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The anisotropic reflectivity technique: theory

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Summary. The reflectivity technique for the calculation of synthetic seismograms from a point source in a horizontally stratified isotropic structure is extended to include weakly anisotropic layers. The formulation is in terms of displacement excitation factors rather than potential functions, which have not yet been specified for wave propagation in anisotropic media. Coupling between vertical, radial and transverse components of motion increases the number of plane-wave reflection and transmission coefficients which must be computed for any problem. These coefficients are calculated by extending Kennett's iterative scheme for the computation of isotropic coefficients to stratified anisotropic structures.

1 Introduction

Theoretical and numerical calculations show that there are fundamental differences between the propagation of seismic waves in isotropic and anisotropic media (Crampin 1977a). Many of these differences are subtle and difficult to detect on observed seismograms, and the calculation of synthetic seismograms is an important technique for determining the effect of anisotropy on seismic wave propagation in any particular Earth structure (Crampin 1981).

The detection and analysis of seismic anisotropy in the Earth's crust and upper mantle is becoming increasingly important as the number of possible applications continues to increase: Crampin (1977b) and Meissner & Flüh (1979) have drawn attention to the possibility of anisotropy in the crystalline upper mantle, which may contain information about the past and present deformation processes and the tectonic history of the lithosphere. Enough anisotropy may be present in shallow sedimentary beds to be important in exploration geophysics (Crampin & Radovich 1982; Gal'perin 1977). Anisotropic effects are displayed by structures containing aligned cracks, and these can be modelled by propagation through homogeneous anisotropic media once appropriate elastic constants have been determined (Crampin 1978). Such crack anisotropy has applications to hot-dry-rock geothermal heat extraction, and to the study of phenomena associated with earthquake dilatancy. Shear wave splitting, diagnostic of anisotropy, has now been observed near the Northern Anatolian Fault in Turkey by Crampin *et al.* (1980).

For a great many investigations of anisotropic structures the seismic effects are too complicated to understand without numerical experiments with synthetic seismograms. Synthetic seismograms for *plane wave* propagation through anisotropic media have been calculated by Keith & Crampin (1977c) and Crampin (1978). However, plane waves have very limited applications in the Earth, where all local and regional arrivals have curved wavefronts. Since the directions of group (energy) and phase propagation generally diverge in anisotropic media (Crampin 1981), there are fundamental differences in behaviour between waves with plane and curved wavefronts in anisotropic propagation (Crampin & McGonigle 1981).

In this paper, we modify the reflectivity technique of Fuchs & Müller (1971), for calculating synthetic seismograms from point sources in layered media, to accept structures where some of the layers may have generally orientated anisotropic symmetries. There are several ways to calculate synthetic seismograms from point sources in layered structures. The particular attraction of modifying the reflectivity technique for anisotropic propagation is that it makes use of propagator matrices (Gilbert & Backus 1966) or reflection and transmission coefficients at interfaces (Kennett 1974), which can be readily calculated for anisotropic structures through the work of Crampin (1970), and Keith & Crampin (1977a, b).

2 Wave propagation in anisotropic media

We shall first indicate some of the fundamental differences between wave propagation in isotropic and anisotropic structure. There are three body waves propagating in every direction in anisotropic media: a quasi-compressional wave, qP, and two quasi-shear waves, $qS1$ and $qS2$, with velocities which vary with direction, and with particle-motion which also varies with direction, but has fixed orthogonal polarizations for any particular direction of phase propagation (Crampin 1981). These polarizations are not in general coincident with the dynamic axes formed by the wavefront and the propagation vector, and, in particular, the polarizations of $qS1$ and $qS2$ are only parallel to SH and SV for propagation in particular symmetry directions of the anisotropy.

It is not possible to describe plane wave propagation in mutlilayered anisotropic media separately in terms of wave motion in the sagittal and transverse directions, as in isotropic layered structures. Consequently, there are differences between the matrix formulations in the reflectivity method which give the plane wave responses of multilayered isotropic, and anisotropic media. The calculation of the plane wave response of a multilayered structure for a wide range of frequencies and horizontal wavenumbers is a fundamental part of the reflectivity technique. The appropriate matrix formulations for multilayered anisotropic media will be described in detail in Section 4.

The variation of body wave velocity with direction means that the direction of group velocity propagation is not in general perpendicular to the plane of constant phase, which propagates at the phase velocity. It is the phase velocity which is represented in the wave equations and in most other analytical expressions, whereas it is the group velocity which is the velocity of propagation of wave energy and is measured in most observations.

