The Analytic Hierarchy Process and the Theory of Measurement

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Abstract

The Analytic Hierarchy Process (Saaty 1977, 1980) is a decision-making procedure for establishing priorities in multi-criteria decision making. Underlying the AHP is the theory of ratio-scale measures developed by psychophysicist Stanley S. Stevens (1946, 1951) in the middle of the last century. It is however well-known that Stevens' original model was flawed in various respects. We reconsider the AHP at the light of the modern theory of measurement based on the so-called separable representations (Narens 1996, Luce 2002). We provide various theoretical and empirical results on the extent to which the AHP can be considered a reliable decision-making procedure in terms of the modern theory of subjective measurement.

Keywords: Decision analysis; Analytic Hierarchy Process; separable representations.

1 Introduction

The Analytic Hierarchy Process (AHP) is a decision-making procedure originally developed by Thomas Saaty (Saaty 1977, 1980, 1986). Its primary use is to offer solutions to decision problems in multivariate environments, in which several alternatives for obtaining given objectives are compared under different criteria. The AHP establishes decision weights for alternatives by organizing objectives, criteria and subcriteria in a hierarchic structure.

Central in the AHP is the process of measurement, in particular measurement on a ratio scale. Decision weights and priorities are obtained from the decision maker's assessments of the way in which each item of a decision problem compares with respect

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to any other item at the same level of the hierarchy. Given a family of $n \geq 2$ items of a decision problem (for example, 3 alternatives) to be compared for a given attribute (for example, one criterion), in the AHP a response matrix $A = [a_{ij}]$ is constructed with the decision maker's assessments a_{ij}, taken to measure on a subjective ratio scale the relative dominance of item i over item j. For all pairs of items i, j, it is assumed:

$$a_{ij} = \frac{w_i}{w_j} \cdot e_{ij} \tag{1}$$

where w_i and w_j are underlying subjective priority weights belonging to a vector $w = (w_1, w_2, ..., w_n)'$, with $w_1 > 0, ..., w_n > 0$ and by convention $\sum w_j = 1$; and where e_{ij} is a multiplicative term introduced to account for errors and inconsistencies in subjective judgments typically observed in practice.

The AHP has spawned a large literature. Critics have concerned both technical and philosophical aspects (see e.g., Dyer 1990a, Smith and Winterfeldt 2004, and references therein). On the more philosophical side, several decision analysts have argued that the AHP lacks of sound normative foundations and is inconsistent with the axioms of utility theory which characterize rational economic behavior. Furthermore, they contended that the comparisons considered by the AHP are ambiguous, especially when they deal with intangibles, because of the difficulty for humans to express subjective estimates on a ratio scale. On the more technical side, debates have concerned the way in which the AHP obtains the priorities w_j from the response matrix $A = [a_{ij}]$. The classical method proposed by Saaty (1977) based on the principal eigenvector (Perron vector) of A has been criticized and other methods based on stronger statistical principles, like the logarithmic least square method, have been proposed (e.g. de Jong 1984, Crawford and Williams 1985).

Defenders of the AHP have always rejected the various criticisms.[1] One argument often put forward is that the normative foundations of the AHP are not in utility theory, but in the theory of measurement (see e.g., Harker and Vargas 1987 and 1990, Saaty 1990, Forman and Gass 2001). Appeal has often been made to the work of psychophysicist Stanley S. Stevens (1946, 1951) and his famous classification of scales of measurement, putting ratio scale measures at the top of all forms of scientific measurement. Furthermore, in line with Stevens' ratio-scaling method, AHP proposers have also vindicated the ability of individuals to perform subjective ratio assessments, which, even if not perfect, are considered sufficiently accurate to be used in AHP analyses. In fact, on the more technical ground, Saaty and co-authors (e.g., Saaty and Vargas 1984, Saaty 2003) have always argued that precisely because subjective ratio assessments are only approximately accurate, the principal eigenvector method is the only method which

[1] Other criticisms against classical AHP are discussed in, e.g. Dyer (1990b). A debated one is the problem of rank reversal (Belton and Gear 1983). Rank reversal may arise in the AHP during the procedures of hierarchic decomposition and aggregation. Extensions of AHP techniques can avoid rank reversal (see e.g., Pérez 1995, for discussion and references). In this paper we will not deal with the issue of rank reversal and only in the conclusion will we make some reference to the implications of the paper on the principles of hierarchic composition.

should be used in the AHP to obtain the priorities w_j, since it is the only method which delivers unambiguous ranking when subjective ratio assessments are near-consistent.

In this paper we reconsider the disputes around the AHP at the light of the newer theory of psychological measurement. Indeed, despite the appeal of the AHP defenders to Stevens' ratio scaling method, it is well known in mathematical psychology that Stevens' theory was flawed in several respects (see e.g., Michell 1999, chapter 7). For mathematical psychologists a major drawback of the theory has always been seen in the lack of rigor and of proper mathematical and philosophical foundations justifying the proposition that, when assessing a ratio judgment, a "subject is, in a scientific sense, 'computing ratios'" (Narens 1996, p. 109).

In recent years, however, there has been an important stream of research clarifying the conditions and giving various sets of axioms that can justify ratio estimations. An important achievement of the recent literature has been the axiomatization of various theories of subjective ratio judgments belonging to a class of so-called *separable representations* (see Narens 1996, 2002, and Luce 2002, 2004). We will show how, within the class of separable representations, equation (1) of the classical AHP approach should be recast as:

$$a_{ij} = W^{-1}\left(\frac{w_i}{w_j}\right) \cdot e_{ij} \tag{2}$$

where $W^{-1}(\cdot)$ is the inverse of a subjective weighting function $W(\cdot)$ relating elicited subjective proportions to numerical ratios. Clearly, when W is the identity, equations (1) and (2) are equivalent. As, however, predicted by mathematical psychologists, the identity and closely related forms like the power model for W have been rejected by various recent psychophysical experiments (see Ellermeier and Faulhammer 2000, Zimmer 2005, Steingrimsson and Luce 2005a, 2005b, and other references in Section 3.3).[2]

We will consider the implications of separable representations for the AHP. After a short review of the ratio-scaling method of classical AHP, we will move on to consider the relationships between the AHP and the modern theory of measurement. We will first of all show how to derive equation (2) from coherent models of subjective ratio judgments. Then we will show how typical inconsistencies often observed in the AHP should be reinterpreted in terms of representation (2). After, we will develop a statistical method to estimate the priority vector $w = (w_1, w_2, ..., w_n)'$ from equation (2) which takes into account possible nonlinearity in the subjective weighting function W; and we will show how to separate in the estimates of $w = (w_1, w_2, ..., w_n)'$ the effects due to random errors (as those due to e_{ij} in equations (1) and (2)) from those due to the psychological distortions carried by W.

Then we will apply the method to some experimental data we have obtained from a subjective ratio estimation experiment. We will compare the results of our method to

[2] Earlier studies testing structural assumptions implicit in direct measurement methods and developing non axiomatic representations for them include Birnbaum and Veit (1974), Birnbaum and Elmasian (1977), Mellers, Davis and Birnbaum (1984). Also notice that the theoretical and experimental research on the nonlinearity of the subjective weighting function in the psychophysical literature parallels the possibly most well-known literature on the nonlinearity of the probability transformation function in utility theory (as for example typified in Prospect Theory, Tversky and Kahneman 1992).

those obtained by the principal eigenvector and by logarithmic least squares. Among other things, the empirical analysis shows that the main inconsistencies in the response data are in fact due to the psychological distortions in W rather than to random errors e_{ij}. We will conclude discussing the implications of the findings for the status and the practice of the AHP.

2 Scaling and prioritization in classical AHP

The explicit words used by Saaty to present the AHP in the title of the article where the approach was firstly set forward (Saaty 1977) were "scaling method for priorities in hierarchical structure". The term scaling was derived from psychophysics. In psychophysical scaling, subjects are asked to relate number names to sensation magnitudes generated by stimuli, which are then treated by the analyst (scaler) as proper mathematical numbers to derive subjective scale measurements of the sensation (feeling, preference, judgment) in question. Much of the present paper will focus on the philosophical and theoretical justifications for the correspondence, also assumed by the AHP, between the number names in the instruction of the subjective scaling procedures and scientific numbers[3]. Before entering such a discussion it is important to review some specific characteristics of the AHP as a scaling method.

2.1 Saaty's "fundamental scale"

In the 1977 paper and subsequent book (Saaty 1980) the AHP was developed as a set of operational procedures without axiomatic foundations. Axioms were later added by Saaty (1986). Central in Saaty's system of axioms is the primitive notion of a "fundamental scale" for pairwise comparisons of alternatives for a finite set of criteria (or attributes or properties). Let $\mathcal{A}$ be a set of alternatives A_i with $i = 1, ..., n$ and n finite; and let C be one among a set of criteria to compare the alternatives. A fundamental scale for criterion C is a mapping P_C, which assigns to every pair $(A_i, A_j) \in \mathcal{A} \times \mathcal{A}$ a positive real number $P_C(A_i, A_j) \equiv a_{ij}$, such that: 1) $a_{ij} > 1$ if and only if A_i dominates (or "is strictly preferred to") A_j according to criterion C; 2) $a_{ij} = 1$ if and only if A_i is equivalent (or "indifferent") to A_j according to criterion C.

Remark that the a_{ij}'s of the definition correspond to the entries of the response matrix A of the Introduction. Thus, the definition not only assumes the existence of the "scale" P_C, but also identifies the scale values $P_C(A_i, A_j)$ as the responses given by the individual in the subjective assessments procedure, so that $A = [a_{ij}] \equiv [P_C(A_i, A_j)]$ (see Saaty 1986, p. 844).

Four axioms are then presented by Saaty to characterize various operations which can be performed with fundamental scales. The first axiom establishes $A = [a_{ij}]$ as a positive reciprocal matrix, that is $a_{ij} = a_{ji}^{-1}$ and $a_{ii} = 1$ all $A_i, A_j \in \mathcal{A}$. The other three axioms

[3]More precise characterizations of the notions of psychophysical *scaling* and scientific *measurement* will be given below. Readers interested in the several different emphases and subtleties which the two terms may assume in psychophysics are referred to Luce and Krumhansl (1988).

are hierarchic axioms. The second, the so-called homogeneity axiom, recommends to aggregate or decompose the items of a decision stimuli into clusters or hierarchy levels so that the stimuli do not differ too much in the property being compared. Otherwise, large errors in judgment may occur. Based on empirical research, the AHP has elaborated various scale models to elicit judgments, including Saaty's famous verbal scale with integers from 1 to 9 for intensities of relative importance. The third and fourth axioms are more controversial and AHP proposers have themselves considered situations in which they may not be applicable.[4]

A fundamental scale P_C does not deliver directly a rank order of alternatives, i.e. a scale of priorities. A scale of priorities is, in Saaty's language, a *derived* scale. A derived scale is a n-dimensional mapping w, from the set of positive reciprocal matrices A to the n-fold Cartesian product of $[0, 1]$. Thus, a scale of priority is a $n-$dimensional vector, $w(A) = (w_1, w_2, .., w_n)'$, with $1 \geq w_i \geq 0$. Now, one question which has taken a lot of debate in the AHP concerns the best prioritization method, that is the best method to obtain the vector $(w_1, w_2, .., w_n)'$ from $A = [a_{ij}]$ so that the $i-$th component of $w(A)$ accurately represents the relative dominance of alternative A_i among the n alternatives in $\mathcal{A}$.

2.2 Principal eigenvalue method for consistent and near consistent matrices

It is shown by Saaty that the answer would be straightforward when the fundamental scale P_C satisfies a property called of (cardinal) consistency, namely when $P_C(A_i, A_j) \cdot P_C(A_j, A_k) = P_C(A_i, A_k)$ for all i, j and k. Only in this case the fundamental scale delivers directly a ratio scale of priorities,[5] that is $P_C(A_i, A_j) = \frac{w_i}{w_j}$. The pairwise response matrix A is also consistent and, in particular, all rows of A are linear transformations of a single row, so that given a row $(a_{i1}, a_{i2}, ..., a_{in})$ of A all other rows can be obtained by the relation $a_{jk} = a_{ik}/a_{ij}$. Thus, in case of consistency, a single row of A is sufficient to deliver the ratio scale of priorities, since the rank of the matrix is $r(A) = 1$.

The AHP does not impose consistency. In fact, the AHP acknowledges that people can be subject to errors and inconsistencies, mainly due to imprecisions and random errors.[6] When subjective ratio assessments concern stimuli of relatively comparable activities (so that the homogeneity axiom is satisfied), the relevant concept is that of near consistent matrix. Consider the positive reciprocally consistent matrix $A_0 = [\frac{w_i}{w_j}]$ with the entries given by the mathematical ratios of priorities. The positive reciprocal response matrix $A = [a_{ij}]$ is a near consistent matrix if it is a small reciprocal multiplicative perturbation of $A_0 = [\frac{w_i}{w_j}]$. This means that A is thought to be given by the Hadamard product $A = A_0 \circ E$, where $E \equiv (e_{ij})$, $e_{ij} = e_{ji}^{-1}$, and $e_{ii} = 1$ all i. Its entries

[4]The third and fourth axioms refer to the principles of hierarchic composition in the AHP and have been discussed and revised at the light of the issue of rank reversal (see footnote 1).

[5]This is quite an interesting result of Saaty which anticipates Narens (1996); see Section 3.1 below.

[6]As we shall see, this notion that errors imply consistency violations is emphasized by Saaty as one important difference of the AHP from standard psychophysical ratio-scaling techniques (see e.g., Saaty 1977, p. 277).

are given by equation (1) of the Introduction, namely:

$$a_{ij} = \frac{w_i}{w_j} \cdot e_{ij}. \tag{1}$$

A small perturbation means that e_{ij} is close to 1. A near consistent matrix remains obviously reciprocally symmetric, that is $a_{ij} = a_{ji}^{-1}$, and has unit diagonal terms $a_{ii} = 1$ all i. It may however violate cardinal consistency $a_{ij} = a_{ik} \times a_{kj}$; or even the weaker requirement of ordinal consistency, namely that when $a_{ij} > 1$ and $a_{jk} > 1$ then also $a_{ik} > 1$.

