The Application of Splines of the Seventh Order Approximation to the Solution of Integral Fredholm Equations

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Abstract: - There are various numerical methods for solving integral equations. Among the new numerical methods, methods based on splines and spline wavelets should be noted. Local interpolation splines of a low order of approximation have proved themselves well in solving differential and integral equations. In this paper, we consider the construction of a numerical solution to the Fredholm integral equation of the second kind using spline approximations of the seventh order of approximation. The support of the basis spline of the seventh order of approximation occupies seven grid intervals. We apply various modifications of the basis splines of the seventh order of approximation at the beginning, the middle, and at the end of the integration interval. It is assumed that the solution of the integral equation is sufficiently smooth. The advantages of using splines of the seventh order of approximation include the use of a small number of grid nodes to achieve the required error of approximation. Numerical examples of the application of spline approximations of the seventh order for solving integral equations are given.

Key-Words: Fredholm integral equation of the second kind, splines of the seventh order of approximation

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1 Introduction

Integral equations often arise in various applications. Many problems of astrophysics, mechanics, viscoelasticity, elasticity, vibrations, plasticity, hydrodynamics, electrodynamics, nuclear physics, biomechanics, geology, medicine problems, and many other problems are formulated in terms of integral equations. The mathematical model for many problems is arising in different industries of natural science, basically formulated using differential and integral equations. The investigation of these equations is conducted with the help of the numerical integration theory, [1]. Mathematical and physics problems are often reduced to solving integral or integro-differential equations.

The (2+1) dimensional Konopelchenko–Dubrovsky equation (2D-KDE) is an integro-differential equation which describes two-layer fluid in shallow water near ocean shores and stratified atmosphere, [2].

In paper [3], the two-dimensional Volterra integro-differential equations for viscoelastic rods and membranes in a bounded smooth domain are studied.

Hypoxy induced angiogenesis processes can be described by coupling an integro-differential kinetic equation of Fokker-Planck type with a diffusion equation for the angiogenic factor, [4].

The development of two numerical techniques for solving the convection-diffusion type partial integro-differential equation (PIDE) with a weakly singular kernel is studied in the paper, [5].

The charged particle motion for certain configurations of oscillating magnetic fields can be simulated by a Volterra integro-differential equation of the second order with time-periodic coefficients, [6].

As it is known, equations of the second and first kinds are distinguished among the integral and integro-differential equations. Usually, the solution of an integral equation is reduced to the solution of a system of algebraic equations.

The compound trapezoidal scheme is often used to solve the first-order linear Fredholm integro-differential equation and the second-order linear Fredholm integro-differential equation, [7], [8].

In paper [9] the trapezoidal rule is used to approximate the integral in linear and nonlinear fractional Fredholm integrodifferential equations.

The B-spline collocation method is used to solve the system of singular integro-differential equations, [10].

On the B-spline basis**,** the Hartree–Fock integro-differential equations are reduced to a computationally eigenvalue problem, [11].

In paper, [12], the approximation of the solution of Fredholm integro-differential equations of the second kind using an exponential spline function was applied.

In paper, [13], the cubic B-spline collocation method is used to solve the stochastic integro-differential equation of fractional order.

In the paper, [14], the advanced multistep and hybrid methods have been used to solve Volterra integral equation.

In paper, [15], the kernel as well as the right part of the equation are initially approximated through Legendre wavelet functions.

In the paper, [16], the forward-jumping methods of hybrid type are used for the construction of the methods with a high order of accuracy.

In the paper, [17], the Fourier integral transform has been employed to reduce the problem of determining the stress component under the contact region of a punch in solving dual integral equations.

In the paper, [18], the method of integral equations is proposed for some problems of electrical engineering (current density, radiative heat transfer, heat conduction). Presented models lead to a system of Fredholm integral equations, integro-differential equations, or Volterra-Fredholm integral equations, respectively.

This paper discusses the solution methods based on the use of local splines of the seventh order of approximation. We use these splines if the kernel and the right side are sufficiently smooth functions and we want to use a small number of grid nodes. To construct an approximate solution at points between grid nodes, we use the interpolation of the same local splines.

2 Problem Formulation

Let $\{x_i\}$ be a grid of nodes on the interval $[a,b]$. Note that the approximations with the splines are constructed separately for each grid interval $[x_k, x_{k+1}]$.

