The analytic rank of $J_0(q)$ and zeros of automorphic L-functions

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Abstract

We study, on average over f, zeros of the L-functions of primitive weight two forms of level q (fixed).

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1 Introduction

This paper is motivated by the conjecture of Birch and Swinnerton-Dyer relating the rank the Mordell-Weil group of an abelian variety defined over a number field with (in its crudest form) the order of vanishing of its Hasse-Weil L-function at the central critical point. Mestre [Mes] started the study of the implications of this conjecture towards providing upper-bounds for the rank, using "explicit formulae" similar to that of Riemann-Weil, assuming the analytic continuation and, more significantly perhaps, the Riemann Hypothesis for those L-functions.

Brumer [Br1] first studied the special case of the Jacobian variety $J_0(q)$ of the modular curve $X_0(q)$, which is defined over $\mathbf{Q}$, of dimension $\gg q$. Here analytic continuation is known, by work of Eichler and Shimura [Sh1], and assuming only the Riemann Hypothesis for the L-functions of automorphic forms (of weight 2 and level q), Brumer proved

$$\operatorname{rank}_a J_0(q) \leq \Big(\frac{3}{2} + o(1)\Big) \dim J_0(q)$$

and conjectured that

$$\operatorname{rank} J_0(q) = \operatorname{rank}_a J_0(q) \sim \frac{1}{2} \dim J_0(q)$$

(based on the fact that the sign of the functional equation for the automorphic L-functions of weight 2 and level q is approximately half the time $+1$ and half the time -1).

Other authors, notably Murty [Mur], considered the same problem, improving the constant 3/2 occuring. Most recently, Luo, Iwaniec and Sarnak [ILS], with the same assumptions, have proved an estimate

$$\operatorname{rank}_a J_0(q) \le (c + o(1)) \dim J_0(q)$$

for some (computable) constant $c < 1$. This turns out to be quite significant in light of the quite general conjectures of Katz-Sarnak [KS] on the distribution of zeros of families of L-functions.

This paper approaches the same problem of the analytic rank of $J_0(q)$ with a different emphasis: we wish to avoid all assumptions about the L-functions involved, and obtain a bound of the correct order of magnitude. Indeed, we prove

Theorem 1. *There exists and absolute and effective constant $C > 0$ such that for any prime number q,*

$$\operatorname{rank}_a J_0(q) \le C \dim J_0(q).$$

If the Birch and Swinnerton-Dyer conjecture holds for $J_0(q)$, then

$$\operatorname{rank} J_0(q) \le C \dim J_0(q).$$

This theorem provides the first known unconditional bound for the analytic rank of a family of L-functions which is of the correct order of magnitude, without using the Generalized Riemann Hypothesis. We were inspired by the unconditional bounds for the analytic rank of twists of elliptic curves obtained by Perelli and Pomykala [PP].

Remark Much progress have been done since in the study of the rank of $J_0(q)$ since the original proof of this result [KM1], [KM2].

1. The constant C can actually be computed; using the methods of this paper, with some improvements to obtain a better result, we have proved [KM3] that we can take $C = 6.5$ in this result. Moreover, work in progress, in collaboration with J. Vanderkam [KMV1], [KMV2], based on different techniques and much more sophisticated arguments, promises to show that $C = 1.18191$ is attainable; observe that this is better than Brumer's original bound using the Riemann Hypothesis.

2. No remotely comparable upper bound for $\operatorname{rank} J_0(q)$ seems to be accessible by algebraic means today (descents, trying to get control of the Selmer group, etc...).

The starting point of this work is the factorization of the Hasse-Weil zeta function of $J_0(q)$, due to Eichler and Shimura [Sh1] (completed by Carayol at the bad primes)

$$L(J_0(q), s) = \prod_{f \in S_2(q)^*} L(f, s + 1/2). \tag{1}$$

where f ranges over the finite set $S_2(q)^*$ of primitive forms (newforms) f of weight 2 and level q, and $L(f, s)$ is the corresponding Hecke L-function, normalized so that the critical line is $\operatorname{Re}(s) = \frac{1}{2}$.

Hence the order of vanishing of the L-function of $J_0(q)$ at $s = \frac{1}{2}$ is the sum of the order of vanishing of the Hecke L-functions at $s = \frac{1}{2}$,

$$\operatorname{rank}_a J_0(q) = \sum_{f \in S_2(q)^*} \operatorname{ord}_{s=\frac{1}{2}} L(f, s)$$

and if the Birch and Swinnerton-Dyer conjecture holds, then

$$\operatorname{rank} J_0(q) = \sum_{f \in S_2(q)^*} \operatorname{ord}_{s=\frac{1}{2}} L(f, s).$$

Thus our main theorem is equivalent with

Theorem 2. *There exists an absolute and effective constant $C > 0$ such that for any prime number q we have*

$$\sum_{f \in S_2(q)^*} \operatorname{ord}_{s=\frac{1}{2}} L(f, s) \leq C|S_2(q)^*|.$$

The strategy that we use is based on the explicit formula, except that a much tighter control of the possible zeros outside the critical line is required. This is obtained by means of the Density Theorem 4 for zeros of automorphic L-functions with imaginary parts as close as $1/\log q$, which is the crucial scale in this problem.[1] This density theorem is analogue to one proved by Selberg [Sel] for Dirichlet characters, and is based on the study of a mollified second moment of values of the L-functions close to the critical line (see below for details).

This proof is carried out in Sections 4 and 5, after some (important) preliminary result in Section 3. In Section 6, we show how the same ideas can be applied to prove a non-vanishing theorem for the central critical value $L(f, \frac{1}{2})$.

Theorem 3. *For any $\varepsilon > 0$ and any q prime large enough (in terms of ε) we have*

$$|\{f \in S_2(q)^* \mid L(f, \tfrac{1}{2}) \neq 0\}| \geq \Big(\frac{1}{6} - \varepsilon\Big)|S_2(q)^*|.$$

Earlier, Duke [Du] had proved that the number of forms f with $L(f, \frac{1}{2}) \neq 0$ was $\gg q/(\log q)^2$. Independently, Vanderkam [Vdk] has also proved that there is a positive proportion (although with smaller constant) of forms with $L(f, \frac{1}{2}) \neq 0$.

This provides a lower-bound for the dimension of the winding quotient of Merel [Me], in particular the work of Kolyvagin-Logachev implies that there is a quotient of $J_0(q)$ defined over $\mathbf{Q}$ with finite Mordell-Weil group and dimension $\geq (1/6 + o(1)) \dim J_0(q)$.

Iwaniec and Sarnak [IS] have proved that $1/6$ could be replaced by $1/4$, and that any constant $> 1/4$ (with some additional lower-bound on $L(f, \frac{1}{2})$, which they prove holds for $1/4$) would prove that Landau-Siegel zeros do not exist for Dirichlet L-functions of quadratic characters.

Notice to the reader and Acknowledgements. The first version of this paper [KM1], [KM2] was written almost two years ago, but publication was unfortunately delayed. The current text is based on [Kow], and contains the results which are the foundations of the more recent works of Vanderkam and the authors. Some density theorems of independent interest which were originally part of [KM2] will be published separately.

As before, we wish to thank É. Fouvry and H. Iwaniec for numerous discussions and suggestions.

2 Notations and preliminaries

Throughout this paper, unless otherwise specified, q is fixed (large) prime.

Let $S_2(q)$ (resp. $S_2(q)^*$) be the space (resp. the finite set) of holomorphic weight 2 cusp forms of level q (resp. primitive weight 2 forms of level q). Recall that

$$\dim S_2(q) = \dim J_0(q) = |S_2(q)^*| \sim \frac{q}{12},$$

the last equality because, as $S_2(1) = 0$, there are no old forms of level q and weight 2 (we use here that q is prime).

[1] This is because we expect that on average the number of zeros of $L(f, s)$ in a neighborhood of $1/2$ of this size is absolutely bounded.

We now list notations and facts which will be used extensively in the sequel. Let $f \in S_2(q)^*$ be given. We write $\lambda_f(n)$ for its Hecke eigenvalues, which also give the Fourier expansion of f at infinity:

$$f(z) = \sum_{n\geq 1} n^{1/2}\lambda_f(n)e(nz), \quad \text{with } \lambda_f(1) = 1,\ \lambda_f(n) \in \mathbf{R}. \tag{2}$$

Deligne's bound (in the particular case of weight 2 this is due to Eichler-Shimura) for the coefficients of holomorphic cusp forms takes the form

$$|\lambda_f(n)| \leq \tau(n). \tag{3}$$

Recall that the Hecke L-function of a primitive form f is defined by

$$L(f,s) = \sum_{n\geq 1} \lambda_f(n)n^{-s} = \prod_p (1 - \lambda_f(p)p^{-s} + \varepsilon_q(p)p^{-2s})^{-1} \tag{4}$$

It satisfies the following functional equation: let

$$\Lambda(f,s) = \Bigl(\frac{\sqrt{q}}{2\pi}\Bigr)^s \Gamma(s+\tfrac{1}{2})L(f,s),$$

then

$$\Lambda(f,s) = \varepsilon_f \Lambda(f, 1-s),\ \varepsilon_f = \pm 1, \tag{5}$$

For q prime (more generally, if q is squarefree), ε_f is given by

$$\varepsilon_f = -\mu(q)q^{1/2}\lambda_f(q). \tag{6}$$

The Euler product representation is equivalent with the following multiplicativity property of the coefficients $\lambda_f(n)$: for any integers $n \geq 1$, $m \geq 1$,

$$\lambda_f(n)\lambda_f(m) = \sum_{d|(n,m)} \varepsilon_q(d)\lambda_f\Bigl(\frac{nm}{d^2}\Bigr). \tag{7}$$

In particular, $n \mapsto \lambda_f(n)$ is multiplicative and $\lambda_f(\delta m) = \lambda_f(\delta)\lambda_f(m)$ if $\delta \mid q$. The formula (7), by Möbius inversion, yields another useful formula

$$\lambda_f(nm) = \sum_{d|(n,m)} \varepsilon_q(d)\mu(d)\lambda_f\Bigl(\frac{n}{d}\Bigr)\lambda_f\Bigl(\frac{m}{d}\Bigr). \tag{8}$$

If p is a prime $\neq q$, we write $1 - \lambda_f(p)X + X^2 = (1-\alpha_p X)(1-\beta_p X)$, so

$$\lambda_f(p) = \alpha_p + \beta_p. \tag{9}$$

The bound (3) is equivalent (for n coprime with the level) with the assertion that $|\alpha_p| = 1$ for all $p \neq q$. For $p = q$, the p-factor of $L(f,s)$ is of degree at most 1, and we let $\alpha_p = \lambda_f(p)$, which is shown to be of modulus at most 1 (actually, smaller), and $\beta_p = 0$.

In addition, we require the Dirichlet series expansion for the logarithmic derivative of $L(f,s)$. From the Euler product, using the factorization of the local factors, it follows

$$-\frac{L'}{L}(f,s) = \sum_{n\geq 1} b_f(n)\Lambda(n)n^{-s} \tag{10}$$

with coefficients given by

$$b_f(n) = \begin{cases} 0, \text{ if } n \text{ is not a power of a prime,} \\ \alpha_p^m + \beta_p^m, \text{ if } n = p^m. \end{cases} \tag{11}$$

We introduce the notation

$$\omega_f = \frac{1}{4\pi(f,f)} \tag{12}$$

where $(\cdot,\cdot)$ denotes the Petersson inner product on $S_2(q)$,

$$(f,g) = \int_{\Gamma_0(q)\backslash\mathbf{H}} f(z)\overline{g(z)}\frac{dxdy}{y^2}$$

for any non-zero cusp form $f \in S_2(q)$ (we call this the "harmonic weight") and define the summation symbol $\sum^h$ by

$$\sum_{f\in S_2(q)^*}^h \alpha_f = \sum_{f\in S_2(q)^*} \omega_f \alpha_f$$

for any family (α_f) of complex numbers. Because (f,f) is of size about $\operatorname{Vol} X_0(q)$, so about $\dim J_0(q)$, this behaves asymptotically like a probability measure, i.e. we have

$$\sum_{f\in S_2(q)^*}^h 1 \sim 1$$

as q tends to infinity. This weight is fundamental to our work, because of the following formula, due to Petersson (see [Iw] for instance), which expresses the so-called Δ-symbol

$$\Delta(m,n) = \sum_{f\in S_2(q)^*}^h \lambda_f(m)\lambda_f(n)$$

in a very convenient way for further analytical manipulations:

$$\Delta(m,n) = \delta(m,n) - 2\pi \sum_{q|c} c^{-1} S(m,n;c) J_1\Big(\frac{4\pi\sqrt{mn}}{c}\Big)$$

where J_1 is the Bessel function and $S(m,n;c)$ is a classical Kloosterman sum. Using the estimates

$$|S(m,n;c)| \leq \tau(c)(m,n,c)^{1/2}\sqrt{c}, \quad J_1(x) \ll x$$

(the first being Weil's bound for Kloosterman sums), we derive

$$\Delta(m,n) = \delta(m,n) + O((m,n,q)(\log(m,n))^2(mn)^{1/2}q^{-3/2}). \tag{13}$$

Similarly, using the formula (6) and multiplicativity we have

$$\Delta'(m,n) = \sum_{f\in S_2(q)^*}^h \varepsilon_f \lambda_f(m)\lambda_f(n) = O\Big(\frac{\sqrt{mn}}{q}(\log q)^2\Big) \tag{14}$$

for m, $n < q$ (see [KM3], [Kow] for instance). We will not need any better bounds in this work (compare [KM3], [IS] for cases where a more precise analysis with the Kloosterman sums is needed).

