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THE AP-DENJOY AND AP-HENSTOCK INTEGRALS REVISITED

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Abstract. The note is related to a recently published paper J. M. Park, J. J. Oh, C.-G. Park, D. H. Lee: The AP-Denjoy and AP-Henstock integrals. Czech. Math. J. 57 (2007), 689–696, which concerns a descriptive characterization of the approximate Kurzweil-Henstock integral. We bring to attention known results which are stronger than those contained in the aforementioned work. We show that some of them can be formulated in terms of a derivation basis defined by a local system of which the approximate basis is known to be a particular case. We also consider the relation between the σ-finiteness of variational measure generated by a function and the classical notion of the generalized bounded variation.

Keywords: approximate Kurzweil-Henstock integral, approximate continuity, local system, variational measure

MSC 2010: 26A39, 26A42, 26A46

1. Introduction

This note is motivated by a recently published paper [9] which concerns a descriptive characterization of the approximate Kurzweil-Henstock integral. The point is that unfortunately none of the results of that work can be found new, moreover, some of them were known before in a reasonably stronger version. We discuss this in more detail in Section 2 where in particular we recall some known results from which the results of [9] follow. Let us remark, by the way, that the paper [9] is in fact repetition, practically word by word, of the earlier papers [7] and [8] by the same group of authors.

Validity of some results in this direction depends essentially on the assumption of measurability of functions or sets under consideration. We consider this problem in Section 3 and show that the a priori assumption of measurability can be dropped

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in some cases. This allows us to generalize one of our results from [12] related to σ-finiteness of the variational measure defined by the approximate derivation basis. Moreover, we formulate this and some other results in a little bit more general setting, namely in terms of a derivation basis defined by a local system (see [14]) of which the approximate basis is known to be a particular case.

In the last Section 4 we consider the relation between the σ-finiteness of the variational measure generated by a function and the classical notion of the generalized bounded variation.

2. Preliminaries. Remarks on the known results

We recall some definitions and notation comparing them with those of [9]. By a *local system* [14] we mean a family $\Delta = \{\Delta(x)\}_{x\in\mathbb{R}}$ such that each $\Delta(x)$ is a nonvoid collection of subsets of $\mathbb{R}$ with the properties

(i) $\{x\} \notin \Delta(x)$,
(ii) if $S \in \Delta(x)$ then $x \in S$,
(iii) if $S \in \Delta(x)$ and $R \supset S$ then $R \in \Delta(x)$,
(iv) if $S \in \Delta(x)$ and $\delta > 0$ then $(x-\delta, x+\delta) \cap S \in \Delta(x)$.

A local system Δ is said to be *bilateral* if, for each $x \in \mathbb{R}$, every set $S \in \Delta(x)$ contains points on either sides of x. A local system Δ is said to be *filtering* if at each point $x \in \mathbb{R}$ we have $S_1 \cap S_2 \in \Delta(x)$ whenever S_1 and S_2 belong to $\Delta(x)$. Any $S \in \Delta(x)$ is called a *path* leading to x. A function $\mathcal{C}$ on $E \subset \mathbb{R}$ such that $\mathcal{C}(x) \in \Delta(x)$ for each $x \in E$ is called a *choice* on E. Given a choice $\mathcal{C}$, we write $(I, x) \in \beta_{\mathcal{C}}$ and say that a *tagged interval* (I, x) is *$\mathcal{C}$-fine* if $x \in I$ and both endpoints of the closed interval I are in $\mathcal{C}(x)$. A family $\{\beta_{\mathcal{C}}\}_{\mathcal{C}}$ where $\mathcal{C}$ runs over all choices, is often called a *derivation basis* (see [13]) generated by the local system Δ. A finite collection of $\mathcal{C}$-fine tagged intervals $\{(I_i, x_i)\}_{i=1}^{n}$ with $I_i \cap \operatorname{int} I_j = \emptyset$ for $i \neq j$ is called a *$\mathcal{C}$-fine division.* It is said to be *tagged* in a set $E \subset \mathbb{R}$ if $x_i \in E$ for each $i = 1, \ldots, n$. It is said to be a *$\mathcal{C}$-fine partition of* an $[a, b]$ if $\bigcup\limits_{i=1}^{n} I_i = [a, b]$.