The possibility of consistency violations poses the problem of estimating the priority vector $w = (w_1, w_2, ..., w_n)'$ in an appropriate way. The estimation or prioritization technique proposed by Saaty in his original paper (Saaty 1977) and defended since then (e.g. Saaty 2003) remains the maximum eigenvalue method (ME). It uses the response matrix $A = [a_{ij}]$ to solve for the column vector of interest $w = (w_1, w_2, ..., w_n)'$ the linear system of equations:

$$Aw = \lambda_{\max} w, \quad \sum_{i=1}^{n} w_i = 1 \tag{3}$$

where $\lambda_{\max} > 0$ is the largest eigenvalue in modulus (the Perron eigenvalue) of A. It is known by the Perron-Frobenius theorem that system (3) has a unique solution, the Perron eigenvector, henceforth denoted by $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)'$. Moreover, if $E = I$, so that A is cardinally consistent with $A = A_0$, ME delivers the correct priority weights $\overline{w}_i = w_i$ for all i, and the maximum eigenvalue is at its minimum $\lambda_{\max} = n$. When A is not cardinally consistent, $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)'$ usually differs from the correct priority vector and $\lambda_{\max} > n$. Therefore, the normalized difference $\mu = (\lambda_{\max} - n)/(n-1)$ is proposed by the AHP as a rule to measure inconsistency. In particular, if the consistency index for a certain response matrix is larger than a given cut-off, the AHP proposes to correct the matrix restarting from the subjective judgments of the individuals until near consistency is reached (various methods are proposed in the AHP to conduct such a revision, e.g., Saaty 2003). Once near consistency is reached, no further adjustment is required by the AHP. In fact, the ME is recommended as the best prioritization method by Saaty and several of his co-authors because in their opinions the ME does not introduce arbitrariness when the slightly perturbed rows of a positive reciprocal near consistent matrix A are used to derive a unique ratio scale of priorities.[7]

2.3 Classical AHP and statistics

Despite the latter consideration, the ME method can be criticized for being a procedure not paying attention to the stochastic structure of the data (see e.g. de Jong 1984). For

[7] In this sense, Saaty and Vargas (1984) have argued that the principal eigenvalue method is the only method which "directly deals with the question of inconsistency and captures the rank order inherent in the inconsistent data" (p. 205). In more technical terms, the argument rests on an averaging effect which Saaty has shown to be possessed by the principal eigenvector calculation, namely it corresponds to finding the dominance of each alternative A_i along all paths of length k, as k goes to infinity (see Saaty 1986, p. 582; or Saaty 2003, for a more recent exposition of the same notion).

this reason, alternative techniques to estimate w have been proposed in the literature. By statistical standards the most important alternative is the logarithmic least squares method (LLSM). It is based on the minimization of the sum of squares $\sum_i \sum_j \varepsilon_{ij}^2$, where $\varepsilon_{ij} = \ln e_{ij}$ are the errors of the log transformation of equation (1):

$$\ln a_{ij} = \ln w_i - \ln w_j + \varepsilon_{ij}, \quad 1 \leq i, j \geq n. \tag{4}$$

Statistical properties of the LLSM estimates of w_i, henceforth denoted with $\overline{\overline{w}}_i$, have been discussed by several authors. Crawford and Williams (1985) have shown that when the ε_{ij} are independently normally distributed, with zero mean and common variance σ^2, the $\overline{\overline{w}}_i$ are in fact maximum likelihood estimators;[8] de Jong (1984) has discussed the statistical properties of the LLSM solution allowing for possible dependence between the error terms ε_{ij}. Important results on the relationships between the ME and the LLSM procedure have been obtained by Genest and Rivest (1994). In particular, under the assumption that every error ε_{ij} is independent of the other errors and has common variance σ^2 that is the asymptotic parameter of the model, they have shown that the ME and the LLSM estimates are similar, in the sense that the $\overline{w}$ and $\overline{\overline{w}}$ are equal within the order of σ: $\overline{\overline{w}} = \overline{w} + O(\sigma^2)$. Moreover they have shown that the unbiased estimate of σ^2 is linearly related to Saaty's consistency index μ. These results imply that when the response matrix $A = [a_{ij}]$ is not too inconsistent and equation (1) holds, the ME and LLSM methods are more or less equivalent in practice. Simulation studies consistent these predictions were anticipated by Zahedi (1986) and Budescu, Zwick and Rapoport (1986).

In the following we discuss other theoretical and empirical reasons to question the validity of equation (1) and propose a more general model of ratio estimation for the AHP, more consistent with the latest developments in the theory of psychological measurement.

3 Stevens, separable representations and the AHP

As is well known, the original idea of developing analytical procedures and experimental techniques for constructing subjective ratio measurement scales in the behavioral sciences is due to Stanley S. Stevens (1946, 1951). The idea came out as a response to the seven years work of a committee of the British Association for the Advancement of Science precisely appointed to deliberate on the possibility of measuring psychological phenomena. Commenting on the work of the committee in his 1946 article in *Science*, Stevens noted that deliberation led only to disagreement, mainly about the very notion of scientific measurement, which for many members of the committee required an observable form of addition, much as in the measurement of weights and lengths. Stevens rejected this position, considering that "measurement, in the broadest sense, is simply

[8] Crawford and Williams (1985) also refer to LLSM as the row geometric mean method (GMM), since with the normalization $\overline{\overline{w}}_1 + \overline{\overline{w}}_2 + ... + \overline{\overline{w}}_n = 1$, $\overline{\overline{w}}_i$ is known to be proportional to the geometric mean of the elements of the $i-$th row of A, namely $\overline{\overline{w}}_i = \left(\prod_k a_{ik}\right)^{1/n} / \sum_j \left(\prod_k a_{jk}\right)^{1/n}$ for $1 \leq i \leq n$.

defined as the assignment of numerals to objects or events according to rules" (1946, p. 677). He went on, especially with his 1951 book, to propose a new measuring method which carefully avoided the use of additive operations. The method, known as Stevens' ratio-scaling approach, can be applied in different forms.

In a ratio estimation, which is a scaling procedure conceptually very similar to the way of obtaining the entries of matrix A in the AHP, an individual is provided with two stimuli z and x, and then is asked to state the value p which corresponds to the subjective ratio of z to x, i.e. that of $t = z/x$.[9] Thus, the experiment provides information about the ratio estimation function $p = p(z, x)$. Stevens thought that the subjective responses recovered in this way could be treated as any other form of scientific measurement and that the subjective estimation function $p(z, x)$ directly represented a ratio-scale measure. In fact, he conjectured that subjective value is a power function of real value, so that Stevens' ratio estimation function $p(z, x)$ can be written as:

$$p(z, x) = \left(\frac{z}{x}\right)^k \tag{5}$$

which corresponds to his famous psychophysical law, that equal physical ratios produce equal psychological ratios.

3.1 Separable representations

A long history of controversies has followed from Stevens' approach.[10] While on the one hand Stevens' ratio scaling method has become a very successful paradigm in applied psychophysics, on the other hand strong criticism has been levied by several authors. The main point of criticism has always been seen in the fact that neither Stevens nor the many ratio-scalers who have followed his approach have ever provided or discussed in a rigorous way the mathematical and philosophical conditions necessary to justify his form of psychological measurement.

In the last 10 years or so, however, a great effort has been done by mathematical psychologists, notably Louis Narens (1996, 2002, 2006) and Duncan Luce (2002, 2004), to comprehend in a deeper perspective the structural assumptions inherent in Stevens' approach and to develop axiomatic theories for direct measurement methods.

A key result of the new development has been the derivation of rigorous conditions that permit one to formulate a representation of subjective estimation in a generalized sense.[11] Formally, we say that a separable representation holds in a ratio estimation if

[9] A dual and perhaps more standard scaling procedure in psychophysics is known as ratio production, in which an observer is required to produce a stimulus x that appears p times more intense than a reference stimulus. (See Luce 2004, and Steingrimsson and Luce 2006, for a deeper discussion of the different notions of ratio production and ratio estimation.)

[10] See e.g., Graham (1958), Anderson (1970), Shepard (1981), Luce and Krumhansl (1988), Michell (1999), for discussions and references on Stevens' approach.

[11] Pioneering non-axiomatic generalizations of direct measurement methods are due to Michael Birnbaum with various co-athours, e.g. Birnbaum and Veit (1974), Birnbaum and Elmasian (1977). (See also below.)

there exist a *psychophysical function* ψ and a *subjective weighting function* W such that the ratio p is in the following relation with z and x:

$$W\left(p\right)=\frac{\psi\left(z\right)}{\psi\left(x\right)}. \tag{6}$$

Equation (6) corresponds to Narens' (1996) original model and incorporates the notion that independent distortions may occur both in the assessment of subjective intensities and in the determination of subjective ratios (see also Luce 2002). Narens developed the model in the tradition of representational measurement theory. Central in his approach is the distinction between numerals, which are the response items p provided by the subject to the experimenter, and scientific numbers.

When uncovering what he called Stevens' assumptions, Narens then argued that the case $W(p) = p$, in which subjects' numerals can be interpreted "in the same manner as is done in science ... is anything more than a coincidence" (Narens 1996, p. 111). In particular, he showed that a given behavioral property, called *multiplicativity*, must hold.[12] The property states that if p is the numeral for the subject's subjective measure of ratio z to x, and q is the numeral for the subject's subjective measure of ratio y to z, then the numeral r for the subject's subjective measure of ratio y to x must satisfy $r = p \cdot q$.

When multiplicativity fails, a weaker behavioral property has been shown by Narens to stay behind representation (6). The condition is called *commutativity.* It states that when p is the numeral for the subject's subjective measure of ratio x to t and of ratio w to y, and q is the numeral for the subject's subjective measure of ratio z to x and of ratio y to t, then $z = w$. In broad terms this means that a subjective proportion, say, of 2 multiplied by a subjective proportion of 3 is equivalent to a subjective proportion of 3 multiplied by a subjective proportion of 2, though neither products of subjective proportions is equivalent to a subjective proportion of 6, for which the full force of the multiplicative property is necessary.

Psychophysical tests of the multiplicative and commutative properties have been conducted by Ellermeier and Faulhammer (2000) and Zimmer (2005) in loudness magnitude production experiments. Both experiments found evidence in favor of commutativity, but against multiplicativity. Other recent evidence in support of separable representations has been obtained by Luce and Steingrimsson (2005a, 2005b) and Bernasconi, Choirat and Seri (2008).

A psychophysical axiomatization of separable representations has also been provided directly by Duncan Luce. In Luce (2002 and 2004) form (6) is derived as a special case of a global theory of psychophysics, developed from empirically testable assumptions

[12] Here we follow the behavioral interpretation of Narens' derivation of form (6). In his 1996 article, Narens also provides a cognitive axiomatization relating the numerical representation to the unobservable sensations. Also note that Narens' original definition of the multiplicative and the following commutative properties are expressed in terms of ratio production tasks, but he postulates that the properties are equally well suited for magnitude estimation (see also Steingrimsson and Luce 2006).

relating sensorial stimuli intensities (like auditory or visual).[13] Luce's approach is also relevant for the functional relationships between the number names used by subjects in the experiments (numerals) and their scientific correspondences, namely the forms of ψ and W. Some properties of these functions, tested experimentally by Steingrimsson and Luce (2006 and 2007), are discussed below.

It is also important to remark that the axiomatic approach underlying separable representations has direct application in the context of utility theory for gambles (see e.g., Luce 2000 and 2002), where ψ is called utility, with domain represented by valued goods and W is a subjective weighting function of probabilities or events.[14] Indeed, a very large literature has accumulated over the years using the nonlinearity of W to generalize the classical expected utility model of von Neumann and Morgenstern (1944) and explain several violations observed against that theory. Tversky and Kahneman's Prospect Theory (Kahneman and Tversky 1979, and Tversky and Kahneman 1992) is probably the best-known model of the list.

3.2 AHP in separable form

Separable representations and ratio estimation have also a natural interpretation in the AHP. The elicited proportions p from a ratio estimation correspond to the entries a_{ij} of a response matrix A in the AHP.[15] In principle, two interpretations are possible for the relationships between the stimuli of a ratio estimation and the priority weights of the AHP. According to the first, the priority weights $w = (w_1, w_2, ..., w_n)'$ could be identified with the stimuli $x_1, x_2, ...$ successively presented to a subject in a ratio estimation (and appropriately normalized to sum up to one). In this case, however, the method could only be applied when the stimuli come from a known scale. A more general and consistent interpretation of ratio estimation in terms of the AHP is that of identifying the priority weights $w = (w_1, w_2, ..., w_n)'$ with the subjective perception of stimuli, that is $w_1 = \psi(x_1)$, $w_2 = \psi(x_2), ...$ Among other things, this is coherent with the objective of the AHP to measure personal judgments, thoughts, feelings and preferences. In fact, as alluded to above, when dealing with preferences between goods,

[13] In particular, since Luce's axiomatization is based on joint presentation of pairs of sensorial stimuli intensities (like two loudness intensities successively presented to the left and to the right ear), the term separable representation is used to denote the special case of a more general model called "subjective-proportion representation" (Luce 2002, p. 523), when one of the stimuli is 0 (namely, at the threshold level).

[14] Consider a gamble (x, p, y) giving x with probability p and y otherwise. Luce (2000) proposes an axiomatization in which its utility is given by:

$$U(x, p, y) = U(x)W(p) + U(y)[1 - W(p)]$$

This representation is called rank-dependent utility. It is consistent with his subjective-proportion representation (Luce 2002). When W is linear, it collapses to expected utility. Moreover, when $y = 0$, the expression $U(x, p, y) = U(x)W(p)$ is a separable form.

[15] More exactly, since Stevens' method does not consider the effect of errors and always imposes consistency, a typical ratio estimation in psychophyiscs needs and produces only one row of a response matrix. (On this see also Saaty 1977, p. 277).

a similar interpretation for the psychophysical function $\psi(\cdot)$ is endorsed by Luce himself, who explicitly refers to $\psi(\cdot)$ as utility (see e.g., Luce 2002, p. 523).

Thus, with this specification, the theory of separable representations implies that the entries a_{ij} of a response matrix A and the priority weights $w = (w_1, w_2, ..., w_n)'$ in the AHP stand in the following relationship:

$$W(a_{ij}) = \frac{w_i}{w_j}. \tag{7}$$

Representation (7) and equation (1) of the Introduction assumed by classical AHP differ in two respects: the ratio form of equation (1) restricts the weighting function $W(\cdot)$ to be the identity; whereas representation (7) ignores considerations of errors e_{ij}. The role of the errors e_{ij} was emphasized by Saaty (1977, 1980) as an important difference from Stevens. In fact, as noted in Section 2, errors may induce possible inconsistencies in the response matrix of classical AHP, giving rise to the issue of prioritization. Below we will re-comprehend the effect of errors in a more general stochastic version of model (7). Firstly, it is important to emphasize the implications of the theory of separable representation for classical AHP, even in the idealized situation in which representation (7) is thought to hold exactly.

3.3 The subjective weighting function $W(\cdot)$ and consistency

In Section 2 we have recalled various properties of the AHP response matrix $A = [a_{ij}]$. We now reconsider the properties in terms of model (7). The first most important property is cardinal consistency, namely that for any three elicited ratio assessments a_{ij}, a_{ik} and a_{kj}, then $a_{ij} = a_{ik} \times a_{kj}$. It should be transparent that in the theory of separable representations, cardinal consistency is equivalent to the multiplicative property which Narens (1996) has shown to be central in Stevens' ratio-scaling approach. In particular, only if cardinal consistency and Narens' multiplicative property hold, then a ratio estimation situation can be seen to provide a ratio-scale measure directly in the sense supposed by Stevens, so that the subjective weighting function $W(\cdot)$ can be chosen as $W(p) = p$. The recent experiments by Ellermeier and Faulhammer (2000) and Zimmer (2005) reject the multiplicative property and the specification $W(p) = p$. In fact, their results also reject the slightly more general specification in which W is a power function $W(p) = p^k$ with $k > 0$ and $W(1) = 1$.[16]

It is interesting that a similar power model was considered by Saaty himself and referred to as the eigenvalue power law (Saaty 1980, p. 189). Saaty thought that the model was relevant as an approximation for pairwise comparisons obtained aggregating or decomposing stimuli into clusters or hierarchy levels. But the evidences alluded to above and also those obtained by Luce and Steingrimsson (2005a, 2005b) and Bernasconi, Choirat and Seri (2008) reject this possibility as well.

[16] It is trivial that the power function $W(p) = p^k$ always satisfies multiplicativity ($W(p) \cdot W(q) = W(p \cdot q)$) when $W(1) = 1$. A subtlety investigated by Steingrimsson and Luce (2007) is that when multiplicativity fails, $W(p)$ may still be a power function with $W(1) \neq 1$. (More on this point below.)

The implications for the AHP are clear: whenever the weighting function is not the identity or the power model, violations of cardinal consistency are inherent in any subjective ratio assessment. At the same time, the importance of separable representations is that when multiplicativity fails, but commutativity holds, ratio estimations may still result into a ratio scale, though it is necessary to pass through the function W to interpret the subjects' subjective measures of ratios as numerical ratios.