In this paper, we consider the application of splines of the seventh order of approximation to solve integral Fredholm equations of the second kind. Different modifications of the splines of the seventh order of approximation are used at the beginning, in the middle, and at the end of the interpolation interval $[a,b]$. The support of the basis spline occupies seven grid intervals.

First, consider the approximation properties of polynomial splines of the seventh order of approximation. Let $\{x_i\}$ be a uniform grid of nodes on the interval $[a,b]$: $a = x_0 <\ldots< x_n = b$ with step h. Let us assume that the values of the function $u(x)$ are given at the grid nodes. The approximation using basis splines is built separately on each grid interval as the sum of the products of the values of the function u at the grid nodes and the basis splines ω_j.

Let r, r_1 be integers, $r + r_1 = 7$, $r \geq 1, r_1 \geq 1$, and the spline ω_k be such that $\operatorname{supp} \omega_k = [x_{k+r}, x_{k+r_1}]$. Following the methodology developed by Professor S. G. Mikhlin, we find the basis functions ω_k by solving the system of approximation relations

$$\sum_{j=k-r_1}^{k+r} x_j^s \omega_j(x) = x^s, \qquad x \in [x_k, x_{k+1}], \qquad s = 0,1,\ldots,6. \tag{1}$$

With different values of the parameters r, r_1, we get basis splines suitable for approximation at the beginning of the interpolation interval (the right basis splines), in the middle of the interpolation interval (the middle basis splines), or at the end of the interpolation interval (the left basis splines).

2.1 Approximation with the Middle Basis Splines

With $r_1 = 3$ and $r = 3$ we get the middle splines. When constructing an approximation on a finite interpolation interval, we use the middle splines, departing 3 grid intervals from the points a and b that are the ends of the interval of integration. At the beginning and the end of the interval $[a,b]$, we apply, respectively, the left and the right splines. For example, on the interval $x \in [x_k, x_{k+1}]$, we construct the approximation with the middle splines at a distance of three grid intervals from the ends of the interval $[a,b]$ in the form:

$$\tilde{u}(x) = \sum_{j=k-3}^{k+3} u(x_j)\omega_j^M(x), x \in [x_k, x_{k+1}]. \tag{2}$$

where the middle basis splines $\omega_j^M(x)$ have the form:

$$\omega_{k-3}^M(x) = c_{k-3}(x)/d_{k-3},$$

where

$$c_{k-3}(x) = (x - x_{k+3})(x - x_{k+2})(x - x_{k+1}) \times (x - x_k)(x - x_{k-1})(x - x_{k-2}),$$

$$d_{k-3} = (x_{k-3} - x_{k+3})(x_{k-3} - x_{k+2}) \times (x_{k-3} - x_{k+1})(x_{k-3} - x_k) \times (x_{k-3} - x_{k-1})(x_{k-3} - x_{k-2});$$

$$\omega_{k-2}^{M}(x)=\frac{c_{k-2}(x)}{d_{k-2}},$$
$c_{k-2}(x)=(x-x_{k+3})(x-x_{k+2})(x-x_{k+1})$
$\times(x-x_k)(x-x_{k-1})(x-x_{k-3}),$
$d_{k-2}=(x_{k-2}-x_{k+3})(x_{k-2}-x_{k+2})$
$\times(x_{k-2}-x_{k+1})(x_{k-2}-x_k)$
$\times(x_{k-2}-x_{k-1})(x_{k-2}-x_{k-3});$

$$\omega_{k-1}^{M}(x)=\frac{c_{k-1}(x)}{d_{k-1}},$$
$c_{k-1}=(x-x_{k+3})(x-x_{k+2})(x-x_{k+1})$
$\times(x-x_k)(x-x_{k-2})(x-x_{k-3}),$
$d_{k-1}=(x_{k-1}-x_{k+3})(x_{k-1}-x_{k+2})$
$\times(x_{k-1}-x_{k+1})(x_{k-1}-x_k)$
$\times(x_{k-1}-x_{k-2})(x_{k-1}-x_{k-3});$

$$\omega_{k}^{M}(x)=\frac{c_k(x)}{d_k},$$
$c_k(x)=(x-x_{k+3})(x-x_{k+2})(x-x_{k+1})$
$\times(x-x_{k-1})(x-x_{k-2})(x-x_{k-3}),$
$d_k=(x_k-x_{k+3})(x_k-x_{k+2})(x_k-x_{k+1})$
$\times(x_k-x_{k-1})(x_k-x_{k-2})(x_k-x_{k-3});$