The deviation of the energy propagation vector from the phase propagation vector in anisotropic media gives rise to differences in the propagation characteristics of waves with plane and spherical wavefronts. Two orthogonally polarized *plane* shear waves will propagate in any direction at the *phase* velocity, with parallel wavefronts which are normal to the phase propagation vector. However, the velocity of arrival of a *spherical* wave from a point source is the *group* velocity of energy propagation along ray which deviates from the direction of the phase propagation vector. At least two shear waves propagate along a ray in

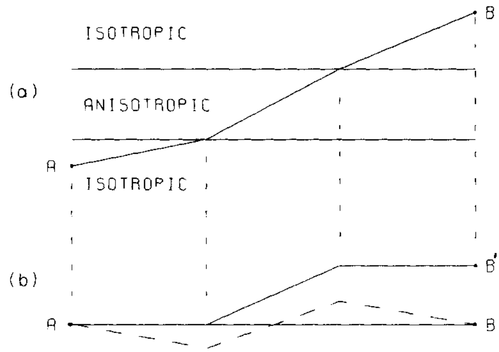

Figure 1. Schematic illustration of a ray path through a plane horizontal layer possessing general anisotropic symmetry, where the sagittal plane is not a plane of symmetry. (a) Sectional view. (b) Plan view. In order that the wave energy may propagate from A to B, rather than B′, the propagation vector at A must diverge from the vertical plane through A and B.

any direction, and there may be additional arrivals if the direction intercepts cusps in the group velocity surface (sometimes called the wave surface). These arrivals will not, in general, have orthogonal polarizations and will be associated with different directions of phase propagation.

Some of the effects of the divergence of the group and phase velocity vectors in a horizontally stratified half-space containing isotropic and anisotropic layers are illustrated by Fig. 1. A schematic representation of a ray path from point A to point B through a stratified anisotropic structure is shown in Fig. 1(a). The plan view in Fig. 1(b) shows incident energy radiating from A in the vertical plane containing A and B, which deviates away from this plane in the anisotropic medium, but returns to propagating parallel to the original plane on re-entering an isotropic medium, and propagates to B′, not B. We would need to consider an incident ray out of the vertical plane in order to determine the travel time of wave energy from A to B (Fig. 1b). This behaviour occurs because the reaction at an interface is controlled by the phase propagation vector, whereas the energy is controlled by the group velocity vector. These vectors may deviate by up to 30° for some directions of propagation in strongly anisotropic solids (Crampin, Stephen & McGonigle 1982), although the deviation is much less in the weakly anisotropic materials which we shall consider in this discussion.

The independence of the sagittal and transverse components of motion in isotropic media allow the wave propagation to be described in terms of wave potentials. The wave energy propagates in the sagittal plane, and the displacements in the near and far field at any azimuth from a general type of point source can be expressed in terms of integrals of cylindrical wave potentials (Hudson 1969). Kennett & Kerry (1979) have generalized the original reflectivity technique of Fuchs & Müller (1971) to construct synthetic seismograms at any point in a horizontally stratified isotropic structure, for a buried, general type of point source.

Wave propagation in anisotropic media does not have a convenient wave potential representation, and a general formulation equivalent to that of Kennett & Kerry has not yet been derived. In order to make the problem tractable, we impose some restrictions on the initial conditions: we consider only the far-field response due to a point source in an isotropic surface layer; we allow only weakly anisotropic media to form the stratified half-space; and as in the original reflectivity technique, free surface reflections above the source are ignored, although their effect may be included at the receiver.

3 The reflectivity technique for anisotropic structures

We use the horizontally stratified model structure in Fig. 2, which is similar to that used for the original isotropic reflectivity technique (Fuchs 1971), but we now permit anisotropic layers in the reflection zone.

A point source of elastic energy lies at the origin of a Cartesian coordinate system immediately below the surface of the plane-layered half-space ($x_3 > 0$). The half-space, which is numbered as in Fig. 2, consists of a surface layer containing the source, and the reflection zone comprised of n homogeneous anisotropic layers. An analytical description of the wavefield from a source in an anisotropic medium has not yet been derived, and we avoid difficulties in the description of the source by stipulating that the source layer is isotropic.

The point source in the isotropic layer generates three possible source wave types j, where $j = 1, 2, 3$ represents P-, SV-, and SH-waves respectively. Six plane wave types $p = 1, 2, \ldots 6$, with the same horizontal wavenumber vector $(k_1, k_2, 0)$ may propagate away from the source in each layer. The wave types $p = 1, 2, 3$ represent qP, $qS1$, and $qS2$ wave propagating downwards, and $p = 4, 5, 6$ their upward travelling equivalents.

