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THE APPELL'S FUNCTION F_2 FOR LARGE VALUES OF ITS VARIABLES

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Abstract. The second Appell's hypergeometric function $F_2(a,b,b',c,c';x,y)$ has a Mellin convolution integral representation in the region $\Re(x+y)<1$ and $a>0$. We apply a recently introduced asymptotic method for Mellin convolution integrals to derive three asymptotic expansions of $F_2(a,b,b',c,c';x,y)$ in decreasing powers of x and y with x/y bounded. For certain values of the real parameters a, b, b', c and c', two of these expansions involve logarithmic terms in the asymptotic variables x and y. Some coefficients of these expansions are given in terms of the Gauss hypergeometric function ${}_3F_2$ and its derivatives.

1. Introduction. The Appell's functions F_1, F_2, F_3 and F_4 are generalizations of the Gauss hypergeometric function ${}_2F_1$ [8, p. 224]. In particular, the second Appell's function F_2 is defined by means of the double infinite sum [14, p. 789]:

$$F_2(a,b,b',c,c';x,y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n m!n!}x^m y^n, \quad |x|+|y|<1. \tag{1.1}$$

Appell's functions have physical applications in several problems of Quantum Mechanics. For example, they appear in the computation of transition matrices in atomic and molecular physics, such as the transitions that involve Coulombic continuum states [7] or ion-atom collisions [6]. They are also representations of the generalized Slater's and Marvin's integrals [16] and the solution of certain ordinary differential equations and partial

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differential equations [17]. There is an extensive mathematical literature devoted to the study of these functions: Sharma has obtained generating functions of the Appell's functions [15]. Some integral representations for F_1 and F_2 have been derived by Manocha [12] and Mittal [13]. The Laplace transforms of these functions have been obtained in [9]. Some reduction formulas for special values of the variables and contiguous relations for Appell's functions have been investigated by Buschman [2], [3]. Carlson has investigated quadratic transformations of Appell's functions [5] and their role on multiple averages [4].

Convergent and asymptotic expansions of $F_1(a,b,c,d;x,y)$ for large values of x and/or y have been obtained in [10] by using the distributional approach introduced by Wong [[18], chap. 6]. The technique used there uses an integral representation of F_1 in the form of a Stieltjes transform. The purpose of this paper is to obtain asymptotic expansions of $F_2(a,b,b'c,c';x,y)$ for large values of x and y with fixed $a,b,b'c,c'$. To this end we do not use the distributional approach here; instead we use a more general asympotic technique valid for any Mellin convolution integral [11].

The starting point is the integral representation [[8], p. 230]

$$F_2(a,b,b',c,c';x,y)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\ \ {}_1F_1(b,c;xs){}_1F_1(b',c';ys)ds, \tag{1.2}$$

where ${}_1F_1(a,b;c)$ is a Confluent Hypergeometric function. This representation is valid for $a>0$ and $\Re(x+y)<1$. It defines the analytical continuation of $F_2(a,b,b',c,c';x,y)$ defined in (1.1) in the x and y complex planes from the region $|x|+|y|<1$ to the region $\Re(x+y)<1$.

After the change of variables $t=s|\tilde{x}|$, with $\tilde{x}=1/|x|$, and defining $\gamma:=\tilde{y}/\tilde{x}:=|x|/|y|$, the above integral reads

$$F_2(a,b,b',c,c';x,y)=\frac{\tilde{x}}{\Gamma(a)}\int_0^\infty e^{-\tilde{x}t}(\tilde{x}t)^{a-1}\ \ {}_1F_1(b,c,te^{i\alpha}){}_1F_1(b',c',te^{i\beta}/\gamma)dt, \tag{1.3}$$

where α and β represent the principal arguments of the respective complex variables x and y: $x=|x|e^{i\alpha}$ and $y=|y|e^{i\beta}$.

We are interested in the approximation of $F_2(a,b,b',c,c';x,y)$ for large x and y uniformly in γ with $0<\gamma_0\le\gamma\le\gamma_1<\infty$. This means that $\tilde{x}$ is small and the remaining real parameters, a, b, b', c, c' and γ, are fixed.