$$\omega_{k+1}^{M}(x)=\frac{c_{k+1}(x)}{d_{k+1}},$$
$c_{k+1}(x)=(x-x_{k+3})(x-x_{k+2})(x-x_k)$
$\times(x-x_{k-1})(x-x_{k-2})(x-x_{k-3}),$
$d_{k+1}=(x_{k+1}-x_{k+3})(x_{k+1}-x_{k+2})$
$\times(x_{k+1}-x_k)(x_{k+1}-x_{k-1})$
$\times(x_{k+1}-x_{k-2})(x_{k+1}-x_{k-3});$

$$\omega_{k+2}^{M}(x)=\frac{c_{k+2}(x)}{d_{k+2}},$$
$c_{k+2}(x)=(x-x_{k+1})(x-x_k)(x-x_{k-1})$
$\times(x-x_{k-2})(x-x_{k-3})(x-x_{k+3}),$
$d_{k+2}=(x_{k+2}-x_{k+1})(x_{k+2}-x_k)$
$\times(x_{k+2}-x_{k-1})(x_{k+2}-x_{k-2})$
$\times(x_{k+2}-x_{k-3})(x_{k+2}-x_{k+3});$

$$\omega_{k+3}^{M}(x)=\frac{c_{k+3}(x)}{d_{k+3}},$$
$c_{k+3}(x)=(x-x_{k+2})(x-x_{k+1})(x-x_k)$
$\times(x-x_{k-1})(x-x_{k-2})(x-x_{k-3}),$
$d_{k+3}=(x_{k+3}-x_{k+2})(x_{k+3}-x_{k+1})$
$\times(x_{k+3}-x_k)(x_{k+3}-x_{k-1})$
$\times(x_{k+3}-x_{k-2})(x_{k+3}-x_{k-3}).$

The plots of the basis splines $\omega_j^M(x), j=k-3,\dots k+3$, are given in Fig. 1, Fig. 2, Fig. 3, Fig. 4, Fig. 5, Fig. 6, Fig. 7.

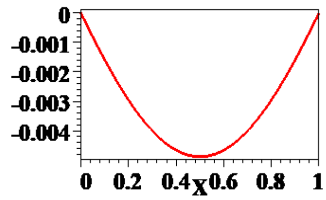

Fig. 1: The plot of the basis spline $\omega_{k-3}^{M}(x)$ when $x_{k+1}=1, x_k=0$.

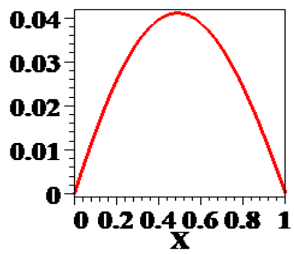

Fig. 2: The plot of the basis spline $\omega_{k-2}^{M}(x)$ when $x_{k+1}=1, x_k=0$.

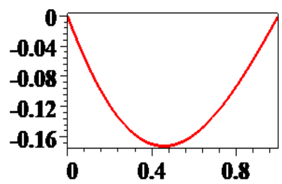

Fig. 3: The plot of the basis spline $\omega_{k-1}^{M}(x)$ when $x_{k+1}=1, x_k=0$.

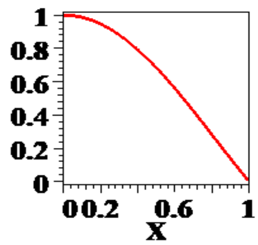

Fig. 4: The plot of the basis spline $\omega_{k}^{M}(x)$ when $x_{k+1}=1, x_k=0$.

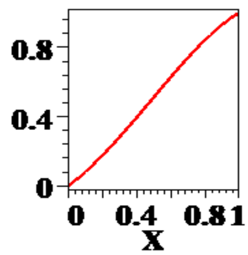

Fig. 5: The plot of the basis spline $\omega_{k+1}^{M}(x)$ when $x_{k+1}=1, x_k=0$.

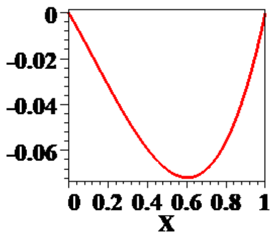

Fig. 6: The plot of the basis spline $\omega^M_{k+2}(x)$ when $x_{k+1} = 1, x_k = 0$.