In this respect, an important property of function $W(\cdot)$ is monotonicity, which follows in the mathematical derivation of separable representations (Luce 2002, p. 522). In terms of the AHP, monotonicity implies ordinal consistency (that if $a_{ij} > 1$ and $a_{jk} > 1$, then also $a_{ik} > 1$; see Section 2.2). Moreover, if $W(\cdot)$ is monotonic, W^{-1} is invertible so that the actual entries of the AHP responses matrix $A = [a_{ij}]$ are given by:

$$a_{ij} = W^{-1}\left(\frac{w_i}{w_j}\right). \tag{8}$$

This is quite important for the AHP, because it implies that if one knows how to estimate the function $W^{-1}(\cdot)$ and $W^{-1}(\cdot)$ is invertible,[17] then one can pinpoint between the elicited numerals a_{ij} and the numerical ratios w_i/w_j in order to obtain the priority weights $w = (w_1, w_2, ..., w_n)'$, which represent the ultimate objectives of the AHP. In Sections 4 and 5 we will show how this can be done in quite a general way, firstly theoretically and then in an actual experiment.

Another important condition on the subjective weighting function is $W(1) = 1$, that is individuals are able to correctly perceive proportions of equal stimuli. This is generally assumed in an ordinary separable model. Notice that the condition is necessary for the AHP to assume $a_{ii} = 1$ for all entries on the diagonal of $A = [a_{ij}]$. Symmetry, that is $a_{ij} = 1/a_{ji}$, instead requires that $W(\cdot)$ is reciprocally symmetric, namely $W(\frac{1}{\cdot}) = \frac{1}{W(\cdot)}$. This is implied by several derivations of separable representations, including a specification developed by Luce (2001, 2002), similar to one which Prelec (1998) proposed in the context of utility theory for risky gambles. A more recent specification proposed by Luce (2004) does not instead impose either $W(1) = 1$ or symmetry. Since, however, the initial evidence on the matter is not conclusive,[18] our main focus in what follows is on the implications of the nonlinearity of $W(\cdot)$ on cardinal consistency in the AHP.

[17] In earlier works by Birnbaum and Veit (1974) and Birnbaum and Elmasian (1977), a function J_R conceptually identical to the inverse function $W^{-1}(\cdot)$ is introduced directly as a monotonic judgmental transformation of a ratio model relating overt magnitude estimation of ratios to subjective impressions of ratios, and is put in connection to a judgmental transformation J_D of a model relating overt rated differences to subjective differences.

[18] In recent psychophysical experiments on loudness production, Steingrimsson and Luce (2007) and Zimmer (2005) have rejected the behavioral hypothesis underlying the specification with $W(1) = 1$ and have accepted one with $W(1) \neq 1$. In an experiment estimating the distance-ratio between cities (similar to the one presented below), Bernasconi, Choirat and Seri (2008) have instead accepted $W(1) = 1$. The same restriction is also accepted by several economic experiments estimating the subjective weighting function $W(\cdot)$ on probabilities, which suggest an inverse s-shaped form for W (see e.g., Wu and Gonzalez 1996, Prelec 1998, and references therein).

4 A separable statistical model for the AHP

The mathematical theories of separable forms discussed in the previous sections "are about idealized situations and do not involve considerations of error" (Narens 1996, p. 109). This is acknowledged as a limitation. People are not like robots. Various elements, including lapses of reason or concentration, states of mind, trembling, rounding effects and computational mistakes imply the obvious notion that no model of human behavior can be thought to hold deterministically.

As emphasized in Section 2, this was also remarked by Saaty in the AHP with the notion of the multiplicative errors e_{ij}. We now re-comprehend the errors in the separable specification of the AHP. In particular, introducing the multiplicative terms e_{ij} to model (8), we obtain the separable statistical model (2) of the Introduction, namely:

$$a_{ij} = W^{-1}\left(\frac{w_i}{w_j}\right) \cdot e_{ij}. \tag{2}$$

Model (2) raises several issues for the AHP. Here we focus on the specification of a procedure for conducting a rigorous mathematical and statistical analysis of model (2), which we then apply to the data of a ratio estimation experiment in the AHP. A different issue concerns the effects of the separable form and of the nonlinearity of the subjective weighting function $W(\cdot)$ on the mathematical behavior of Saaty's maximum-eigenvalue method. We do not conduct a technical analaysis of the latter issue here. Still, in the empirical analysis, we will show how it is possible to separate the effects of the distortions due to $W(\cdot)$ from those due to the noise e_{ij} on the subjective assessments a_{ij}, and thus to indicate the type of mistakes which may be done using the maximum-eigenvalue to estimate the priority vector.

We proceed first showing how to obtain a regression model from equation (2) (in Section 4.1), and then presenting an inference method to obtain the unknown parameters of the model (in Section 4.2). We remark that the procedure must be considered to apply on an individual basis, in the sense that the responses a_{ij} for which we propose the analysis must be considered as given by a single individual.[19]

4.1 Regression model

As a first step to transform equation (2) into a regression model amenable to statistical analysis, we apply the log transformation:

$$\ln a_{ij} = \ln W^{-1}\left[\exp\left(\ln w_i - \ln w_j\right)\right] + \varepsilon_{ij} \tag{9}$$

[19] This is standard in psychophysical experiments of separable representations (see e.g., all experimental studies cited in Section 3). The AHP considers instead also the possibility to be applied to group decision making (Saaty 1980, Saaty and Aczél 1983), treating two possible approaches: the aggregation of individual judgments or the aggregation of individual priorities. A large literature has discussed which approach is the best and under which conditions. In terms of that literature, an interesting issue opened up by separable representations (which is not considered in the present paper) concerns the question about the implication for group decision making of the nonlinearity of the individuals' weighting functions W.

where $\varepsilon_{ij} = \ln e_{ij}$. We now assume that the deterministic function $\ln W^{-1}[\exp(\cdot)]$ can be approximated through a polynomial in its arguments. This is generally possible: according to the Weierstrass Approximation Theorem, any continuous function on a compact domain can be approximated to any desired degree of accuracy by a polynomial in its arguments. Here we stop to an approximation of the third order, which yields the expression:

$$\ln W^{-1}[\exp(z)] \simeq \beta_0 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 \tag{10}$$

with $z = \ln w_i - \ln w_j$. We emphasize that an approximation to the third order is sufficient to characterize all the various restrictions discussed in the previous sections for the AHP.[20] In particular, notice that:

- the restriction $W(1) = 1$ (from $a_{ii} = 1$) implies $\beta_0 = 0$;
- the fact that W is reciprocally symmetric (from $a_{ji} = \frac{1}{a_{ij}}$) implies $\beta_2 = 0$;
- the classical AHP where W is the identity (or the power model $W(p) = p^k$ with $k > 0$ and $W(1) = 1$, see Section 3.3) restricts $\beta_1 = 1$ and $\beta_3 = 0$;
- at last, the case in which $\beta_1 = 1$ and β_3 is left free to vary corresponds to a (third order) log approximation of the inverse separable model (8) of Section 3.3, namely:

$$\ln W^{-1}[\exp(z)] \simeq z + \beta_3 z^3. \tag{11}$$

Substituting in equation (9), we finally obtain the statistical inference model:

$$\ln a_{ij} \simeq (\ln w_i - \ln w_j) + \beta_3 (\ln w_i - \ln w_j)^3 + \varepsilon_{ij} \tag{12}$$

which, when $\beta_3 = 0$, collapses to the AHP log form (4) analyzed with the logarithmic least squares method (LLSM) by many previous authors (see Section 2.3).

4.2 Statistical analysis

In fact, we now propose a method to conduct the statistical inference in model (12) which can be viewed as a generalization of the LLSM method (in particular, of the analysis of Genest and Rivest 1994). We derive the estimators $\widehat{w}_i$ and $\widehat{\beta}_3$ minimizing the sum of squares:[21]

$$\left(\widehat{w}_1, \ldots, \widehat{w}_n, \widehat{\beta}_3\right) = \arg \min_{(w_1,\ldots,w_n,\beta_3)} \sum_{i,j=1}^{n} \varepsilon_{ij}^2$$

[20] More generally, in the statistical approach presented below the order of the approximation can be extended to any desired degree (provided of course that one has enough data to estimate the model) and then the order of the polynomial can be selected using various techniques, for example the statistical theory of model selection, as we do in Bernasconi, Choirat and Seri (2008).

[21] The precise statement of the results of this Section and the full derivation of the asymptotic theory of estimator $(\widehat{w}_1, \ldots, \widehat{w}_n, \widehat{\beta}_3)$ is in an Electronic Companion Appendix.

where

$$\varepsilon_{ij} = \ln a_{ij} - (\ln w_i - \ln w_j) - \beta_3 (\ln w_i - \ln w_j)^3 ,$$

under the constraint that $\sum_{i=1}^{n} \widehat{w}_i = 1$. As in Genest and Rivest (1994), we assume that the errors ε_{ij} for $1 \leq i \leq n$, $1 \leq j < i$ are independent with common variance σ^2. Also recall that $\varepsilon_{ij} = -\varepsilon_{ji}$ and $\varepsilon_{ii} = 0$. Under suitable hypotheses it is possible to show that the estimator $\left(\widehat{w}_1, \ldots, \widehat{w}_n, \widehat{\beta}_3\right)$ is consistent and asymptotically normal when $\sigma \downarrow 0$, but the formulas of its asymptotic variance depend on the unknown parameter σ. It is slightly more complicated to find an estimator of σ^2, but if we define the residuals

$$\widehat{\varepsilon}_{ij} = \ln a_{ij} - (\ln \widehat{w}_i - \ln \widehat{w}_j) - \widehat{\beta}_3 (\ln \widehat{w}_i - \ln \widehat{w}_j)^3 ,$$

the estimator

$$\widehat{\sigma}^2 = \frac{1}{n^2 - 3n} \cdot \sum_{i,j=1}^{n} \widehat{\varepsilon}_{ij}^2$$

is asymptotically unbiased (in the sense that $\lim_{\sigma \downarrow 0} \frac{\mathbb{E}\widehat{\sigma}^2}{\sigma^2} = 1$). The problem is that this estimator is not consistent. Also the LLSM estimator of Genest and Rivest (1994) is not consistent. This is not dramatic: it only implies that when $\sigma \downarrow 0$ and σ^2 is replaced by its estimator $\widehat{\sigma}^2$, the classical t-tests on coefficients are $t(\frac{n^2-3n}{2})$-distributed (in classical regression theory they are $\mathcal{N}$). Tests and confidence intervals can however be built even if asymptotic theory is nonstandard.

5 The experiment

We consider the data of an experiment with 69 individuals and $n = 5$ alternatives. Participants were asked to estimate the relative distances of 5 Italian cities from Milan (between brackets the distances normalized to sum up to 1): Turin (0.051559), Venice (0.102703), Rome (0.204158), Naples (0.273597), Palermo (0.367983).

In 10 pairwise comparisons, subjects were asked to indicate: firstly, which city in each pair they considered more distant from Milan; and then to quantify with a number chosen in the interval of integers from 1 to 9 how many times the more distant city was according to them actually more distant from Milan than the less distant city. The interval of integers 1-9 was used in accordance with Saaty's verbal "scale of relative importance". Also notice that the integers 1-9 cover the proportions between the physical stimuli, so that the design satisfies Saaty's homogeneity axiom. It may still be objected that quantification on a discrete interval could by itself induce rounding errors and consistency violations. In this respect we emphasize that in the experiment presented in Bernasconi, Choirat, Seri (2008), we have conducted a distance experiment similar to the one described here, but with quantifications on the continuous space of real numbers, and we have obtained results similar to those presented below. The discrete interval 1-9 is used here to to be closer to classical AHP.[22]

[22] Indeed, distance experiments similar to the one presented here have been conducted in various occasions by Saaty as demonstrations of the AHP (see e.g., Saaty 1977).

Participants were undergraduates students in Economics at the University of Insubria in Italy. It was decided to use a monetary reward as an incentive for subjects to perform the experiment as well as possible. Subjects were explained at the beginning of the experiment how the payment would be calculated. Namely, payment was proportional to good estimates in the experiment.

5.1 Empirical estimates

We have estimated the priority vector $(w_1, \ldots, w_n,)'$ for every individual participating in the experiment using the various methods of estimation discussed in the paper: Saaty's maximum eigenvalue-eigenvector method (ME), the logarithmic least squares method (LLSM) proposed by de Jong (1984), Crawford and Williams (1985), Genest and Rivest (1994), and our theory of polynomial approximation based on the joint estimation of vector $(\widehat{w}_1, \ldots, \widehat{w}_n)'$ and parameter $\widehat{\beta}_3$.

In Fig. 1 we plot the estimates $(\widehat{w}_1, \ldots, \widehat{w}_n)'$ obtained by our method against the corresponding true values 0.05155925, 0.10270270, 0.20415800, 0.27359667, 0.36798337, and compare them with the ME and the LLSM estimates.[23] In the individual graphs, the black solid piecewise lines represent our estimates; the black dashed lines give the confidence intervals at 95%; the light-grey lines represent the estimates as obtained by Saaty's ME method; the dark-grey lines are the estimates obtained by the LLSM method. It turns out that the ME and LLSM estimates are very similar, in fact indistinguishable in the diagrams, confirming previous results and theoretical expectations (see Zahedi 1986, Budescu, Zwick and Rapoport 1986, Genest and Rivest 1994). Estimates based on our method are for many subjects more different than those from the other two methods. Confidence intervals are often (but not always) quite small.

In Fig. 2 we show the functions $W^{-1}(x) \simeq x \cdot \exp\left\{\beta_3 \ln^3 x\right\}$ estimated for each subject (remark that we do not plot W, but its inverse). In each graph, the grey solid lines represent the identity function; the black solid lines represent the estimated functions; the black dashed lines show the confidence intervals at 95%. The confidence intervals allow us to conduct graphically for each subject the test of the null hypothesis that W is the identity: in 47 cases out of 69 the hypothesis is rejected at 95%, in 22 cases it cannot be rejected. The diagrams also indicate that most subjects substantially underestimate the ratios (fitted W^{-1} is below the 45^0 line, with only 4 subjects showing a tendency for overestimation). Moreover, a concave shape of function $W^{-1}(x)$ is estimated for most subjects, indicating that the tendency to underestimate ratios increases as the ratios get larger and larger above one.

We take these results as a clear confirmation of modern measurement theory, that subjects' subjective measures of ratios are affected by systematic distortions and cannot be interpreted as numerical ratios. This is in line with the various recent experiments referred to in Section 3.

It is here also of interest to notice the confirmation of the implications of the subjec-

[23] We give here a graphical representation of the results. The individual data are available in the Electronic Companion Appendix.