3 From harmonic average to natural average, I

3.1 Averages

We will be dealing quite extensively with sums over $f \in S_2(q)^*$. The following notations are designed to emphasize the underlying structure. We usually suppose given a family $\alpha = (\alpha_f)$ of complex numbers, defined for all forms $f \in S_2(q)^*$, q being any level, or maybe restricted to squarefree or prime levels. We then introduce the "natural" averaging operator

$$A[\alpha] = \sum_{f \in S_2(q)^*} \alpha_f$$

where we only sum over forms of a fixed level, and consider the behavior of $A[\alpha]$ as a function of the level q, asymptotically as q gets large.

Similarly, we define the "harmonic" averaging operator

$$A^h[\alpha] = \sum_{f \in S_2(q)^*}^{h} \alpha_f.$$

Suppose we have a family $\alpha = (\alpha_f)$ of complex numbers, for all $f \in S_2(q)^*$ with prime level q, and that we know the behavior of the weighted sum

$$A^h[\alpha] = \sum_{f \in S_2(q)^*}^{h} \alpha_f$$

(for instance, we have an asymptotic formula for q going to infinity), but wish to obtain the same information for the natural sum

$$A[\alpha] = \sum_{f \in S_2(q)^*} \alpha_f.$$

Since, by Petersson's formula (13) for $m = n = 1$, $A^h[1] = 1 + O(q^{-3/2})$, we expect that when α is well-distributed and not biased against the Petersson inner-product (or, what amounts to the same thing, against the value of the symmetric square at $s = 1$), we should have

$$A[\alpha] \sim \dim J_0(q) A^h[\alpha]$$

meaning that $L(\mathrm{Sym}^2 f, 1)$ and α_f act here as independent random variables would, with the average of $L(\mathrm{Sym}^2 f, 1)$ equal to the obvious constant factor $\zeta_q(2)$ (which is equivalent to $\zeta(2)$ as q tends to infinity).

In this section we build a method to approach this problem, and – using the mean-value estimate established before – prove a result which solves part of the problem for quite general vectors α. This reduces to another estimate which has to be supplied independently in each case.

3.2 The symmetric square

The harmonic weight ω_f, which is required to express the Δ-symbol of the modular forms by Petersson's formula, is related to the special value of the symmetric square L-function at $s = 1$, which is the edge of the critical strip (in our "analytic" normalization). This is essentially due to Shimura [Sh3].

The symmetric square L-function of f is the Dirichlet series $L(\mathrm{Sym}^2 f, s)$ defined by

$$L(\mathrm{Sym}^2 f, s) = \zeta_q(2s) \sum_{n\geq 1} \lambda_f(n^2) n^{-s}. \tag{15}$$

We write $\rho_f(n)$ for the coefficients of this Dirichlet series. The relation we seek is given by the following formula

$$4\pi(f,f) = \frac{\dim J_0(q)}{\zeta(2)} L(\mathrm{Sym}^2 f, 1) + O((\log q)^3) \tag{16}$$

(uniformly in f as the prime q tends to infinity; for a proof see [Kow] for instance).

The following lemma summarizes some properties of the coefficients $\rho_f(n)$.

Lemma 1. *For any $n \geq 1$ we have*

$$\rho_f(n) = \sum_{\ell m^2 = n} \varepsilon_q(m) \lambda_f(\ell^2) \tag{17}$$

$$\lambda_f(n^2) = \sum_{\ell m^2 = n} \mu(m) \varepsilon_q(m) \rho_f(\ell) \tag{18}$$

and in particular $\rho_f(n) = \lambda_f(n^2)$ for n squarefree. Moreover, $L(\mathrm{Sym}^2 f, s)$ has an Euler product expansion of degree 3

$$L(\mathrm{Sym}^2 f, s) = \prod_{(p,q)=1} (1 - \alpha_p^2 p^{-s})^{-1} (1 - p^{-s})^{-1} (1 - \alpha_p^{-2} p^{-s})^{-1} \prod_{p|q} (1 - \alpha_p^2 p^{-s})^{-1}$$

where α_p is as in (9). Finally, for all $n \geq 1$ we have

$$|\rho_f(n)| \leq \tau(n)^2. \tag{19}$$

The last estimate is proved using Deligne's bound $|\lambda_f(n)| \leq \tau(n)$ and the Euler product.

Lemma 2. *For all q prime and all $f \in S_2(q)^*$, we have*

$$L(\mathrm{Sym}^2 f, 1) \ll (\log q)^3 \tag{20}$$

and

$$L(\mathrm{Sym}^2 f, 1) \gg (\log q)^{-1} \tag{21}$$

where the implied constants are absolute in both cases.

The (deeper) lower-bound is the main result of [GHL]; the fact that q is prime ensures that f is not a monomial form. The upper bound is much easier and well-known. In particular we have uniformly for $f \in S_2(q)^*$

$$\omega_f \ll \frac{\log q}{q}. \tag{22}$$

We require a property of "almost-orthogonality" of the coefficients of the symmetric square L-functions of the forms $f \in S_2(q)^*$. It is implicitly contained in the second part of [D-K], where it was developed for other applications.

Proposition 1. *Let $q \geq 1$ be any squarefree integer, and let $N \geq q^9$ a real number. The inequality*

$$\sum_{f \in S_2(q)^*} \Big| \sum_{n \leq N} a_n \rho_f(n) \Big|^2 \ll N(\log N)^{15} \sum_{n \leq N} |a_n|^2 \tag{23}$$

holds for any finite family $(a_n)_{1 \leq n \leq N}$ of complex numbers, with an absolute implied constant.

We deduce the following corollary:

Corollary 1. *Let $N \geq q^9$ be a real number and $(a(n))_{n \sim N}$ any complex numbers which satisfy*

$$a(n) \ll \frac{(\tau(n) \log n)^A}{n}$$

for some constant $A > 0$. There exists a constant $D = D(A) \geq 0$ such that

$$\sum_{f \in S_2(q)^*} \Big| \sum_{n \sim N} a(n) \lambda_f(n^2) \Big|^2 \ll (\log N)^D$$

(with an absolute implied constant).

Proof. The point is, of course, that the assumption on the a_n means that we are essentially "on the line $\mathrm{Re}\,(s) = 1$" (or beyond), and in this region the symmetric square behaves as the series

$$\sum_{n \geq 1} \lambda_f(n^2) n^{-s}.$$

In exacting details, we have from (17)

$$\begin{aligned}
\sum_{f \in S_2(q)^*} \Big| \sum_{n \sim N} a(n) \lambda_f(n^2) \Big|^2 &= \sum_{f \in S_2(q)^*} \Big| \sum_{n \sim N} \sum_{\ell m^2 = n} \mu(m) \varepsilon_q(m) \rho_f(\ell) a(n) \Big|^2 \\
&= \sum_{f \in S_2(q)^*} \Big| \sum_{\ell \leq 2N} \rho_f(\ell) \tilde{a}(\ell) \Big|^2
\end{aligned}$$

where

$$\tilde{a}(\ell) = \sum_{\sqrt{\frac{N}{\ell}} < m \leq \sqrt{\frac{2N}{\ell}}} \mu(m) \varepsilon_q(m) a(\ell m^2).$$

Now we derive from the assumption a bound

$$\tilde{a}(\ell) \ll (N\ell)^{-1/2} (\log \ell)^D,$$

(for some $D \geq 0$, with an absolute implied constant), hence the result on applying the mean-value estimate of Proposition 1 to the coefficients $\tilde{a}(\ell)$.

□

3.3 Removing the harmonic weight

We assume that $\alpha = (\alpha_f)$ satisfies the conditions

$$A^h[|\alpha_f|] \ll (\log q)^A \qquad \text{(for some absolute } A > 0\text{)} \tag{24}$$

$$\operatorname*{Max}_{f \in S_2(q)^*} |\omega_f \alpha_f| \ll q^{-\delta} \qquad \text{(for some } \delta > 0\text{)} \tag{25}$$

as the level q (prime) tends to infinity.

Remark Neither of these conditions is very restrictive in practice: the first one is interpreted as saying that $|\alpha_f|$ is "almost" bounded, and can often be achieved by some normalization. If this is true, the second condition is fairly reasonable since we have shown in (22) that $\omega_f \ll (\log q)q^{-1}$. In other words, by normalizing if necessary, both conditions can be expected to hold whenever the size of α_f doesn't increase or oscillate wildly.

We write the unweighted average as a weighted one and replace the Petersson inner product by the special value of the symmetric square (16):

$$\begin{aligned} A[\alpha] &= \sum_{f \in S_2(q)^*}^{h} 4\pi (f, f) \alpha_f \\ &= \frac{\dim J_0(q)}{\zeta(2)} \sum_{f \in S_2(q)^*}^{h} L(\mathrm{Sym}^2 f, 1)\alpha_f + O((\log q)^3 A^h[|\alpha_f|]). \end{aligned} \tag{26}$$

We wish to replace the value of the symmetric square by a partial sum of the Dirichlet series. This can be done by a long enough sum, of length y say. Then the sum above is essentially a finite sum of averages over the α_f, twisted by symmetric square coefficients $\rho_f(n)$:

$$\sum_{n \leq y} \frac{1}{n} A^h[\rho_f(n)\alpha_f].$$

If by any chance the methods which give us control over the average $A^h[\alpha_f]$ (corresponding to $n = 1$) also apply to the twisted ones, *in the range* $n < y$, then we are done. Unfortunately, in applications this will only be the case for very small values of y, say $y = q^\delta$ for very small $\delta > 0$. On the Riemann hypothesis (more precisely, the Lindelöf hypothesis suffices for this purpose) we can recover the L-function from such a short sum but individually we can only do this with y much larger ($y = q^2$ or maybe $y = q$), and indeed too large for our applications.

But we can exploit the average over f involved, by means of the mean-value estimate of Proposition 1. The fact that this requires also a long sum in n is not a problem here because we are looking at the symmetric square at a point on the edge of the critical strip, where the Dirichlet series almost converges absolutely. Then the "extra length" needed to enter the effective range of n for the mean-value estimate will not matter, much as the partial sums

$$\sum_{n < q^\delta} n^{-1}$$

of the harmonic series are of the same size as q tends to infinity for any fixed $\delta > 0$.

Now we implement this idea. Let therefore $\alpha = (\alpha_f)_{f \in S_2(q)^*}$ be given for all q prime, satisfying the conditions (24). Since the conductor of $\mathrm{Sym}^2 f$ for $f \in S_2(q)^*$ is q^2, the functional equation and the usual estimates give the approximation

$$L(\mathrm{Sym}^2 f, 1) = \omega_f(y) + O(q^2 y^{-1}) \tag{27}$$

(with an absolute implied constant), where

$$\omega_f(y) = \sum_{n \le y} \rho_f(n) n^{-1}.$$

We assume $\log y = O(\log q)$, say $y < q^{10}$.

Now let $x < y$ be given. The partial sum is further decomposed as

$$\omega_f(y) = \omega_f(x) + \omega_f(x, y)$$

where

$$\omega_f(x, y) = \sum_{x < n \le y} \rho_f(n) n^{-1}.$$

We consider here the weighted average built with the tail, namely

$$A^h[\omega_f(x, y)\alpha_f] = \sum_{f \in S_2(q)^*}^{h} \omega_f(x, y)\alpha_f.$$

We will use Hölder's inequality to separate $\omega_f(x, y)$ and α_f. The former is handled by the following lemma.

Lemma 3. *Let $r \ge 1$ be an integer, such that $x^r \ge q^{11}$. There exists a positive constant $C = C(r) > 0$ such that*

$$A[\omega_f(x, y)^{2r}] \ll (\log q)^C$$

where the implied constant is absolute.

The proof starts with some other lemmas. We say that an integer n is squarefull if for any prime p dividing n, p^2 divides n; in other words, for all p dividing n, the valuation of p in n is at least 2. Notice that

$$\sum_{n \text{ squarefull}} n^{-s} = \prod_p (1 + p^{-2s} + p^{-3s} + \ldots)$$

which converges absolutely for $\mathrm{Re}\,(s) > \frac{1}{2}$, hence we have

$$\sum_{\substack{n \text{ squarefull} \\ n > z}} n^{-1} \ll z^{-1/2} \tag{28}$$

with an absolute implied constant.

Lemma 4. *For any integer $r \ge 1$ and any $f \in S_2(q)^*$, we can write*

$$\omega_f(x, y)^r = \sum_{x^r < mn \le y^r} \lambda_f(m^2) \frac{c(m, n)}{mn} \tag{29}$$

with $c(m,n)=0$ unless n can be written

$$n = dn_1, \ \text{with } d \mid m, \ n_1 \text{ squarefull.} \tag{30}$$

and there exists $\gamma = \gamma(r) > 0$ such that

$$|c(m,n)| \leq \tau(mn)^{\gamma}.$$

Moreover, the coefficients c depend on r, x and y but not on the form f.

Proof. We proceed by induction on r. For $r=1$, we write by (17)

$$\begin{aligned}\omega_f(x,y) &= \sum_{x<n\leq y} \frac{1}{n} \sum_{\ell m^2 = n} \varepsilon_q(m)\lambda_f(\ell^2) \\ &= \sum_{x<\ell m^2\leq y} \lambda_f(\ell^2)\frac{\varepsilon_q(m)}{\ell m^2}\end{aligned}$$

so we can take $c(\ell,m)=0$ unless m is square and $c(\ell,m^2)=\varepsilon_q(m)$.