The wave potential of a curved wavefront in an isotropic medium can be expressed as an integral of plane, or cylindrical wave potential solutions to the isotropic equations of motion through the Weyl, or Sommerfeld integrals, respectively. Wave propagation in anisotropic media can be described in terms of plane wave displacements (Keith & Crampin 1977a). The far-field component displacement spectra at $(x_1, 0, x_3)$ in an anisotropic layer m due to a point-source at $(0, 0, 0)$ is then written as a superposition of plane wave displacements in an expression which corresponds to the Weyl integral in isotropic media:

$$u_i(\omega) = F(\omega) \sum_{j=1}^{3} \sum_{p=1}^{6} \int_0^\infty \int_0^\infty S_j \{ f_j^m(p)\, a_i^m(p) \exp[-i\omega q_p^m (x_3 - d_m)] \times \exp(ik_1 x_1) \} \, dk_1 dk_2; \quad (3.1)$$

where $F(\omega)$ is the source spectrum, and the summations include all possible source wave types $j = 1, 2, 3$ and all waves propagating in the receiver layer, $p = 1, 2, \ldots 6$. The part of the integrand in curly brackets represents the anisotropic plane wave displacements, and the integration is over both components of the horizontal wavenumber vector, k_1 and k_2.

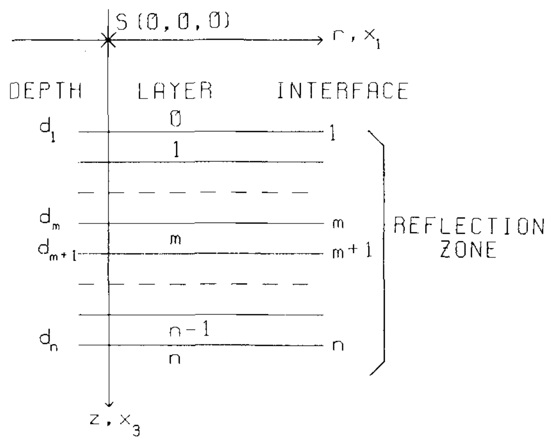

Figure 2. Configuration of the reflection zone. The source, S, is located at the top of an isotropic layer above the multilayered reflection zone.

The variable q_p^m is the vertical slowness of wave-type p in layer m. In isotropic media, where $p = 1, 2, 3$ represents downward propagating *P*-, *SV*-, and *SH*-waves respectively, we can write q_p^m explicitly as:

$$\begin{aligned} q_j^m &= [(k_j^m)^2 - k^2]^{1/2}/\omega, \qquad && k < k_j^m \\ &= -i[k^2 - (k_j^m)^2]^{1/2}/\omega, \qquad && k > k_j^m; \\ q_{j+3}^m &= -q_j^m, \qquad \text{for} && j = 1, 2, 3; \end{aligned} \tag{3.2}$$

where k_j^m is the wavenumber in the direction of propagation of the wave type j in layer m, and k is the horizontal wavenumber in the direction of the receiver. The $a_i^m(p)$ terms represent the direct cosines of the unit amplitude polarization vector of each wave type p in the component directions x_i, $i = 1, 2, 3$, in layer m. They are given by simple expressions in terms of k, k_p^m, and q_p^m in an isotropic layer (Table 1).

In an anisotropic layer, q_p^m and $a_i^m(p)$ cannot be written explicitly. They are found by solving either a sextic polynomial for q_p^m and three simultaneous equations for the polarization vector (Crampin 1970), or a linear eigenvalue problem, where the eigenvalues are the squares of the velocities and the eigenvectors are the polarizations (Taylor & Crampin 1978; Crampin 1981).

The $f_j^m(p)$ terms are excitation factors which give the relative effect of each plane wave to the total displacement in layer m for a source wave type j. When evaluated in isotropic media, these displacement excitation factors are equivalent to, though not in general equal to, the excitation factors defined in terms of wave potentials which have been used by Fuchs (1971), Kennett (1974) and many others. The excitation factors $f_j^0(p)$, $p = 4, 5, 6$ and the factors $f_j^n(p)$, $p = 1, 2, 3$ correspond to reflection and transmission coefficients, respectively, for the whole layer sequence. They include the effects of all reverberations and mode conversions in the reflection zone, with the associated path-dependent phasing of the response. Thus they are a function of frequency, horizontal wavenumber and the elastic constants of the reflection zone. The technique for the computation of the $f_j^m(p)$ in an anisotropic reflection zone is described in detail in the next section.