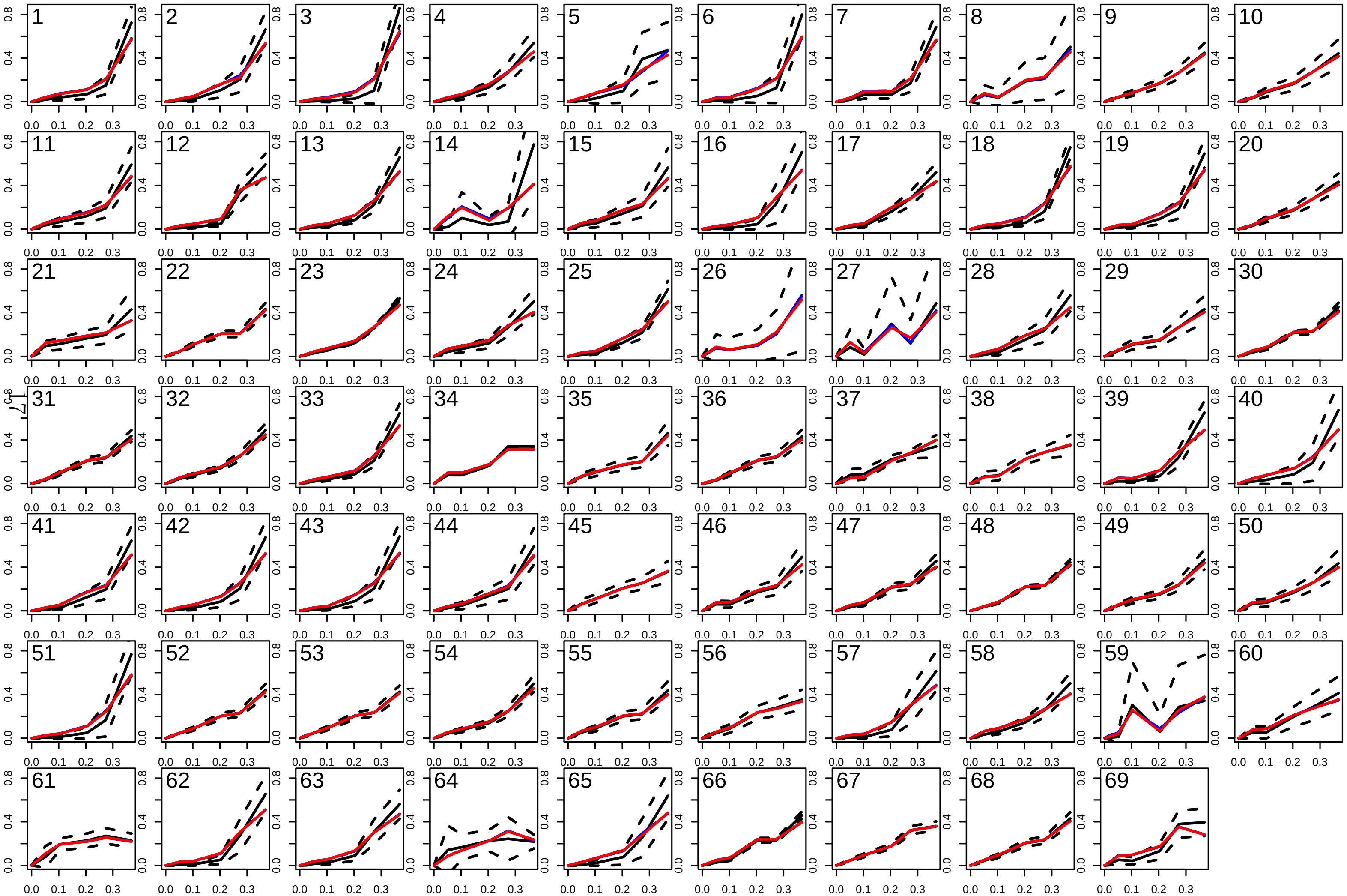

FIGURE 1: Individual estimates of priority weights obtained by various methods.

tive weighting function for the relationships between the various methods of estimating the priority vector. In particular notice that only when the subjective weighting function is linear, then our statistical theory of obtaining the priority vector yields estimates similar to Saaty's ME and LLSM method. This in particular applies to the 22 subjects with a linear $W^{-1}(x)$ estimated in Fig. 2, who also have plots of the priority vectors in Fig. 1 generally indistinguishable from those obtained by the ME and LLSM methods;[24] whereas for the 47 subjects with a nonlinear $W^{-1}(x)$ the estimates of the priority vector in Fig. 1 are less close to the estimates obtained by the ME and LLSM methods.

The diagrams in Fig. 2 also allows to conduct some tests of ordinal consistency. In particular, recall that if W^{-1} is not an invertible function, then the subjects' ratio assessments violate ordinal consistency. In our estimation procedure we have decided not to impose invertibility. Nevertheless, we find that only for 5 subjects out of 69 the estimated function W^{-1} is not invertible.[25]

5.2 The effects of W and errors ε_{ij}

Having confirmed that the subjective weighting function is for many subjects nonlinear, perhaps the most empirically relevant question for the AHP is whether the distortions due to W (and thus β_3) are larger or smaller than the ones due to the noise e_{ij} (or ε_{ij}). In this respect, the estimates of the priority vectors and of the subjective weighting functions allow to conduct the comparisons between Saaty's eigenvector $\overline{w} = (\overline{w}_1, ..., \overline{w}_n)'$ and eigenvalue $\overline{\lambda}$ computed from the matrix of the elicited responses $A = [a_{ij}]$ only taking account of the distortions due to the noise, with the eigenvalue-eigenvector obtained firstly removing from the matrix the effect of the noise, thus computing w and λ on the basis of $\widehat{W}^{-1}\left(\frac{\widehat{w}_i}{\widehat{w}_j}\right)$ (with $\widehat{W}^{-1}$ and $\widehat{w} = (\widehat{w}_1, \ldots, \widehat{w}_n)'$ estimated according to our method); and then removing also the effect of the distortions due to W, thus computing w and λ directly on the basis of matrix $[\frac{\widehat{w}_i}{\widehat{w}_j}]$.

The results of the decomposition are shown in Fig. 3. The six subgraphs display the empirical cdf (cumulative distributions) of the 69 values for the five components of the maximum eigenvectors obtained from the different matrices indicated above, followed by the empirical cdf of the 69 values of the maximum eigenvalue computed for the various matrices. In particular, the thin-black lines refer to the cdf of the five components of Saaty's eigenvector $\overline{w} = (\overline{w}_1, ..., \overline{w}_n)'$ and eigenvalue $\overline{\lambda}$; the grey curves represent the empirical cdf of the 69 values of the corresponding quantities (eigenvector and eigenvalue) computed for each individual on the basis of the matrix with generic element $\frac{\widehat{w}_i}{\widehat{w}_j} \cdot \exp\left\{\widehat{\beta}_3 \cdot \ln^3 \frac{\widehat{w}_i}{\widehat{w}_j}\right\} = a_{ij}/\widehat{\varepsilon}_{ij}$; the thick-black curves represent the empirical cdf of the 69 values of the same quantities computed for each individual on the basis of the matrix with generic element $\frac{\widehat{w}_i}{\widehat{w}_j}$. The dashed vertical lines represent the real values. Summing up: the thin-black curves are based on the elicited data, the grey curves on the data

[24]The 22 subjects with a presumably linear $W^{-1}(x)$ are: subjects 8, 9, 10, 14, 20, 22, 26, 27, 29, 35, 37, 38, 45, 49, 50, 52, 53, 56, 59, 63, 67, 68.

[25]The subjects are: 21, 34, 60, 61, 69.

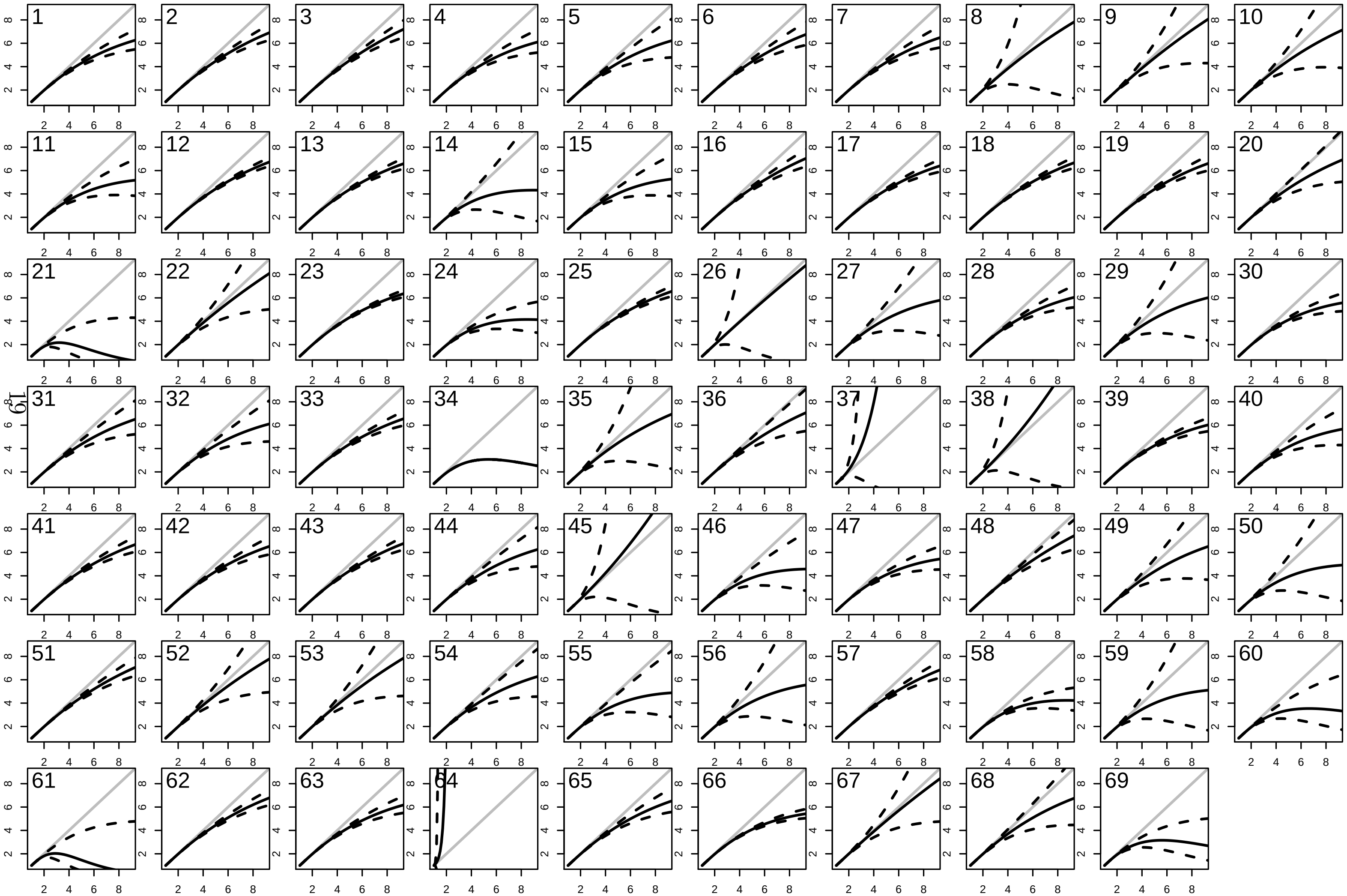

FIGURE 2: Individual estimates of the inverse subjective weigthing function W^{-1}

without the noise, the thick-black curves on the data without the noise and W, and the dashed lines are the real values.

Looking at the diagrams, it is apparent that the distance between the grey and the thin-black curves is less than the distance between the thick-black and the grey curves. This indicates that the distortions due to the errors ε_{ij} have a definitively smaller effect on the elicited responses than the distortions due to W. The important implication is that the errors ε_{ij} have only a limited impact on consistency violations. This is demonstrated by the last graph for the eigenvalue which shows that the errors ε_{ij} have a moderate effect to generate inconsistencies in the data, since the grey line computed with the data cleaned by the noise is only a bit lower than the thin-black line for the elicited data (recall that perfect consistency holds when the Perron eigenvalue $\lambda_{\max} = 5$), whereas inconsistencies are completely removed only when the effect of W is also taken away from elicited responses (see the thick-black line).

The above analysis has important implications also under a practical perspective. Saaty has always warned "that once near consistency is attained, it becomes uncertain which coefficients should be perturbed to transform a near consistent matrix to a consistent one. If such perturbations were forced, they could be arbitrary and thus distort the validity of the derived priority vector in representing the underlying decision" (Saaty 2003, p. 85). The present analysis indicates how one could operate in order to avoid arbitrariness of corrections. The subjective weighting function $W(\cdot)$ is clearly not arbitrary, but "is the scientific way of interpreting the subjective measures of ratios as numerical ratios" (Narens 1996, p. 110). The forms of the weighting function estimated provide important information for classical AHP, which could lead to two possibilities of revision. First of all, the estimated subjective weighted functions could be used to obtain a proper *ratio scale* $\frac{\widehat{w}_i}{\widehat{w}_j}$ from the elicited responses a_{ij} (as in the procedure underlying the thick-black lines in Fig. 3). Alternatively, the estimated function could be used to provide suggestions to the decision makers about how to improve the scientific accuracy of their elicited responses. For example, it is clear that in a repetition of the present experiment, participants should be advised of their tendency to increasingly underweight the ratios as the ratios get larger above one (see again the plots in Fig. 2).

In all cases, the results of the experiment confirm that the responses obtained by the AHP cannot be taken to provide a ratio-scale measure directly, nor that the hypothesis of multiplicative perturbations (or random error) is sufficient to characterize the observed departures from a ratio scale. AHP proposers must be aware of this and corrections must be chosen accordingly.

6 Conclusion

The AHP is a problem solving technique for establishing priorities in multivariate environments. Quarrels have concerned both its normative and descriptive status. Quoting Smith and von Winterfeldt (2004): "our view is that if the AHP is truly intended as a descriptive model, then one should test it to see how well it describes actual decision making behavior. Though we do not know of such tests, we are confident that

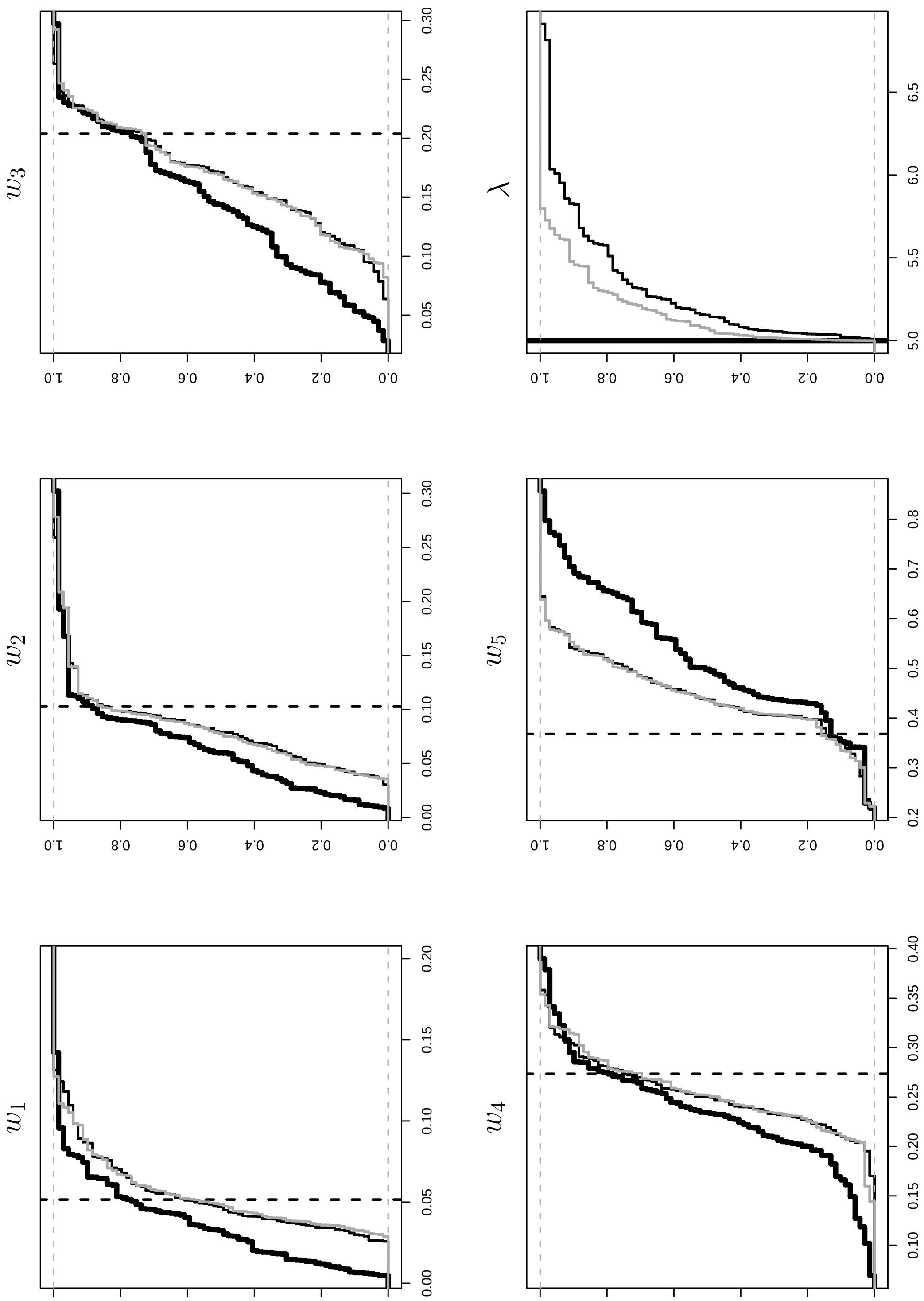

FIGURE 3: Empirical cumulative distributions of eigenvector and eigenvalue estimates.

the AHP would not do a very good job in predicting decision making behavior, just as the expected utility model has limited descriptive power. The appeal of the AHP as a prescriptive theory remains a matter of disagreement. While many in the decision analysis community (ourselves included) follow Dyer (1990a) in believing the AHP to be fundamentally unsound, others (including Saaty, Harker, and Vargas) disagree and the AHP is still widely used in practice today" (p. 568).