The integral (1.3) is of the form of the Mellin integrals considered in [11]:

$$F_2(a,b,b',c,c';x,y)=\frac{\tilde{x}}{\Gamma(a)}\int_0^\infty h(\tilde{x}t)f(t)dt, \tag{1.4}$$

with

$$f(t):={}_1F_1(b,c;te^{i\alpha}){}_1F_1(b',c';te^{i\beta}/\gamma)\quad\text{and}\quad h(t):=e^{-t}t^{a-1}.$$

The functions $f(t)$ and $h(t)$ are locally integrable on $[0,\infty)$, a property required by the technique introduced in [11] to obtain an asymptotic expansion of the integral (1.4) for small $\tilde{x}$. That technique also requires the asymptotic behaviours of these functions given below and is valid for $\Re x<0$ and $\Re y<0$ ($\pi/2<|\alpha|,|\beta|<\pi$).

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(i) A power asymptotic expansion of $h(t)$ at $t=0$:

$$h(t)=\sum_{k=0}^{m-1} A_k t^{k+a-1}+h_m(t), \quad A_k:=(-1)^k/k!, \tag{1.5}$$

with $h_m(t)=\mathcal{O}(t^{m+a-1})$ when $t\to 0^+$.
(ii) An inverse power asymptotic expansion of $f(t)$ at $t=\infty$:

$$f(t)=\sum_{k=0}^{n-1} B_k t^{-k-b-b'}+f_n(t), \tag{1.6}$$

with

$$B_k:=\frac{\Gamma(c)\Gamma(c')(-1)^k(x/y)^{b'}}{\Gamma(c-b)\Gamma(c'-b')}\sum_{j=0}^{k}\frac{(b)_j(1+b-c)_j(b')_{k-j}(1+b'-c')_{k-j}(x/y)^{k-j}}{j!(k-j)!}$$

and $f_n(t)=\mathcal{O}(t^{-b-b'-n})$ when $t\to\infty$. The above expansion follows from the following expression for ${}_1F_1(a,b;z)$ given in [1, eq. 13.5.1]:

$${}_1F_1(a,b;z)=\frac{\Gamma(b)e^{i\pi a}}{\Gamma(b-a)z^a}\left[\sum_{k=0}^{n-1}\frac{(a)_k(1+a-b)_k}{k!z^k}+\mathcal{O}(z^{-n})\right], \quad z\to\infty, \quad \Re z<0.$$

(iii) $h(t)=\mathcal{O}(t^{-N})$ when $t\to\infty$ for any natural number N as large as we wish and $f(t)=\mathcal{O}(1)$ when $t\to 0^+$.

Apart from the restriction $a>0$ needed for the convergence of (1.3) at $t=0$, the asymptotic method given in [11] also requires the restriction $b+b'>0$.

2. Asymptotic expansions of $F_2(a,b,b',c,c';x,y)$**.** The analytic form of the asymptotic expansion of $F_2(a,b,b',c,c';x,y)$ for large x and y depends crucially on the integer/non-integer character of the real number $a-b-b'$.

2.1. *The non-logarithmic case.* When $b+b'-a$ is not an integer number, from [11, Theorem 1], we have the following asymptotic expansion of $F_2(a,b,b',c,c';x,y)$ for small $\tilde{x}$. For any $n\in\mathbb{N}$,

$$\begin{aligned}\Gamma(a)F_2(a,b,b',c,c';x,y)=&\sum_{k=0}^{n-1}B_kM[h;1-k-b-b']\tilde{x}^{k+b+b'}\\&+\sum_{k=0}^{m-1}A_kM[f;k+a]\tilde{x}^{k+a}+R_n(\tilde{x}),\end{aligned} \tag{2.1}$$

where $m=n+\lfloor 1-a+b+b'\rfloor$ and the remainder $R_n(\tilde{x})=\mathcal{O}(\tilde{x}^{n+b+b'})$ when $\tilde{x}\to 0$. The symbol $M[g;z]$ denotes the Mellin transform of a function $g\in L_{Loc}(0,\infty)$: $M[g;z]=\int_0^\infty g(t)t^{z-1}dt$ when this integral exists, or its analytic continuation as a function of z. Computing the Mellin transforms involved in (2.1) and after some calculations, we obtain that, with $\Re x<0$, $\Re y<0$, $a>0$ and $b+b'>0$,

$$F_2(a,b,b',c,c';x,y)=\left[\sum_{k=0}^{n-1}\frac{\hat{B}_k(x/y)}{x^{k+b+b'}}+\sum_{k=0}^{m-1}\frac{\hat{A}_k(x/y)}{x^{k+a}}+R_n(x)\right], \tag{2.2}$$