The source function S_j gives the amplitude directivity of each of the three possible wave types *P, SV,* and *SH* transmitted from the point source. It is a function of horizontal wavenumber and the elastic constants of the source layer. Since the propagation of energy at the group velocity diverges from the phase propagation vector in anisotropic media, S is a very complicated function in anisotropic media, and so we confine the source to an isotropic layer at the top of the half-space.

An evaluation of the wave energy arriving from a point source at any point in an anisotropic reflection zone will require a superposition of plane waves whose propagation vectors are not in general confined to the sagittal plane. Thus the computation of the far-field displacement spectra of a curved wavefront from an anisotropic reflection zone requires a double integration over the sagittal and transverse components of the horizontal wave-

Table 1. Relative values of $a_i(p)$ for an isotropic layer in terms of wavenumbers k, k_1, k_2, ν_1 and ν_2 where $\nu_i = \omega q_i$, and the superscript m is omitted.

i	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
1	k/k_1	ν_2/k_2	0	k/k_1	ν_2/k_2	0
2	0	0	1	0	0	1
3	ν_1/k_1	$-k/k_2$	0	$-\nu_1/k_1$	k/k_2	0

number vectors, as in (3.1). The terms $S_j, f_j^m(p), a_i^m(p)$, and q_p^m are functions of k_1 and k_2, and $f_j^m(p)$ is also a function of ω.

In an isotropic reflection zone, the displacement spectra due to a point source may be obtained by appropriate differentiations of the Sommerfeld integral representations of the wave potentials (Müller 1969). The expression for the far-field component displacement spectra at $(x_1, 0, x_3)$ can be written in terms of a single integral over k_1:

$$u(\omega) = F(\omega) \sum_{j=1}^{3} \sum_{p=1}^{6} S_j J_0(k_1 x_1) f_j^m(p)\, a_i^m(p) \exp[-i\omega q_p^m (x_3 - d_m)]\, dk_1; \tag{3.3}$$

where J_0 is the Bessel function of the first kind and zero order. The terms $S_j, f_j^m(p), a_i^m(p)$ and q_p^m refer to the same parameters as the equivalent terms in (3.1), but in an isotropic reflection zone their dependence on wavenumber is confined to the sagittal component of the horizontal wavenumber vector, k_1. Expressions for q_p^m and $a_i^m(p)$ have been given in (3.2) and Table 1, and S_j is given in Table 2 for the simple types of point source shown in Fig. 3. All the wave energy arriving at the receiver travels in the sagittal plane, and a single integration over k_1 gives the displacement spectra of each wavefront.

The evaluation of the excitation factors $f_j^m(p)$ over the required range of frequencies and horizontal wavenumbers, and the subsequent integration and Fourier transformation to the synthetic seismograms, is expensive in computer time even for isotropic media. In anisotropic media, the $f_j^m(p)$ are intrinsically more difficult to compute, since the anisotropic equations of motion must be solved for the q_p^m and $a_i^m(p)$ in each layer for each horizontal wavenumber. In addition, the propagator matrix formulations yielding the $f_j^m(p)$, which are described in the next section, cannot be simplified by treating the wave motion in the

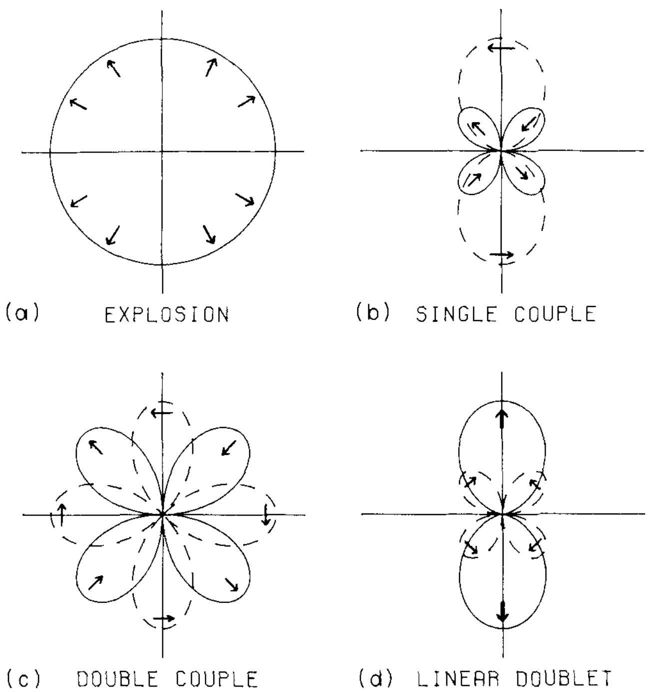

Figure 3. Amplitude directivity patterns for P- and SV-waves (solid and dashed lines, respectively) generated by (a) explosion, (b) single couple, (c) double couple and (d) linear doublet.