Our opinion is that much of the disagreement about both the descriptive and normative status of the AHP derives from the confusion and imprecision which characterizes its relationships with the theory of measurement. Most of the obscurity derives from the appeal of AHP supporters to rely on a *direct* method of measurement similar to Stevens' ratio-scaling approach, which is itself ill-founded. In particular, the error of Saaty and of the AHP in following Stevens is to believe that the numerals obtained by subjects in ratio estimation procedures can be treated as scientific numbers. This is what we believe most of the AHP opposers have always contended, implicitly or explicitly, in their critics. Rightly so: "direct measurement methodologies that rely on assigning a number to a stimulus because it corresponds to a number named in the instructions should be looked with incredulity" (Narens 2006, p. 298).

From this, however, we also believe that most of the critics to the AHP have derived the belief that no ratio-scale measures of subjective judgments, feelings, preferences are possible. This may be equally wrong. In recent years mathematical psychology has provided various axiomatizations based on different psychological primitives that have made explicit the structural assumptions inherent in representing direct measurement data. If on the one hand tests of these properties have confirmed the difficulty to use ratio-scaling methods to obtain ratio-scale measures directly, on the other hand they have shown that the data produced by direct measurement instructions can still be an important tool in rigorous behavioral applications if analyzed by appropriate methodologies. In particular, support for separable representations and their underlying properties "suggests that ratio scales obtained through the use of numbers names in instructions and proper measurement techniques may still be a valuable tool for basic psychophysics" (Narens 2006, p. 299). The key for a rigorous analysis is the subjective weighting function $W(\cdot)$, which allows one to pinpoint on the basis of normatively justified arguments and descriptively supported hypotheses, between the subjects' perception of proportions and their underlying scientific ratio-scale representation.

In this paper we have shown how the same notions of separable representation and of subjective weighting function can be extended to the AHP. Our analysis has focused on subjective ratio assessments more directly applicable to single-level hierarchies. In complex, multi-level hierarchies, the AHP obtains subjective ratio assessments at each level of the hierarchy, which are then weighted and added to obtain an overall ratio scale of alternatives. The problem of inconsistencies originated by the subjective weighting function could then become even more severe, since the distortions in the assessments occurred at each level of the hierarchy multiply with each other possibly increasing the distortions in the final assessment of alternatives.

Our method of investigating the subjective weighting function can be extended to multi-level hierarchies simply replicating the analysis to measure inconsistencies at all hierachic levels. This could also contribute to provide the AHP with more descriptive and prescriptive credibility.

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A Asymptotic theory of estimator $\left(\widehat{w}_1, \ldots, \widehat{w}_n, \widehat{\beta}_3\right)$

The model is:

$$\begin{aligned} aij &= \frac{w_i}{w_j} \cdot \exp\left[\beta_3 \cdot \left(\ln \frac{w_i}{w_j}\right)^3\right] \cdot \exp \varepsilon_{ij} \\ \ln a_{ij} &= \ln w_i - \ln w_j + \beta_3 \cdot (\ln w_i - \ln w_j)^3 + \varepsilon_{ij}. \end{aligned}$$

We want to obtain estimates of $\mathbf{w} = (w_i, \ldots, w_n)'$ and β_3. In order to do so, we minimize the following objective function:

$$Q(\mathbf{w}, \beta_3) \triangleq \sum_{i \neq j=1}^{n} \varepsilon_{ij}^2 = \sum_{i \neq j=1}^{n} \left[\ln a_{ij} - \ln w_i + \ln w_j - \beta_3 \cdot (\ln w_i - \ln w_j)^3\right]^2. \quad (13)$$

We will indicate the objective function as $Q^{(\sigma)}$ in order to stress the dependence on the asymptotic parameter σ; $Q^{(0)}$ is the objective function when $\sigma = 0$, while $Q_0^{(\sigma)}$ and $Q_0^{(0)}$ are the previous quantities when evaluated at the true parameters $\mathbf{w}_0$ and $\beta_{3,0}$; it is clear that $Q_0^{(0)} \equiv 0$. When needed, we will write $\boldsymbol{\theta} = (\mathbf{w}, \beta_3)$.

We make the following assumptions.

Ass. 1 The estimator $\widehat{\boldsymbol{\theta}}$ is obtained minimizing the function (13) under the constraint $\sum_{i=1}^n w_i = 1$.

Ass. 2 Let $\mathbf{E}_0$ be the skew-symmetric matrix applying when $\mathbf{w} = \mathbf{w}_0$ and $\beta_3 = \beta_{3,0}$, with generic (i,j) −element $\varepsilon_{0,ij}$. Let $\boldsymbol{\varepsilon}_0 = \widetilde{v}(\mathbf{E}_0)$ the $\left(\left(\frac{n^2-n}{2}\right) \times 1\right)$ vector obtained stacking the subdiagonal elements of $\mathbf{E}_0$; $\boldsymbol{\varepsilon}_0$ is such that $\sigma^{-1}\boldsymbol{\varepsilon}_0 \xrightarrow{\mathcal{D}} \mathcal{N}\left(\mathbf{0}, \mathbf{I}_{\left(\frac{n^2-n}{2}\right)}\right)$. Asymptotic results are stated for $\sigma \downarrow 0$.

Ass. 3 The parameter w_i takes its value in the interval $[\varepsilon_i, 1-\varepsilon_i]$ for some $\varepsilon_i > 0$; the parameter β belongs to a compact interval $[\beta_L, \beta_U]$; the weights respect the equality $\sum_{i=1}^n w_i = 1$. The parameter space is $\Theta = \{\prod_{i=1}^n [\varepsilon_i, 1-\varepsilon_i] \cap \{\sum_{i=1}^n w_i = 1\}\} \times [\beta_L, \beta_U]$; moreover $\boldsymbol{\theta}_0 \in \mathrm{relint}\Theta = \{\prod_{i=1}^n (\varepsilon_i, 1-\varepsilon_i) \cap \{\sum_{i=1}^n w_i = 1\}\} \times (\beta_L, \beta_U)$.

We remark that the asymptotic parameter is σ and not n. This implies that even if these results seem standard, they aren't. In particular, normalizations through functions of n are very important since different normalizations (e.g. by n and by $n-1$) do not lead to the same asymptotic behavior. We also remark that the requirement that $w_i \in [\varepsilon_i, 1-\varepsilon_i]$ is in line with Axiom 2 of Saaty (1986).

Consider the indexes $1 \leq i, j \leq n$ with $i > j$, and let $k = (j-1) \cdot n + i - {}^{j(j+1)}/_2$.

Then we will need the $\left((n+1)\times\left(\frac{n^2-n}{2}\right)\right)$ matrix $\mathbf{Q}_0$ given by:

$$\begin{aligned}[\mathbf{Q}_0]_{(h,k)} &= 4\cdot\mathbf{1}_{i=h}\cdot\left(-\frac{1}{w_{0,i}}-\frac{3\beta_0}{w_{0,i}}\cdot(\ln w_{0,i}-\ln w_{0,j})^2\right)\\ &\quad +4\cdot\mathbf{1}_{j=h}\cdot\left(\frac{1}{w_{0,j}}+\frac{3\beta_0}{w_{0,j}}\cdot(\ln w_{0,i}-\ln w_{0,j})^2\right),\end{aligned}$$

for $1\le h\le n,\quad 1\le k\le\left(\frac{n^2-n}{2}\right)$,

$$[\mathbf{Q}_0]_{(n+1,k)} = 4\cdot(\ln w_{0,j}-\ln w_{0,i})^3$$

for $1\le k\le\left(\frac{n^2-n}{2}\right)$.

Matrix $\mathbf{Q}_0$ is such that $\dot{Q}_0^{(\sigma)}=\mathbf{Q}_0\cdot\boldsymbol{\varepsilon}_0$, where $\boldsymbol{\varepsilon}_0$ is defined in Ass 2 (it can be checked that the $k-$th element of $\boldsymbol{\varepsilon}_0$ is $\varepsilon_{0,ij}$, where $k=(j-1)\cdot n+i-{}^{j(j+1)}/_2$ and $i>j$).

The following Proposition shows that consistency and asymptotic normality (when σ^2 is known) hold for this estimator.

Proposition A.1. *Under Ass. 1 and 3, the estimator $\widehat{\boldsymbol{\theta}}$ is weakly consistent for $\boldsymbol{\theta}_0$ as $\sigma\downarrow 0$. Under Ass. 1-3 it has the following asymptotic distribution:*

$$\sigma^{-1}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(\mathbf{0},16\cdot\mathbf{D}_n\left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1}\mathbf{D}_n'\right)$$

where $\mathbf{D}_n$ is the $((n+1)\times n)$ matrix $\mathbf{D}_n=\begin{bmatrix}-\left(\mathbf{e}_{n-1}',0\right)\\ \mathbf{I}_n\end{bmatrix}$.

Now we consider the situation in which σ is replaced by an estimator.

Proposition A.2. *Under Ass. 1-3, the estimator:*

$$\widehat{\sigma}^2=\frac{1}{n^2-3n}\cdot\sum_{i\neq j=1}^{n}\widehat{\varepsilon}_{ij}^2$$

is asymptotically unbiased (in the sense that $\mathbb{E}\frac{\widehat{\sigma}^2}{\sigma^2}\to 1$) and has asymptotic distribution:

$$\left(\frac{n^2-3n}{2}\right)\cdot\frac{\widehat{\sigma}^2}{\sigma^2}\to_{\mathcal{D}}\chi^2\left(\frac{n^2-3n}{2}\right).$$

Consider a full row rank $(m\times(n+1))$ matrix $\boldsymbol{\Gamma}$. Then:

$$\frac{\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right)'\boldsymbol{\Gamma}'\left\{\boldsymbol{\Gamma}\mathbf{D}_n\left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1}\mathbf{D}_n'\boldsymbol{\Gamma}'\right\}^{-1}\boldsymbol{\Gamma}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right)}{16\widehat{\sigma}^2 m}\to_{\mathcal{D}}F\left(m,\frac{n^2-3n}{2}\right).$$

When $m=1$, in particular:

$$\frac{\boldsymbol{\Gamma}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right)}{4\widehat{\sigma}\left\{\boldsymbol{\Gamma}\mathbf{D}_n\left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1}\mathbf{D}_n'\boldsymbol{\Gamma}'\right\}^{1/2}}\to_{\mathcal{D}}t\left(\frac{n^2-3n}{2}\right).$$

Consider the residual $\widehat{\varepsilon}_{ij}$ with $i > j$. Let $\mathbf{e}_k$ be the $((n+1)\times 1)$ vector with 1 in the $k-$th position and all other components equal to 0. Then the following result holds with $k = (j-1)\cdot n + i - j(j+1)/2$:

$$\frac{\widehat{\varepsilon}_{ij}}{\widehat{\sigma}\cdot\sqrt{1-\mathbf{e}_k'\mathbf{Q}_0'\mathbf{D}_n\left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1}\mathbf{D}_n'\mathbf{Q}_0\mathbf{e}_k}} \to_{\mathcal{D}} t\left(\frac{n^2-3n}{2}\right).$$

Proof of Proposition A.1. We start with a result of consistency. When $\sigma = 0$, $\ln a_{ij} = \ln w_{0,i} - \ln w_{0,j} - \beta_0\cdot(\ln w_{0,i} - \ln w_{0,j})^3$; thus when $\sigma \downarrow 0$, we have

$$\begin{aligned} Q^{(\sigma)}(\mathbf{w},\beta) - Q^{(0)}(\mathbf{w},\beta) &= 2\sum_{i\neq j=1}^{n}\varepsilon_{0,ij}\cdot\left[\ln w_{0,i} - \ln w_{0,j} + \beta_0\cdot(\ln w_{0,i} - \ln w_{0,j})^3\right] \\ & -2\sum_{i\neq j=1}^{n}\varepsilon_{0,ij}\cdot\left[\ln w_i - \ln w_j + \beta\cdot(\ln w_i - \ln w_j)^3\right] \\ & +\sum_{i\neq j=1}^{n}\varepsilon_{0,ij}^2. \end{aligned}$$

This converges uniformly in probability to 0 under Ass. 3 when as $\sigma \downarrow 0$. $Q^{(0)}$ is continuous on a compact space and is uniquely minimized at $w_i = w_{0,i}$ and $\beta = \beta_0$. Therefore Theorem 2.1 in Newey and McFadden (1994) applies.