Assume that (29) holds for some r and s as claimed, with coefficients c (for r) and c' (for s). Then

$$\begin{aligned}\omega_f(x,y)^{r+s} &= \sum_{\substack{x^r<m_1n_1\leq y^r\\ x^s<m_2n_2\leq y^s}} \lambda_f(m_1^2)\lambda_f(m_2^2)\frac{c(m_1,n_1)c'(m_2,n_2)}{m_1n_1m_2n_2} \\ &= \sum_{\substack{x^r<m_1n_1\leq y^r\\ x^s<m_2n_2\leq y^s}} \sum_{\substack{d\mid m_1^2\\ d\mid m_2^2}} \lambda_f\Big(\frac{m_1^2m_2^2}{d^2}\Big)\frac{\varepsilon_q(d)c(m_1,n_1)c'(m_2,n_2)}{m_1n_1m_2n_2}\end{aligned}$$

by multiplicativity for λ_f.

Now d can be written uniquely as $d=d_1d_2^2$ with d_1 squarefree and then we have $d \mid m^2$ if and only if $d_1d_2 \mid m$. Therefore we can write

$$\begin{cases} m_1 = d_1d_2m_1' \\ m_2 = d_1d_2m_2' \end{cases}$$

and then

$$\omega_f(x,y)^{r+s} = \sum_{\substack{x^r<d_1d_2m_1'n_1\leq y^r\\ x^s<d_1d_2m_2'n_2\leq y^s}} \lambda_f\big((d_1m_1'm_2')^2\big)\frac{\varepsilon_q(d_1d_2)c(d_1d_2m_1',n_1)c'(d_1d_2m_2',n_2)}{(d_1d_2)^2m_1'm_2'n_1n_2}.$$

Now write $m_0 = d_1m_1'm_2'$, $n_0 = d_1d_2^2n_1n_2$. By the induction hypothesis we see that if $c(m_1,n_1)\neq 0$ and $c'(m_2,n_2)\neq 0$, then n_0 can be written as $\delta n_0'$ with $\delta \mid m_0$ and n_0' squarefull (this is not absolutely obvious because m_1m_2 does not divide m_0, but the extra prime divisors can be pushed to the squarefull part).

Estimating rather trivially the multiplicity of representation of m_0, we find the desired representation. This immediately concludes the induction. □

Lemma 5. *Let $z \geq 1$ be given and the coefficients $c(m,n)$ be as in lemma 4 for r. Then there exists $A = A(r) > 0$ such that*

$$\sum_{\substack{x^r < mn \leq y^r \\ n > z}} \lambda_f(m^2) \frac{c(m,n)}{mn} = O\big(z^{-1/2} (\log qz)^A\big).$$

Proof. By Deligne's bound we have

$$\sum_{\substack{x^r < mn \leq y^r \\ n > z}} \lambda_f(m^2) \frac{c(m,n)}{mn} \leq \sum_{x^r < m \leq y^r} \frac{\tau(m)}{m} \sum_{\substack{x^r m^{-1} < n \leq y^r m^{-1} \\ n > z}} \frac{|c(m,n)|}{n}$$

but using the condition on the support of $c(m,n)$, the inner sum is

$$\sum_{\substack{x^r m^{-1} < n \leq y^r m^{-1} \\ n > z}} \frac{|c(m,n)|}{n} \leq \tau(m)^\gamma \sum_{d|m} \frac{1}{d} \sum_{\substack{n \text{ squarefull} \\ dn > z}} \frac{\tau(n)^\gamma}{n}$$
$$\ll \tau(m)^{\gamma+1} z^{-1/2} (\log z)^A$$

(by (28) and the result follows. □

Lemma 6. *There exists a real number M such that $x^r z^{-1} < M \leq y^r z$, and real numbers $c(m)$ such that we have*

$$\sum_{f \in S_2(q)^*} \omega_f(x,y)^{2r} \ll (\log qz)^B \sum_{f \in S_2(q)^*} \Big| \sum_{m \sim M} \lambda_f(m^2) \frac{c(m)}{m} \Big|^2 + O(qz^{-1/2} (\log qz)^B)$$

and

$$|c(m)| \leq \tau(m)^C (\log qm)^C$$

for some $C > 0$.

Proof. By the previous lemma

$$\omega_f(x,y)^{2r} = \sum_{n \leq z} \Big| \sum_{x^r < mn \leq y^r} \lambda_f(m^2) \frac{c(m,n)}{mn} \Big| + O(qz^{-1/2} (\log qz)^A).$$

Write $\xi_n = \operatorname{sign}\Big(\sum_{x^r < mn \leq y^r} \lambda_f(m^2) \frac{c(m,n)}{mn}\Big)$, split the summation over dyadic intervals in m, then use Cauchy's inequality and sum over f: the result follows for some M with

$$c(m) = \sum_{x^r m^{-1} < n \leq z} \xi_n \frac{c(m,n)}{n} \ll \tau(m)^C (\log qm)^C$$

for some $C > 0$, as desired. □

This now easily implies Lemma 3: we take $z = q^2$, then the assumption $x^r \geq q^{11}$ implies that $M \geq q^9$ and we may appeal to the mean-value estimate of Corollary 1 to bound the first term, with $\log M \ll \log q$.

Proposition 2. *Let (α_f) be complex numbers satisfying conditions (24), and $x = q^\kappa$ for some $\kappa > 0$. There exists an absolute constant $\gamma = \gamma(\kappa, \delta) > 0$ (δ the exponent in (24)) such that*

$$A^h[\omega_f(x,y)\alpha_f] \ll q^{-\gamma}$$

and

$$A[\alpha_f] = \frac{\dim J_0(q)}{\zeta(2)} A^h[\omega_f(x)\alpha_f] + O(q^{1-\gamma}).$$

Proof. Let $r \geq 1$ be any integer. By Hölder's inequality we have (with s the complementary exponent to $2r$, $(2r)^{-1} + s^{-1} = 1$)

$$\begin{aligned}
A^h[\omega_f(x,y)\alpha_f] &= \sideset{}{^h}\sum_{f\in S_2(q)^*} \omega_f(x,y)\alpha_f \\
&= \sum_{f\in S_2(q)^*} \omega_f\omega_f(x,y)\alpha_f \\
&\leq A[\omega_f(x,y)^{2r}]^{\frac{1}{2r}} \Big(\sum_{f\in S_2(q)^*} (\omega_f|\alpha_f|)^s \Big)^{\frac{1}{s}} \\
&\leq A^{\frac{1}{2r}} A[\omega_f(x,y)^{2r}]^{\frac{1}{2r}} A^h[|\alpha_f|]^{\frac{1}{s}}
\end{aligned}$$

where we have denoted

$$A = \operatorname*{Max}_{f\in S_2(q)^*} \omega_f|\alpha_f|.$$

Take now r large enough so that $x^r \geq q^{11}$ ($r = [11\kappa^{-1}] + 1$ suffices). Then Lemma 3 gives

$$A[\omega_f(x,y)^{2r}]^{\frac{1}{2r}} \ll (\log q)^D$$

for some $D = D(\kappa) > 0$, while we have, from (25) and (24) respectively,

$$\begin{aligned}
A^{\frac{1}{2r}} &\ll q^{-\gamma_0} \quad \text{for some } \gamma_0 = \gamma_0(\kappa,\delta) > 0, \\
A^h[|\alpha_f|] &\ll (\log q)^C \quad \text{for some absolute constant } C > 0.
\end{aligned}$$

Hence the proposition, the last equality being an immediate corollary of the formula

$$A[\alpha_f] = \frac{\dim J_0(q)}{\zeta(2)} A^h[L(\mathrm{Sym}^2 f, 1)\alpha_f] + O((\log q)^3 A^h[|\alpha_f|])$$

and the decomposition

$$L(\mathrm{Sym}^2 f, 1) = \omega_f(x) + \omega_f(x,y) + O(q^2 y^{-1})$$

applied with $y = q^3$. □

4 Upper bound for the analytic rank of $J_0(q)$

In this section we prove Theorem 1, via its equivalent form of Theorem 2.

4.1 Reduction to the density theorem

The explicit formulae, discovered in essence by Riemann, and later extended and formalized by Weil, have been used first by Mestre in studying abelian varieties. We use the following variant (compare [Br1], [PP], [Kow]).

Proposition 3. *Let* $\psi : \mathbf{R} \longrightarrow \mathbf{R}$ *be a* C^∞ *even function with compact support, and* $\hat{\psi} := \int_{\mathbf{R}} \phi(x) e^{sx} dx$ *its Laplace transform, which is an entire function. Then for any primitive form* $f \in S_2(q)^*$

$$\sum_\rho \hat{\psi}(\rho - \tfrac{1}{2}) = \psi(0)\log q - 2\sum_{n\geq 1} \frac{b_f(n)}{\sqrt{n}}\Lambda(n)\psi(\log n) + \frac{1}{2i\pi}\int_{(1/2)} 2\Big(\frac{\Gamma'}{\Gamma}(s+\tfrac{1}{2}) - \log 2\pi\Big)\hat{\psi}(s-\tfrac{1}{2})ds \quad (31)$$

the summation on the left-hand side being extended over all zeros ρ *of* $L(f,s)$ *in the critical strip – those with* $0 \leq \mathrm{Re}\,(s) \leq 1$ *– counted with multiplicity. The coefficients* $b_f(n)$ *are defined in (11).*

In this chapter, ρ will always designate such a "non-trivial" zero of $L(f,s)$, and we always write

$$\rho = \beta + i\gamma$$

so $\gamma = \mathrm{Im}\,(\rho)$, $\beta = \mathrm{Re}\,(\rho)$. For any α with $0 \leq \alpha \leq 1$, and any real numbers $t_1 \leq t_2$, we define $N(f;\alpha,t_1,t_2)$ to be the number of zeros $\rho = \beta + i\gamma$ of $L(f,s)$, counted with multiplicity, such that

$$\beta \geq \alpha,\ t_1 \leq \gamma \leq t_2,$$

and for any $t > 0$ we let

$$N(f;\alpha,t) = N(f;\alpha,-t,t), \quad N(f,t) = N(f;0,t).$$

Fix once for all a smooth function ϕ, even, non negative, compactly supported in $[-1,1]$, such that $\phi(0) = 1$, and such that $\hat{\phi}(s)$ satisfies for all $s \in \mathbf{C}$ with $|\mathrm{Re}\,(s)| \leq 1$,

$$\mathrm{Re}\,(\hat{\psi}(s)) \geq 0. \quad (32)$$

Now let $\phi_\lambda(x) = \phi(x/\lambda)$ so that

$$\hat{\psi}_\lambda(s) = \lambda\hat{\psi}(\lambda s).$$

We will take $\lambda = \theta \log q$ with $\theta > 0$ a fixed parameter (small enough) to be determined later.

The crucial assumption on ϕ is of course (32). Such test functions were constructed by Poitou and others for the purpose of obtaining lower-bounds for the discriminant of number fields [Poi].

Remark In [KM1], a specific test function F is used, which had been constructed previously by Perelli and Pomykala [PP]. However in our situation, it is not actually necessary to use it (this was observed by Pomykala).

The parameter λ will be used to effect a localization in detecting the zeros around $\frac{1}{2}$ in the explicit formula.

By integration by parts one infers that for any integer $k \geq 1$,

$$\hat{\psi}(s) \ll_k \frac{1}{(1+|\mathrm{Im}\,(s)|)^k} e^{\mathrm{Re}\,(s)} \tag{33}$$

where the implied constant depends only on k (and on the specific choice of ψ).

Let q be prime and $f \in S_2(q)^*$ a primitive form of level q. Applying the explicit formula (31) to f with the test function ψ_λ, we obtain

$$\sum_\rho \hat{\psi}_\lambda(\rho - \tfrac{1}{2}) = \log q - 2\sum_{n\geq 1} \frac{b_f(n)}{\sqrt{n}} \Lambda(n)\psi_\lambda(n) + O(1)$$

after having estimated the integral in (31) by

$$\frac{1}{2i\pi} \int_{(1/2)} 2\Big(\frac{\Gamma'}{\Gamma}(s+\tfrac{1}{2}) - \log 2\pi\Big)\hat{\psi}_\lambda(s-\tfrac{1}{2})ds \ll \lambda \int_{-\infty}^{+\infty} (1+|u|)\hat{\psi}(\lambda iu)du \ll 1$$

uniformly in λ (we have used $\Gamma'/\Gamma(s) \ll \log|s|$ for $\mathrm{Re}\,(s) = 1$, and (33)).