Table 2. Values of S_j for each wave type for the point sources in Fig. 3 with the same notation as in Table 1, and superscripts $m = 1$ omitted. The couples may be rotated by θ about the x_3-axis, and the doublet by ϕ about the x_1-axis to generate *SH* motion. $D = -1/4\pi\rho\omega^2$.

Source type	$P: j = 1$	$SV: j = 2$	$SH: j = 3$
Explosion (Fuchs & Müller 1971)	$\dfrac{-k\,k_1}{\nu_1}$	0	0
Single couple (Müller 1969)	$\dfrac{Dkk_1^3}{\nu_1}\left(\dfrac{k\nu_1}{k_1^2}\right)\cos\theta$	$\dfrac{Dkk_2^3}{\nu_2}\left(\dfrac{\nu_2}{k_2}\right)^2\cos\theta$	$\dfrac{Dkk_2^3}{\nu_2}\left(\dfrac{\nu_2}{k_2}\right)\sin\theta$
Double couple (Kind & Müller 1975)	$\dfrac{Dkk_1^3}{\nu_1}\left(\dfrac{2k\nu_1}{k_1^2}\right)\cos\theta$	$\dfrac{Dkk_2^3}{\nu_2}\left(\dfrac{\nu_2^2-k^2}{k_2^2}\right)\cos\theta$	$\dfrac{Dkk_2^3}{\nu_2}\left(\dfrac{\nu_2}{k_2}\right)\sin\theta$
Linear doublet (Pilant 1979)	$\dfrac{Dkk_1^3}{\nu_1}\left(\dfrac{\nu_1}{k_1}\right)^2\cos^2\phi$	$\dfrac{-Dkk_2^3}{\nu_2}\left(\dfrac{k\nu_2}{k_2^2}\right)\cos^2\phi$	$\dfrac{Dkk_2^3}{\nu_2}\left(\dfrac{\nu_2}{k_2}\right)\sin\phi\cos\phi$

sagittal and transverse directions separately. The evaluation of the displacement spectra in (3.1), involves the computation of the integrand over a range of frequencies and *two* wavenumber ranges, corresponding to the radial and transverse components of the horizontal wavenumber vector. These calculations would require unacceptably long computing times with the techniques available to us at present.

However, a good approximation to the true seismograms may be obtained with a single integration over wavenumber, if we only consider *weakly* anisotropic media for which we can assume that the wave energy does not diverge significantly from the sagittal plane. This is equivalent to assuming that the integrand in (3.1) is only weakly azimuthally dependent, and then the expression (3.3) becomes a reasonable approximation to (3.1). Then the $f_j^m(p)$, $a_i^m(p)$ and q_p^m in the anisotropic reflection zone need only be calculated over a range of values of the *sagittal* component of the horizontal wavenumber vector. Note that the assumption that S_j is only weakly azimuthally dependent implies that the approximate expressions for the displacement spectra cannot be used near a node in the azimuthal direction of the source directivity pattern, where there is a rapid variation of signal amplitude with direction.

The matrix formulations which have been developed to calculate the plane wave excitation factors in horizontally stratified anisotropic structures are described in the next section.

4 Calculation of the plane wave response

4.1 THE DIRECT METHOD

The plane wave decomposition in (3.1) and (3.3) allows the use of the propagator matrix formulations of Gilbert & Backus (1966), adapted for anisotropic media by Crampin (1970) and Keith & Crampin (1977b), to describe wave propagation through the sequence of layers. The excitation factors in any layer m may be defined as phased with respect to the top of the layer as in equations (3.1) and (3.3), or the bottom of the layer, and the corresponding 6-vectors are written $\mathbf{f}^m$ and $\bar{\mathbf{f}}^m$ respectively, dropping the subscript j. These vectors are related by the equation:

$$\tilde{\mathbf{f}}^m = \mathsf{D}^m \mathbf{f}^m; \tag{4.1}$$

where

$$\mathsf{D}^m = \mathrm{diag}\,[i\omega q_p^m (d_{m+1} - d_m)], \qquad p = 1, 2 \ldots, 6.$$