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with

$$\hat{A}_k\left(\frac{x}{y}\right) := \frac{(-1)^{-k-a}}{\Gamma(a)}\Gamma(c')A_k \times \left\{\frac{\Gamma(b'-k-a)\Gamma(k+a)}{\Gamma(b')\Gamma(c'-k-a)}\left(\frac{x}{y}\right)^{k+a} {}_3F_2\left(\begin{matrix} b, k+a, 1-c'+k+a \\ c, 1-b'+k+a \end{matrix}\middle| -\frac{x}{y}\right) + \frac{\Gamma(c)\Gamma(b+b'-k-a)\Gamma(k+a-b')}{\Gamma(b)\Gamma(c'-b')\Gamma(b'+c-k-a)}\left(\frac{x}{y}\right)^{b'} {}_3F_2\left(\begin{matrix} b', 1+b'-c', b+b'-k-a \\ 1+b'-k-a, b'+c-k-a \end{matrix}\middle| -\frac{x}{y}\right)\right\} \tag{2.3}$$

and

$$\hat{B}_k\left(\frac{x}{y}\right) := \frac{\Gamma(a-b-b'-k)(-1)^{-b-b'-k}B_k}{\Gamma(a)}. \tag{2.4}$$

The remainder term $R_n(x)$ verifies $R_n(x) = \mathcal{O}(x^{-n-b-b'})$ when $|x| \to \infty$ uniformly in $|y|$ with x/y bounded.

2.2. *The logarithmic case I.* When $1+b+b'-a$ is a natural number, from [11, Theorem 1] we have the following asymptotic expansion for small $\tilde{x}$. For any $n \in \mathbb{N}$,

$$\Gamma(a)F_2(a,b,b',c,c';x,y) = \sum_{k=0}^{b+b'-a-1} A_k M[f;k+a]\tilde{x}^{k+a} + \sum_{k=0}^{n-1} \tilde{x}^{k+b+b'}\Big\{-B_k A_{k-a+b+b'}\log\tilde{x} + lim_{z\to 0}\big[B_k M[h;z+1-k-b-b'] + A_{k-a+b+b'}M[f;z+k+b+b']\big]\Big\} + R_n(\tilde{x}), \tag{2.5}$$

where $m = n-a+b+b'$ and the remainder $R_n(\tilde{x}) = \mathcal{O}(\tilde{x}^{n+b+b'}\log\tilde{x})$ when $\tilde{x} \to 0$. Computing the Mellin transforms and the limit involved in (2.5) and after some calculations, we obtain that, with $\Re x < 0$, $\Re y < 0$, $a > 0$ and $b+b' > 0$,

$$F_2(a,b,b',c,c';x,y) = \left[\sum_{k=0}^{b+b'-a-1} \frac{\hat{A}_k(x/y)}{x^{k+a}} + \sum_{k=0}^{n-1} \frac{\tilde{C}_k(x/y) - \tilde{B}_k(x/y)\log(-x)}{x^{k+b+b'}} + R_n(x)\right], \tag{2.6}$$

with $\hat{A}_k(x/y)$ given in (2.3),

$$\tilde{B}_k\left(\frac{x}{y}\right) := \frac{(-1)^{-k-b-b'}B_k A_{k-a+b+b'}}{\Gamma(a)} \tag{2.7}$$

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and

$$\tilde{C}_k\left(\frac{x}{y}\right) := \frac{(-1)^{-k-b-b'}}{\Gamma(a)}\left\{\frac{B_k(-1)^{k-a+b+b'}\Psi(k-a+b+b'+1)}{(k-a+b+b')!}\right.$$
$$+ A_{k-a+b+b'}\Gamma(c')(x/y)^{b'}\left[\frac{(x/y)^{b+k}\Gamma(-k-b)\Gamma(b+b'+k)}{\Gamma(b')\Gamma(-b-b'+c'-k)}\right.$$
$$\times {}_3F_2\left(\left.\begin{matrix} b, b+b'+k, 1+b+b'-c+k \\ c, 1+b+k \end{matrix}\right| -\frac{x}{y}\right)$$
$$\left.\left.+ \frac{(-1)^k\Psi(k+1)\Gamma(c)\Gamma(b+k)}{\Gamma(b)\Gamma(c'-b')\Gamma(-b+c-k)k!}\left[{}_3F_2\left(\left.\begin{matrix} b', 1+b'-c', -k \\ 1-k-b, c-k-b \end{matrix}\right| -\frac{x}{y}\right)\right]\right]\right\}, \tag{2.8}$$

where $\psi(z)$ is the digamma function.