Then we isolate the multiplicity of the zero at $\frac{1}{2}$, and further distinguish among the remaining zeros ρ between those which are close to $\frac{1}{2}$, precisely those with $|\beta - \frac{1}{2}| \leq \lambda^{-1}$, and the others. On the other side we use the fact that Λ is supported on powers of primes, and put the primes apart from the squares and higher powers. This way we rewrite the outcome of the explicit formula:

$$\lambda\hat{\psi}(0)\mathrm{ord}_{s=\frac{1}{2}} L(f,s) \;+\; \Xi_1(f,\lambda) \;+\; \Xi_2(f,\lambda) \;=\; \log q \;-\; 2S_1(f,\lambda) \;-\; 2S_2(f,\lambda) \;+\; O(1) \tag{34}$$

with:

$$\Xi_1(f,\lambda) = \lambda \sum_{|\beta-\frac{1}{2}|\leq\lambda^{-1}} \hat{\psi}(\lambda(\rho-\tfrac{1}{2})), \qquad \Xi_2(f,\lambda) = \lambda \sum_{|\beta-\frac{1}{2}|>\lambda^{-1}} \hat{\psi}(\lambda(\rho-\tfrac{1}{2})) \tag{35}$$

$$S_1(f,\lambda) = \sum_p \frac{\lambda_f(p)}{\sqrt{p}}(\log p)\psi_\lambda(\log p), \qquad S_2(f,\lambda) = \sum_{n\geq 2}\sum_p \frac{\lambda_f(p^n)}{p^{n/2}}(\log p)\psi_\lambda(\log p^n). \tag{36}$$

Each term will be treated separately. First, since ψ has compact support in $[-1,1]$, and $|b_f(n)| \leq 2$ for all n, we have

$$S_2(f,\lambda) \ll \sum_{p\leq\exp(\lambda/2)} \frac{\log p}{p} + \sum_{3\leq n\leq\lambda}\;\sum_{\log p\leq\lambda/n} \frac{\log p}{p^{n/2}} \ll \lambda.$$

Now, in (34), we take the real part. For a zero ρ appearing in $\Xi_1(f,\lambda)$, we have $|\mathrm{Re}\,\lambda(\rho-\frac{1}{2})| \leq 1$, hence

$$\Xi_1(f,\lambda) \geq 0$$

by the positivity property (32) of the test function ψ. Therefore we can drop this term by positivity and get

$$\lambda\hat{\psi}(0)\mathrm{ord}_{s=\frac{1}{2}} L(f,s) \leq \log q - 2S_1(f,\lambda) + \mathrm{Re}\,(\Xi_2(f,\lambda)) + O(\lambda).$$

Again, intuitively, this application of positivity should not affect the chances of proving the result being sought, since the number of zeros dropped in the sum $\Xi_1(f,\lambda)$, on average over f, should be bounded.

Performing the average over f, we have consequently

$$\lambda\hat{\psi}(0) \sum_{f\in S_2(q)^*} \mathrm{ord}_{s=\frac{1}{2}} L(f,s) \le (\log q)\dim J_0(q) - 2\sum_{f\in S_2(q)^*} S_1(f,\lambda) \\ + \sum_{f\in S_2(q)^*} \mathrm{Re}\,(\Xi_2(f,\lambda)) + O(\lambda q). \tag{37}$$

If $\theta < 1$, we have by [Br1, 3]

$$\sum_{f\in S_2(q)^*} S_1(f,\lambda) \ll q^{1-\delta}. \tag{38}$$

for some $\delta = \delta(\theta) > 0$.

Remark As a variant, one can use the technique of Section 3 to prove this estimate, using the Petersson formula instead of the trace formula, as in [Br1]. See [Kow] for the details.

Thus it only remains to estimate the contribution of Ξ_2, the sum over zeros not too close to $\frac{1}{2}$. Of course, on the Generalized Riemann Hypothesis, those do not exist, and we see, taking the upper bound above (37) immediately implies a weak form of Brumer's result, namely

$$\sum_{f\in S_2(q)^*} \mathrm{ord}_{s=\frac{1}{2}} L(f,s) \ll \dim J_0(q)$$

for q prime. Indeed, up to this point, the treatment is basically the same as Brumer's. But handling Ξ_2 without appealing to the Riemann Hypothesis is precisely the crux of the matter. It will be possible to show that if there are zeros in the region $|\beta - \frac{1}{2}| > \lambda^{-1}$, then they are very few in number, in a very precise sense, which we now describe.

Theorem 4. *Let q be a prime number. There exists an absolute constant $A > 0$ such that for any real numbers t_1, t_2 with*

$$t_1 < t_2$$
$$t_2 - t_1 \ge \frac{1}{\log q},$$

for any $\alpha \ge \frac{1}{2} + (\log q)^{-1}$ and any c, $0 < c < \frac{1}{4}$, it holds

$$\sum_{f\in S_2(q)^*} N(f;\alpha,t_1,t_2) \ll (1+|t_1|+|t_2|)^A q^{1-c(\alpha-\frac{1}{2})}(\log q)(t_2-t_1), \tag{39}$$

the implied constant depending only on c.

The bulk of this section will be devoted to proving this result.

Remark In this density theorem, only the q-aspect is taken into consideration, and this statement is indeed trivial with respect to $T := |t_1| + |t_2|$. However, it is important (as the deduction of the upper bound from the density theorem shows) that the bounds obtained be at most polynomial in the imaginary part T. Thus, in the rest of this chapter, inequalities of the form

$$f(q,T) \ll (1+T)^B g(q)$$

will often be encountered; the constant $B \ge 0$ may appear, or its value may change, from line to line without further comment.

Assuming Theorem 4, we can now estimate Ξ_2. By the symmetry of the zeros, it is enough to consider those in the first quadrant. Subdividing the region $[\lambda^{-1}, \frac{1}{2}] \times \mathbf{R}^+$ into small squares of side λ^{-1}

$$R(m,n) = \Big[\frac{m}{\lambda}, \frac{m+1}{\lambda}\Big] \times \Big[\frac{n}{\lambda}, \frac{n+1}{\lambda}\Big]$$

with $1 \leq m \leq \lambda$, $0 \leq n$, we estimate the contribution Ξ_2^1 of those zeros:

$$\begin{aligned}\sum_{f \in S_2(q)^*} \mathrm{Re}\,(\Xi_2^1(f,\lambda)) &\leq \lambda \sum_{m=1}^{\lambda} \sum_{n\geq 0} N(f; \tfrac{1}{2} + \tfrac{n}{\lambda}, \tfrac{m}{\lambda}, \tfrac{m+1}{\lambda}) \sup_{s \in R(m,n)} |\hat{\psi}(\lambda s)| \\ &\ll \lambda \sum_{m=1}^{\lambda} \sum_{n \geq 0} (1 + \tfrac{n+1}{\lambda})^A q^{1-c\frac{n}{\lambda}} \log q \lambda^{-1} e^{m+1} (1+n)^{-k} \\ &\ll q(\log q)\end{aligned}$$

if we choose $\theta < c$, and $k \geq A+3$. hence from (37), dividing out by λ, we deduce Theorem 2.

4.2 Reduction to second moment estimates

Theorem 4 is the analogue of a result of Selberg [Sel, Th. 4] for Dirichlet characters. We will borrow the general principle from this paper (with some simplifications also found in [Luo]), starting with a crucial lemma which will reduce the theorem to some estimates of a mollified second moment of values of $L(f,s)$, $f \in S_2(q)^*$.

Lemma 7. *(Selberg, [Sel, Lemma 14]). Let h be a function holomorphic in the region*

$$\{s \in \mathbf{C} \mid \mathrm{Re}\,(s) \geq \alpha,\ t_1 \leq \mathrm{Im}\,(s) \leq t_2\}$$

satisfying

$$h(s) = 1 + o\Big(\exp(-\frac{\pi}{t_2 - t_1}\mathrm{Re}\,(s)\Big) \tag{40}$$

in this region, uniformly as $\mathrm{Re}\,(s) \to +\infty$. Denoting the zeros of f (in the interior of this region) by $\rho = \beta + i\gamma$, we have

$$\begin{aligned}2(t_2 - t_1)\sum_{\rho} \sin\Big(\pi\frac{\gamma - t_1}{t_2 - t_1}\Big)\sinh\Big(\pi\frac{\beta-\alpha}{t_2-t_1}\Big) &= \int_{t_1}^{t_2} \sin\Big(\pi\frac{t-t_1}{t_2-t_1}\Big)\log|h(\alpha+it)|dt \\ &+ \int_{\alpha}^{+\infty} \sinh\Big(\pi\frac{\sigma-\alpha}{t_2-t_1}\Big)(\log|h(\sigma+it_1)| + \log|h(\sigma+it_2)|)d\sigma\end{aligned}$$

(where the zeros are also summed with multiplicity).

This lemma will be applied to the functions $1 - (M(f,s)L(f,s)-1)^2$, where $M(f,s)$ is a suitable mollifier for which (40) holds, for α equal to $\frac{1}{2} + (\log q)^{-1}$. This means that $M(f,s)$ must approximate quite closely the inverse of $L(f,s)$.

Lemma 8. *The inverse $L(f,s)^{-1}$ is given by the Dirichlet series*

$$L(f,s)^{-1} = \sum_{m,n \geq 1} \varepsilon_q(n)\mu(m)\mu(mn)^2\lambda_f(m)(mn^2)^{-s}$$

which is absolutely convergent for $\mathrm{Re}\,(s) > 1$.

Proof. This is an immediate consequence of the Euler product expansion

$$L(f,s)^{-1} = \prod_p (1 - \lambda_f(p)p^{-s} + \varepsilon_q(p)p^{-2s})$$

by multiplicativity (every integer $\ell \geq 1$ has a unique expression as $\ell = mn^2r$ with m, n, r coprime in pairs, m and n squarefree and r cubefull). $\square$

We also define, for every $M \geq 1$, a function g_M by

$$g_M(x) = \begin{cases} 1, & \text{if } x \leq \sqrt{M} \\ \dfrac{\log M/x}{\log \sqrt{M}}, & \text{if } \sqrt{M} \leq x \leq M \\ 0, & \text{if } x > M. \end{cases} \tag{41}$$

Then for M fixed and any integer $1 \leq m \leq M$, we let

$$x_m(s) = \frac{\mu(m)}{m^{s-\frac{1}{2}}} \sum_{n \geq 1} \frac{\varepsilon_q(n)\mu(mn)^2}{n^{2s}} g_M(mn) \tag{42}$$

and we define the mollifier

$$M(f,s) = \sum_{m \leq M} \frac{x_m(s)}{\sqrt{m}} \lambda_f(m). \tag{43}$$

We observe that $M(f,s)$ is a Dirichlet polynomial of length at most M, with coefficients

$$c_f(\ell) = \sum_{mn^2=\ell} \varepsilon_q(n)\mu(m)\mu(mn)^2\lambda_f(m)g_M(mn) \tag{44}$$

and by Deligne's bound, they are bounded by

$$|c_f(\ell)| \leq \sum_{m|\ell} \tau(m) \leq \tau(\ell)^2. \tag{45}$$

¿From the definition, it follows easily that namely for $M = q^{\Delta}$ with $\Delta > 0$, we have

$$M(f,s)L(f,s) = 1 + O((\log q)^{15} q^{\Delta(1-\sigma)/2}) \tag{46}$$

uniformly for $\mathrm{Re}\,(s) = \sigma \to +\infty$.

The density theorem follows from a good estimate for the average of the second moment of $M(f,s)L(f,s)$, $\mathrm{Re}\,(s) \geq \frac{1}{2} + (\log q)^{-1}$.

Proposition 4. *Let $M = q^{\Delta}$ with $\Delta < \frac{1}{4}$, and let c be any positive real number with $c < \Delta$. Then there exists a constant $B > 0$ such that for all q prime large enough*

$$\sum_{f \in S_2(q)^*} |M(f,\beta+it)L(f,\beta+it) - 1|^2 \ll (1+|t|)^B q^{1-c(\beta-\frac{1}{2})}. \tag{47}$$

uniformly for $\beta \geq \frac{1}{2} + (\log q)^{-1}$ and $t \in \mathbf{R}$, the implied constant depending only on Δ and c.

Assuming this proposition, the proof of Theorem 4 can be completed, again following Selberg's argument.

4.3 The harmonic second moment

Proposition 4 will be proved by the method of Section 3, going through a corresponding weighted result first.

Proposition 5. *Let $M = q^{\Delta}$ with $\Delta < \frac{1}{4}$, and $\beta = \frac{1}{2} + b(\log q)^{-1}$, where $b > 0$ is any constant. For all q prime large enough we have*

$$\sum_{f \in S_2(q)^*}^{h} |M(f, \beta + it)L(f, \beta + it)|^2 \ll (1 + |t|)^B \tag{48}$$

for some absolute constant $B > 0$. The implied constant depends only on b and Δ.

We write $\beta = \frac{1}{2} + \delta$ and assume only $\delta = b(\log q)^{-1}$. Then we define for simplicity

$$M_2(\delta) = \sum_{f \in S_2(q)^*}^{h} |M(f, \beta + it)L(f, \beta + it)|^2 \tag{49}$$

which we consider as a quadratic form in the coefficients $x_m = x_m(\beta + it)$ of the mollifier. To emphasize this viewpoint, it will be convenient to simply write x_m and $M(f)$ while performing transformations to facilitate the ultimate estimations.

Let $f \in S_2(q)^*$ and $\beta = \frac{1}{2} + \delta$ with $0 < \delta < \frac{1}{2}$ be given.

Choose an integer $N \geq 1$ (which will have to be large enough, $N = 2$ works already) and a real polynomial G satisfying

$$G(-s) = G(s), \tag{50}$$

$$G(-N) = \ldots = G(-1) = 0 \tag{51}$$

and having no zeros for $-\frac{1}{2} \leq \mathrm{Re}\,(s) \leq \frac{1}{2}$.