The *approximate derivation basis* is generated by a particular local system, namely by the *density* local system Δ_{ap} defined by the notion of the density of a set at a point: $\Delta_{\mathrm{ap}}(x)$ for each $x \in \mathbb{R}$ is the collection of all $D \subset \mathbb{R}$ such that there is Lebesgue measurable $E \subset D$ with $x \in E$ and $d(E, x) = 1$, where $d(E, x)$ stands for the density of E at x ($\bar{d}(E, x)$ and $\underline{d}(E, x)$ will analogously mean the upper and the lower density of E at x).

Let Δ be a local system, $F\colon \mathbb{R} \to \mathbb{R}$. The value

$$\omega_{\Delta}(F, x) = \inf_{S\in\Delta(x)} \sup_{y\in S} |F(x) - F(y)|$$

is called the Δ-*oscillation* of F at x. We say F is Δ-*continuous* at x if $\omega_\Delta(F,x)=0$. We say F is Δ-*differentiable* at x if $\omega_\Delta(Q^x_F,x)=0$, where $Q^x_F(y)=(F(y)-F(x))\times(y-x)^{-1}$, $y\neq x$, $Q^x_F(x)=d$ for some $d\in\mathbb{R}$; d is then the Δ-*derivative* $D_\Delta F(x)$ *of* F *at* x. Let $E\subset\mathbb{R}$. By the Δ-*variational measure* of E generated by F we understand the value

$$V^\Delta_F(E)=\inf_{\mathcal{C}}\sup_{D}\sum_{i=1}^{n}|F(b_i)-F(a_i)|,$$

where sup is taken over all $\mathcal{C}$-fine divisions $D=\{([a_i,b_i],x_i)\}_{i=1}^n$ tagged in E and inf is taken over all choices $\mathcal{C}$. We say V^Δ_F is σ-*finite* on a set $E\subset\mathbb{R}$ if $E=\bigcup_{n=1}^{\infty}E_n$ and $V^\Delta_F(E_n)<\infty$ for each n. We say V^Δ_F is *absolutely continuous* if $V^\Delta_F(E)=0$ for each nullset E.

The class of functions generating an absolutely continuous variational measure is widely used in the Kurzweil-Henstock theory of integration. In the case of approximate basis the authors of [7]–[9] use the term “AL functions (approximate Lusin functions)” to name functions of this class.

Another class which also plays an important role in this theory is the class of functions of generalized absolute continuity with respect to a basis. For the case of the basis generated by a local system Δ the definition is as follows (see [3]). A function F is said to be AC_Δ on a set E if for any $\varepsilon>0$ there exist $\delta>0$ and a choice $\mathcal{C}$ on E such that $\sum_{i=1}^{n}|F(b_i)-F(a_i)|<\varepsilon$ for any $\mathcal{C}$-fine division $D=\{([a_i,b_i],x_i)\}_{i=1}^n$ tagged in E with $\sum_{i=1}^{n}(b_i-a_i)<\delta$. F is said to be ACG_Δ on E if $E=\bigcup_{k=1}^{\infty}E_k$ and F is AC_Δ on E_k for each k.

Among almost everywhere (in the sense of Lebesgue measure) Δ-differentiable functions, for a wide class of local systems Δ the class ACG_Δ is known to coincide with the class of functions generating an absolutely continuous Δ-variational measure. In the case of approximate basis this is Lemma 2.2 of [9]. But it was known earlier for a more general case (see [5, Theorem 5.1]).

To define, following [15], a Kurzweil-Henstock type integral with respect to a local system Δ we assume that Δ is bilateral, filtering and has the *partitioning property*: for each choice $\mathcal{C}$ and each $[a,b]$ there is a $\mathcal{C}$-fine partition of $[a,b]$.

We say a function $f\colon [a,b]\to\mathbb{R}$ is Δ-*integrable* if there is a number A, the value of the Δ-integral, such that for each $\varepsilon>0$ we can find a choice $\mathcal{C}$ with the property that $\left|\sum_{i=1}^{n}f(x_i)|I_i|-A\right|<\varepsilon$ holds provided $\{(I_i,x_i)\}_{i=1}^n$ is a $\mathcal{C}$-fine partition of $[a,b]$. In this case we write $A=(\Delta)\int_a^b f$. The value of the integral is unique because Δ is assumed to be filtering. In these terms the approximate Kurzweil-Henstock integral

(AP-Henstock integral as in [9]) is the Δ_{ap}-integral. Note that the local system Δ_{ap} has the partitioning property (see [6, Lemma 3]).