We define a stress-displacement 6-vector $\mathbf{v}^m$, whose elements are the displacements and the vertical components of stress, σ_{j3}:

$$\mathbf{v}^m = (u_1, u_2, u_3, \sigma_{13}, \sigma_{23}, \sigma_{33})^{\mathrm{T}}; \tag{4.2}$$

where $\mathbf{v}^m$ is related to the excitation vector $\mathbf{f}^m$ in layer m by

$$\mathbf{v}^m = \mathsf{E}^m \mathbf{f}^m; \tag{4.3}$$

and E^m is given by Keith & Crampin (1977a). The continuity of the stress-displacement vector across any interface m gives:

$$\mathbf{v}^m = \mathsf{E}^{m-1} \tilde{\mathbf{f}}^{m-1}; \tag{4.4}$$

so that, from (4.3), the excitation factors on either side of the mth interface are related by

$$\mathbf{f}^m = (\mathsf{E}^m)^{-1} \mathsf{E}^{m-1} \tilde{\mathbf{f}}^{m-1}. \tag{4.5}$$

The stress-displacement vectors at the top and bottom of any anisotropic layer m are related by:

$$\mathbf{v}^m = \mathsf{A}^m \mathbf{v}^{m-1}; \tag{4.6}$$

where the anisotropic propagator matrix A^m is given by

$$\mathsf{A}^m = \mathsf{D}^m (\mathsf{E}^m)^{-1}; \tag{4.7}$$

Keith & Crampin (1977b). Thus the excitation factors in the layers above and below the reflection zone are related by:

$$\mathbf{f}^n = (\mathsf{E}^n)^{-1} \mathsf{A}^{n-1} \ldots \mathsf{A}^1 \mathsf{E}^1 \tilde{\mathbf{f}}^0. \tag{4.8}$$

Application of the appropriate boundary conditions to the reflection zone for each incident wave type j, and the use of equations (4.3) and (4.6), allows us to solve for the excitation factors in any layer of the multilayered structure. Note that when all the layers are isotropic, the wave propagation in the sagittal and transverse planes decouples, and the sagittal and transverse components of the wave motion may be described separately in terms of 4×4 and 2×2 propagator matrices.

Straightforward calculation of the excitation factors by direct application of the propagator matrices can lead to significant loss of numerical precision. Kind (1976) overcomes these problems by using a system of reduced matrices, the elements of which are derived from all the possible subdeterminants of the isotropic propagator matrices. This method is efficient when applied to the 4×4 and 2×2 isotropic propagator matrices, but it is unsuited for application to the full 6×6 anisotropic propagator matrices.

Kennett (1974) adapted the Gilbert-Backus propagator formulation to produce a convenient iterative method of solution for the excitation factors on either side of a stratified isotropic reflection zone. This technique, which avoids the inaccuracy problem, is capable of easy physical interpretation and is easily extended to an anisotropic reflection zone. Kennett & Kerry (1979) have extended the iterative technique to the calculation of the plane wave response when both source and receiver are located in the reflection zone, with all free surface effects being included. An anisotropic formulation of the work of Kennett & Kerry has not yet been developed, but we shall show how excitation factors due to a source outside an anisotropic reflection zone may be calculated at a receiver located outside or inside the reflection zone, using the work of Kennett (1974) and Stephen (1977).

4.2 THE ITERATIVE METHOD

We partition the matrix D^m (equation 4.1) and the vectors $\mathbf{f}^m$, $\tilde{\mathbf{f}}^m$ into *U*pward and *D*ownward propagating components:

$$\mathsf{D}^m = \begin{pmatrix} D^m_\mathrm{D} & 0 \\ 0 & D^m_\mathrm{U} \end{pmatrix} ; \tag{4.9}$$

and we can write $\mathbf{f}^m = (\mathbf{f}^m_\mathrm{D}, \mathbf{f}^m_\mathrm{U})^\mathrm{T}$.

Following Kennett (1974), we shall be describing the propagation through a sequence of layers in terms of the matrices R^m_U, R^m_D, T^m_U and T^m_D, where the matrix elements are the *R*eflection and *T*ransmission coefficients for *U*pward and *D*ownward propagation for the *m*th interface. Kennett used 2×2 coefficient matrices and two-component excitation-factor vectors $\mathbf{f}^m_\mathrm{D}$, $\mathbf{f}^m_\mathrm{U}$ in his formulation, to describe *P*- and *SV*-wave motion in an isotropic reflection zone. This formulation can be extended to an anisotropic reflection zone by substituting 3×3 coefficient matrices and three-component excitation vectors in the matrix equations.