Let $t \in \mathbf{R}$ be a fixed real number. Define the entire function $Z(f, s)$ by

$$Z(f, s) = \Lambda(f, s + \tfrac{1}{2} + it)\Lambda(f, s + \tfrac{1}{2} - it)$$

which satisfies the functional equation

$$Z(f, s) = Z(f, -s).$$

Since the Fourier coefficients $\lambda_f(n)$ of f are real, we have

$$|\Lambda(f, \beta + it)|^2 = Z(f, \beta). \tag{52}$$

We now consider the complex integral

$$\begin{aligned} I_\delta &= \frac{1}{2i\pi} \int_{(2)} Z(f, s)G(s + it)G(s - it) \frac{ds}{s - \delta} \\ &= \frac{1}{2i\pi} \int_{(2)} L(f, s + \tfrac{1}{2} + it)L(f, s + \tfrac{1}{2} - it)H(s)\Big(\frac{\sqrt{q}}{2\pi}\Big)^{2s+1} \frac{ds}{s - \delta} \end{aligned}$$

(defined, as a function of δ, for all $\delta \in \mathbf{R}$) with

$$H(s) = G(s+it)G(s-it)\Gamma(s+1+it)\Gamma(s+1-it).$$

This integral is absolutely convergent (by stirling formula). From (51), zeros of the polynomial G cancel the first poles of the Γ function, so H is holomorphic for $\mathrm{Re}\,(s) > -N-1$. Next, normalizing G, we can assume that $H(\delta) = 1$. The Gamma function has exponential decay in vertical strips, while G has polynomial growth; in fact, by Stirling's formula, H satisfies (uniformly in vertical strips)

$$H(s) \ll (1+|t|+|\mathrm{Im}\,(s)|)^B e^{-\pi|\mathrm{Im}\,(s)|} \text{ for some constant } B > 0.$$

We can shift the contour of integration to the line $\mathrm{Re}\,(s) = -2$; only a simple pole at $s = \delta$ appears while shifting, with residue

$$\mathrm{Res}_{s=\delta}\, Z(f,s)G(s+it)G(s-it)\frac{1}{s-\delta} = \Big(\frac{q}{4\pi^2}\Big)^{\beta} H(\delta)|L(f,\beta+it)|^2$$

by (52).

On the line $\mathrm{Re}\,(s) = -2$, the integral is seen to be

$$\frac{1}{2i\pi}\int_{(-2)} Z(f,s)G(s+it)G(s-it)\frac{ds}{s-\delta} = -I_{-\delta}$$

by the change of variable $s \mapsto -s$, using the functional equation of $Z(f,s)$ and the symmetry $G(s) = G(-s)$. Hence we have the formula

$$\Big(\frac{q}{4\pi^2}\Big)^{\beta} H(\delta)|L(f,\beta+it)|^2 = I_\delta + I_{-\delta}. \tag{53}$$

On the other hand, using the Hecke relation (7) one has in the region of absolute convergence the identity

$$L(f,s+it)L(f,s-it) = \zeta_q(2s)\sum_{n\geq 1}\lambda_f(n)\eta_t(n)n^{-s},$$

where

$$\eta_t(n) = \sum_{ab=n}\Big(\frac{a}{b}\Big)^{it}. \tag{54}$$

We may then integrate term by term to obtain

$$I_\delta = \Big(\frac{q}{4\pi^2}\Big)^{1/2}\sum_n \frac{\lambda_f(n)}{n^{1/2}}\eta_t(n)W_\delta\Big(\frac{4\pi^2 n}{q}\Big)$$

where

$$W_\delta(y) = \frac{1}{2i\pi}\int_{(2)} H(s)\zeta_q(1+2s)y^{-s}\frac{ds}{s-\delta} \tag{55}$$

Finally one gets

$$\Big(\frac{q}{4\pi^2}\Big)^{\delta} H(\delta)|L(f,\beta+it)|^2 = \sum_{n\geq 1}\frac{\lambda_f(n)}{\sqrt{n}}\eta_t(n)U\Big(\frac{4\pi^2 n}{q}\Big) \tag{56}$$

where now

$$U(y) = W_\delta(y) + W_{-\delta}(y) = \frac{1}{2i\pi}\int_{(2)} H(s)\zeta_q(1+2s)y^{-s}\frac{2sds}{(s-\delta)(s+\delta)}. \tag{57}$$

We conclude this section by listing the basic properties of the test function U and the arithmetic function η_t. These should be skipped and consulted when referred to later.

Lemma 9. *For $\delta \neq 0$, we have*

$$U(y) = H(\delta)\zeta_q(1+2\delta)y^{-\delta} + H(-\delta)\zeta_q(1-2\delta)y^{\delta} + O(y^N(1+|t|)^B) \tag{58}$$

for $0 \leq y \leq 1$ and

$$U(y) \ll_j y^{-j}(1+|t|)^B, \text{ for all } j \geq 1 \tag{59}$$

for $y \geq 1$, (B depending on j).

Proof. This follows easily by shifting the line of integration either to the right to $\mathrm{Re}\,(s) = j$, or to the left to $\mathrm{Re}\,(s) = -N-\delta$ and by computing the residues. □

Lemma 10. *For all $t \in \mathbf{R}$, the arithmetic function η_t is real valued. It satisfies the identities*

$$\eta_t(n)\eta_t(m) = \sum_{d|(n,m)} \eta_t\Big(\frac{nm}{d^2}\Big) \tag{60}$$

$$\eta_t(nm) = \sum_{d|(n,m)} \mu(d)\eta_t\Big(\frac{n}{d}\Big)\eta_t\Big(\frac{m}{d}\Big) \tag{61}$$

$$\sum_{n\geq 1}\eta_t(n)n^{-s} = \zeta(s-it)\zeta(s+it) \tag{62}$$

$$\sum_{n\geq 1}\eta_t(n)^2 n^{-s} = \frac{\zeta(s-2it)\zeta(s)^2\zeta(s+2it)}{\zeta(2s)} \tag{63}$$

$$\sum_{n\geq 1}\eta_t(n^2)n^{-s} = \frac{\zeta(s-2it)\zeta(s)\zeta(s+2it)}{\zeta(2s)} \tag{64}$$

and the estimate

$$|\eta_t(n)| \leq \tau(n). \tag{65}$$

Proof. Everything can be checked elementarily by direct computations, but it may as well be deduced from the fact that $\eta_t(n)$ is a Hecke eigenvalue for the operator $T(n)$ acting on the derivative at $s = \frac{1}{2}$ of the non-holomorphic Eisenstein series $E(z,s)$ of level 1. □

We now come to the mollifier $M(f)$. By multiplicativity of the coefficients $\lambda_f(n)$, once more, we have

$$|M(f)|^2 = \sum_b \frac{1}{b}\sum_{m_1,m_2}\frac{\lambda_f(m_1m_2)}{\sqrt{m_1m_2}}x_{bm_1}\overline{x_{bm_2}}$$

so that by (56), the second moment $M_2(\delta)$ is given by

$$\Big(\frac{q}{4\pi^2}\Big)^\delta H(\delta)M_2(\delta) = \sum_b \frac{1}{b}\sum_{n\geq 1}\sum_{m_1,m_2}\frac{\eta_t(n)}{\sqrt{m_1m_2n}}x_{bm_1}\overline{x_{bm_2}}U\Big(\frac{4\pi^2 n}{q}\Big)\Delta(m_1m_2,n)$$

where Δ is the Delta-symbol. We have (13)

$$\Delta(m,n) = \delta(m,n) + O((mn)^{1/2}(\log q)^2 q^{-3/2})$$

for m, $n \le q$, where the implied constant is absolute.

Using (42) to estimate that

$$x_m \ll \zeta(1+2\delta)m^{-\delta}$$

the contribution to $M_2(\delta)$ of the remainder term of $\Delta(m,n)$ is at most

$$\begin{aligned} &\frac{(\log q)^2}{q^{3/2}} \sum_b \frac{1}{b} \Big| \sum_{bm \le M} \tau(m) x_{bm} \Big|^2 \Big| \sum_{n \ge 1} U\Big(\frac{4\pi^2 n}{q}\Big) \Big| \\ &\ll_\epsilon (1+|t|)^B (qM)^\epsilon \frac{M^2}{q^{1/2}} \end{aligned} \tag{66}$$

We now study the "diagonal contribution" where $n = m_1 m_2$, namely the sum $M'(\delta)$ defined by the equality

$$\Big(\frac{q}{4\pi^2}\Big)^\delta H(\delta) M'(\delta) = \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{\eta_t(m_1 m_2)}{m_1 m_2} x_{bm_1} \overline{x_{bm_2}} U\Big(\frac{4\pi^2 m_1 m_2}{q}\Big).$$

Inserting (58), we have

$$\Big(\frac{q}{4\pi^2}\Big)^\delta H(\delta) M'(\delta) = \Big(\frac{q}{4\pi^2}\Big)^\delta H(\delta) M''(\delta) + O_\epsilon((1+|t|)^B (qM)^\epsilon q^{-1/2} M^2) \tag{67}$$

where the sum $M''(\delta)$ is given by

$$\begin{aligned} \Big(\frac{q}{4\pi^2}\Big)^\delta H(\delta) M''(\delta) = &\Big(\frac{q}{4\pi^2}\Big)^\delta H(\delta) \zeta_q(1+2\delta) \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{\eta_t(m_1 m_2)}{(m_1 m_2)^{1+\delta}} x_{bm_1} \overline{x_{bm_2}} \\ &+ \Big(\frac{q}{4\pi^2}\Big)^{-\delta} H(-\delta) \zeta_q(1-2\delta) \sum_b \frac{1}{b} \sum_{m_1,m_2} \frac{\eta_t(m_1 m_2)}{(m_1 m_2)^{1-\delta}} x_{bm_1} \overline{x_{bm_2}} \end{aligned} \tag{68}$$

and the error term has been estimated by

$$(1+|t|)^B \frac{1}{\sqrt{q}} \sum_{b \le M} \frac{1}{b} \Big| \sum_{bm \le M} \frac{\tau(m)}{\sqrt{m}} x_{bm} \Big|^2 \ll (1+|t|)^B (qM)^\epsilon q^{-1/2} M^2)$$

4.4 Diagonalization and estimation of the second moment

First, m_1 and m_2 can be separated in (68) by means of the Möbius inversion formula (61), so

$$\begin{aligned} &\Big(\frac{q}{4\pi^2}\Big)^\delta H(\delta) M''(\delta) = \\ &\Big(\frac{q}{4\pi^2}\Big)^\delta H(\delta) \zeta_q(1+2\delta) \sum_b \frac{1}{b} \sum_a \frac{\mu(a)}{a^{2(1+\delta)}} \sum_{m_1,m_2} \frac{\eta_t(m_1)\eta_t(m_2)}{(m_1 m_2)^{1+\delta}} x_{abm_1} \overline{x_{abm_2}} \\ &+ \Big(\frac{q}{4\pi^2}\Big)^{-\delta} H(-\delta) \zeta_q(1-2\delta) \sum_b \frac{1}{b} \sum_a \frac{\mu(a)}{a^{2(1-\delta)}} \sum_{m_1,m_2} \frac{\eta_t(m_1)\eta_t(m_2)}{(m_1 m_2)^{1-\delta}} x_{abm_1} \overline{x_{abm_2}} \end{aligned}$$

and we can collect the single variable $k = ab$, introducing the arithmetic function

$$\nu_\delta(k) = \sum_{ab=k} \frac{\mu(a)}{a^{1+2\delta}}$$

to derive

$$\begin{aligned}\left(\frac{q}{4\pi^2}\right)^\delta H(\delta) M''(\delta) = \left(\frac{q}{4\pi^2}\right)^\delta H(\delta)\zeta_q(1+2\delta) \sum_k \frac{\nu_\delta(k)}{k} \Big|\sum_m \frac{\eta_t(m)}{m^{1+\delta}} x_{km}\Big|^2 \\ + \left(\frac{q}{4\pi^2}\right)^{-\delta} H(-\delta)\zeta_q(1-2\delta) \sum_k \frac{\nu_{-\delta}(k)}{k} \Big|\sum_m \frac{\eta_t(m)}{m^{1-\delta}} x_{km}\Big|^2.\end{aligned} \tag{69}$$

Following Selberg, we notice that for $0 < \delta < \frac{1}{2}$ the inequalities hold

$$-\zeta_q(1-2\delta) \geq 0,\ \ H(-\delta) = |\Gamma(1-\delta+it)G(-\delta+it)|^2 > 0\ \ \nu_{-\delta}(k) = \prod_{p|k}(1-p^{-1+2\delta}) \geq 0.$$

Hence, by positivity

$$M''(\delta) \leq \zeta_q(1+2\delta) \sum_k \frac{\nu_\delta(k)}{k} \Big|\sum_m \frac{\eta_t(m)}{m^{1+\delta}} x_{km}\Big|^2. \tag{70}$$

Let

$$y_k = \sum_m \frac{\eta_t(m)}{m^{1+\delta}} x_{km} \tag{71}$$

(which is supported on squarefree integers $k \leq M$).

Proposition 6. *Assume $\delta = b(\log q)^{-1}$ for some (absolute) constant $b > 0$ and $M = q^\Delta$ with $\Delta < \frac{1}{4}$. Then for k squarefree, $k \leq M$, we have*

$$k^{\delta+it}\xi(k) y_k \ll \frac{1}{\log q}$$

where

$$\xi(k) = \prod_{p|k}(1-p^{-1/2}).$$

Remark This saving of a factor $\log q$ is the critical moment. It will come essentially from cancellation due to the oscillations of the Möbius function, or in other words, from the Prime Number Theorem.