For the Δ-integral many of the usual properties, known also for more general classes of bases, hold. In particular (see [1]–[3], [6], [15]):

P1) If a function f is Δ-integrable on $[a,b]$, then it is also Δ-integrable on each subinterval of $[a,b]$. Therefore the indefinite Δ-integral $F(x) = (\Delta)\int_a^x f$ is defined for any $x \in [a,b]$.

P2) The Δ-indefinite integral F of f is Δ-continuous at each point of $[a,b]$ and it is Δ-differentiable a.e. with $D_\Delta F(x) = f(x)$ a.e. on $[a,b]$.

P3) A function F, $F(a) = 0$, is the indefinite Δ-integral of a function f on $[a,b]$ if and only if F generates an absolutely continuous Δ-variational measure and F is Δ-differentiable a.e. with $D_\Delta F(x) = f(x)$ a.e. on $[a,b]$.

P4) A function F, $F(a) = 0$, is the indefinite Δ-integral of a function f on $[a,b]$ if and only if F is an ACG_Δ-function Δ-differentiable a.e. with $D_\Delta F(x) = f(x)$ a.e. on $[a,b]$.

The properties P3) and P4) are examples of the so-called partial descriptive characterizations of the indefinite integral (see [10]). A certain drawback of these characterizations is that Δ-differentiability a.e. of the functions in the class of primitives is included into the characterization as an additional assumption. That is why they are called "partial" to distinguish them from a "full descriptive characterization" in which the differentiability a.e. of all the functions in the class of primitives is implied by the main characteristic of the class.

The main results given in [7] and [9] can be summarized (in our language) as follows: *the class of indefinite Δ_{ap}-integrals coincides with the class of all a.e. approximately differentiable functions with an absolutely continuous $\mathrm{V}_F^{\Delta_{\text{ap}}}$*. So it is a partial descriptive definition of the Δ_{ap}-integral and a particular case of the above property P3) which was known earlier in the local system setting (see [15]).

A deeper result giving full descriptive characterization of the Δ_{ap}-integral is

Theorem 1. *The class of indefinite Δ_{ap}-integrals coincides with the class of all functions F generating an absolutely continuous $\mathrm{V}_F^{\Delta_{\text{ap}}}$.*

This result can be easily obtained from [5, Theorem 5.1]. In our paper [12] this theorem was formulated as a corollary of a more general result. Making use of Vasile Ene's lemma (Lemma 2, next section), we have pointed out in [12] that even an assumption weaker than absolute continuity of the approximate variational measure $\mathrm{V}_F^{\Delta_{\text{ap}}}$ generated by F, namely finiteness of $\mathrm{V}_F^{\Delta_{\text{ap}}}$ on each nullset, implies that F is almost everywhere approximately differentiable. In the next section we show that the assumption of finiteness in the last statement can be replaced by an assumption of σ-finiteness of $\mathrm{V}_F^{\Delta_{\mathrm{ap}}}$ on nullsets. As we shall see, the problem of measurability is involved in the proof.

3. Problems of measurability

We recall that an $F\colon E \to \mathbb{R}$, $E \subset \mathbb{R}$, is a VBG-function if $E = \bigcup_{n=1}^{\infty} E_n$ and F is VB on each E_n. If a VBG-function F on a measurable set E is measurable, then it is approximately differentiable (i.e., Δ_{ap}-differentiable) at almost every $x \in E$. This is the so-called Denjoy-Khintchine theorem [11, Chapter 7 (4.3)].