The boundary conditions at interface m for a wave incident from the $(m-1)$th layer are:

$$\tilde{\mathbf{f}}^{m-1}_\mathrm{U} = R^m_\mathrm{D}\, \tilde{\mathbf{f}}^{m-1}_\mathrm{D};$$

and (4.10)

$$\mathbf{f}^m_\mathrm{D} = T^m_\mathrm{D}\, \tilde{\mathbf{f}}^{m-1}_\mathrm{D};$$

where

$$R^m_\mathrm{D} = \begin{pmatrix} r_{PP} & r_{1P} & r_{2P} \\ r_{P1} & r_{11} & r_{21} \\ r_{P2} & r_{12} & r_{22} \end{pmatrix}^m_\mathrm{D} \quad \text{and} \quad T^m_\mathrm{D} = \begin{pmatrix} t_{PP} & t_{1P} & t_{2P} \\ t_{P1} & t_{11} & t_{21} \\ t_{P2} & t_{12} & t_{22} \end{pmatrix}^m_\mathrm{D} ;$$

and $(r_{1P})^m_\mathrm{D}$, for example, is the interface coefficient for a *r*eflected *qP*-wave from a *qS*1-wave incident *D*ownwards on interface m and phased with respect to the bottom of the $(m-1)$th layer. There are similar relationships for upward propagating incident waves on interface m phased with respect to the top of the mth layer:

$$\mathbf{f}^m_\mathrm{D} = R^m_\mathrm{U}\, \mathbf{f}^m_\mathrm{U};$$

and (4.11)

$$\tilde{\mathbf{f}}^{m-1}_\mathrm{U} = T^m_\mathrm{U}\, \mathbf{f}^m_\mathrm{U}.$$

The interface coefficients are independent of frequency when expressed in this way with the phase of the excitation factors related to the interface where the transformation takes place.

They are obtained by solving (4.5) with the appropriate boundary conditions (4.10) and (4.11) for each incident wave type. The reflection and transmission coefficients correspond to the appropriate upward and downward excitation factors for an incident wave of unit amplitude.

The various R and T matrices are the coefficients for individual interfaces. We write the overall reflection coefficient matrix for the sequence of layers from m to n for *D*ownward propagating incident waves as ${}^m\boldsymbol{R}_{\mathrm{D}}^n$, where the first superscript refers to the layer containing the incident wavefield, and ${}^m\tilde{R}_{\mathrm{D}}^n$ is phased with respect to the bottom of layer m. Thus at the bottom of layer m, we have the relationship:

$$\tilde{\mathbf{f}}_{\mathrm{U}}^m = {}^m\tilde{R}_{\mathrm{D}}^n \tilde{\mathbf{f}}_{\mathrm{D}}^m. \tag{4.12}$$

Phasing the excitation factors with respect to the top of layer m, we have

$$\mathbf{f}_{\mathrm{U}}^m = (D_{\mathrm{U}}^m)^{-1} \, {}^m\tilde{R}_{\mathrm{D}}^n \, D_{\mathrm{D}}^m \, \mathbf{f}_{\mathrm{D}}^m;$$

and (4.13)

$${}^m\boldsymbol{R}_{\mathrm{D}}^n = (D_{\mathrm{U}}^m)^{-1} \, {}^m\tilde{R}_{\mathrm{D}}^n \, D_{\mathrm{D}}^m;$$

and similar expressions for ${}^n\boldsymbol{R}_{\mathrm{U}}^m$, ${}^m\boldsymbol{T}_{\mathrm{D}}^n$ and ${}^n\boldsymbol{T}_{\mathrm{U}}^m$; where the $\boldsymbol{T}$ are the overall transmission coefficient matrices.