2.3. *The logarithmic case II.* When $a-b-b'$ is a natural number, from [11, Theorem 1] we have the following asymptotic expansion for small $\tilde{x}$. For any $n \in \mathbb{N}$,

$$\Gamma(a)F_2(a,b,b',c,c';x,y) = \sum_{k=0}^{a-b-b'-1} B_k M[h;1-k-b-b']\tilde{x}^{k+b+b'}$$
$$+ \sum_{k=0}^{m-1} \tilde{x}^{k+a}\left\{-A_k B_{k+a-b-b'}\log\tilde{x} + lim_{z\to 0}\left[B_{k+a-b-b'}M[h;z+1-a-k]\right.\right. \tag{2.9}$$
$$\left.\left.+ A_k M[f;z+k+a]\right]\right\} + R_n(\tilde{x}),$$

where $n = m+a-b-b'$ and the remainder $R_n(\tilde{x}) = \mathcal{O}(\tilde{x}^{n+b+b'}\log\tilde{x})$ when $\tilde{x} \to 0$. Computing the Mellin transforms and the limit involved in (2.9) and after some calculations we obtain that, with $\Re x < 0$, $\Re y < 0$, $a > 0$ and $b+b' > 0$,

$$F_2(a,b,b',c,c';x,y) = \left[\sum_{k=0}^{a-b-b'-1}\frac{\hat{B}_k(x/y)}{x^{k+b+b'}}\right.$$
$$\left.+ \sum_{k=0}^{m-1}\frac{\tilde{C}_{k+a-b-b'}(x/y) - \tilde{B}_{k+a-b-b'}(x/y)\log(-x)}{x^{k+a}} + R_n(x)\right], \tag{2.10}$$

with $\hat{B}_k(x/y)$, $\tilde{B}_k(x/y)$ and $\tilde{C}_k(x/y)$ given in (2.4), (2.7) and (2.8), respectively. In (2.6) and (2.10), the remainder term $R_n(x)$ verifies $R_n(x) = \mathcal{O}(x^{-m-a}\log x)$ when $|x| \to \infty$ uniformly in $|y|$ with x/y bounded.

3. Numerical experiments. The following numerical experiments show the accuracy of expansion (2.2), (2.6) and (2.10). The symbols AE($n=1$) or AE($n=2$) mean the value of the approximations (2.2), (2.6) or (2.10) for $n=1$ and $n=2$ respectively and $u := 1+i$.

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TABLE 1

x, y	F_2	$AE(n=1)$	$AE(n=2)$	$R.error1$	$R.error2$
$-10, -10$	0.32358	0.34930	0.31225	0.07949	0.03501
$-100, -100$	0.10941	0.11046	0.10929	0.00960	0.00110
$-1000, -1000$	0.03489	0.03493	0.03489	0.00115	$< E-5$
$-10u, -10u$	0.26293 $-0.09793i$	0.27136 $-0.11240i$	0.26293 $-0.09205i$	0.05969	0.02096
$-100u, -100u$	0.08555 $-0.03497i$	0.08581 $-0.03554i$	0.08555 $-0.03490i$	0.00678	0.00076
$-1000u, -1000u$	0.02713 $-0.01122i$	0.02714 $-0.01124i$	0.02713 $-0.01122i$	0.00076	$< E-5$
$-10, -12$	0.31055	0.33330	0.30104	0.07326	0.03062
$-100, -120$	0.10449	0.10540	0.10438	0.00871	0.00105
$-1000, -1200$	0.03330	0.03333	0.03330	0.00090	$< E-5$
$-10u, -12u$	0.25185 $-0.09446i$	0.25894 $-0.10726i$	0.25160 $-0.08953i$	0.05440	0.01835
$-100u, -120u$	0.08165 $-0.03341i$	0.08188 $-0.03392i$	0.08165 $-0.03336i$	0.00634	0.00057
$-1000u, -1200u$	0.02589 $-0.01071i$	0.02589 $-0.01073i$	0.02589 $-0.01071i$	0.00071	$< E-5$

Parameter values: $a = 1/2$, $b = 1$, $b' = 1$, $c = 2$, $c' = 2$ (non-logarithmic case).