The easiest proof of Theorem 1 rests on two facts. The first is that if a variational measure $\mathrm{V}_F^{\Delta_{\mathrm{ap}}}$, defined with respect to the density local system Δ_{ap}, is absolutely continuous, then F is approximately continuous and so measurable. The second is

Lemma 2 (see [4, Theorem 3]). *A measurable $F\colon [a,b] \to \mathbb{R}$ is* VBG *if and only if it is so on each nullset.*

So F, as a measurable VBG-function, is a.e. approximately differentiable, by Denjoy-Khintchine theorem, and Theorem 1 follows. Let us stop for a moment and remark that the measurability assumption in Lemma 2 is essential. This is established by the following simple example.

Example. There is a function $F\colon [0,1] \to [0,1]$ which is not VBG, but it is so on each null subset of $[0,1]$.

Construction. We will use the continuum hypothesis and transfinite induction. Let Ω be the first uncountable ordinal. Arrange all reals of $[0,1]$ into a transfinite sequence $\{p_\alpha\}_{\alpha<\Omega}$. Arrange so all $\mathcal{G}_\delta$ null subsets of $[0,1]$: $\{G_\alpha\}_{\alpha<\Omega}$. Put $H_0 = G_0$ and for each $\alpha < \Omega$ take an $\tilde{\alpha} < \Omega$ such that

$$G_{\tilde{\alpha}} \supset \bigcup_{\beta<\alpha} H_\beta \cup G_\alpha$$

and such that

$$G_{\tilde{\alpha}} \setminus \bigcup_{\beta<\alpha} H_\beta \quad \text{is uncountable.} \tag{1}$$

Define $H_\alpha = G_{\tilde{\alpha}}$. Clearly $\{H_\alpha\}_{\alpha<\Omega}$ is ascending and $\bigcup_{\alpha<\Omega} H_\alpha = \bigcup_{\alpha<\Omega} G_\alpha = [0,1]$. Put $F(x) = p_\alpha$ if $x \in H_\alpha \setminus \bigcup_{\beta<\alpha} H_\beta$. Of course F is VBG on each nullset $D \subset [0,1]$. Indeed, $D \subset G_\alpha \subset H_\alpha$ for some $\alpha < \Omega$. Thus the image of D under F is contained in the countable set $\{p_\beta\}_{\beta\leqslant\alpha}$.

Due to (1), F takes on each $x \in [0,1]$ as a value uncountably many times. Such a function cannot be VBG. Indeed, suppose $[0,1] = \bigcup_{n=1}^{\infty} E_n$ and F is VB on each E_n. Denote $I_n = \{p_\alpha\colon\ E_n \cap F^{-1}(p_\alpha) \text{ is infinite}\}$, $n \in \mathbb{N}$. Then $|I_n| > 0$ for some n, for if not, from $[0,1] = \bigcup_{n=1}^{\infty} I_n$ we would have $|[0,1]| = 0$ (here and below $|\cdot|$ stands for the Lebesgue outer measure). Fix this n and consider the indicatrix function I of $F \restriction E_n$. It is a consequence of the Banach indicatrix theorem [11, Chapter 9 (6.4)] that

$$\int_{-\infty}^{\infty} I \leqslant V(F \restriction E_n) < \infty.$$

On the other hand, $I(y) = \infty$ for each $y \in I_n$, whence

$$\int_{-\infty}^{\infty} I \geqslant \infty \cdot |I_n| = \infty,$$

a contradiction. □

We aim at discussing both the steps of the already described proof of Theorem 1. We consider under what assumptions put on the local system Δ each of the above steps can follow. This will lead us to some generalizations.

First, we consider the question when measurability of $F\colon\ \mathbb{R} \to \mathbb{R}$ will follow from σ-finiteness of V_F^Δ on nullsets. We assume until the end of this section that a local system $\Delta = \{\Delta(x)\}_{x\in\mathbb{R}}$ has the following two extra properties, the latter being a strengthening of the condition (iv):

(a) if $S \in \Delta(x)$ then there exists a measurable set $R \subset S$ such that

$$\underline{d}(R,x) > 0;$$

(b) if $S \in \Delta(x)$ and $d(R,x) = 1$, $R \ni x$, then $S \cap R \in \Delta(x)$.