Proof. We proceed as in [Luo]. From the definition (42), for $s = \beta + it = \frac{1}{2} + \delta + it$, we have

$$x_{km} = \frac{\mu(k)}{k^{\delta+it}} \times \frac{\mu(m)}{m^{\delta+it}} \sum_n \frac{\mu(kmn)^2}{n^{1+2\delta+2it}} g_M(kmn)$$

(there is no $\varepsilon_q(n)$ since $n \leq kmn \leq M < q$). Therefore

$$y_k = \frac{\mu(k)}{k^{\delta+it}} \sum_{m,n} \frac{\mu(kmn)^2\mu(m)\eta_t(m)n^{-it}}{(mn)^{1+2\delta+it}} g_M(kmn).$$

Assume first that $1 \leq k \leq \sqrt{M}$ (and of course k is squarefree). We use the following integral formula: for all $\ell \geq 1$

$$g_M(k\ell) = \frac{1}{2i\pi} \int_{(2)} \frac{(\sqrt{M}/k)^s (M^{s/2} - 1)}{\log \sqrt{M}} \ell^{-s} \frac{ds}{s^2}. \tag{72}$$

which follows from

$$\frac{1}{2i\pi} \int_{(2)} y^s \frac{ds}{s^2} = \begin{cases} \log y, & \text{if } y \geq 1, \\ 0, & \text{if } 0 < y \leq 1. \end{cases}$$

Hence

$$k^{\delta+it} y_k = \frac{1}{2i\pi} \int_{(2)} L_k(s + 1 + 2\delta + it) \frac{(\sqrt{M}/k)^s (M^{s/2} - 1)}{\log \sqrt{M}} \frac{ds}{s^2} \tag{73}$$

with the *ad-hoc* Dirichlet series

$$L_k(s) = \sum_{\ell \geq 1} \mu(k\ell)^2 \Bigl(\sum_{mn=\ell} \mu(m) \eta_t(m) n^{-it} \Bigr) \ell^{-s}$$

which is easily computed. Indeed, the inner sum is the coefficient of ℓ^{-s} in the product

$$\begin{aligned} \zeta(s+it) \sum_{m \geq 1} \mu(m) \eta_t(m) m^{-s} &= \prod_p (1 - p^{-s-it})^{-1} (1 - p^{-s}(p^{it} + p^{-it})) \\ &\qquad \text{(by multiplicativity and the definition of } \eta_t) \\ &= \prod_p \Bigl(1 - p^{-s+it} (1 - p^{-s-it})^{-1} \Bigr) \\ &= \prod_p (1 - p^{-s+it} \sum_{j \geq 0} p^{-j(s+it)}) \end{aligned}$$

and $L_k(s)$ is obtained from this Dirichlet series by taking the subseries restricted to integers prime to k and squarefree (this is the effect of inserting $\mu(k\ell)^2$ in a Dirichlet series). This gives the very simple answer

$$L_k(s) = \zeta_k(s - it)^{-1}.$$

¿From the theorems of Hadamard and de la Vallée-Poussin, $\zeta(s)$ has no zeros on the line $\mathrm{Re}\,(s) = 1$ and more precisely the estimate

$$\zeta(s)^{-1} \ll \log(2 + |\mathrm{Im}\,(s)|) \tag{74}$$

holds with an absolute implied constant (see [Tit, ch. 3]) uniformly for

$$\mathrm{Re}\,(s) \geq 1 - \frac{D}{\log(2 + |\mathrm{Im}\,(s)|)}$$

($D > 0$ being another absolute constant).

Let r be small enough so that the circle $|s| \leq r$ is included in this zero-free region, and $0 < r < \frac{1}{2}$ (of course, any $r < \frac{1}{2}$ will do, the Riemann Hypothesis being numerically valid in such a range!). In (73), we shift the integration to the contour C consisting of the vertical line $\mathrm{Re}\,(s) = 0$ from $-i\infty$ to $-ir$, followed by the half-circle $s = re(x)$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, and then again the line $\mathrm{Re}\,(s) = 0$

from ir to $i\infty$. By the choice of r, this is permissible; the contour shift passes through a unique simple pole at $s=0$ (simple because of the zero of $s \mapsto M^{s/2}-1$), and from the residue and the formula for $L_k(s)$ we get

$$k^{\delta+it} y_k = \zeta_k(1+2\delta)^{-1} + \frac{1}{2i\pi}\int_C \frac{\zeta(s+1+2\delta)^{-1}}{\prod_{p|k}(1-p^{-(s+1+2\delta)})}\frac{(\sqrt{M}/k)^s(M^{s/2}-1)}{\log\sqrt{M}}\frac{ds}{s^2}.$$

The integral over C is now estimated. Using (74), the part from ir to $i\infty$ is dominated by

$$\frac{1}{\log M}\Bigl|\int_r^{+\infty}\frac{\zeta(1+2\delta+iu)^{-1}}{\prod_{p|k}(1-p^{-(1+2\delta+iu)})}(\sqrt{M}/k)^{iu}(M^{iu/2}-1)\frac{du}{u^2}\Bigr| \ll \frac{1}{\log q}\frac{1}{\xi(k)}$$

since clearly

$$\prod_{p|k}(1-p^{-1-2\delta})^{-1} \le \xi(k)^{-1}.$$

The same holds without change for the other vertical half-line. For the semi-circle, we use the fact that $k \le \sqrt{M}$ so that

$$(\sqrt{M}/k)^s(M^{s/2}-1) \ll 1$$

on this semi-circle where $\mathrm{Re}\,(s) < 0$, and similarly the product over primes dividing k is dominated by its value at $s=-r$ which is

$$\prod_{p|k}(1-p^{r-1-\delta})^{-1} \le \xi(k)^{-1}$$

since $r < \frac{1}{2}$. Hence the same bound holds again. Now we collect the results and we use the assumption that $\delta = b(\log q)^{-1}$, which implies

$$\zeta(1+2\delta)^{-1} \ll (\log q)^{-1}.$$

Hence for $k \le \sqrt{M}$ we obtain immediately the desired bound

$$\xi(k)k^{1+\delta}y_k \ll \frac{1}{\log q}.$$

This is also true in the case $\sqrt{M} \le k \le M$: we use a similar reasoning, replacing (72) by the other formula

$$g_M(k\ell) = \frac{1}{2i\pi}\int_{(2)} \frac{(M/k)^s}{\log\sqrt{M}}\ell^{-s}\frac{ds}{s^2}$$

and using the same contour shift. This finishes the proof. □

¿From the previous proposition and (70) we have

$$\begin{aligned}M''(\delta) &\le \zeta_q(1+2\delta)\sum_{k\le M}\frac{\nu_\delta(k)}{k}|y_k|^2 \ll \frac{\zeta_q(1+2\delta)}{(\log q)^2}\sum_{k\le M}\frac{\mu(k)^2\nu_\delta(k)}{\xi(k)^2}k^{-(1+2\delta)}\\ &\ll \frac{1}{\log q}\sum_{k\le M}\frac{\mu(k)^2\nu_\delta(k)}{\xi(k)^2}k^{-(1+2\delta)}\end{aligned}$$

Now the above last sum is a partial sum for a Dirichlet serie admitting analytic continuation to $\mathrm{Re}\,(s) \geq 7/8$ with a simple pole as $s = 1$ and therefore

$$\sum_{k \leq M} \frac{\mu(k)^2}{\xi(k)^2} k^{-(1+2\delta)} \ll \log q.$$

To conclude the proof of Proposition 5 we look back to the error terms (66) and (67) introduced in going from the original second moment $M_2(\delta)$ to $M'(\delta)$ and then to $M''(\delta)$, and we see that they bring a total contribution which is

$$\ll q^{-\gamma}(1+|t|)^B$$

for some $\gamma = \gamma(\Delta)$ if $M = q^\Delta$ with $\Delta < \frac{1}{4}$.

4.5 Harmonic average and natural average, II

Having estimated $A^h[|M(f, \beta + it)L(f, \beta + it)|^2]$, we now apply Proposition 2 to study

$$A[|M(f, \beta + it)L(f, \beta + it)|^2].$$

The notations and assumptions are the same as at the beginning of the previous Section: recall that $\beta = \frac{1}{2} + \delta$, and $M = q^\Delta$ with $\Delta < \frac{1}{4}$.

First we check the conditions; (24) is contained in Proposition 5, while for (25), we have

Lemma 11. *For all $f \in S_2(q)^*$, it holds*

$$\omega_f |M(f, \beta + it)L(f, \beta + it)|^2 \ll q^{-\frac{1}{4}}(1+|t|)^2$$

for all β with $\beta \geq \frac{1}{2}$, the implied constant being absolute.

Proof. Using (45), the trivial bound for $M(f, \beta + it)$ is

$$M(f, \beta + it) \ll \sqrt{M}(\log q)^3$$

while the convexity bound for $L(f, s)$ on the critical line gives

$$L(f, \beta + it) \ll_\varepsilon q^{\frac{1}{4}+\varepsilon}(1+|t|)^{\frac{1}{2}+\varepsilon}$$

for $\beta \geq \frac{1}{2}$. Since on the other hand we have $\omega_f \ll (\log q)q^{-1}$ from (22), the result follows. $\square$

Hence Proposition 2 with $x = q^\kappa$, for any $\kappa > 0$, gives the equality

$$\begin{aligned} A[|M(f, \beta + it)L(f, \beta + it)|^2] = \frac{\dim J_0(q)}{\zeta(2)} & A^h[\omega_f(x)|M(f, \beta + it)L(f, \beta + it)|^2] \\ & + O((1+|t|)^B q^{1-\gamma}) \end{aligned} \tag{75}$$

for some $\gamma = \gamma(\Delta, \kappa) > 0$ (the dependence in t of the error term has to be checked by looking back at the proof of the proposition).

We let

$$\begin{aligned} \mathcal{M}_2(\delta) &= A^h[\omega_f(x)|M(f, \beta + it)L(f, \beta + it)|^2] \\ &= \sum_{d\ell^2 \leq x} \frac{1}{d\ell^2} \sum_{f \in S_2(q)^*}^h \lambda_f(d^2)|M(f, \beta + it)L(f, \beta + it)|^2. \end{aligned}$$

Computing as before we get

$$\Big(\frac{q}{4\pi^2}\Big)^{\delta} H(\delta)\mathcal{M}_2(\delta) = \sum_b \frac{1}{b}\sum_{n\geq 1}\sum_{m_1,m_2} \frac{\eta_t(n)}{\sqrt{m_1m_2n}} x_{bm_1}\overline{x_{bm_2}}U\Big(\frac{4\pi^2 n}{q}\Big)\Delta^n(m_1m_2,n)$$

where now

$$\begin{aligned}\Delta^n(m,n) &= \sum_{\ell\leq x} A^h[\rho_f(\ell)\lambda_f(m)\lambda_f(n)] &(76)\\ &= \sum_{d\ell^2\leq x}\ \sideset{}{^h}\sum_{f\in S_2(q)^*} \lambda_f(d^2)\lambda_f(m)\lambda_f(n)\\ &= \sum_{d\ell^2\leq x}\frac{1}{d\ell^2}\sum_{r\mid (d^2,m)} \delta\Big(\frac{md^2}{r^2},n\Big) + O\Big((\log q)^3\frac{x\sqrt{mn}}{q^{3/2}}\Big) &(77)\end{aligned}$$

by (13) again. The error term yields a contribution which, by the same computation as in (66), is at most

$$\ll_\epsilon (qMx)^\epsilon \frac{xM^{2(1-\delta)}}{\sqrt{q}}(1+|t|)^B \ll_\kappa (1+|t|)^B q^{-\gamma} \tag{78}$$

for some $\gamma>0$, if κ is taken small enough.

The diagonal contribution $n=md^2r^{-2}$ is

$$\sum_b \frac{1}{b}\sum_{m_1,m_2}\frac{x_{bm_1}\overline{x_{bm_2}}}{m_1m_2}\sum_{d\ell^2\leq x}(d\ell)^{-2}\sum_{r\mid (m_1m_2,d^2)} r\eta_t\Big(\frac{m_1m_2d^2}{r^2}\Big)U\Big(\frac{4\pi^2m_1m_2d^2}{qr^2}\Big)$$

and we use (58) to get

$$\begin{aligned}\Big(\frac{q}{4\pi^2}\Big)^{\delta} H(\delta)\mathcal{M}_2(\delta) &= \Big(\frac{q}{4\pi^2}\Big)^{\delta} H(\delta)\zeta_q(1+2\delta)\mathcal{M}(\delta) &(79)\\ &+\Big(\frac{q}{4\pi^2}\Big)^{-\delta} H(-\delta)\zeta_q(1-2\delta)\mathcal{M}(-\delta) + O((1+|t|)^B q^{-\gamma})\end{aligned}$$

for some $\gamma>0$, where the sum $\mathcal{M}(\delta)$ is

$$\mathcal{M}(\delta) = \sum_b \frac{1}{b}\sum_{m_1,m_2}\frac{x_{bm_1}\overline{x_{bm_2}}}{(m_1m_2)^{1+\delta}}\sum_{d\ell^2\leq x}\frac{1}{d^{2+2\delta}\ell^2}\sum_{r\mid (m_1m_2,d^2)} r^{1+2\delta}\eta_t\Big(\frac{m_1m_2d^2}{r^2}\Big). \tag{80}$$

We will first compute the inner sum, showing in particular that we can now again extend the summation over d, ℓ to all integers, and then we compute this complete series.

We define a function $u(s,r)$ for $s\in \mathbf{C}$ and $r\geq 1$ an integer by

$$u(s,r) = \sum_{ab=r}\mu(a)b^s = \prod_{p\mid r}(p^s-1)$$

and a function $v_x(s,r)$, supported on cubefree integers r, by

$$v_x(s,r) = \sum_{\substack{d\ell^2\leq x\\ r\mid d^2}} \ell^{-2}d^{-2s}\eta_t\Big(\frac{d^2}{r}\Big) \tag{81}$$

we also denote by $v(s,r)$ the function obtained by removing the constraint $d\ell^2 \leq x$ in the definition of $v(s,r)$.