The iterative schemes which give the overall reflection and transmission coefficient for a sequence of layers are given by equations (30) and (31) of Kennett (1974). The iterative expression which gives ${}^{m-1}\boldsymbol{R}_{\mathrm{D}}^n$ in terms of ${}^m\boldsymbol{R}_{\mathrm{D}}^n$ is:

$${}^{m-1}\boldsymbol{R}_{\mathrm{D}}^n = (D_{\mathrm{U}}^{m-1})^{-1} \, [R_{\mathrm{D}}^m + T_{\mathrm{U}}^m \, {}^m\boldsymbol{R}_{\mathrm{D}}^n \, (I - R_{\mathrm{U}}^m \, {}^m\boldsymbol{R}_{\mathrm{D}}^n)^{-1} \, T_{\mathrm{D}}^m] \, D_{\mathrm{D}}^{m-1}. \tag{4.14}$$

All reverberations and associated mode-conversions in the iterative schemes are contained in the 3×3 matrix terms of the form $(I-X)^{-1}$. We may calculate the complete response of a sequence of layers by evaluating (4.14) with matrix inversions, or obtain only the response for the direct waves, or the direct waves and the first reverberations by expanding $(I-X)^{-1}$ and truncating the series expansion (Kennett 1974). This facility allows the reflectivity technique to be used for a wide range of applications, where, for example, propagation in a non-attenuative structure with a wave guide would result in a long train of reverberations and lead to time-aliasing problems unless the number of reverberations is truncated in this way.

The iterative equation (4.14) is applied successively until we have the response ${}^0\boldsymbol{R}_{\mathrm{D}}^n$ for the whole layered sequence for a source in layer 0. The appropriate excitation factors to evaluate (3.3) at a receiver at the surface from a surface source are then:

$$\mathbf{f}_{\mathrm{U}}^0 = {}^0\boldsymbol{R}_{\mathrm{D}}^n \, \mathbf{f}_{\mathrm{D}}^0. \tag{4.15}$$

When we wish to calculate seismograms at a receiver which is located in a layer within the reflection zone, the excitation factors for upward *and* downward waves must be calculated. Stephen (1977) obtained the following expressions for the excitation factors of upward and downward propagating waves in layer m within the reflection zone:

$$\mathbf{f}_{\mathrm{U}}^m = {}^m\boldsymbol{R}_{\mathrm{D}}^n \, (I - {}^m\boldsymbol{R}_{\mathrm{U}}^0 \, {}^m\boldsymbol{R}_{\mathrm{D}}^n)^{-1} \, {}^0\boldsymbol{T}_{\mathrm{D}}^m \, \mathbf{f}_{\mathrm{D}}^0;$$

and (4.16)

$$\mathbf{f}_{\mathrm{D}}^m = (I - {}^m\boldsymbol{R}_{\mathrm{U}}^0 \, {}^m\boldsymbol{R}_{\mathrm{D}}^n)^{-1} \, {}^0\boldsymbol{T}_{\mathrm{D}}^m \, \mathbf{f}_{\mathrm{D}}^0.$$

These expressions (4.16) are also valid for an anisotropic reflection zone when 3×3 coefficient submatrices and three-component excitation factor vectors are used. The expressions contain an inverse matrix of the same form as that in (4.14), and its series expansion may be similarly truncated to give the response for direct waves only, or direct waves and the first reverberations.

5 Computational procedure

The terms in the integrand of (3.3) are calculated for a range of frequencies covering the source spectrum $F(\omega)$, and a range of horizontal wavenumbers (or equivalently, slownesses) covering the principal wave arrivals of interest (Fuchs & Müller 1971). Synthetic seismograms at $(x_1, 0, x_3)$ are obtained by integration and Fourier transformation of the spectra given by (3.3). Although we have ignored free surface reflections above the source, their effect at a receiver located on the free surface can easily be included (Červený & Ravindra 1971), as follows: conversion coefficients given by Červený & Ravindra are substituted for the components of the amplitude polarization vectors $a_i^0(p)$ of each upgoing wave type at the receiver. These conversion coefficients include the contributions to the overall displacement at the receiver from waves reflected downwards at the free surface. Their signs correspond to those of the appropriate $a_i^0(p)$ for upgoing waves in Table 1.

The effect of attenuation in an anisotropic stratified structure may be included by representing the attenuation coefficient $1/Q$ by appropriate imaginary parts of the otherwise real elastic constants (Crampin 1981). Since all the other variables in the program are already complex, very little alteration is required to introduce attenuation into the calculations. The provision for attenuation is necessary when the attenuation of a structure is known, but it is also convenient to prevent overflow and time-aliasing problems, for example, when the displacements in a low-velocity channel are calculated. For a channel without attenuation, the calculation may involve integration through the poles of the secular function which correspond to guided channel waves. This secular function is given by the determinant of the matrix of the form $(I-X)$ in equations (4.16). The poles are shifted off the path of integration along the real wavenumber axis when the channel is made slightly attenuative. Thus the full effect of the low-velocity channel in creating channel waves is retained, and numerical overflow problems do not arise.

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