Lemma 3. *The set $\mathcal{C}_F^\Delta$ of all points at which F is Δ-continuous is a measurable subset of $\mathbb{R}$. Moreover, $F \restriction \mathcal{C}_F^\Delta$ is a measurable function.*

Proof. Fix $x \in \mathcal{C}_F^\Delta$ and $n \in \mathbb{N}$. There is a path $S_x^n \in \Delta(x)$ such that $|F(x) - F(y)| < 1/n$ for each $y \in S_x^n$. By (a) there is a measurable $R_x^n \subset S_x^n$, $R_x^n \ni x$, with $\underline{d}(R_x^n, x) > 0$. By virtue of the Lebesgue density theorem we can assume that R_x^n has density 1 at each $y \in R_x^n$, $y \neq x$. We claim that the set

$$S^n = \bigcup_{x\in\mathcal{C}_F^\Delta} R_x^n$$

is measurable. Suppose not. Let E be a measurable hull, D a measurable kernel of S^n; we have $|S^n \setminus D| = |E \setminus D| > 0$. Pick an $x \in S^n \setminus D$ which is a density point of $E \setminus D$. By the definition of S^n there is a measurable set $R^n_{x'} \subset S^n$ with positive lower density at x. So, also $(E \setminus D) \cap R^n_{x'} \subset S^n \setminus D$ has positive lower density at x and consequently is not a nullset, a contradiction with the fact that D is a measurable kernel of S^n. So S^n is measurable. To get the measurability of $\mathcal{C}^{\Delta}_F$ it is enough to check that

$$\mathcal{C}^{\Delta}_F = \bigcap_{n=1}^{\infty} S^n.$$

By definition $\mathcal{C}^{\Delta}_F \subset \bigcap\limits_{n=1}^{\infty} S^n$. The converse inclusion follows from the fact that $\omega_{\Delta}(F, y) \leqslant 2/n$ for each $y \in R^n_x$, $x \in \mathcal{C}^{\Delta}_F$ (see the property (b)).

Let $\alpha \in \mathbb{R}$. We will show the set $E = \{x \in \mathcal{C}^{\Delta}_F \colon F(x) < \alpha\}$ is measurable. Since $\mathcal{C}^{\Delta}_F$ is measurable and F is Δ-continuous on it, for almost every $x \in E$ there is a measurable $S \subset E$ with $\underline{d}(S, x) > 0$. Hence, in order to show measurability of E, we can pattern the argument of the first part of the proof. □

Theorem 4. *Assume V^{Δ}_F is σ-finite on each nullset. Then $F \colon \mathbb{R} \to \mathbb{R}$ is measurable.*

P r o o f. It is enough to show F is nearly everywhere Δ-continuous, see Lemma 3. Suppose the set $\mathcal{D}^{\Delta}_F$ (of all points at which F is not Δ-continuous) is uncountable. Since it is measurable (Lemma 3 again) we can pick an uncountable nullset $D \subset \mathcal{D}^{\Delta}_F$. Let $D = \bigcup\limits_{m=1}^{\infty} D_m$, where $V^{\Delta}_F(D_m) < \infty$ for each m. For at least one m, say m_0, the set D_m is also uncountable. So, there is an $n \in \mathbb{N}$ such that the set $E = \bigl\{x \in D_{m_0} \colon \omega_{\Delta}(F, x) > 1/n\bigr\}$ is infinite. Pick a sequence $\{x_k\}_{k=1}^{\infty} \subset E$. As V^{Δ}_F is a metric outer measure, it becomes a measure if restricted to the class of Borel sets, and so

$$V^{\Delta}_F(D_{m_0}) \geqslant V^{\Delta}_F\bigl(\{x_k\}_{k=1}^{\infty}\bigr) = \sum_{k=1}^{\infty} V^{\Delta}_F(\{x_k\}) \geqslant \sum_{k=1}^{\infty} \omega_{\Delta}(F, x_k) \geqslant \sum_{k=1}^{\infty} \frac{1}{n} = \infty,$$

a contradiction with $V^{\Delta}_F(D_{m_0}) < \infty$. □

In [12] we have proved that if the measure $\mathrm{V}^{\Delta_{\mathrm{ap}}}_F$ generated by a *measurable* function F on $[a, b]$ is σ-finite on each nullset, then it is σ-finite on $[a, b]$. In turn, this implies F is almost everywhere approximately differentiable. Combining this with Theorem 4 we finally get

Theorem 5. *Assume $V^{\Delta_{\mathrm{ap}}}_F$ is σ-finite on each nullset. Then $V^{\Delta_{\mathrm{ap}}}_F$ is σ-finite and F is approximately differentiable a.e.*

The second step in proving Theorem 1 was to establish that the σ-finiteness of $\mathrm{V}_F^{\Delta_{\mathrm{ap}}}$ implies the VBG property of F. We extend this result to the case of more general local systems.

