The Application of Optimal Linear Regulator Theory to a Problem in Water Pollution

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***Abstract*—At present, many metropolitan sewer systems do not meet existing and proposed standards on water pollution. Existing systems were designed to overflow at prescribed locations in order to protect the sewage treatment plants whenever severe overload conditions exist (usually during storms). This discharge of untreated overflows into natural receiving waters is of growing concern to water pollution control authorities. The model considered in this paper is representative of the combined storm–sewer systems in cities such as Minneapolis–St. Paul, Minn., Seattle, Wash., and San Francisco, Calif. The objective is to utilize the total storage capacity available in the system in such a manner as to minimize the water pollution resulting from overflows at individual points within the system. In addition, it is required that no abrupt changes in control be admitted, as this is likely to lead to undesirable surges.**

The nonlinear model is shown to fit within the framework of an optimal regulator problem with derivative constraints. The optimal feedback control law is derived and compared with the optimal bang-bang controller. The solution technique that is presented may be applied to many combined storm–sewer systems in which the flows through the systems to the treatment plants may be controlled. It may be used by city engineers to determine necessary modifications to existing systems in order to meet the new standards regarding water pollution.

I. Introduction

A MAJOR PROBLEM presently existing in many urban areas is that of water pollution caused by direct overflow from combined storm–sewer systems to natural receiving waters. Combined storm–sewer systems are sewer systems in which both sanitary waste and storm runoff are handled simultaneously. The original design of such systems was such that not all of the storm runoff could be carried in addition to the normal sanitary flow. The excess flow was diverted to receiving waters at numerous points throughout the system. It was considered that the stormwater was clean and would sufficiently dilute the sanitary waste so that there would be no pollution problems associated with the direct discharge of the excess flow into receiving waters.

The practice in the past has been to increase the number of overflow locations as the total load on the system has increased due to growth in the urban area. This principle may have been appropriate when the total sewer overflow and the overflow locations did not contribute seriously to water pollution. Recent studies [1] have shown, however, that the storm runoff itself may be heavily polluted, especially during the early phases of runoff when it picks up pollutants from the surface. Also, as the urban areas expanded and more of the surface area within a drainage basin was covered by construction, the frequency of occurrence of overflows has increased. In addition, due to a decrease in the percentage of permeable ground cover that results from urban expansion, the magnitude of the overflows has tended to increase.

As a result of more stringent water quality standards, and also due to the fact that there are 1329 municipal jurisdictions with combined storm–sewer systems affecting 36 million people in the United States [2], there are many studies being conducted at present to determine methods of minimizing the effect of pollution from the overflows themselves. Methods proposed for reducing the effects of the overflows include screening/dissolved air flotation and microscreening; these appear to reduce pollutant concentrations [3], [4]. In addition, separation of storm and sanitary sewers has been considered, but this is very expensive [5] and still does not consider the problem of pollution from stormwater runoff. Reduction of overflows themselves may be accomplished by modification of the hydrology of the watershed basin, by improving the design and maintenance of the regulator structures, or through utilizing the entire storage capacity of the sewer system. By installing flow-control devices within the system, it is possible to store more of the inflows (sanitary waters plus stormwater) until they can be routed through the normal treatment facilities. This has been accomplished in Minneapolis–St. Paul, Minn., and significant reductions in overflows have been demonstrated [5]. Use of in-system storage is also planned for Seattle, Wash., San Francisco, Calif., Chicago, Ill., and Detroit, Mich. The method usually employed in these systems is to equip each regulator–outfall station with automatic controls which use water conditions at the station as control references. It is still essentially a single-unit control strategy, with the control at each overflow point being determined by conditions at that point and not necessarily by conditions at all other control points in the system.