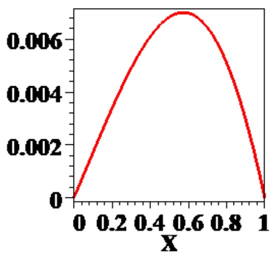

Fig. 7: The plot of the basis spline $\omega^M_{k+3}(x)$ when $x_{k+1} = 1, x_k = 0$.

2.2 Approximation with the Left Basis Splines

Let us consider the approximation with the left basis splines. We get the left basis splines when $r_1 = 5$, $r = 1$. In this case, formula (1) on the interval $[x_k, x_{k+1}]$ takes the form:

$$\tilde{u}(x) = \sum_{j=k-5}^{k+1} u(x_j)\omega^L_j(x)\,, x \in [x_{k,}x_{k+1}]. \quad (3)$$

where the basis splines $\omega^L_k(x)$ have the form

$$\omega^L_{k-5}(x) = c_{k-5}(x)/d_{k-5},$$
$$c_{k-5}(x) = (x - x_{k+1})(x - x_k)(x - x_{k-1}) \times (x - x_{k-2})(x - x_{k-3})(x - x_{k-4}),$$
$$d_{k-5} = (x_{k-5} - x_{k+1})(x_{k-5} - x_k) \times (x_{k-5} - x_{k-1})(x_{k-5} - x_{k-2}) \times (x_{k-5} - x_{k-3})(x_{k-5} - x_{k-4});$$

$$\omega^L_{k-4}(x) = \frac{c_{k-4}(x)}{d_{k-4}},$$
$$c_{k-4}(x) = (x - x_{k+1})(x - x_k)(x - x_{k-1}) \times (x - x_{k-2})(x - x_{k-3})(x - x_{k-5}),$$
$$d_{k-4} = (x_{k-4} - x_{k+1})(x_{k-4} - x_k) \times (x_{k-4} - x_{k-1})(x_{k-4} - x_{k-2}) \times (x_{k-4} - x_{k-3})(x_{k-4} - x_{k-5});$$

$$\omega^L_{k-3}(x) = \frac{c_{k-3}(x)}{d_{k-3}},$$
$$c_{k-3}(x) = (x - x_{k+1})(x - x_k)(x - x_{k-1}) \times (x - x_{k-2})(x - x_{k-4})(x - x_{k-5}),$$
$$d_{k-3} = (x_{k-3} - x_{k+1})(x_{k-3} - x_k) \times (x_{k-3} - x_{k-1})(x_{k-3} - x_{k-2}) \times (x_{k-3} - x_{k-4})(x_{k-3} - x_{k-5});$$

$$\omega^L_{k-2}(x) = \frac{c_{k-2}(x)}{d_{k-2}},$$
$$c_{k-2}(x) = (x - x_{k+1})(x - x_k)(x - x_{k-1}) \times (x - x_{k-3})(x - x_{k-4})(x - x_{k-5}),$$
$$d_{k-2} = (x_{k-2} - x_{k+1})(x_{k-2} - x_k) \times (x_{k-2} - x_{k-1})(x_{k-2} - x_{k-3}) \times (x_{k-2} - x_{k-4})(x_{k-2} - x_{k-5});$$

$$\omega^L_{k-1}(x) = \frac{c_{k-1}(x)}{d_{k-1}},$$
$$c_{k-1}(x) = (x - x_{k+1})(x - x_k)(x - x_{k-2}) \times (x - x_{k-3})(x - x_{k-4})(x - x_{k-5}),$$
$$d_{k-1} = (x_{k-1} - x_{k+1})(x_{k-1} - x_k) \times (x_{k-1} - x_{k-2})(x_{k-1} - x_{k-3}) \times (x_{k-1} - x_{k-4})(x_{k-1} - x_{k-5});$$

$$\omega^L_k(x) = \frac{c_k(x)}{d_k},$$
$$c_k(x) = (x - x_{k+1})(x - x_{k-1})(x - x_{k-2}) (x - x_{k-3})(x - x_{k-4})(x - x_{k-5}),$$
$$d_k = (x_k - x_{k+1})(x_k - x_{k-1})(x_k - x_{k-2}) (x_k - x_{k-3})(x_k - x_{k-4})(x_k - x_{k-5});$$

$$\omega^L_{k+1}(x) = \frac{c_{k+1}(x)}{d_{k+1}},$$
$$c_{k+1}(x) = (x - x_k)(x - x_{k-1})(x - x_{k-2}) \times (x - x_{k-3})(x - x_{k-4})(x - x_{k-5}),$$
$$d_{k+1} = (x_{k+1} - x_k)(x_{k+1} - x_{k-1}) \times (x_{k+1} - x_{k-2})(x_{k+1} - x_{k-3}) \times (x_{k+1} - x_{k-4})(x_{k+1} - x_{k-5}).$$