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TABLE 2

x, y	F_2	$AE(n=1)$	$AE(n=2)$	$R.error1$	$R.error2$
$-10, -10$	0.04203	0.07410	0.03410	0.76303	0.18867
$-100, -100$	0.00197	0.00234	0.00194	0.18782	0.01523
$-1000, -1000$	0.00007	0.00007	0.00007	$< E-5$	$< E-5$
$-10u, -10u$	0.01561 $-0.02385i$	0.01686 $-0.04070i$	0.01686 $-0.02070i$	0.59276	0.11889
$-100u, -100u$	0.00053 $-0.00110i$	0.00053 $-0.00129i$	0.00053 $-0.00109i$	0.15561	0.00819
$-1000u, -1000u$	0.00002 $-0.00004i$	0.00002 $-0.00004i$	0.00002 $-0.00004i$	$< E-5$	$< E-5$
$-10, -12$	0.03754	0.06453	0.03120	0.71897	0.16889
$-100, -120$	0.00173	0.00204	0.00171	0.17919	0.01156
$-1000, -1200$	0.00006	0.00006	0.00006	$< E-5$	$< E-5$
$-10u, -12u$	0.01368 $-0.02130i$	0.01468 $-0.03545i$	0.01470 $-0.01878i$	0.56036	0.10739
$-100u, -120u$	0.00046 $-0.00096i$	0.00046 $-0.00112i$	0.00046 $-0.00095i$	0.15030	0.00939
$-1000u, -1200u$	0.00001 $-0.00003i$	0.00001 $-0.00004i$	0.00001 $-0.00003i$	0.31623	$< E-5$

Parameter values: $a = 3/2$, $b = 1$, $b' = 1$, $c = 2$, $c' = 2$ (non-logarithmic case).

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TABLE 3

x, y	F_2	$AE(n=1)$	$AE(n=2)$	$R.error1$	$R.error2$
$-10, -10$	0.01228	0.01061	0.01220	0.13599	0.00651
$-100, -100$	0.00108	0.00106	0.00108	0.01852	$< E-5$
$-1000, -1000$	0.00011	0.00011	0.00011	$< E-5$	$< E-5$
$-10u, -10u$	0.00530 $-0.00616i$	0.00531 $-0.00531i$	0.00531 $-0.00610i$	0.10461	0.00748
$-100u, -100u$	0.00053 $-0.00054i$	0.00053 $-0.00053i$	0.00053 $-0.00054i$	0.01322	$< E-5$
$-1000u, -1000u$	0.00005 $-0.00005i$	0.00005 $-0.00005i$	0.00005 $-0.00005i$	$< E-5$	$< E-5$
$-10, -12$	0.00931	0.00800	0.00924	0.14071	0.00752
$-100, -120$	0.00081	0.00080	0.00081	0.01235	$< E-5$
$-1000, -1200$	0.00008	0.00008	0.00008	$< E-5$	$< E-5$
$-10u, -12u$	0.00400 $-0.00466i$	0.00400 $-0.00400i$	0.00400 $-0.00462i$	0.10747	0.00651
$-100u, -120u$	0.00040 $-0.00041i$	0.00040 $-0.00040i$	0.00040 $-0.00041i$	0.01746	$< E-5$
$-1000u, -1200u$	0.00004 $-0.00004i$	0.00004 $-0.00004i$	0.00004 $-0.00004i$	$< E-5$	$< E-5$

Parameter values: $a=1$, $b=1/2$, $b'=5/2$, $c=1$, $c'=1$ (logarithmic case I).

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TABLE 4

x, y	F_2	$AE(n=1)$	$AE(n=2)$	$R.error1$	$R.error2$
$-10, -10$	0.01682	0.01592	0.01671	0.05351	0.00654
$-100, -100$	0.00160	0.00160	0.00160	$< E-5$	$< E-5$
$-1000, -1000$	0.00016	0.00016	0.00016	$< E-5$	$< E-5$
$-10u, -10u$	0.00794 $-0.00841i$	0.00796 $-0.00796i$	0.00796 $-0.00836i$	0.03895	0.00466
$-100u, -100u$	0.00080 $-0.00080i$	0.00080 $-0.00080i$	0.00080 $-0.00080i$	$< E-5$	$< E-5$
$-1000u, -1000u$	0.00008 $-0.00008i$	0.00008 $-0.00008i$	0.00008 $-0.00008i$	$< E-5$	$< E-5$
$-10, -12$	0.01529	0.01453	0.01519	0.04970	0.00654
$-100, -120$	0.00146	0.00145	0.00146	0.00685	$< E-5$
$-1000, -1200$	0.00015	0.00015	0.00015	$< E-5$	$< E-5$
$-10u, -12u$	0.00725 $-0.00765i$	0.00726 $-0.00726i$	0.00726 $-0.00760i$	0.03702	0.00484
$-100u, -120u$	0.00073 $-0.00073i$	0.00073 $-0.00073i$	0.00073 $-0.00073i$	$< E-5$	$< E-5$
$-1000u, -1200u$	0.00007 $-0.00007i$	0.00007 $-0.00007i$	0.00007 $-0.00007i$	$< E-5$	$< E-5$