Assume that the local system Δ satisfies, beside (i)–(iv) by definition, also the so-called intersection condition

(c) for each choice $\{\mathcal{C}(x)\}_{x\in\mathbb{R}}$ there is a $\delta\colon \mathbb{R}\to(0,\infty)$ such that $\mathcal{C}(x)\cap\mathcal{C}(y)\cap[x,y]\neq\emptyset$ if $|x-y|\leqslant\min\{\delta(x),\delta(y)\}$.

Theorem 6. *Let $F\colon\mathbb{R}\to\mathbb{R}$ and let Δ be as above. Suppose the variational measure V_F^{Δ} is σ-finite on $E\subset\mathbb{R}$. Then F is* VBG *on E.*

Proof. There is a sequence $\{E_n\}_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} E_n=E$ and $V_F^{\Delta}(E_n)<\infty$ for each n. It means that for a fixed n there is a choice $\{\mathcal{C}(x)\}_{x\in E_n}$ such that $\sum\limits_{i=1}^{r}|F(b_i)-F(a_i)|<V_F^{\Delta}(E_n)+1$ for each $\mathcal{C}$-fine division $\{([a_i,b_i],x_i)\}_{i=1}^{r}$ tagged in E_n. Pick a function δ according to (c) for $\{\mathcal{C}(x)\}_{x\in E_n}$ and define $E_{kln}=\bigl\{x\in E_n\colon \delta(x)\geqslant 1/k\bigr\}\cap\bigl[(l-1)/k,l/k\bigr]$, $k\in\mathbb{N}$, $l\in\mathbb{Z}$. Consider any collection of pairwise nonoverlapping intervals $\{[c_j,d_j]\}_{j=1}^{s}$ with all $c_j,d_j\in E_{kln}$. Since $d_j-c_j\leqslant 1/k\leqslant\min\{\delta(c_j),\delta(d_j)\}$, $j=1,\dots,s$, there is $\xi_j\in[c_j,d_j]\cap\mathcal{C}(c_j)\cap\mathcal{C}(d_j)$. The collection

$$\bigl\{([c_j,\xi_j],c_j),([\xi_j,d_j],d_j)\bigr\}_{j=1}^{s}$$

is thus a $\mathcal{C}$-fine division tagged in E_n, whence

$$\sum_{j=1}^{s}|F(d_j)-F(c_j)|\leqslant\sum_{j=1}^{s}|F(\xi_j)-F(c_j)|+\sum_{j=1}^{s}|F(d_j)-F(\xi_j)|<V_F^{\Delta}(E_n)+1<\infty.$$

We have proved F is VB on each E_{kln}, thus VBG on E. □

We will show now that the assumption related to the form of the intersection condition cannot be weakened in the above theorem by replacing it with the so-called external intersection condition of the form $\mathcal{C}(x)\cap\mathcal{C}(y)\neq\emptyset$, even in case of a continuous function. As an example it is enough to consider some $\mathcal{P}$-adic path system with a sequence $\mathcal{P}$ rapidly growing to ∞. We need more terminology.

Let $\mathcal{P}=\{p_j\}_{j=0}^{\infty}$ be a fixed sequence of integers with $p_j>1$ for all j. We set $m_0=1$, $m_k=p_0p_1\dots p_{k-1}$ for $k\geqslant 1$. We call the closed intervals

$$I_r^{(k)}=\left[\frac{r}{m_k},\frac{r+1}{m_k}\right],\quad r\in\mathbb{Z},$$

for fixed $k=0,1,\dots$, the *$\mathcal{P}$-intervals of rank k*.

