\{z: |z| \leqslant r_i\} \to \{z: |z| \leqslant r_1\}$. The inverse image of zero under this map is an infinite set. Hence, $\hat{\alpha}$ is not even a light map.

Note one other thing about this example. The algebra $\mathfrak{A}(\alpha)$ contains a non-trivial projection corresponding to each of the sets $\{z: |z| = r_i\}$. The version of the Shilov idempotent Theorem given in 4.9 detects these projections since each $\{z: |z| = r_i\}$ is a component of $\operatorname{Sp}(\alpha, X)$. However, $\operatorname{Sp}_{\mathfrak{A}(\alpha)}(\alpha) = \{z: |z| = r_1\}$ is connected and does not indicate the existence of these projections.

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