The problem of minimizing the effects of pollution from overflows as well as minimizing the overflows themselves has been considered in [6], and the results have been applied to a representative situation in [7]. In these papers it was considered that overflows from different control points within a system would have different polluting effects on the receiving waters. Therefore, the problem that was considered was to find the control strategy that would minimize the weighted sum of overflows from all control points

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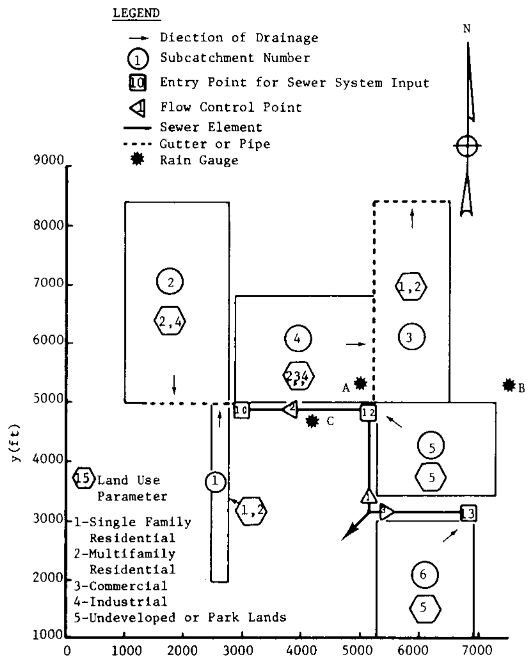

Fig. 1. Physical system.

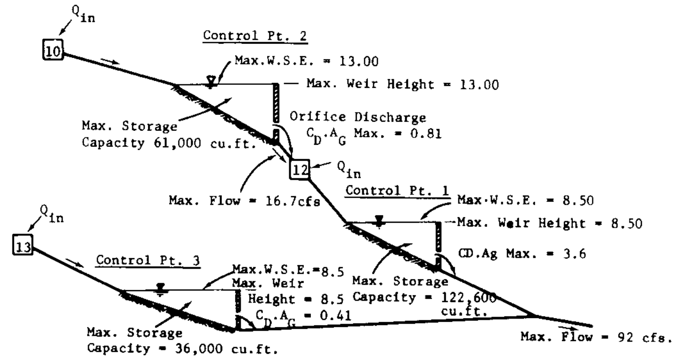

Fig. 2. Control program analog of physical system.

within a system. The necessary conditions of optimal control theory were utilized to determine the form of the optimal control for systems with up to and including three control points. This has potential importance for large-scale systems because the data handling and computational requirements for large-scale systems may be so great that it would be advantageous to share the control task among several local controllers, with each group of controllers acting in an optimal manner.

Although it may be possible to extend these results to systems with many control points, the results may not be practical in that the optimal control strategy was determined to be bang-bang. The effects of rapid changes in the control variables in a hydrological system may not be tolerable due to undesirable surges.

This paper presents a suboptimal control strategy that may be applied to a system with an arbitrary number of control points and that avoids the problem associated with rapid changes in control variables. The optimal control problem for the nonlinear system is transformed to a linear regulator problem for which the solution is well known.

II. Problem Formulation

A pictorial representation of a watershed basin is shown in Fig. 1. The system consists of the six subcatchment areas with runoff from the subcatchments feeding into the system at the three points numbered 10, 12, and 13 on Fig. 1. The combined storm–sewer system has the four conduits shown. Three of the conduits have control points (numbered 1, 2, and 3 on the figure) at which flow may be stored or diverted to the receiving waters. The backwater storage locations may be interpreted as reservoirs where the dimensions of each reservoir are determined by the dimensions of the conduit directly upstream from each control point. This three-reservoir analogy is shown on Fig. 2. The flow through the system is controlled by the orifice openings and the overflows are controlled by variable-height weirs. The objective function selected on which to determine the control logic was to minimize the weighted overflows from the system. If the different weighting factors are chosen on the basis of pollutant concentrations within the systems, combined with a knowledge of the receiving waters, then the overall effect on the receiving waters may be minimized. For example, overflow from a control point in a subcatchment located in an industrial region with high traffic density may contain a higher percentage of pollutants than that from a residential section.