The points r/m_k where $r \in \mathbb{Z}$ and $k = 0, 1, \ldots$, constitute the set of all $\mathcal{P}$*-adic rationals* on $\mathbb{R}$. Its complementary set on $\mathbb{R}$ is the set of all $\mathcal{P}$*-adic irrationals* on $\mathbb{R}$. For each $\mathcal{P}$-adic irrational point x there exists only one $\mathcal{P}$-interval $I_x^{(k)} = [a_x^{(k)}, b_x^{(k)}]$ of rank k containing x so that $\{x\} = \bigcap\limits_{k=0}^{\infty} [a_x^{(k)}, b_x^{(k)}]$. We say that the sequence $\{[a_x^{(k)}, b_x^{(k)}]\}_k$ of nested $\mathcal{P}$-intervals is the *basic sequence of* $\mathcal{P}$*-intervals convergent to* x. If x is a $\mathcal{P}$-adic rational point, then there exist two descending sequences of $\mathcal{P}$-intervals for which x is a common endpoint starting with some k; i.e., for such a point we have two basic sequences convergent to x: the left and the right one. Now we define the system of $\mathcal{P}$-adic paths. If x is a $\mathcal{P}$-adic irrational we denote by $P_-(x)$ and $P_+(x)$ the sequences $\{a_x^{(k)}\}_k$ and $\{b_x^{(k)}\}_k$ convergent to x, respectively, which are given by the definition of the basic sequence of $\mathcal{P}$-intervals. Then the set $P_x = P_-(x) \cup P_+(x) \cup \{x\}$ is the $\mathcal{P}$*-adic path at* x. In the case of a $\mathcal{P}$-adic rational x, we denote by $P_-(x)$ (by $P_+(x)$) the sequence of the left (right) endpoints of the intervals from the left (right) basic sequence. The definition of the $\mathcal{P}$-adic path P_x at x is the same as in the case of a $\mathcal{P}$-adic irrational. We denote by $\Delta_{\mathcal{P}}$ the local system generated by these $\mathcal{P}$-adic paths; i.e., $\Delta_{\mathcal{P}}(x) = \{P_x \cap (x - \delta, x + \delta) \colon \delta > 0\}$, $x \in \mathbb{R}$.

Example. There is a sequence $\mathcal{P}$ and a continuous function $F\colon [0,1] \to \mathbb{R}$ such that $\mathrm{V}_F^{\Delta_{\mathcal{P}}}$ is σ-finite, but F is not VBG on some nullset $A \subset [0,1]$.

Construction. We consider the sequence $\mathcal{P}$ with $p_0 = m_1 = 24$ and $p_k = k m_k$ for $k \geqslant 1$. For each $I_r^{(k)}$ we define a subinterval

$$J_r^{(k)} = \left[\frac{r}{m_k} + \frac{1}{4m_k}, \frac{r+1}{m_k} - \frac{1}{4m_k}\right].$$

Note that it can be represented as a union of $\frac{1}{2}p_k = \frac{1}{2}k m_k$ $\mathcal{P}$-intervals of rank $k+1$.

For each $k \geqslant 1$ we define on $[0,1]$ the following continuous and piecewise linear functions:

$$F_k(x) = \begin{cases} 0 & \text{if } x = 0 \ \text{ or } \ x = 1, \\ \dfrac{1}{m_{k-1}} & \text{if } x \in I_{1+6j}^{(k)},\ j = 0, 1, \ldots, \frac{1}{6}m_k - 1, \\ -\dfrac{1}{m_{k-1}} & \text{if } x \in I_{4+6j}^{(k)},\ j = 0, 1, \ldots, \frac{1}{6}m_k - 1, \\ \text{linear} & \text{on the closure of each interval contiguous to the above.} \end{cases}$$

We also define the sequence of sets

$$A_1 = J_0^{(0)} = \left[\frac{1}{4}, \frac{3}{4}\right], \quad A_{k+1} = A_k \cap \bigcup_{j=0}^{m_k/6-1} (J_{1+6j}^{(k)} \cup J_{4+6j}^{(k)}).$$