For the asymptotic distribution, we reason in terms of $\tilde{\boldsymbol{\theta}} = \begin{bmatrix} \mathbf{0}_{n\times 1} & \mathbf{I}_n \end{bmatrix}\cdot\boldsymbol{\theta}$, that is $\boldsymbol{\theta}$ without the first component; remark that $\boldsymbol{\theta}$ can be recovered as $\boldsymbol{\theta} = \begin{bmatrix} 1 \\ \mathbf{0}_{n\times 1}\end{bmatrix} + \begin{bmatrix} -\left(\mathbf{e}_{n-1}',0\right) \\ \mathbf{I}_n \end{bmatrix}\tilde{\boldsymbol{\theta}}$. Therefore, the gradient and the Hessian are $\frac{\partial Q^{(\sigma)}}{\partial\tilde{\boldsymbol{\theta}}} = \mathbf{D}_n'\cdot\frac{\partial Q^{(\sigma)}}{\partial\boldsymbol{\theta}}$ and $\frac{\partial^2 Q^{(\sigma)}}{\partial\tilde{\boldsymbol{\theta}}\partial\tilde{\boldsymbol{\theta}}'} = \mathbf{D}_n'\cdot\frac{\partial^2 Q^{(\sigma)}}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\cdot\mathbf{D}_n$. A limited development of the first order conditions $\frac{\partial Q^{(\sigma)}}{\partial\tilde{\boldsymbol{\theta}}}\left(\widehat{\tilde{\boldsymbol{\theta}}}\right) = \mathbf{0}$ yields:

$$\begin{aligned} \frac{\partial Q_0^{(\sigma)}}{\partial\tilde{\boldsymbol{\theta}}} + \frac{\partial^2 Q^{(\sigma)}}{\partial\tilde{\boldsymbol{\theta}}\partial\tilde{\boldsymbol{\theta}}'}(\boldsymbol{\theta}^\star)\cdot\left(\widehat{\tilde{\boldsymbol{\theta}}} - \tilde{\boldsymbol{\theta}}_0\right) &= 0 \\ \sigma^{-1}\left(\widehat{\tilde{\boldsymbol{\theta}}} - \tilde{\boldsymbol{\theta}}_0\right) &= -\left(\frac{\partial^2 Q^{(\sigma)}}{\partial\tilde{\boldsymbol{\theta}}\partial\tilde{\boldsymbol{\theta}}'}(\boldsymbol{\theta}^\star)\right)^{-1}\cdot\sigma^{-1}\frac{\partial Q_0^{(\sigma)}}{\partial\tilde{\boldsymbol{\theta}}}. \end{aligned}$$

Therefore the distribution of $\sigma^{-1}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)$ can be obtained as:

$$\sigma^{-1}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) = -\mathbf{D}_n\left(\mathbf{D}_n'\cdot\frac{\partial^2 Q^{(\sigma)}}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}(\boldsymbol{\theta}^\star)\cdot\mathbf{D}_n\right)^{-1}\mathbf{D}_n'\cdot\sigma^{-1}\frac{\partial Q_0^{(\sigma)}}{\partial\boldsymbol{\theta}}.$$

In the following we will indicate respectively the gradient and the Hessian with respect to $\boldsymbol{\theta}$ with one and two dots. We start from the behavior of the gradient.

The gradient is:

$$
\begin{aligned}
\frac{\partial Q^{(\sigma)}(\mathbf{w},\beta)}{\partial\beta} &= -2\cdot\sum_{i\neq j=1}^{n}\left(\ln a_{ij}-\ln w_i+\ln w_j-\beta\cdot(\ln w_i-\ln w_j)^3\right)\cdot(\ln w_i-\ln w_j)^3\\
\frac{\partial Q^{(\sigma)}(\mathbf{w},\beta)}{\partial w_i} &= -4\cdot\sum_{j\in\{1,\ldots,n\}\backslash i}\frac{\left(\ln a_{ij}-\ln w_i+\ln w_j-\beta\cdot(\ln w_i-\ln w_j)^3\right)}{w_i}\\
&\quad\cdot\left[1+3\beta\cdot(\ln w_i-\ln w_j)^2\right].
\end{aligned}
$$

Now we consider $\dot{Q}_0^{(\sigma)}$:

$$
\begin{aligned}
\frac{\partial Q^{(\sigma)}(\mathbf{w}_0,\beta_0)}{\partial\beta} &= -2\cdot\sum_{i\neq j=1}^{n}\varepsilon_{ij,0}\cdot(\ln w_{i,0}-\ln w_{j,0})^3\\
\frac{\partial Q^{(\sigma)}(\mathbf{w}_0,\beta_0)}{\partial w_i} &= -4\cdot\sum_{j\in\{1,\ldots,n\}\backslash i}\frac{\varepsilon_{ij,0}}{w_{i,0}}\cdot\left[1+3\beta_0\cdot(\ln w_{i,0}-\ln w_{j,0})^2\right].
\end{aligned}
$$

The fact that $\dot{Q}_0^{(\sigma)}=\mathbf{Q}_0\cdot\boldsymbol{\varepsilon}_0$ implies that $\mathbb{V}\left(\sigma^{-1}\dot{Q}_0^{(\sigma)}\right)=\mathbf{Q}_0\mathbf{Q}_0'$ and $\sigma^{-1}\dot{Q}_0^{(\sigma)}=\mathbf{Q}_0\cdot\sigma^{-1}\boldsymbol{\varepsilon}_0\to_{\mathcal{D}}\mathcal{N}\left(\mathbf{0},\mathbf{Q}_0\mathbf{Q}_0'\right)$

The asymptotic Hessian for $\sigma\downarrow 0$ is:

$$
\begin{aligned}
\frac{\partial^2 Q^{(\sigma)}(\mathbf{w},\beta,\lambda)}{\partial\beta^2} &\to \frac{\partial^2 Q^{(0)}(\mathbf{w},\beta,\lambda)}{\partial\beta^2}=2\cdot\sum_{i\neq j=1}^{n}\left(\ln\frac{w_i}{w_j}\right)^6\\
\frac{\partial^2 Q^{(\sigma)}(\mathbf{w},\beta,\lambda)}{\partial w_i^2} &\to \frac{\partial^2 Q^{(0)}(\mathbf{w},\beta,\lambda)}{\partial w_i^2}=4\cdot\sum_{j\in\{1,\ldots,n\}\backslash i}\frac{1}{w_i^2}\cdot\left(1+3\beta\cdot\left(\ln\frac{w_i}{w_j}\right)^2\right)^2\\
\frac{\partial^2 Q^{(\sigma)}(\mathbf{w},\beta,\lambda)}{\partial w_i\partial w_j} &\to \frac{\partial^2 Q^{(0)}(\mathbf{w},\beta,\lambda)}{\partial w_i\partial w_j}=-4\cdot\frac{1}{w_iw_j}\cdot\left[1+3\beta\cdot\left(\ln\frac{w_i}{w_j}\right)^2\right]^2\\
\frac{\partial^2 Q^{(\sigma)}(\mathbf{w},\beta,\lambda)}{\partial w_i\partial\beta} &\to \frac{\partial^2 Q^{(0)}(\mathbf{w},\beta,\lambda)}{\partial w_i\partial\beta}=4\cdot\sum_{j\in\{1,\ldots,n\}\backslash i}\frac{\left(\ln\frac{w_i}{w_j}\right)^3}{w_i}\cdot\left[1+3\beta\cdot\left(\ln\frac{w_i}{w_j}\right)^2\right]
\end{aligned}
$$

.

Under Ass. 2 and 3 convergence is uniform. Therefore, we have $\mathbb{V}\left(\dot{Q}_0^{(\sigma)}\right)=\sigma^2\cdot\mathbf{Q}_0\mathbf{Q}_0'=4\sigma^2\cdot\ddot{Q}_0^{(0)}$, that is the variance of the gradient is $4\sigma^2$ the limiting Hessian. Therefore:

$$
\begin{aligned}
\sigma^{-1}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right) &= -\mathbf{D}_n\left(\mathbf{D}_n'\cdot\frac{\partial^2 Q^{(\sigma)}}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}(\boldsymbol{\theta}^\star)\cdot\mathbf{D}_n\right)^{-1}\mathbf{D}_n'\cdot\sigma^{-1}\frac{\partial Q_0^{(\sigma)}}{\partial\boldsymbol{\theta}}\\
&\sim -4\cdot\mathbf{D}_n\left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1}\mathbf{D}_n'\mathbf{Q}_0\cdot\frac{\boldsymbol{\varepsilon}_0}{\sigma}\\
\sigma^{-1}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0\right)\to_{\mathcal{D}} &\quad \mathcal{N}\left(\mathbf{0},16\cdot\mathbf{D}_n\left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1}\mathbf{D}_n'\right).
\end{aligned}
$$

Remark that $\boldsymbol{\theta}_0$ can be replaced by any consistent estimator such as $\widehat{\boldsymbol{\theta}}$.

Proof of Proposition A.2. Let $\mathbf{E}$ be the skew-symmetric matrix with generic (i,j) −element $\varepsilon_{ij} = \ln a_{ij} - \ln w_i + \ln w_j - \beta \cdot (\ln w_i - \ln w_j)^3$; consider the $\left(\left(\frac{n^2-n}{2}\right) \times 1\right)$ vector $\boldsymbol{\varepsilon} = \widetilde{v}(\mathbf{E})$. A limited development around $\boldsymbol{\varepsilon}_0$ can then be obtained as $\boldsymbol{\varepsilon} \sim \boldsymbol{\varepsilon}_0 + \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{\theta}'} \cdot (\boldsymbol{\theta} - \boldsymbol{\theta}_0)$. Through direct computation, it is simple to show the equality $\frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{\theta}'} = \frac{1}{4} \cdot \mathbf{Q}_0$, so that through Proposition A.1 $\widehat{\boldsymbol{\varepsilon}} \sim \boldsymbol{\varepsilon}_0 + \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{\theta}'} \cdot \left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)$ becomes:

$$\begin{aligned}
\widehat{\boldsymbol{\varepsilon}} &\sim \boldsymbol{\varepsilon}_0 - \mathbf{Q}_0 \mathbf{D}_n \left(\mathbf{D}_n' \mathbf{Q}_0 \mathbf{Q}_0' \mathbf{D}_n\right)^{-1} \mathbf{D}_n' \mathbf{Q}_0 \cdot \boldsymbol{\varepsilon}_0 \\
&= \left\{\mathbf{I}_{\frac{n(n-1)}{2}} - \mathbf{Q}_0' \mathbf{D}_n \left(\mathbf{D}_n' \mathbf{Q}_0 \mathbf{Q}_0' \mathbf{D}_n\right)^{-1} \mathbf{D}_n' \mathbf{Q}_0\right\} \cdot \boldsymbol{\varepsilon}_0
\end{aligned}$$

This implies:

$$\mathbb{V}(\widehat{\boldsymbol{\varepsilon}}) \sim \sigma^2 \cdot \left\{\mathbf{I}_{\frac{n(n-1)}{2}} - \mathbf{Q}_0' \mathbf{D}_n \left(\mathbf{D}_n' \mathbf{Q}_0 \mathbf{Q}_0' \mathbf{D}_n\right)^{-1} \mathbf{D}_n' \mathbf{Q}_0\right\}.$$

On the other hand:

$$\sum_{i \neq j = 1}^{n} \frac{\widehat{\varepsilon}_{ij}^2}{\sigma^2} = \frac{2 \widehat{\boldsymbol{\varepsilon}}' \widehat{\boldsymbol{\varepsilon}}}{\sigma^2} \sim 2 \cdot \frac{\boldsymbol{\varepsilon}_0'}{\sigma} \cdot \left\{\mathbf{I}_{\frac{n(n-1)}{2}} - \mathbf{Q}_0' \mathbf{D}_n \left(\mathbf{D}_n' \mathbf{Q}_0 \mathbf{Q}_0' \mathbf{D}_n\right)^{-1} \mathbf{D}_n' \mathbf{Q}_0\right\} \cdot \frac{\boldsymbol{\varepsilon}_0}{\sigma}.$$

Therefore:

$$\begin{aligned}
\mathbb{E}\left(\sum_{i \neq j = 1}^{n} \frac{\widehat{\varepsilon}_{ij}^2}{\sigma^2}\right) &\sim 2 \cdot \mathbb{E}\mathrm{tr}\left\{\mathbf{I}_{\frac{n(n-1)}{2}} - \mathbf{Q}_0' \mathbf{D}_n \left(\mathbf{D}_n' \mathbf{Q}_0 \mathbf{Q}_0' \mathbf{D}_n\right)^{-1} \mathbf{D}_n' \mathbf{Q}_0\right\} \\
&= n(n-1) - 2\mathrm{tr}\left(\mathbf{Q}_0' \mathbf{D}_n \left(\mathbf{D}_n' \mathbf{Q}_0 \mathbf{Q}_0' \mathbf{D}_n\right)^{-1} \mathbf{D}_n' \mathbf{Q}_0\right) \\
&= n^2 - 3n.
\end{aligned}$$

We consider the estimator:

$$\widehat{\sigma}^2 = \frac{1}{n^2 - 3n} \cdot \sum_{i \neq j = 1}^{n} \widehat{\varepsilon}_{ij}^2.$$

Clearly this estimator is asymptotically unbiased.

The matrix $\mathbf{A}_1 = \mathbf{I}_{\frac{n(n-1)}{2}} - \mathbf{Q}_0' \mathbf{D}_n \left(\mathbf{D}_n' \mathbf{Q}_0 \mathbf{Q}_0' \mathbf{D}_n\right)^{-1} \mathbf{D}_n' \mathbf{Q}_0$ is idempotent and has rank equal to its trace, that is $\frac{n^2-3n}{2}$. The asymptotic distribution is:

$$\begin{aligned}
\left(\frac{n^2 - 3n}{2}\right) \cdot \frac{\widehat{\sigma}^2}{\sigma^2} &= \\
\frac{1}{2} \sum_{i \neq j = 1}^{n} \frac{\widehat{\varepsilon}_{ij}^2}{\sigma^2} &\sim \frac{\boldsymbol{\varepsilon}_0'}{\sigma} \cdot \left\{\mathbf{I}_{\frac{n(n-1)}{2}} - \mathbf{Q}_0' \mathbf{D}_n \left(\mathbf{D}_n' \mathbf{Q}_0 \mathbf{Q}_0' \mathbf{D}_n\right)^{-1} \mathbf{D}_n' \mathbf{Q}_0\right\} \cdot \frac{\boldsymbol{\varepsilon}_0}{\sigma} \to_{\mathcal{D}} \chi^2\left(\frac{n^2 - 3n}{2}\right).
\end{aligned}$$

Now we work out the distribution of a vector of linear combinations of the regression parameters. Consider the k combinations $\sigma^{-1} \boldsymbol{\Gamma}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)$ of the regression parameters $\widehat{\boldsymbol{\theta}}$

represented by the $(m \times (n+1))$ matrix $\mathbf{\Gamma}$, of full row rank. Remark that the quadratic form $\left(\frac{n^2-3n}{2}\right) \cdot \frac{\widehat{\sigma}^2}{\sigma^2}$ is asymptotically independent of the linear form $\sigma^{-1}\mathbf{\Gamma}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)$.