2.3 Approximation with the Right Basis Splines

Consider the approximation with the right basis splines. Let $r_1 = 0$, $r = 6$, in this case, on the interval $[x_k, x_{k+1}]$ formula (1) takes the form:

$$\tilde{u}(x) = \sum_{j=k}^{k+6} u(x_j)\omega^R_j(x), \qquad x \in [x_k, x_{k+1}]\ , (4)$$

where the right basis splines $\omega^R_j(x)$ have the form:

$$\omega^R_k(x) = \frac{c_k(x)}{d_k},$$
$$c_k(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+4}) \times (x - x_{k+3})(x - x_{k+2})(x - x_{k+1}),$$
$$d_k = (x_k - x_{k+6})(x_k - x_{k+5})(x_k - x_{k+4}) \times (x_k - x_{k+3})(x_k - x_{k+2})(x_k - x_{k+1});$$

$$\omega_{k+1}^R(x) = \frac{c_{k+1}(x)}{d_{k+1}},$$

$$c_{k+1}(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+4}) \times (x - x_{k+3})(x - x_{k+2})(x - x_k),$$

$$d_{k+1} = (x_{k+1} - x_{k+6})(x_{k+1} - x_{k+5}) \times (x_{k+1} - x_{k+4})(x_{k+1} - x_{k+3}) \times (x_{k+1} - x_{k+2})(x_{k+1} - x_k);$$

$$\omega_{k+2}^R(x) = \frac{c_{k+2}(x)}{d_{k+2}},$$

$$c_{k+2}(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+4}) \times (x - x_{k+3})(x - x_{k+1})(x - x_k),$$

$$d_{k+2} = (x_{k+2} - x_{k+6})(x_{k+2} - x_{k+5}) \times (x_{k+2} - x_{k+4})(x_{k+2} - x_{k+3}) \times (x_{k+2} - x_{k+1})(x_{k+2} - x_k);$$

$$\omega_{k+3}^R(x) = \frac{c_{k+3}(x)}{d_{k+3}},$$

$$c_{k+3}(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+4}) \times (x - x_{k+2})(x - x_{k+1})(x - x_k),$$

$$d_{k+3} = (x_{k+3} - x_{k+6})(x_{k+3} - x_{k+5}) \times (x_{k+3} - x_{k+4})(x_{k+3} - x_{k+2}) \times (x_{k+3} - x_{k+1})(x_{k+3} - x_k);$$

$$\omega_{k+4}^R(x) = \frac{c_{k+4}(x)}{d_{k+4}},$$

$$c_{k+4}(x) = (x - x_{k+6})(x - x_{k+5})(x - x_{k+3}) \times (x - x_{k+2})(x - x_{k+1})(x - x_k),$$

$$d_{k+4} = (x_{k+4} - x_{k+6})(x_{k+4} - x_{k+5}) \times (x_{k+4} - x_{k+3})(x_{k+4} - x_{k+2}) \times (x_{k+4} - x_{k+1})(x_{k+4} - x_k);$$

$$\omega_{k+5}^R(x) = \frac{c_{k+5}(x)}{d_{k+5}},$$

$$c_{k+5}(x) = (x - x_{k+6})(x - x_{k+4})(x - x_{k+3}) \times (x - x_{k+2})(x - x_{k+1})(x - x_k),$$

$$d_{k+5} = (x_{k+5} - x_{k+6})(x_{k+5} - x_{k+4}) \times (x_{k+5} - x_{k+3})(x_{k+5} - x_{k+2}) \times (x_{k+5} - x_{k+1})(x_{k+5} - x_k);$$

$$\omega_{k+6}^R(x) = \frac{c_{k+6}(x)}{d_{k+6}},$$

$$c_{k+6}(x) = (x - x_{k+5})(x - x_{k+4})(x - x_{k+3}) \times (x - x_{k+2})(x - x_{k+1})(x - x_k),$$

$$d_{k+6} = (x_{k+6} - x_{k+5})(x_{k+6} - x_{k+4}) \times (x_{k+6} - x_{k+3})(x_{k+6} - x_{k+2}) \times (x_{k+6} - x_{k+1})(x_{k+6} - x_k).$$

2.4 Approximation Theorem

Further, we will use the norm of the vector of the form:

$$\| u \|_{[a,b]} = \max_{x\in[a,b]} | u(x)|$$

When approximating a function with the splines of the 7th order of approximation, the next Theorem is valid.