We put $A = \bigcap_{k=1}^{\infty} A_k$. It is clear that A is a perfect set of measure zero containing only $\mathcal{P}$-adic irrationals. We define

$$F = \sum_{k=1}^{\infty} F_k \chi_{A_k}.$$

This series is uniformly convergent and so F is a continuous function which is piecewise linear (with a countable number of linear pieces) on each interval contiguous to A. We show that F is not VBG on A. To this end, it is enough to prove that F is VB on no nonempty portion of A. Note that any such portion P contains a portion of the type $A \cap J_r^{(k)}$ for any $k \geqslant k_0$ and for a suitable r. Fix such a portion for a certain k. It has nonempty intersection with $\frac{1}{12}p_k = \frac{1}{12}km_k$ intervals of the type $I_{1+6j}^{(k+1)}$ and $\frac{1}{12}km_k$ intervals of the type $I_{4+6j}^{(k+1)}$. We can choose $\frac{1}{12}km_k$ intervals $[\alpha_j, \beta_j]$ with endpoints in $A \cap J_r^{(k)}$ such that $\alpha_j \in A \cap I_{1+6j}^{(k+1)}$ and $\beta_j \in A \cap I_{4+6j}^{(k+1)}$. As $\sum_{s=1}^{k} F_s$ is constant on each $A \cap J_r^{(k)} \subset I_r^{(k)}$, we have

$$\begin{aligned} F(\alpha_j) - F(\beta_j) = {} & F_{k+1}(\alpha_j)\chi_{A_{k+1}}(\alpha_j) + \sum_{s=k+2}^{\infty} F_s(\alpha_j)\chi_{A_s}(\alpha_j) \\ & - F_{k+1}(\beta_j)\chi_{A_{k+1}}(\beta_j) - \sum_{s=k+2}^{\infty} F_s(\beta_j)\chi_{A_s}(\beta_j) \\ & \geqslant \frac{2}{m_k} - \sum_{s=k+2}^{\infty} \frac{2}{m_{s-1}} \geqslant \frac{2}{m_k} - 2\frac{2}{m_{k+1}} = \frac{2}{m_k} - 2\frac{2}{km_k^2} \geqslant \frac{1}{m_k}. \end{aligned}$$

So

$$\sum_{j=1}^{km_k/12} |F(\alpha_j) - F(\beta_j)| \geqslant \frac{k}{12}$$

and F is not VB on the portion $P \supset A \cap J_r^{(k)}$ and hence is not VBG on $[0,1]$.

Now we estimate the lower and upper $\Delta_{\mathcal{P}}$-derivatives of F at $x \in A$: $\underline{D}_{\mathcal{P}}F(x)$ and $\overline{D}_{\mathcal{P}}F(x)$. Let $x \in A$. Then $\{x\} = \bigcap_{k=0}^{\infty} [a_x^{(k)}, b_x^{(k)}]$. We have $x - a_x^{(k)} \geqslant \frac{1}{4}m_k^{-1}$ and $b_x^{(k)} - x \geqslant \frac{1}{4}m_k^{-1}$ for each k. Note that $\sum_{s=1}^{k} F_s\chi_{A_s}$ is constant on $[a_x^{(k)}, b_x^{(k)}]$ if $x \in A$ and $F(a_x^{(k)}) = F(b_x^{(k)}) = \sum_{s=1}^{k} F_s(x)\chi_{A_s}(x)$. Then

$$\left| \frac{F(x) - F(a_x^{(k)})}{x - a_x^{(k)}} \right| \leqslant \sum_{s=k+1}^{\infty} \frac{|F_s(x)\chi_{A_s}(x)|}{|x - a_x^{(k)}|} \leqslant \frac{\sum_{s=k+1}^{\infty} (m_{s-1})^{-1}}{(4m_k)^{-1}} \leqslant \frac{2}{m_k} 4m_k = 8$$

and in the same way,

$$\left|\frac{F(b_x^{(k)}) - F(x)}{b_x^{(k)} - x}\right| \leqslant 8.$$

So,

$$-8 \leqslant \underline{D}_{\mathcal{P}} F(x) \leqslant \overline{D}_{\mathcal{P}} F(x) \leqslant 8$$

if $x \in A$. This implies that $\mathrm{V}_F^{\Delta_{\mathcal{P}}}(A) = 0$. □

Problem. Assume the variational measure of $F\colon \mathbb{R} \to \mathbb{R}$ related to a local system Δ is σ-finite on each nullset. Prove (perhaps under some extra condition) that V_F^{Δ} is σ-finite on $\mathbb{R}$.

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