From:

$$\begin{aligned} & \sigma^{-2}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)' \mathbf{\Gamma}' \left\{\mathbb{V}\left[\sigma^{-1} \cdot \mathbf{\Gamma}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)\right]\right\}^{-1} \mathbf{\Gamma}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) \\ \sim \; & \frac{1}{16\sigma^2} \cdot \left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)' \mathbf{\Gamma}' \left\{\mathbf{\Gamma}\mathbf{D}_n \left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1} \mathbf{D}_n'\mathbf{\Gamma}'\right\}^{-1} \mathbf{\Gamma}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) \rightarrow_{\mathcal{D}} \chi^2(k) \end{aligned}$$

we get:

$$\begin{aligned} & \frac{\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)' \mathbf{\Gamma}' \left\{\mathbf{\Gamma}\mathbf{D}_n \left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1} \mathbf{D}_n'\mathbf{\Gamma}'\right\}^{-1} \mathbf{\Gamma}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)}{16\widehat{\sigma}^2 m} \\ \rightarrow & {}_{\mathcal{D}} \frac{\chi^2(m)/m}{\chi^2\left(\frac{n^2-3n}{2}\right)/\left(\frac{n^2-3n}{2}\right)} = F\left(m, \frac{n^2-3n}{2}\right). \end{aligned}$$

When $m = 1$, in particular:

$$\frac{\left\{\mathbf{\Gamma}\mathbf{D}_n \left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1} \mathbf{D}_n'\mathbf{\Gamma}'\right\}^{-1/2} \mathbf{\Gamma}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)}{4\widehat{\sigma}} \rightarrow_{\mathcal{D}} t\left(\frac{n^2-3n}{2}\right).$$

At last, we develop the distribution of the residuals. Consider $\widehat{\varepsilon}_{ij} = \mathbf{e}_k'\widehat{\boldsymbol{\varepsilon}}$ where $\mathbf{e}_k$ is a vector of zeros with 1 at position $k = (j-1) \cdot n + i - j(j+1)/2$:

$$\begin{aligned} \sigma^{-1} \cdot \widehat{\varepsilon}_{ij} &\sim \mathcal{N}\left(0, 1 - \mathbf{e}_k'\mathbf{Q}_0'\mathbf{D}_n \left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1} \mathbf{D}_n'\mathbf{Q}_0\mathbf{e}_k\right) \\ \frac{\widehat{\varepsilon}_{ij}}{\sigma \cdot \sqrt{1 - \mathbf{e}_k'\mathbf{Q}_0'\mathbf{D}_n \left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1} \mathbf{D}_n'\mathbf{Q}_0\mathbf{e}_k}} &\rightarrow_{\mathcal{D}} \mathcal{N}(0,1). \end{aligned}$$

If we replace σ with $\widehat{\sigma}$, we get:

$$\frac{\widehat{\varepsilon}_{ij}}{\widehat{\sigma} \cdot \sqrt{1 - \mathbf{e}_k'\mathbf{Q}_0'\mathbf{D}_n \left(\mathbf{D}_n'\mathbf{Q}_0\mathbf{Q}_0'\mathbf{D}_n\right)^{-1} \mathbf{D}_n'\mathbf{Q}_0\mathbf{e}_k}} \rightarrow_{\mathcal{D}} t\left(\frac{n^2-3n}{2}\right).$$

B Estimates of the Experiment

Table 1: Estimates by various methods of the subjects' priority vector $(w_1, \ldots, w_n,)$

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
1	ME	0.04129063	0.0708945	0.1106584	0.2036664	0.5734901	
	LLSM	0.04172535	0.07419514	0.1124588	0.2052478	0.5663729	
	SRPA	0.01916008	0.03929491	0.06892905	0.1491650	0.723451	-0.03566354
		(0.00516022)	(0.01029210)	(0.01680887)	(0.03197533)	(0.05581976)	(0.00468128)
							[0.00061943]

Table 1: continued

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
2	ME	0.02628716	0.05072505	0.1641858	0.2226889	0.5361131	
	LLSM	0.02570832	0.04985564	0.1595814	0.2418801	0.5229745	
	SRPA	0.006490734	0.02097762	0.1079738	0.2023028	0.662255	-0.0268643
		(0.00245435)	(0.00696277)	(0.02666896)	(0.04466031)	(0.06218403)	(0.00339927)
							[0.00052191]
3	ME	0.02583643	0.03622918	0.08717453	0.2057127	0.6450472	
	LLSM	0.02691077	0.04206622	0.09409599	0.2146092	0.6223179	
	SRPA	0.004766087	0.008671295	0.02841957	0.1019029	0.8562402	-0.02315544
		(0.00243196)	(0.00455861)	(0.01454140)	(0.04677840)	(0.06284450)	(0.00354627)
							[0.00126041]
4	ME	0.03954093	0.06899994	0.1628576	0.2706986	0.4579029	
	LLSM	0.03912404	0.06669853	0.1602265	0.2753927	0.4585582	
	SRPA	0.01841276	0.04332897	0.1342935	0.2662853	0.5376794	-0.03794797
		(0.00516649)	(0.00988510)	(0.02211231)	(0.03728890)	(0.05014453)	(0.00559486)
							[0.00105957]
5	ME	0.03959574	0.0825392	0.1585573	0.2907122	0.4285957	
	LLSM	0.03703162	0.0765722	0.1445817	0.2752335	0.4665809	
	SRPA	0.006164041	0.03141557	0.09944917	0.3900263	0.4729449	-0.03624931
		(0.00424253)	(0.01825205)	(0.04228243)	(0.09480283)	(0.09990390)	(0.00917698)
							[0.01085047]
6	ME	0.03424897	0.03961011	0.1187271	0.211999	0.5954148	
	LLSM	0.03642007	0.04338934	0.1251956	0.2141063	0.5808887	
	SRPA	0.01025982	0.01192563	0.05334017	0.1271143	0.79736	-0.02886232
		(0.00513872)	(0.00600363)	(0.02487535)	(0.05367914)	(0.07966364)	(0.00505655)
							[0.00230541]
7	ME	0.03602147	0.0914084	0.09467922	0.2100932	0.5677977	
	LLSM	0.0361625	0.09567082	0.0977082	0.2124507	0.5580078	
	SRPA	0.01816898	0.06282722	0.06562521	0.1695388	0.6838398	-0.03252466
		(0.00461637)	(0.01355458)	(0.01405381)	(0.03070741)	(0.05192648)	(0.0049152)
							[0.00118624]
8	ME	0.07810686	0.04265432	0.1980524	0.2266181	0.4545683	
	LLSM	0.06641183	0.04325354	0.1961923	0.2155291	0.4786132	
	SRPA	0.06101148	0.03823877	0.1881922	0.2109741	0.5015834	-0.01561739
		(0.03464129)	(0.02786233)	(0.06978067)	(0.07484325)	(0.14375560)	(0.06298271)
							[0.81402440]
9	ME	0.04380688	0.08279949	0.1712654	0.2653895	0.4367387	
	LLSM	0.04421443	0.08348458	0.1709491	0.2652866	0.4360653	
	SRPA	0.04053150	0.08014656	0.1673619	0.2644088	0.4475512	-0.01284905
		(0.00796735)	(0.01038654)	(0.01618932)	(0.02137739)	(0.03435552)	(0.02207040)
							[0.58570480]

Table 1: continued

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
10	ME	0.03965914	0.09661405	0.1725403	0.2735888	0.4175977	
	LLSM	0.03986529	0.09670542	0.1723873	0.2758946	0.4151474	
	SRPA	0.03246532	0.0880511	0.1637548	0.2734765	0.4422523	-0.02393819
		(0.00994344)	(0.01603886)	(0.02421823)	(0.03463748)	(0.04916443)	(0.02120898)
							[0.31025730]
11	ME	0.05542534	0.09114355	0.1488258	0.219417	0.4851884	
	LLSM	0.05682476	0.09340581	0.1535360	0.2197052	0.4765282	
	SRPA	0.0351334	0.06451847	0.1198285	0.1906863	0.5898333	-0.05286325
		(0.00881806)	(0.01461234)	(0.02336027)	(0.03313632)	(0.06198701)	(0.01045049)
							[0.00390515]
12	ME	0.03102907	0.04571794	0.09514932	0.3582529	0.4698508	
	LLSM	0.03096251	0.04743282	0.09326362	0.3571193	0.4712217	
	SRPA	0.008669132	0.01651892	0.04786312	0.3342732	0.5926756	-0.02912672
		(0.00188644)	(0.00362494)	(0.00865897)	(0.03490794)	(0.03790180)	(0.00190965)
							[0.00002197]
13	ME	0.03439497	0.04945492	0.1281071	0.2585371	0.529506	
	LLSM	0.03442574	0.05015303	0.1276154	0.2638769	0.523929	
	SRPA	0.01352831	0.02404054	0.08411951	0.2220423	0.6562693	-0.03111255
		(0.00249105)	(0.00421156)	(0.01162079)	(0.02530579)	(0.03453735)	(0.00240081)
							[0.00048764]
14	ME	0.118193	0.1956172	0.0786651	0.1953216	0.412203	
	LLSM	0.1002870	0.205355	0.09467973	0.1909118	0.4087665	
	SRPA	0.01919027	0.1006695	0.03702765	0.06936455	0.773748	-0.06911407
		(0.01399486)	(0.09252098)	(0.03073148)	(0.06317346)	(0.19233930)	(0.03331963)
							[0.09273423]
15	ME	0.05526368	0.07712407	0.177081	0.2290141	0.4615172	
	LLSM	0.0560352	0.07668478	0.1739160	0.2294835	0.4638806	
	SRPA	0.03296243	0.0519529	0.1436594	0.2093331	0.5620922	-0.05112254
		(0.01002867)	(0.01437588)	(0.02984952)	(0.03955660)	(0.06817072)	(0.01144384)
							[0.00659601]
16	ME	0.02631965	0.03993277	0.1049874	0.2902212	0.538539	
	LLSM	0.02676726	0.04153865	0.1027419	0.2883822	0.5405699	
	SRPA	0.004885648	0.01054933	0.04489413	0.2347508	0.7049201	-0.02514762
		(0.00238254)	(0.00520409)	(0.01844219)	(0.07011547)	(0.08014366)	(0.00363817)
							[0.00097154]
17	ME	0.03419073	0.05062252	0.1991733	0.278945	0.4370685	
	LLSM	0.03433243	0.05066689	0.1967061	0.2814804	0.4368141	
	SRPA	0.01430529	0.02698342	0.1613546	0.2791007	0.518256	-0.03366923
		(0.00313376)	(0.00521186)	(0.01723731)	(0.02596455)	(0.03311462)	(0.00286178)
							[0.00007803]

Table 1: continued

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
18	ME	0.03582386	0.04708124	0.1056337	0.2332736	0.5781877	
	LLSM	0.03583778	0.04892269	0.1120414	0.2389275	0.5642706	
	SRPA	0.01189003	0.02036089	0.0586934	0.1612880	0.7477676	-0.02999754
		(0.00254683)	(0.00433972)	(0.01119177)	(0.02640995)	(0.03846036)	(0.00245904)
							[0.00000654]
19	ME	0.03680229	0.04439121	0.1384403	0.2379025	0.5424637	
	LLSM	0.03674372	0.04409448	0.1431017	0.2425886	0.5334715	
	SRPA	0.01336694	0.02037039	0.09324938	0.1822217	0.6907915	-0.03079098
		(0.00359147)	(0.00532217)	(0.01904024)	(0.03313657)	(0.04956805)	(0.00337846)
							[0.00026628]
20	ME	0.0379017	0.09820258	0.1797497	0.2772508	0.4068952	
	LLSM	0.03817232	0.09913269	0.1804774	0.276235	0.4059826	
	SRPA	0.02910284	0.09026181	0.172718	0.2757089	0.4322084	-0.02670483
		(0.00575042)	(0.00995761)	(0.01547555)	(0.02168919)	(0.02951034)	(0.01109777)
							[0.06114102]
21	ME	0.1243080	0.1427924	0.1884158	0.2164329	0.3280509	
	LLSM	0.1243080	0.1427924	0.1884158	0.2164329	0.3280509	
	SRPA	0.09558736	0.1133592	0.1654336	0.1964584	0.4291615	-0.246865
		(0.01742286)	(0.02087584)	(0.02810807)	(0.03147202)	(0.07273991)	(0.06898243)
							[0.01589728]
22	ME	0.04697142	0.1131730	0.2076236	0.2076236	0.4246084	
	LLSM	0.04676154	0.1126123	0.2076814	0.2076814	0.4252634	
	SRPA	0.04360901	0.1100425	0.2063675	0.2063675	0.4336135	-0.01259874
		(0.00530785)	(0.00769640)	(0.01159768)	(0.01159768)	(0.02134560)	(0.01675300)
							[0.48590660]
23	ME	0.04366988	0.0751404	0.1425024	0.2676959	0.4709914	
	LLSM	0.04339587	0.0755566	0.1426641	0.2693748	0.4690087	
	SRPA	0.02899676	0.05969662	0.1239616	0.2574097	0.5299354	-0.03434922
		(0.00118869)	(0.00192361)	(0.00318029)	(0.00538030)	(0.00807965)	(0.00164328)
							[0.00046417]
24	ME	0.0711349	0.09426032	0.1474914	0.2817728	0.4053406	
	LLSM	0.07061233	0.09317353	0.1495885	0.2824493	0.4041763	
	SRPA	0.04211116	0.06736587	0.1220367	0.2665333	0.501953	-0.07312354
		(0.00836280)	(0.01217722)	(0.01788685)	(0.03074827)	(0.04583395)	(0.01105655)
							[0.00118908]
25	ME	0.03421998	0.04908188	0.1668968	0.2469100	0.5028913	
	LLSM	0.03420684	0.04942891	0.1677349	0.2507546	0.4978747	
	SRPA	0.01459324	0.02637093	0.1266597	0.2187408	0.6136354	-0.03153449
		(0.00242082)	(0.00399123)	(0.01324903)	(0.02044830)	(0.02981471)	(0.00235705)
							[0.00041742]

TABLE 1: continued

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
26	ME	0.08639348	0.06220666	0.1090458	0.2230819	0.5192722	
	LLSM	0.07642455	0.06134916	0.1018316	0.2073463	0.5530483	
	SRPA	0.07440574	0.05988523	0.09989507	0.2049374	0.5608766	-0.005273608
		(0.04860630)	(0.04390933)	(0.05721311)	(0.08543718)	(0.19929680)	(0.11580110)
							[0.96543960]
27	ME	0.1313191	0.03033239	0.2633019	0.1701323	0.4049143	
	LLSM	0.1274945	0.03797291	0.2854408	0.1312711	0.4178207	
	SRPA	0.08293897	0.01630948	0.2972459	0.1187186	0.484787	-0.04267262
		(0.0644448)	(0.02167591)	(0.16642160)	(0.08387341)	(0.20481290)	(0.02586954)
							[0.15995240]
28	ME	0.03724576	0.0672826	0.1934432	0.2536117	0.4484167	
	LLSM	0.03741405	0.06680075	0.194327	0.256416	0.4450422	
	SRPA	0.01390006	0.03619523	0.1549768	0.2374581	0.5574698	-0.03885975
		(0.00467339)	(0.01009202)	(0.02916613)	(0.04048124)	(0.05698635)	(0.00536051)
							[0.00078007]
29	ME	0.05947937	0.1136486	0.1540947	0.2672144	0.405563	
	LLSM	0.0601973	0.1136630	0.1535542	0.2673534	0.405232	
	SRPA	0.05059017	0.1066557	0.1444181	0.2668051	0.4315310	-0.03926962
		(0.01241821)	(0.01709790)	(0.02042155)	(0.03066031)	(0.04753027)	(0.03283786)
							[0.28536400]
30	ME	0.05256206	0.08387636	0.2226623	0.2330976	0.4078017	
	LLSM	0.05218784	0.0837867	0.2234816	0.2336812	0.4068626	
	SRPA	0.03594672	0.06948397	0.2169497	0.2239887	0.4536309	-0.04617491
		(0.00300314)	(0.00419048)	(0.00848960)	(0.00869274)	(0.01437468)	(0.00478404)
							[0.00020256]
31	ME	0.04132681	0.1017848	0.2112218	0.2396299	0.4060367	
	LLSM	0.04127759	0.1015226	0.2123121	0.2387972	0.4060905	
	SRPA	0.03030898	0.09157548	0.2074739	0.2334032	0.4372385	-0.03225136
		(0.00421862)	(0.00712069)	(0.01248610)	(0.01362360)	(0.02102073)	(0.00770109)
							[0.00858921]
32	ME	0.05539319	0.08967754	0.1540355	0.2534363	0.4474574	
	LLSM	0.05522681	0.08953515	0.1538905	0.2529578	0.4483897	
	SRPA	0.04321040	0.07804835	0.1417718	0.2486166	0.4883529	-0.03796704
		(0.00511773)	(0.00707535)	(0.01026570)	(0.01491932)	(0.02495058)	(0.00990710)
							[0.01222328]
33	ME	0.03735165	0.06065903	0.1170628	0.2504509	0.5344757	
	LLSM	0.03734766	0.06139029	0.1195450	0.2524533	0.5292637	
	SRPA	0.01936088	0.03813114	0.08747908	0.2102903	0.6447387	-0.03169708
		(0.00323886)	(0.00576379)	(0.01112909)	(0.02213795)	(0.03371001)	(0.00323822)
							[0.00018939]