Theorem. If $\operatorname{supp} \omega_k = [x_{k-1}, x_{k+6}]$, then the following inequalities are valid:

$$|u(x) - \tilde{u}(x)|_{x\in[x_k,x_{k+1}]} \leq 95.842 \cdot h^7 \frac{\| u^{(7)} \|_{[x_{k-5},x_{k+1}]}}{7!}.$$

If $\operatorname{supp} \omega_k = [x_{k-3}, x_{k+4}]$, then the following approximation estimate is valid:

$$|u(x) - \tilde{u}(x)|_{x\in[x_k,x_{k+1}]} \leq 12.359 \cdot h^7 \frac{\| u^{(7)} \|_{[x_{k-3},x_{k+3}]}}{7!}.$$

If $\operatorname{supp} \omega_k = [x_{k-6}, x_{k+1}]$, then the following approximation estimate is valid:

$$|u(x) - \tilde{u}(x)|_{x\in[x_k,x_{k+1}]} \leq 95.842 \cdot h^7 \frac{\| u^{(7)} \|_{[x_k,x_{k+6}]}}{7!}.$$

Proof. In the case of approximating the function u on the interval $[x_k, x_{k+1}]$ near the left end of the interval $[a, b]$, we use the right basis splines:

$$\tilde{u}(x) = \sum_{j=k}^{k+6} u(x_j)\omega_j^R(x)dx, x \in [x_k, x_{k+1}].$$

Let us estimate the approximation error on the interval $[x_k, x_{k+1}]$ when the right basis splines were used. Using the formula of the remainder term of the interpolation polynomial that solves the Lagrange interpolation problem, we obtain the relation

$$u(x) - \tilde{u}(x) = \frac{u^{(7)}(\xi)}{7!}(x - x_k) \dots (x - x_{k+6}), \quad \xi \in [x_{k-5}, x_{k+1}].$$

There is a product $(x - x_k) \dots (x - x_{k+6})$ in the error estimate. Let the ordered grid of nodes $\{x_k\}$ be uniform with step h. Let us estimate the product of factors $(x - x_k) \dots (x - x_{k+6})$.

Thus, estimating the maximum of the expression $\frac{u^{(7)}(\xi)}{7!}(x - x_k) \dots (x - x_{k+6})$, where $\xi \in [x_k, x_{k+6}]$, we obtain

$$\| u(x) - \tilde{u}(x) \|_{[x_k,x_{k+1}]} \leq K h^7 \| u^{(7)} \|_{[x_k, x_{k+6}]},$$

Similarly, we obtain an approximation estimate on the grid interval $[x_k, x_{k+1}]$ with the left and middle splines.

This completes the proof of the theorem.

3 The Application of the Local Splines of the Seventh Order of Approximation to Calculate Integrals

First of all, we note how to apply local splines of the seventh order of approximation to calculate integrals over the interval $[a,b]$. As already noted, the spline approximation of the function is applied separately for each grid interval. Applying the estimates given in the theorem, we should calculate the integral $\int_a^b f(s)ds$ as followed.

Let s_j be the nodes of the set on the interval $[a,b]$:

$$a = s_0 < s_1 < \cdots < s_n = b.$$

We represent the integral in the form:

$$\int_a^b f(s)ds = \sum_{k=0}^{n-1} \int_{s_k}^{s_{k+1}} f(s)ds\,.$$

The function $f(s) = K(x,s)u(s), s \in [s_k, s_{k+1}]$, can be approximated with the expression: $f(s) \approx \tilde{f}(s) = K(x,s)\tilde{u}(s)$. Let us denote $c_j = u(s_j)$, and let α_m and $\beta_m, m = 1,2,3$, determine the type of spline: the left, the right, or the middle spline. On the intervals $[s_k, s_{k+1}]$, $k = 0,1,2$, we put $\alpha_1 = k, \beta_1 = k+6$. On the intervals $[s_k, s_{k+1}], k = 3, \ldots, n-4$, we put $\alpha_2 = k-3, \beta_2 = k+3$. On the intervals $[s_k, s_{k+1}], k = n-3, \ldots, n-1$, we put $\alpha_3 = k-5, \beta_3 = k+1$.