The functions $F_1(t)$, $F_2(t)$, and $F_3(t)$ represent the inflows to the sewer systems (corresponding to points 10, 12, and 13 on Fig. 1). The variables V_1, V_2, and V_3 represent the overflows to be diverted to the receiving waters, while q_1, q_2, and q_3 represent the outflow volumes through the controllable area orifices. The flows over the variable-height weirs are governed by

$$\dot{V}_i = c_{w_i} h_i^{3/2}$$

where c_{w_i} is an empirical coefficient and h_i represents the depth of flow over weir i. The flows through the orifices are governed by

$$\dot{q}_i = c_{o_i} r_i^2 (d_i)^{1/2}$$

where c_{o_i} is an empirical coefficient, r_i is the radius of the opening, and d_i is the depth of water above the center line of the orifice. The depth d_i is governed by

$$\dot{d}_i = \frac{F_i - c_{o_i} r_i^2 (d_i)^{1/2} - c_{w_i} h_i^{3/2}}{A_i(d_i)}$$

where $A_i(d_i)$ is an area–depth relationship determined by the dimensions of the conduit immediately upstream of the control point.

The objective is to determine $r_i(t), h_i(t)$ to minimize the functional

$$J = \int_{t_0}^{t_f} \left\{ \sum_{i=1}^{N} z_i c_{w_i} h_i^{3/2}(t) \right\} dt$$

where z_i represents the weighting factor, t_f represents the final time, which may be specified (the predicted duration

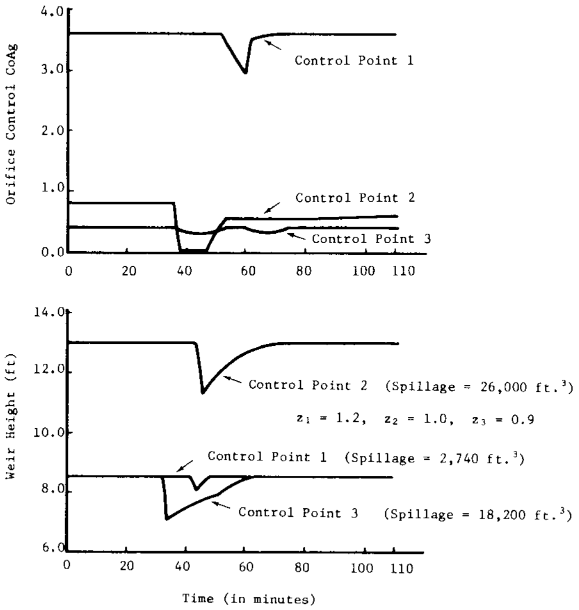

Fig. 3. Computed optimal control.

of a storm, for example) or may be arbitrary, and t_0 represents the initial time. In addition to satisfying the state equations, the following state and control variable inequality constraints must be satisfied:

$$h_i(h_i - d_i) \leq 0 \tag{1}$$

this states that the depth of flow over the weir cannot be less than zero nor greater than the depth of water in the reservoir;

$$r_i(r_i - R_i) \leq 0 \tag{2}$$

this states that the orifice radius cannot be negative nor exceed a specified maximum value;

$$d_i(d_i - D_i) \leq 0 \tag{3}$$

this provides that the depth of water above the control point cannot be negative nor greater than some preset maximum value. In addition to the above constraints, it may be necessary to include a flow constraint that provides that flow rates downstream of a control point (or control points) cannot exceed maximum limits. For example, a constraint of the form

$$\dot q_2 + \dot q_3 - Q \leq 0$$

may be included.

The usual necessary conditions may be used to determine the optimal control strategy and the resulting overflows for given inputs $F_i(t)$. The form of the optimal control is presented in [6] and the particular control for a representative system is shown in [7] and presented here in Fig. 3 for convenience. The orifice and weir controls experience rapid changes which would lead to undesirable surges in a real system and which are not reflected in the relatively simple mathematical model used in this analysis. In addition, it would be quite a complex and time-consuming task to determine the optimal control for a system with a large number of control points. The following transformation from the nonlinear model to a linear regulator problem circumvents these difficulties.

III. Regulator Model

By considering a modified cost function and by making some standard transformations, the nonlinear optimal control problem may be restated as a linear regulator problem.