¿From the formula (61), we have for every integers m and n

$$\sum_{r|(m,n)} r^s \eta_t\Big(\frac{mn}{r^2}\Big) = \sum_{r|(m,n)} u(s,r)\eta_t\Big(\frac{m}{r}\Big)\eta_t\Big(\frac{n}{r}\Big)$$

hence

$$\sum_{d\ell^2\leq x} \frac{1}{d^{2+2\delta}\ell^2} \sum_{r|(m_1m_2,d^2)} r^{1+2\delta}\eta_t\Big(\frac{m_1m_2d^2}{r^2}\Big) = \sum_{r|m_1m_2} \eta_t\Big(\frac{m_1m_2}{r}\Big)u(1+2\delta,r)v_x(1+\delta,r). \tag{82}$$

We define two multiplicative functions N and M by

$$N(r) = \prod_{p|r} p, \quad M(r) = \prod_{p||r} p.$$

Lemma 12. *For all cubefree integers $r \geq 1$, and s with $\mathrm{Re}\,(s) = \sigma > \frac{1}{2}$, we have*

$$v_x(s,r) = v(s,r) + O\Big(\frac{(\log x)^3\tau(r)}{N(r)^{2\sigma-\frac{1}{2}}\sqrt{x}}\Big). \tag{83}$$

Moreover

$$v(s,1) = \frac{\zeta(2)\zeta(2s)\zeta(2s+2it)\zeta(2s-2it)}{\zeta(4s)}$$

and for all $r \geq 1$

$$v(s,r) = v(s,1)N(r)^{-2s}\prod_{p||r}\frac{\eta_t(p)}{1+p^{-2s}}$$

Proof. The point is that for a cubefree integer r and any $d \geq 1$, we have $r \mid d^2$ if and only if $N(r) \mid d$. Since

$$r = M(r)\frac{N(r)^2}{M(r)^2} = \frac{N(r)^2}{M(r)}$$

we can write

$$\begin{aligned} v_x(s,r) &= \sum_{\substack{d\ell^2\leq x\\ N(r)|d}} \ell^{-2}d^{-2s}\eta_t\Big(\frac{d^2}{r}\Big) \\ &= N(r)^{-2s}\sum_{d\ell^2\leq x/N(r)} \ell^{-2}d^{-2s}\eta_t(M(r)d^2). \end{aligned}$$

and similarly without constraint for $v(s,r)$. Now, putting $y = x/N(r)$

$$\begin{aligned} \sum_{d\ell^2>x/N(r)} \ell^{-2}d^{-2s}\eta_t(M(r)d^2) &\ll \tau(M(r))\Big(\sum_{\ell^2<y}\ell^{-2}\sum_{d>y/\ell^2}\tau(d^2)d^{-2\sigma} + \sum_{\ell^2>y}\ell^{-2}\Big) \\ &\ll \tau(r)(\log x)^3y^{-1/2} \end{aligned}$$

and this gives the first formula.

To compute $v(s,r)$ (which is a kind of "non-primitive" symmetric square for η_t), we define

$$v'(s,r) = \sum_{d\geq 1} \eta_t(M(r)d^2)d^{-2s}$$

so that $v(s,r) = \zeta(2)N(r)^{-2s}v'(s,r)$. We denote by $Z(s)$ the full symmetric square given by (64), and by Z_p its p-factor.

Every integer d has a unique expression $d = d_1d_2$ with $d_1 \mid M(r)^\infty$ and $(d_2, M(r)) = 1$ so by multiplicativity we get

$$\begin{aligned} v'(s,r) &= \Big(\sum_{(d,M(r))=1} \eta_t(d^2)d^{-2s}\Big)\Big(\sum_{d\mid M(r)^\infty} \eta_t(M(r)d^2)d^{-2s}\Big) \\ &= Z(2s)\prod_{p\|r} Z_p(2s)^{-1} \times \prod_{p\|r}\sum_{k\geq 0} \eta_t(p^{2k+1})p^{-2ks}. \end{aligned}$$

Again by multiplicativity,

$$\eta_t(p^{2k+1}) = \eta_t(p)\eta_t(p^{2k}) - \eta_t(p^{2k-1})$$

for $k \geq 1$, so that

$$(1 + p^{-2s})\sum_{k\geq 0} \eta_t(p^{2k+1})p^{-2ks} = \eta_t(p)Z_p(2s)$$

which yields

$$v'(s,r) = Z(2s)\prod_{p\|r} \frac{\eta_t(p)}{1 + p^{-2s}}.$$

This gives the lemma, since

$$v(s,1) = Z(2s) = \frac{\zeta(2)\zeta(2s)\zeta(2s+2it)\zeta(2s-2it)}{\zeta(4s)}$$

from (64). $\square$

Let now $w_x(s,m)$ be the function defined for $s \in \mathbf{C}$ and $m \geq 1$ by

$$w_x(s,r) = \sum_{r\mid m} \eta_t\Big(\frac{m}{r}\Big)u(2s-1,r)v_x(s,r), \tag{84}$$

and let $w(s,m)$ be the same with $v(s,r)$ replacing $v_x(s,r)$. Then from (82) and (80) comes the formula

$$\mathcal{M}(\delta) = \sum_b \frac{1}{b}\sum_{m_1,m_2} \frac{w_x(1+\delta,m)}{(m_1m_2)^{1+\delta}} x_{bm_1}\overline{x_{bm_2}}. \tag{85}$$

Lemma 13. *Assume that $\delta = b(\log q)^{-1}$ for any constant $b > 0$. Then*

$$\mathcal{M}(\delta) = \sum_b \frac{1}{b}\sum_{m_1,m_2} \frac{w(1+\delta,m_1m_2)}{(m_1m_2)^{1+\delta}} x_{bm_1}\overline{x_{bm_2}} + O(q^{-\gamma})$$

for some $\gamma = \gamma(\kappa,\Delta) > 0$.

Proof. Since m_1 and m_2 are squarefree, the product m_1m_2, and any divisor thereof, is always cubefree. So we use (83) to replace $v_x(1+\delta,r)$ by $v(1+\delta,r)$. This gives first

$$w(1+\delta,m)=w_x(1+\delta,m)+O\Big(\frac{\tau(m)^3(\log x)^3}{\sqrt{x}}\Big)$$

because the error term is bounded by

$$\sum_{r|m}\tau\Big(\frac{m}{r}\Big)|u(1+2\delta,r)|\frac{(\log x)^3\tau(r)}{N(r)^{\frac{3}{2}+2\delta}\sqrt{x}}\ll\frac{\tau(m)^3(\log x)^3}{\sqrt{x}}$$

by the estimate

$$|u(1+2\delta,r)|=\Big|\prod_{p|r}(p^{1+2\delta}-1)\Big|\leq N(r)^{1+2\delta}.$$

Then inserting this inside $\mathcal{M}(\delta)$ gives the result. □

4.6 End of evaluation of the second moment

After Lemma 13, we now want to "diagonalize" the quadratic form $\mathcal{M}(\delta)$ (more precisely, its main term). This is done by the following transformations (compare [KM3], [Kow], where more complicated expressions involving functions which are "less multiplicative" than v occur).

We have seen that $v(1+\delta,r)$ is the product of a constant and a multiplicative function, and by Dirichlet convolution it follows that $w(1+\delta,m)$ is also:

$$w(1+\delta,m)=v(1+\delta,1)\overline{w}(m)$$

with $\overline{w}$ multiplicative.

We extract the common divisor of m_1 and m_2 and remove the ensuing coprimality condition by Möbius inversion:

$$\begin{aligned}\sum_b\frac{1}{b}\sum_{m_1,m_2}\frac{w(1+\delta,m_1m_2)}{(m_1m_2)^{1+\delta}}x_{bm_1}\overline{x_{bm_2}}&=v(1+\delta,1)\sum_b\frac{1}{b}\sum_{m_1,m_2}\frac{\overline{w}(m_1m_2)}{(m_1m_2)^{1+\delta}}x_{bm_1}\overline{x_{bm_2}}\\&=v(1+\delta,1)\sum_b\frac{1}{b}\sum_a\frac{\overline{w}(a^2)}{a^{2(1+\delta)}}\sum_{(m_1,m_2)=1}\frac{\overline{w}(m_1)\overline{w}(m_2)}{(m_1m_2)^{1+\delta}}x_{abm_1}\overline{x_{abm_2}}\\&=v(1+\delta,1)\sum_b\frac{1}{b}\sum_a\frac{\overline{w}(a^2)}{a^{2(1+\delta)}}\sum_d\frac{\mu(d)\overline{w}(d)^2}{d^{2(1+\delta)}}\sum_{m_1,m_2}\frac{\overline{w}(m_1)\overline{w}(m_2)}{(m_1m_2)^{1+\delta}}x_{adbm_1}\overline{x_{adbm_2}}\\&=v(1+\delta,1)\sum_k\frac{\tilde{\nu}_\delta(k)}{k}\Big|\sum_m\frac{\overline{w}(m)}{m^{1+\delta}}x_{km}\Big|^2\end{aligned}\tag{86}$$

with

$$\tilde{\nu}_\delta(k)=\sum_{abd=k}\frac{\mu(d)\overline{w}(d)^2\overline{w}(a^2)}{(ad)^{1+2\delta}}.$$

Remember that $\tilde{\nu}(k)$ also depends on t (through η_t involved in $\overline{w}$).

Lemma 14. *There exists an absolute constant $b > 0$ such that if $|\delta| \leq b(\log q)^{-1}$ then*

$$\tilde{\nu}_\delta(k) \geq 0$$

for all $t \in \mathbf{R}$ and all $k < q$, and

$$v(1+\delta, 1) \gg 1$$

for all $t \in \mathbf{R}$.

Proof. By multiplicativity it is enough to consider $k = p$ prime, $p < q$. Then $e^{-2b} \leq p^{2\delta} \leq e^{2b}$. We have

$$\nu_\delta(p) = 1 + \frac{1}{p^{1+2\delta}}(\overline{w}(p^2) - \overline{w}(p)^2)$$

and by direct computation, from Lemma 12 and the definition of $w(1+\delta, m)$

$$\begin{aligned}\overline{w}(p) &= \eta_t(p) + (p^{1+2\delta} - 1)p^{-2(1+\delta)}\frac{\eta_t(p)}{1 + p^{-2(1+\delta)}} \\ &= \eta_t(p)\frac{p^{2+2\delta} + p^{1+2\delta}}{p^{2+2\delta} + 1}\end{aligned}$$

and similarly

$$\begin{aligned}\overline{w}(p^2) &= \eta_t(p^2) + \eta_t(p)^2\frac{p^{1+2\delta} - 1}{p^{2+2\delta} + 1} + \frac{p^{1+2\delta} - 1}{p^{2+2\delta}} \\ &= \eta_t(p)^2\frac{p^{2+2\delta} + p^{1+2\delta}}{p^{2+2\delta} + 1} - 1 + \frac{p^{1+2\delta} - 1}{p^{2+2\delta}}\end{aligned}$$

hence

$$\overline{w}(p^2) - \overline{w}(p)^2 = -\eta_t(p)^2\frac{p^{2+2\delta} + p^{1+2\delta}}{p^{2+2\delta} + 1}\frac{p^{1+2\delta} - 1}{p^{2+2\delta} + 1} - 1 + \frac{p^{1+2\delta} - 1}{p^{2+2\delta}}.$$

For p large enough, the result is now clear, uniformly in δ, and we leave to the reader the choice of her argument for dealing with small primes. For instance, note that for $p = 2$, $\delta = 0$, we obtain

$$\overline{w}(p^2) - \overline{w}(p)^2 \geq -4\frac{6}{25} - 1 + \frac{1}{4} > -2,$$

and the result is now clear by continuity. □

We can now conclude this part of the argument.

Proposition 7. *Assume that $\delta = b(\log q)^{-1}$ with $b > 0$ a fixed constant such that the previous lemma applies. Then*

$$\sum_{f \in S_2(q)^*} |M(f, \beta + it)L(f, \beta + it)|^2 \ll q(1 + |t|)^B$$

for some absolute constant $B \geq 0$. The implied constant depends now only on Δ.