Let us denote $c_j \approx u(x_j)$. Now we use the following approximations of the function $u(s)$ at the first three grid intervals of the interval $[a,b]$:

$$\tilde{u}_R(s) = \sum_{j=k}^{k+6} c_j\, \omega_j^R(s), \;\; s \in [s_k, s_{k+1}], k = 0,1,2,$$

We use the following approximations of the function $u(s)$ in the middle of the interval $[a,b]$:

$$\tilde{u}_M(s) = \sum_{j=k-3}^{k+3} c_j\, \omega_j^M(s), \;\; s \in [s_k, s_{k+1}],$$
$$k = 3, \ldots, n-4.$$

We use the following approximations of the function $u(s)$ at the last three grid intervals of the interval $[a,b]$:

$$\tilde{u}_L(s) = \sum_{j=k-5}^{k+1} c_j\, \omega_j^L(s), \;\; s \in [s_k, s_{k+1}],$$
$$k = n-3, \ldots, n-1.$$

We assume that the integral $\int_{s_k}^{s_{k+1}} K(x,s)\omega_j(s)ds$ can be computed exactly. Otherwise, we can use the quadrature formulas. In this case, it is necessary to take into account the error of the applied quadrature formula. We can use, for example, Simpson's compound formula.

4 The Application of the Local Splines of the Seventh Order of Approximation which is used to Calculate a Solution of an Integral Equation

First, we discuss the solution of the integral equation of the second kind

$$Au \equiv u(x) - \int_a^b K(x,s)u(s)ds = g(x),$$
$$x \in [a,b].$$

We assume that the kernel $K(x,s)$ and the right side of the equation $g(x)$ are continuous. In addition, we assume that the equation is uniquely solvable and the estimate for the norm of the inverse operator in the space C is known: $\| A^{-1} \| \leq B$.

Let us choose an integer $n \geq 10$. We build a uniform grid with a step $h = \frac{b-a}{n}$.

Using the results from the third section, we can reduce the integral equation to the solution of a system of linear algebraic equations. To do this, we put $x = x_m$, $m = 0, \ldots, n-1$, (x_m, takes the same values as s_j, $j = 0, \ldots, n-1$) in the equation

$$u(x) - \sum_{k=0}^{2} \sum_{j=k}^{k+6} c_j \int_{s_k}^{s_{k+1}} K(x,s)\omega_j^R(s)ds + \sum_{k=3}^{n-4} \sum_{j=k-3}^{k+3} c_j \int_{s_k}^{s_{k+1}} K(x,s)\omega_j^M(s)ds + \sum_{k=n-3}^{n-1} \sum_{j=n-3}^{n-1} c_j \int_{s_k}^{s_{k+1}} K(x,s)\omega_j^L(s)ds = g(x).$$

And now we have to solve the system of linear algebraic equations

$$u(x_m) - \sum_{k=0}^{2} \sum_{j=k}^{k+6} c_j \int_{s_k}^{s_{k+1}} K(x_m,s)\omega_j^R(s)ds + \sum_{k=3}^{n-4} \sum_{j=k-3}^{k+3} c_j \int_{s_k}^{s_{k+1}} K(x_m,s)\omega_j^M(s)ds + \sum_{k=n-3}^{n-1} \sum_{j=n-3}^{n-1} c_j \int_{s_k}^{s_{k+1}} K(x_m,s)\omega_j^L(s)ds = g(x_m).$$

$$m = 0, \ldots, n.$$

First, let us consider some examples of the application of splines of the seventh order of approximation in the example of solving the Fredholm integral equation of the second kind.

Example 1. Consider the equation

$$y(x) + \int_0^1 \sin(x - s)e^{x-s}y(s)ds = g(x),$$
$$x \in [0,1].$$

Note that the right side of this equation was constructed according to the exact solution, which has the form $y(x) = \sin(10\,x)$. The plot of the function $g(x)$ is given in Fig.8.