In the differential equation for the depth presented earlier, the area–depth relationship may be assumed linear in many cases [7]. Considering the state variables to be the depth of water at a control point and the flow through the orifice at each control point, the state equations take the form

$$d_i \dot d_i = \frac{1}{k_i}\{F_i(t) - c_{o_i} r_i^2 \sqrt{d_i} - c_{w_i} h_i^{3/2}\}$$

$$\dot q_i = c_{o_i} r_i^2 \sqrt{d_i}.$$

On making the transformation

$$y_i = \frac{d_i^2}{2}$$

the state equations may be written as

$$\dot y_i = \tilde F_i(t) - \frac{1}{k_i} \tilde c_{o_i} r_i^2 (y_i)^{1/4} - \frac{1}{k_i} c_{w_i} h_i^{3/2}$$

$$\dot q_i = \tilde c_{o_i} r_i^2 (y_i)^{1/4}$$

where

$$\tilde F_i(t) = \frac{F_i}{k_i} \qquad \tilde c_{o_i} = c_{o_i}(2)^{1/4}.$$

One obvious suggestion is to define r^2 and $h^{3/2}$ as the controls, in which case the state equations are nonlinear. An alternative suggestion is to define the control components in terms of fluid flows as follows. Define the control components by

$$u_i = \frac{c_{w_i}(h_i)^{3/2}}{k_i}$$

$$u_{i+N} = \frac{\tilde c_{o_i} r_i^2 (y_i)^{1/4}}{k_i}$$

where N represents the number of control points in the system. Then the state equations assume the form

$$\dot y_i = \tilde F_i - u_i - u_{i+N}$$

$$\dot q_i = k_i u_{i+N}$$

which represents a dynamical system in the form

$$\dot x = Fx + Gu$$

where the system matrix is identically zero and the inputs $\tilde F_i$ represent disturbances. It should now be clear why the alternative formulation of the control components is a good one. Once optimal control trajectories u_i^* and u_{i+N}^* are determined, the optimal values of the adjustable parameters

h_i^* and r_i^* are recoverable as follows:

$$h_i^* = \left[\frac{k_i u_i^*}{c_{w_i}}\right]^{2/3}$$

$$r_i^* = \left[\frac{k_i u_{i+N}^*}{\tilde{c}_{0_i} y_i^{*1/4}}\right]^{1/2}.$$

Notice that r_i^* is expressed as a function of the optimal state trajectory y_i^*. Since this state variable is readily measured in such a system, no real difficulty is presented.

A cost function that represents the original cost function but that also penalizes rapid changes in control variables may be defined as

$$J = \int_{t_0}^{t_f} \{x^T Q x + u^T R u + \dot{u}^T S \dot{u}\}\, dt$$

where $x^T = \{y^T q^T\}$ and Q and S may be selected according to how much of a penalty is desired with respect to depth or rate of change of control. The R matrix represents the relative weighting factors applied to overflows from the different control points.

This problem with derivative constraints may be rewritten as a standard regulator problem by means of the transformations [8]

$$x_1^T = \{x^T \quad u^T\} \qquad \tilde{u}_1 = \dot{u} \qquad R_1 = S$$

$$F_1 = \begin{bmatrix} F & G \\ 0 & 0 \end{bmatrix} \qquad G_1 = \begin{bmatrix} 0 \\ I \end{bmatrix} \qquad Q_1 = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}.$$

The system dynamics are now given by

$$\dot{x}_1 = F_1 x_1 + G_1 \tilde{u}_1, \qquad x_1(t_0) \text{ given}$$

and the cost function is

$$J = \int_{t_0}^{t_f} \{x_1^T Q_1 x_1 + \tilde{u}_1^T R_1 \tilde{u}_1\}\, dt.$$

The problem is now stated as a standard regulator problem for which the solution is well known. The optimal control may be easily determined [8] (assuming, of course, various controllability and observability conditions). In fact, provided that $G^T G$ is positive definite, the optimal control may be defined in terms of proportional-plus-integral state feedback as shown on Fig. 4. The gains are computed in terms of solutions to the Riccati equation