Proof. ¿From Lemma 14, and the computation of $\mathcal{M}(\delta)$ and the subsequent diagonalization of the main term, we see that for q large enough we have

$$\mathcal{M}(-\delta) \geq 0$$

hence, using the same trick as before that $\zeta_q(1-2\delta) \leq 0$, we get by positivity the inequality

$$\mathcal{M}_2(\delta) \leq \zeta(1+2\delta)\mathcal{M}(\delta) \ll v(1+\delta,1)\zeta(1+2\delta)\sum_k \frac{\tilde{\nu}_\delta(k)}{k}\Big|\sum_m \frac{\overline{w}(m)}{m^{1+\delta}}x_{km}\Big|^2.$$

Now, in terms of the linear forms y_k introduced in (71), we can write

$$\sum_m \frac{\overline{w}(m)}{m^{1+\delta}}x_{km} = \sum_{a,b} \frac{\eta_t(a)\eta_t(b)u(1+\delta,b)}{(ab)^{1+\delta}(b^{2(1+\delta)}+1)}x_{abk} = \sum_b \frac{u(1+\delta,b)\eta_t(b)}{b^{1+\delta}(b^{2(1+\delta)}+1)}y_{bk}$$

since for squarefree n we have $N(n)=n$. But $u(1+\delta,b) \leq b^{1+2\delta}$ for b squarefree, and Proposition 6 gives immediately

$$\sum_m \frac{\overline{w}(m)}{m^{1+\delta}}x_{km} \ll \frac{1}{k^{(1+\delta)}\xi(k)}\frac{1}{\log q}$$

and then the proof is completed as for the harmonic average, going back to (75) to conclude. □

Proposition 47 is an easy consequence of this estimate near the critical line. Indeed, it first immediately provides the bound

$$\sum_{f\in S_2(q)^*} |M(f,\beta+it)L(f,\beta+it)-1|^2 \ll q(1+|t|)^B \tag{87}$$

for $\beta = \frac{1}{2}+b(\log q)^{-1}$. On the other hand, for $\mathrm{Re}\,(s)=\sigma>1$, we have

$$\sum_{f\in S_2(q)^*} |M(f,s)L(f,s)-1|^2 \ll q^{1-\Delta(1-\sigma)}(\log q)^{30} \tag{88}$$

as a consequence of the trivial individual bound of Lemma 46.

Finally by means of (87), (88) and a simple (and well-known) extension of the classical convexity principle of Phragmen-Lindelöf one deduces Proposition 4.

5 Non-vanishing of central values

The main step in the previous section was the evaluation of the mollified second moment

$$M_2(\delta) = \sum_{f\in S_2(q)^*} |M(f)L(f,\tfrac{1}{2}+\delta+it)|^2.$$

Here $\delta>0$, but a slight adaptation of the method yields a similar formula for the second moment of central critical values

$$M_2 = \sum_{f\in S_2(q)^*} |M(f)L(f,\tfrac{1}{2})|^2$$

for any mollifier of the form

$$M(f) = \sum_{m \le M} \frac{x_m}{\sqrt{m}} \lambda_f(m). \tag{89}$$

Such an estimate has implications to the problem of estimating how many forms $f \in S_2(q)^*$ satisfy $L(f, \frac{1}{2}) \neq 0$. Indeed, if we let

$$M_1 = \sum_{f \in S_2(q)^*} M(f) L(f, \tfrac{1}{2})$$

be the corresponding first moment, we have by Cauchy's inequality (compare [Du], [KM3], etc...)

$$|\{f \in S_2(q)^* \mid L(f, \tfrac{1}{2}) \neq 0\}| \ge \frac{M_1^2}{M_2} \tag{90}$$

provided $M_2 \neq 0$. We extend our method here to deduce Theorem 3.

5.1 The first moment

We consider the mollifier (89) in general, with real coefficients (x_m). For simplicity we assume they are supported on squarefree numbers and are not too large, namely

$$x_m \ll (\tau(m) \log m)^A \tag{91}$$

for some (absolute) constant $A \ge 0$.

Proposition 8. *Let (x_m) be as above. For $M = q^\Delta$ and $\Delta < 1/4$, there exists some $\delta = \delta(\Delta) > 0$ such that*

$$M_1 = \sum_{f \in S_2(q)^*} M(f) L(f, \tfrac{1}{2}) = |S_2(q)^*| \zeta(2) \sum_m \frac{x_m d_{-1}(m)}{m} + O(q^{-\delta}), \tag{92}$$

where

$$d_{-1}(n) = \sum_{d|n} d^{-1}.$$

Proof. Since this is similar but simpler than the second moment, we only give a sketch. The method of Section 3 is applicable so we need only prove that the formula holds for

$$M_1(x) = \sum_{f \in S_2(q)^*}^{h} \omega_f(x) M(f) L(f, \tfrac{1}{2})$$

where $x = q^\kappa$, for some $\kappa > 0$, except for the factor $|S_2(q)^*| \zeta(2)^{-1}$.

Considering the integral expression

$$J = \frac{1}{2i\pi} \int_{(3)} \Lambda(f, s) G(s) \frac{ds}{s - 1/2}$$

and shifting contour and using the functional equation (5) and the Dirichlet series expansion we obtain

$$L(f, \tfrac{1}{2}) = (1 + \varepsilon_f) \sum_{n \ge 1} \frac{\lambda_f(n)}{\sqrt{n}} W\Big(\frac{\sqrt{2\pi} n}{\sqrt{q}}\Big)$$

where

$$W(y) = \frac{1}{2i\pi}\int_{(3)} \Gamma(s+1)G(s+\tfrac{1}{2})y^{-s}\frac{ds}{s}.$$

This test function has fast decay for y large and is $= 1 + O(y)$ for $y \to 0$.

Summing now over f we obtain the desired expression, using (13) and (14), together with (76) and (77) (see the previous computation of $\mathcal{M}_2(\delta)$). One has to take κ small enough, depending on Δ, to obtain the required error term. □

5.2 The second moment

The second moment of central critical values

$$M_2 = \sum_{f \in S_2(q)^*} |M(f)L(f, \tfrac{1}{2})|^2$$

corresponds to the case $\delta = 0$ of the computations done in Section 5. There are minor differences, due to a second order pole at 0 in the integral giving the corresponding test function: the latter is now

$$U(y) = 2\frac{1}{2i\pi}\int_{(3)} H(s)\zeta_q(1+2s)y^{-s}\frac{ds}{s}$$

which satisfies

$$U(y) = -\frac{\varphi(q)}{q}\log y + C_q + O(y^{1/2})$$

as $y \to 0$ ($C_q = c_0 + O(q^{-1}\log q)$ for some explicitly computable but unimportant constant c_0).

The same computations leading to (79) now will show that for $M = q^{\Delta}$ with $\Delta < 1/4$ we have

$$M_2 = \frac{\varphi(q)}{q}\sum_b \frac{1}{b}\sum_{m_1,m_2}\Bigl(\log\frac{q}{4\pi^2 m_1 m_2}\Bigr)\frac{x_{bm_1}x_{bm_2}}{m_1 m_2}w(m_1m_2) + R(x) + O(q^{-\delta}) \qquad (93)$$

for some $\delta > 0$, where $w(n)$ is the arithmetic function denoted by $w(1,n)$ in Section 4. As for $R(x)$, it is a quadratic form, whose contribution to the second moment will turn out, after choosing a mollifier (x_m), to be of smaller order of magnitude than the one displayed. It arises from the non-multiplicativity of the logarithm function coming from U near 0, specifically we decompose

$$\log\frac{qr^2}{4\pi^2 m_1 m_2 d^2} = \log\frac{q}{4\pi^2 m_1 m_2} + 2\log r - 2\log d$$

and R contains what comes from the two extra terms. Dealing with $R(x)$ will not be done here; see [KM3] or [Kow] for complete details of the principles involved.

Note that

$$w(n) = \sum_{r|n}\tau\Bigl(\frac{m}{r}\Bigr)u(r)v(r)$$

with (see Lemma 12)

$$u(r) = \sum_{ab=n}\mu(a)b = \prod_{p|n}(p-1), \quad v(r) = \frac{\zeta(2)^4}{\zeta(4)}N(r)^{-2}\prod_{p||r}\frac{2}{1+p^{-2}}.$$

We now diagonalize the quadratic form $Q(x)$ above as in Section 4 and we find that

$$Q(x) = \frac{\varphi(q)}{q}\frac{\zeta(2)^4}{\zeta(4)}\sum_k \tilde{\nu}(k)\sum_{m_1,m_2}\frac{\overline{w}(m_1)\overline{w}(m_2)}{m_1m_2}x_{km_1}x_{km_2}\Big(\log\frac{q}{4\pi^2 m_1m_2}+2\tilde{\psi}(k)\Big) \tag{94}$$
$$= \frac{\varphi(q)}{q}\frac{\zeta(2)^4}{\zeta(4)}(Q_1(x)+Q_2(x))$$

where

$$\tilde{\nu}(k) = \frac{1}{k}\sum_{abd=k}\frac{\mu(d)\overline{w}(d)^2\overline{w}(a^2)}{ad},$$

and

$$\tilde{\psi}(k) = \frac{1}{\tilde{\nu}(k)}\sum_{abd=k}\frac{\mu(d)\overline{w}(d)^2\overline{w}(a^2)}{b}\log\frac{1}{ad} = \sum_{p|k}\Big(\frac{1}{p\tilde{\nu}(p)}-1\Big)\log p = O(\log\log k).$$

We will now choose a vector (x_m) with $m \le M$, satisfying the required conditions, and normalized so that

$$\sum_m \frac{x_m d_{-1}(m)}{m} = 1 \tag{95}$$

which minimizes the first term $Q_1(x)$ (not involving $\tilde{\psi}$) of the quadratic form Q; then we evaluate and estimate the other terms, which will prove the non-vanishing theorem by (90).

We let

$$y_k = \sum_m \frac{\overline{w}(m)}{m}x_{km},$$
$$y'_k = \sum_m \frac{\overline{w}(m)}{m}x_{km}\log m = \sum_\ell \frac{\overline{w}(\ell)}{\ell}\Lambda(\ell)y_{k\ell}. \tag{96}$$

so that

$$Q_1(x) = \Big(\log\frac{q}{4\pi^2}\Big)\sum_k \tilde{\nu}(k)y_k^2 - 2\sum_k \tilde{\nu}(k)y_ky'_k,$$
$$Q_2(x) = 2\sum_k \tilde{\nu}(k)\tilde{\psi}(k)y_k^2.$$

Let j be the arithmetic function such that

$$M_1 = \sum_k j(k)y_k + \text{error term}$$

namely j is the Dirichlet convolution

$$j = g \star d_{-1},$$

g being the Dirichlet convolution inverse of $\overline{w}$.

If we choose to optimize only the first term of Q_1, subject to (95), the optimal choice is

$$y_k = \frac{1}{J}\frac{\mu(k)^2 j(k)}{\tilde{\nu}(k)}$$

where

$$J = \sum_k \frac{\mu(k)^2 j(k)^2}{\tilde{\nu}(k)}.$$

After some computations, we find that

$$j(p) = -A(p^{-1}), \text{ where } A = \frac{(1-X)(1+X)^2}{1+X^2}$$

$$\tilde{\nu}(p) = B(p^{-1}), \text{ where } B + \frac{(1-X^2)^3}{(1+X^2)^2}$$

hence for squarefree k we have

$$\frac{j(k)^2}{\tilde{\nu}(k)} = \frac{1}{k}\prod_{p|k}\frac{1+p^{-1}}{1-p^{-1}}.$$

We find that

$$\sum_{k\geq 1}\frac{\mu(k)^2 j(k)^2}{\tilde{\nu}(k)}k^{-s} = \prod_p \Bigl(1 + \frac{2p^{-(s+1)}}{(1+p^{-(s+1)})(p-1)}\Bigr)$$

which has analytic continuation to $\mathrm{Re}\,(s) > -1$ with a simple pole at $s = 0$ with residue

$$R = \frac{1}{\zeta(2)}\prod_p \Bigl(1 + \frac{2}{p^2-1}\Bigr) = \frac{\zeta(2)}{\zeta(4)}$$

hence $J = \zeta(2)/\zeta(4)\log M + O(1)$ (see [Kow]). Using this one checks easily the growth condition (91).

It remains to evaluate the other term of Q_1 and Q_2 for this specific choice of x_m. This is done using (96) and the formula

$$\frac{\overline{w}(p)j(p)}{p\tilde{\nu}(p)} = -\frac{2}{p} + O(p^{-2})$$

for p prime to evaluate y'_k (compare [KM3], [Kow]) as follows:

$$\begin{aligned} y'_k &= \sum_{\ell\leq M/k}\frac{\overline{w}(\ell)}{\ell}\Lambda(\ell)y_{k\ell} \\ &= \frac{2}{J}\frac{\mu(k)^2 j(k)^2}{\tilde{\nu}(k)}\sum_{\substack{p\leq M/k\\(p,k)=1}}\frac{j(p)^2}{\tilde{\nu}(p)}\log p \\ &= -2y_k\Bigl(\log\frac{M}{k}\Bigr) + O(j(k)/\tilde{\nu}(k)). \end{aligned}$$

Now we have, using the definition of y_k and this formula

$$\begin{aligned} \sum_{k\leq M}\tilde{\nu}(k)y_k^2 &= \frac{1}{J} \text{ by definition} \\ \sum_{k\leq M}\tilde{\nu}(k)y_k y'_k &= -\frac{2}{J}\Bigl\{(\log M)\sum_{k\leq M}\tilde{\nu}(k)y_k^2 - \sum_{k\leq M}\tilde{\nu}(k)y_k^2\log k\Bigr\} + \text{error term} \\ &= -2\Bigl\{\frac{\log M}{J} - \frac{1}{2}\frac{\log M}{J}\Bigr\} \\ &= -\frac{\log M}{J}, \end{aligned}$$

by summation by parts.

One estimates now directly that Q_2 satisfies

$$Q_2(x) = O\Big(Q_1(x)\frac{\log\log q}{\log q}\Big)$$

for this choice of (x_m). It follows that

$$M_1 = \zeta(2)^2|S_2(q)^*| + O(q^{-\delta})$$
$$M_2 = 2\zeta(2)^4|S_2(q)^*|\Big(1+\frac{\log q}{2\log M}\Big)\Big(1+O\Big(\frac{\log\log q}{\log q}\Big)\Big)$$

for some $\delta = \delta(\Delta) > 0$ if $\Delta < 1/4$. The non-vanishing theorem follows by (90), letting $\Delta \to 1/4$.

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