Parameter values: $a = 3$, $b = 1/2$, $b' = 1/2$, $c = 1$, $c' = 1$ (logarithmic case I).

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TABLE 5

x, y	F_2	$AE(n=1)$	$AE(n=2)$	$R.error1$	$R.error2$
$-10, -10$	0.03517	0.03183	0.03525	0.09497	0.00227
$-100, -100$	0.00326	0.00318	0.00325	0.02454	0.00031
$-1000, -1000$	0.00032	0.00032	0.00032	$< E-5$	$< E-5$
$-10u, -10u$	0.01663 $-0.01791i$	0.01592 $-0.01592i$	0.01654 $-0.01790i$	0.08645	0.00371
$-100u, -100u$	0.00160 $-0.00163i$	0.00159 $-0.00159i$	0.00160 $-0.00163i$	0.01805	$< E-5$
$-1000u, -1000u$	0.00016 $-0.00016i$	0.00016 $-0.00016i$	0.00016 $-0.00016i$	$< E-5$	$< E-5$
$-10, -12$	0.02931	0.02642	0.02902	0.09860	0.00989
$-100, -120$	0.00270	0.00264	0.00270	0.02222	$< E-5$
$-1000, -1200$	0.00027	0.00026	0.00026	0.00370	0.00370
$-10u, -12u$	0.01375 $-0.01491i$	0.01321 $-0.01321i$	0.01368 $-0.01472i$	0.08794	0.00998
$-100u, -120u$	0.00133 $-0.00135i$	0.00132 $-0.00132i$	0.00133 $-0.00135i$	0.016687	$< E-5$
$-1000u, -1200u$	0.00013 $-0.00013i$	0.00013 $-0.00013i$	0.00013 $-0.00013i$	$< E-5$	$< E-5$

Parameter values: $a = 1$, $b = 1/2$, $b' = 3/2$, $c = 1$, $c' = 1$ (logarithmic case II).

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TABLE 6

x, y	F_2	$AE(n=1)$	$AE(n=2)$	$R.error1$	$R.error2$
$-10, -10$	0.03517	0.03183	0.03525	0.09500	0.00227
$-100, -100$	0.00326	0.00318	0.00325	0.02454	0.00307
$-1000, -1000$	0.00032	0.00032	0.00032	$< E-5$	$< E-5$
$-10u, -10u$	0.01663 $-0.0179i$	0.01592 $-0.01592i$	0.01654 $-0.01790i$	0.08609	0.00368
$-100u, -100u$	0.00160 $-0.00163i$	0.00159 $-0.00159i$	0.00160 $-0.00163i$	0.01805	$< E-5$
$-1000u, -1000u$	0.00016 $-0.00016i$	0.00016 $-0.00016i$	0.00016 $-0.00016i$	$< E-5$	$< E-5$
$-10, -12$	0.03198	0.02906	0.03192	0.09131	0.00188
$-100, -120$	0.00297	0.00291	0.00297	0.02020	$< E-5$
$-1000, -1200$	0.00029	0.00029	0.00029	$< E-5$	$< E-5$
$-10u, -12u$	0.01512 $-0.01626i$	0.01453 $-0.01453i$	0.01505 $-0.01619i$	0.08232	0.00446
$-100u, -120u$	0.00146 $-0.00149i$	0.00145 $-0.00145i$	0.00146 $-0.00148i$	0.01976	0.00479
$-1000u, -1200u$	0.00015 $-0.00015i$	0.00015 $-0.00015i$	0.00015 $-0.00015i$	$< E-5$	$< E-5$

Parameter values: $a = 2$, $b = 1/2$, $b' = 1/2$, $c = 1$, $c' = 1$ (logarithmic case II).

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