$$-\dot{P} = PF_1 + F_1^T P - PG_1 R_1^{-1} G_1^T P + Q_1, \quad P(T,T) = 0.$$

Denoting $\bar{P} = \lim_{t \to \infty} P(t,T)$, and partitioning $\bar{P}$ as

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_{22} \end{bmatrix}$$

the optimal control is given in the form [8]

$$u(t) = K_3^T x(t) + \int_{t_0}^{t} K_1^T x(t)\, dt + u(t_0) - K_3^T x(t_0)$$

where

$$K_1^T = -S^{-1} \bar{P}_{12}^T$$

$$K_2^T = -S^{-1} \bar{P}_{22}$$

$$K_3^T = K_2^T (G^T G)^{-1} G^T.$$

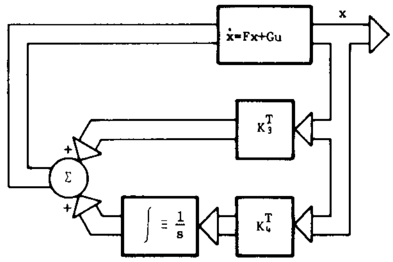

Fig. 4. Proportional-plus-integral state feedback control strategy.

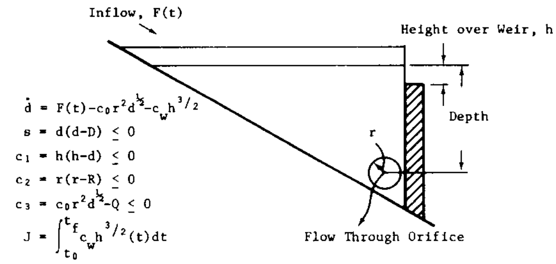

Fig. 5. Single-reservoir operation.

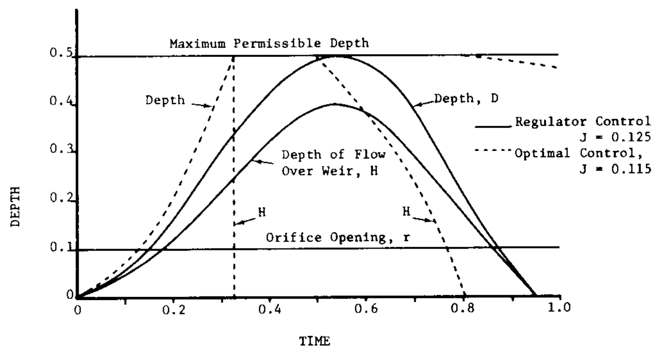

Fig. 6. Optimal control for single-reservoir operation.

We stress again that in this section we have formulated a plant model and performance index so that linear optimal control results are relevant. From one point of view it is surprising that nonlinear state equations with a nonquadratic index can be so manipulated, but when one thinks in terms of fluid flows rather than weir heights and orifice radii, it is not so surprising that the problem formulation simplifies.

IV. Sample Cases

The regulator formulation has been applied to the three-reservoir model presented in [7]. Before examining this problem, consider, for simplicity, the single-reservoir model represented by Fig. 5. The form of the optimal bang-bang control was presented in [6] and is shown for a particular example in Fig. 6. The differential equation for depth is

$$\dot{d} = F(t) - c_o r^2 \sqrt{d} - c_w h^{3/2}$$

where $A(d)$ has been taken as unity, as this is meant for illustrative purposes only. The control is

$$u = u_1 + u_2$$

where

$$u_1 = c_w h^{3/2} \qquad u_2 = c_o r^2 \sqrt{d}.$$

The cost function is

$$J = \int_{t_o}^{t_f} \{u^T R u + \dot{u}^T S \dot{u} + x^T Q x\}\, dt$$

where

$$x = d \qquad R = [\alpha_1] \qquad S = [\alpha_2] \qquad Q = [\alpha_3].$$

The state equation is

$$\dot{x} = Fx + Gu$$

where

$$F = [0] \qquad G = [-1].$$

The optimal control is

$$u = u(t_0) - K_3 x(t_0) + K_3 x(t) + \int_{t_o}^{t} K_4 x(t)\, dt$$

where

$$K_3 = \sqrt{\alpha_1/\alpha_2 + 2\alpha_3} \qquad K_4 = \alpha_3.$$

The results are presented in Fig. 6. The shape of the response can be adjusted by adjusting the coefficients α_i.