Table 1: continued

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
34	ME	0.0985663	0.0985663	0.1763024	0.3132825	0.3132825	
	LLSM	0.09863931	0.09863931	0.1758347	0.3134433	0.3134433	
	SRPA	0.07749908	0.07749908	0.1626110	0.3411956	0.3411953	-0.117799
		(0.00000047)	(0.00000047)	(0.00000055)	(0.00000101)	(0.00000101)	(0.00000102)
							[0.00000000]
35	ME	0.07009935	0.1088047	0.1752197	0.2037169	0.4421594	
	LLSM	0.06978948	0.1083025	0.1738778	0.2044941	0.4435361	
	SRPA	0.06440575	0.1035366	0.1704232	0.2002767	0.4613577	-0.02638769
		(0.01127913)	(0.01359067)	(0.01807811)	(0.01987816)	(0.04299553)	(0.03930063)
							[0.53171210]
36	ME	0.03574621	0.09848842	0.2133273	0.2470735	0.4053646	
	LLSM	0.03573016	0.09884118	0.2143809	0.246259	0.4047887	
	SRPA	0.02717614	0.09058508	0.2098656	0.2398813	0.4324919	-0.02481461
		(0.00454401)	(0.00801239)	(0.01434302)	(0.01580614)	(0.02371722)	(0.00878543)
							[0.03691183]
37	ME	0.04986753	0.05904872	0.2106950	0.2813668	0.399022	
	LLSM	0.0504573	0.05934178	0.2115656	0.2791625	0.3994729	
	SRPA	0.07894669	0.08757942	0.2216829	0.2703329	0.3414582	0.2572728
		(0.02088205)	(0.01978443)	(0.01369105)	(0.01499456)	(0.04041080)	(0.33350180)
							[0.47531130]
38	ME	0.06005359	0.06900756	0.2260894	0.287187	0.3576624	
	LLSM	0.06050393	0.06950076	0.2255538	0.2869629	0.3574786	
	SRPA	0.06457479	0.07375251	0.2266955	0.2852187	0.3497585	0.02361672
		(0.01861330)	(0.01795956)	(0.01766713)	(0.02128090)	(0.03769352)	(0.11015600)
							[0.83870960]
39	ME	0.05098402	0.04685101	0.1199987	0.2890309	0.4931354	
	LLSM	0.05260404	0.04815507	0.121153	0.2910396	0.4870483	
	SRPA	0.02020586	0.01908973	0.06918481	0.2406134	0.6509062	-0.03917061
		(0.00429399)	(0.00405688)	(0.01230212)	(0.03225512)	(0.04162073)	(0.00331436)
							[0.00007634]
40	ME	0.04510774	0.07856545	0.1394286	0.2399333	0.4969649	
	LLSM	0.04675109	0.0776699	0.136988	0.246197	0.492394	
	SRPA	0.01801508	0.03367796	0.08453871	0.1910886	0.6726797	-0.04481222
		(0.00816371)	(0.01509117)	(0.03340520)	(0.06498136)	(0.09883743)	(0.00966985)
							[0.00566214]
41	ME	0.02964577	0.05384828	0.1717498	0.2296569	0.5150992	
	LLSM	0.02940533	0.05436467	0.1754407	0.2337654	0.5070239	
	SRPA	0.01007403	0.02676122	0.1252396	0.1958673	0.6420579	-0.02990095
		(0.00292669)	(0.00677522)	(0.02335486)	(0.03369758)	(0.05052168)	(0.00338752)
							[0.00030910]

Table 1: continued

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
42	ME	0.03335678	0.05675491	0.1321911	0.2501153	0.527582	
	LLSM	0.03320838	0.05707763	0.1352357	0.2545456	0.5199327	
	SRPA	0.01117187	0.02639897	0.08565307	0.203935	0.6728411	-0.03201326
		(0.00339976)	(0.00750210)	(0.01992268)	(0.04064223)	(0.05693059)	(0.00392342)
							[0.00044927]
43	ME	0.02990993	0.04213677	0.1473065	0.2505858	0.530061	
	LLSM	0.03046376	0.0435927	0.1492476	0.2565225	0.5201734	
	SRPA	0.009149535	0.01596293	0.08953594	0.2028206	0.682531	-0.028705
		(0.00262503)	(0.00444129)	(0.01860254)	(0.03658470)	(0.04948919)	(0.00285688)
							[0.00016709]
44	ME	0.04153540	0.06615878	0.1603843	0.2219270	0.5099946	
	LLSM	0.04254179	0.06830013	0.1606527	0.2298890	0.4986163	
	SRPA	0.02549878	0.04786479	0.1379374	0.2004536	0.5882455	-0.03575434
		(0.00809625)	(0.01305157)	(0.02861683)	(0.03804588)	(0.06539195)	(0.00929047)
							[0.01201946]
45	ME	0.06140871	0.1099535	0.20776	0.2559947	0.3648831	
	LLSM	0.06209589	0.1106923	0.2090064	0.2543041	0.3639013	
	SRPA	0.06548954	0.1127750	0.2095016	0.2536183	0.3586155	0.02167581
		(0.01561218)	(0.01510286)	(0.01975992)	(0.02200422)	(0.03738973)	(0.10067060)
							[0.83802960]
46	ME	0.07736575	0.0791885	0.1875765	0.2319334	0.4239359	
	LLSM	0.07773487	0.07939028	0.1867385	0.2326259	0.4235104	
	SRPA	0.06059107	0.05887027	0.1713362	0.2148802	0.4943223	-0.06387362
		(0.01218636)	(0.01196106)	(0.02287556)	(0.02656211)	(0.05092063)	(0.01817074)
							[0.01700615]
47	ME	0.05197311	0.07875992	0.2198869	0.2457468	0.4036332	
	LLSM	0.0517144	0.07838438	0.2214543	0.2452753	0.4031717	
	SRPA	0.03302500	0.0615924	0.2145860	0.2345289	0.4562677	-0.04833492
		(0.00459371)	(0.00645725)	(0.01380052)	(0.01475431)	(0.02313145)	(0.00635565)
							[0.00062450]
48	ME	0.04246682	0.08008809	0.2233581	0.2318353	0.4222517	
	LLSM	0.04244599	0.08014545	0.2235258	0.2318269	0.4220558	
	SRPA	0.03627382	0.07405743	0.2203308	0.2276673	0.4416707	-0.02029807
		(0.00246653)	(0.00325978)	(0.00632393)	(0.00646015)	(0.01103379)	(0.00591974)
							[0.01865811]
49	ME	0.05646834	0.1045575	0.1595529	0.2407516	0.4386697	
	LLSM	0.05677564	0.1043773	0.1582063	0.2397960	0.4408448	
	SRPA	0.04842568	0.09603027	0.1506398	0.236304	0.4686003	-0.03215048
		(0.00809881)	(0.01125355)	(0.01489357)	(0.01999069)	(0.03522961)	(0.02017440)
							[0.17190060]

Table 1: continued

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
50	ME	0.07701049	0.08687491	0.1804432	0.2588493	0.3968221	
	LLSM	0.07738633	0.08775351	0.1796905	0.2571315	0.3980382	
	SRPA	0.06545127	0.07455711	0.1682683	0.2564771	0.4352462	-0.05742063
		(0.01363891)	(0.01435853)	(0.02071041)	(0.02658462)	(0.04726433)	(0.03421988)
							[0.15419040]
51	ME	0.02619319	0.03972608	0.1091969	0.2424197	0.5824641	
	LLSM	0.02675197	0.04151492	0.1128038	0.2484714	0.5704579	
	SRPA	0.005480701	0.01154233	0.04915994	0.1661737	0.7676434	-0.02490111
		(0.00261880)	(0.00552472)	(0.02054254)	(0.05891411)	(0.07553016)	(0.00377406)
							[0.00120198]
52	ME	0.04867407	0.09002774	0.2039273	0.2315724	0.4257984	
	LLSM	0.04874113	0.08988917	0.2038626	0.2311734	0.4263337	
	SRPA	0.04449563	0.08621614	0.2015158	0.2286105	0.4391619	-0.01612111
		(0.00541855)	(0.00684143)	(0.01134404)	(0.012263050)	(0.02183840)	(0.01596779)
							[0.35901900]
53	ME	0.04930247	0.09547731	0.2065928	0.2354418	0.4131856	
	LLSM	0.04945857	0.09537517	0.2068634	0.2345761	0.4137267	
	SRPA	0.04568355	0.09195999	0.2050354	0.2325483	0.4247728	-0.01536264
		(0.00592496)	(0.00742020)	(0.01191965)	(0.01289063)	(0.02240267)	(0.01862329)
							[0.44697430]
54	ME	0.05478022	0.08545023	0.1517679	0.2496663	0.4583353	
	LLSM	0.05449409	0.08518374	0.1518488	0.2496017	0.4588718	
	SRPA	0.04265518	0.07472437	0.1395797	0.2445896	0.4984511	-0.03546789
		(0.00586220)	(0.008014250)	(0.01187927)	(0.01720338)	(0.02934512)	(0.01120431)
							[0.02493896]
55	ME	0.06785552	0.1021570	0.2057980	0.2260366	0.3981529	
	LLSM	0.06760769	0.1011599	0.2071423	0.22464	0.3994501	
	SRPA	0.05277203	0.08989176	0.2008313	0.2195110	0.4369939	-0.05809331
		(0.00850703)	(0.01071121)	(0.01706134)	(0.01816203)	(0.03260006)	(0.01930451)
							[0.02977906]
56	ME	0.05707625	0.09748253	0.2363964	0.2698964	0.3391485	
	LLSM	0.0576019	0.09795583	0.2358996	0.2709775	0.3375651	
	SRPA	0.04584295	0.0885473	0.2349682	0.278719	0.3519225	-0.0465344
		(0.01348691)	(0.01508112)	(0.02436956)	(0.02814947)	(0.03566497)	(0.03376442)
							[0.22661900]
57	ME	0.02899733	0.03771295	0.149102	0.3044785	0.4797092	
	LLSM	0.03018718	0.03970696	0.1452655	0.2982098	0.4866305	
	SRPA	0.005840981	0.009639123	0.07725994	0.2953153	0.6119446	-0.02768686
		(0.00267854)	(0.00446383)	(0.02338546)	(0.06267345)	(0.07005612)	(0.00360897)
							[0.00059968]

Table 1: continued

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
58	ME	0.06527027	0.0910018	0.1768715	0.2623762	0.4044802	
	LLSM	0.06492536	0.0907432	0.1754231	0.2658918	0.4030165	
	SRPA	0.03384953	0.05971756	0.1466381	0.2584412	0.5013535	-0.07106378
		(0.00608018)	(0.00953986)	(0.01709514)	(0.02591097)	(0.03711853)	(0.00801698)
							[0.00030384]
59	ME	0.04093548	0.2590007	0.06375906	0.2566673	0.3796375	
	LLSM	0.05272539	0.2525224	0.08865427	0.2334321	0.3726658	
	SRPA	0.01245671	0.3020141	0.05842126	0.2855809	0.341527	-0.05404574
		(0.02157024)	(0.15371160)	(0.06061038)	(0.14935060)	(0.16341240)	(0.03914380)
							[0.22590080]
60	ME	0.07568192	0.08585216	0.2115315	0.2722892	0.3546452	
	LLSM	0.0779401	0.0874453	0.2105879	0.2778724	0.3461543	
	SRPA	0.05321041	0.05362012	0.1978015	0.2850018	0.4103662	-0.09278795
		(0.02104277)	(0.02116285)	(0.03555432)	(0.04801921)	(0.06095434)	(0.02309015)
							[0.01013575]
61	ME	0.1096824	0.1938939	0.2193648	0.2576942	0.2193648	
	LLSM	0.1106379	0.1926317	0.2212757	0.2541790	0.2212757	
	SRPA	0.07952625	0.1930009	0.2284706	0.2705317	0.2284706	-0.2947341
		(0.03851621)	(0.02142326)	(0.02495234)	(0.02771683)	(0.02495234)	(0.09127065)
							[0.02322567]
62	ME	0.03269941	0.03914843	0.1140874	0.3051881	0.5088767	
	LLSM	0.03337364	0.04037852	0.1106154	0.3040078	0.5116247	
	SRPA	0.006949039	0.01107536	0.05270659	0.2746094	0.6546596	-0.02845276
		(0.00269183)	(0.00435183)	(0.01640982)	(0.05781842)	(0.06466492)	(0.00332706)
							[0.00036006]
63	ME	0.03844505	0.05584371	0.1369972	0.3025894	0.4661246	
	LLSM	0.03862487	0.05676696	0.1327747	0.3011241	0.4707094	
	SRPA	0.01459137	0.02665779	0.09053726	0.3079189	0.5602947	-0.03693221
		(0.00414683)	(0.00722789)	(0.01783269)	(0.04349846)	(0.05133930)	(0.00406733)
							[0.00027101]
64	ME	0.0869807	0.1388629	0.2268645	0.3107637	0.2365281	
	LLSM	0.0891365	0.1383261	0.2273738	0.3177899	0.2273738	
	SRPA	0.1422412	0.1676223	0.2271497	0.2441337	0.2188530	5.990984
		(0.08644416)	(0.04511194)	(0.03884423)	(0.07676963)	(0.02478278)	(36.88026)
							[0.87731700]
65	ME	0.03230626	0.0683695	0.1387820	0.2783662	0.4821760	
	LLSM	0.03144746	0.0664103	0.1329390	0.29329	0.4759132	
	SRPA	0.0064899	0.02309557	0.07802701	0.2547423	0.6376453	-0.0320199
		(0.00312160)	(0.01031742)	(0.02781419)	(0.06807117)	(0.08332007)	(0.00544111)
							[0.00201309]

Table 1: continued

Subj.	Model	w_1	w_2	w_3	w_4	w_5	β_3
66	ME	0.04890132	0.07312151	0.2399679	0.2399679	0.3980414	
	LLSM	0.04862182	0.07275175	0.2411323	0.2411323	0.3963617	
	SRPA	0.02682540	0.05159574	0.2305677	0.2305680	0.4604432	-0.04860295
		(0.00224534)	(0.00337201)	(0.00844658)	(0.00844659)	(0.01289868)	(0.00252430)
							[0.00000697]
67	ME	0.04743928	0.09487856	0.1791066	0.3203622	0.3582133	
	LLSM	0.04748162	0.09496324	0.1793072	0.3196335	0.3586144	
	SRPA	0.04534373	0.09312256	0.1779058	0.3224452	0.3611828	-0.008949557
		(0.00568909)	(0.00666839)	(0.00937088)	(0.01409979)	(0.01623938)	(0.02002046)
							[0.67355870]
68	ME	0.05116532	0.09628559	0.2083451	0.2376999	0.4065041	
	LLSM	0.05128775	0.09629617	0.2088610	0.2368413	0.4067138	
	SRPA	0.04362916	0.08945698	0.2051869	0.2325557	0.4291713	-0.02889702
		(0.00559622)	(0.00730799)	(0.01208610)	(0.01315172)	(0.02215205)	(0.01447903)
							[0.10248990]
69	ME	0.0891051	0.09826153	0.1767079	0.3533593	0.2825662	
	LLSM	0.09159645	0.09702114	0.1729501	0.3541452	0.2842871	
	SRPA	0.05170118	0.04194808	0.1322603	0.3789344	0.395156	-0.1121476
		(0.01639990)	(0.01288315)	(0.02924278)	(0.04825235)	(0.04892556)	(0.02198803)
							[0.00376891]

In brackets standard errors; in square brackets p-values for estimates of β_3.