The three-reservoir problem shown in Fig. 2 may be described by

$$\dot{d}_1 = \frac{F_1(t) + c_{o_2} r_2^2 \sqrt{d_2} - c_{o_1} r_1^2 \sqrt{d_1} - c_{w_1} h_1^{3/2}}{A_1(d_1)}$$

$$\dot{d}_2 = \frac{F_2(t) - c_{o_2} r_2^2 \sqrt{d_2} - c_{w_2} h_2^{3/2}}{A_2(d_2)}$$

$$\dot{d}_3 = \frac{F_3(t) - c_{o_3} r_3^2 \sqrt{d_3} - c_{w_3} h_3^{3/2}}{A_3(d_3)}$$

$$\dot{q}_1 = c_{o_1} r_1^2 \sqrt{d_1}$$

$$\dot{q}_2 = c_{o_2} r_2^2 \sqrt{d_2}$$

$$\dot{q}_3 = c_{o_3} r_3^2 \sqrt{d_3}.$$

The inputs $F_i(t)$ have been determined for this case by means of a rainfall–runoff simulation program in which a simulated rainstorm is moved across the drainage basin (Fig. 1) and the corresponding runoff from the subcatchments is calculated and fed into the sewer system. On making the transformations presented earlier and using a linear area–depth relationship, the equations may be written as

$$\dot{x} = Fx + Gu$$

where

$$F = [0] \qquad G = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_3 \end{bmatrix}.$$

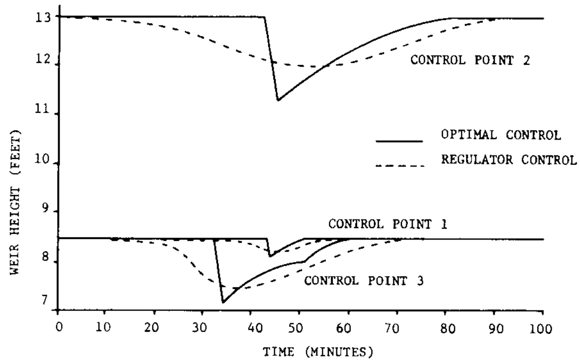

Fig. 7. Weir control for three-reservoir system.

The solution to the Riccati equation has been obtained by the method presented in [9] and the solution for the weir control is presented in Fig. 7. The limiting solution of the Riccati equation has been used; this is considered satisfactory due to the relative times involved in reaching the limiting solution and the flow in the physical system. This suboptimal control strategy results in a slightly greater spillage than that obtained through use of the optimal bang-bang controller, but part of this spillage is due to a tendency on the part of the regulator to spill regardless of the magnitude of the disturbance. This could be corrected in an operational environment by the inclusion of a deadband. Also, the weighting matrices have been selected by trial and error and have not been optimized. It is a straightforward parameter optimization problem to adjust the coefficients in the weighting matrices to minimize the cost function and still satisfy the constraints. The weighting matrices may also be considered to be time varying, so that high-intensity or long-duration storms may be handled differently from those with low intensity or short duration. There exists the possibility of selecting parameters which would lead to an oscillatory solution. Since negative depth is not meaningful, this would necessarily have to be examined for any given system in order to determine an admissible range of values for the coefficients in the weighting matrices.

V. Conclusions

The problem of reducing pollution from combined storm–sewer systems has been presented as a standard regulator problem for which the optimal solution may be readily obtained. The solution is obtained as a feedback control law and has the advantage over previously determined control strategies that involved a prediction of the disturbances. This formulation has assumed no time delay between control points in the system and has also assumed no backflow. Hence, this would necessarily have to be modified for systems in which those factors would have to be taken into account. The time delay may be readily taken into account by using a discrete time model, so this would present no great difficulty. The approach would remain the same with the appropriate modifications appearing in the model. It is recognized that the model presented here is not precise, but it is expected that through adjustment of the model parameters it could be made representative of many given systems.

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