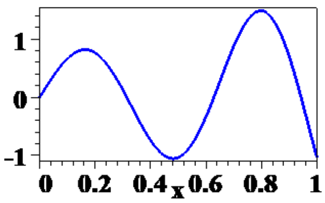

Fig. 8: The plot of the function $g(x)$.

Using splines of the seventh order of approximation we construct the system of equations.
Solving the system of equations with the number of grid nodes ($n = 16$), we obtain the solution error that is shown in Fig. 9. The nodes are marked along the abscissa axis. For comparison, in Fig.10 a plot of the absolute values of the solution error when using splines of the second order of approximation is given (see, [19], [20]). Programs for solving the integral equations were developed in the Maple system.

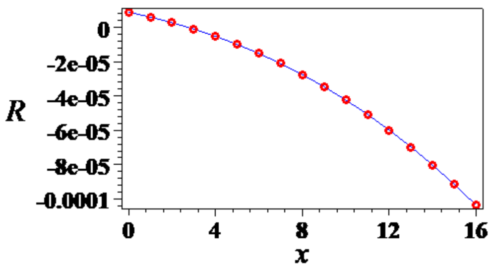

Fig. 9: The application of the splines of the seventh order of approximation

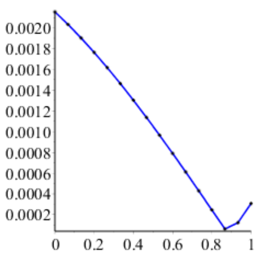

Fig. 10: The application of the composite quadrature formula of trapezoids

In Fig. 10, Fig. 12, along the abscissa axis, the integration interval is marked, and in other figures, grid nodes are plotted along the abscissa axis.

Example 2. Consider the equation

$$y(x) + \int_0^1 e^{x-s}y(s)ds = g(x),\ x \in [0,1].$$

The exact solution of this equation is $y(x) = e^{3x}\sin(3\,x)$. The right side of the integral equation was constructed according to the exact solution.
Using splines of the seventh order of approximation we construct the system of equations. Solving the system of equations with the number of grid nodes ($n = 16$), we obtain the solution error that is shown in Fig. 11. Fig. 12 shows the error in solving the equation using the composite quadrature formula of trapezoids ($n = 16$).

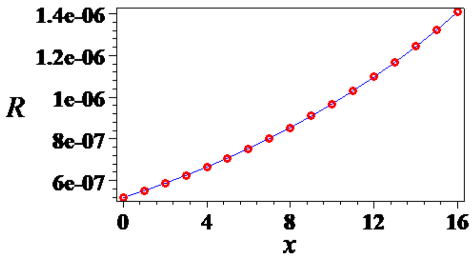

Fig. 11: The application of the splines of the seventh order of approximation

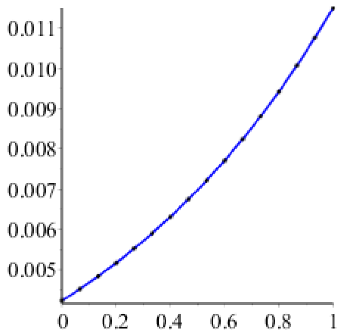

Fig. 12: The application of the composite quadrature formula of trapezoids

Note that to achieve the order of error of 10^{-6} using the trapezoid formula, the number of grid nodes $n = 512$ was required, and the computation time was 125 seconds. Using the splines we can construct formulas for approximating the derivatives of the function with the given error.

Example 3. Consider the equation

$$y(x) + \int_0^1 \sin(x\,s)y(s)ds = g(x),\ x \in [0,1].$$

The exact solution of this equation is $y(x) = \sin(x^2)$. The right side of the integral equation was constructed according to the exact solution. Solving the equation using splines of the seventh order of approximation with the number of grid nodes ($n = 16$), we obtain the solution error that is shown in Fig. 13.

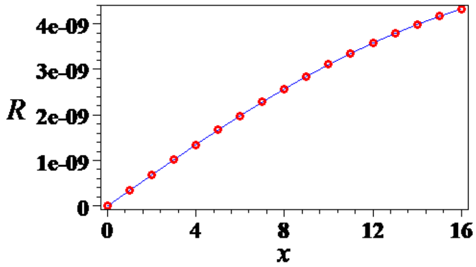

Fig. 13: The application of the splines of the seventh order of approximation ($n = 16$).

Solving the equation using splines of the seventh order of approximation with the number of grid nodes ($n = 10$), we obtain the solution error that is shown in Fig. 14.

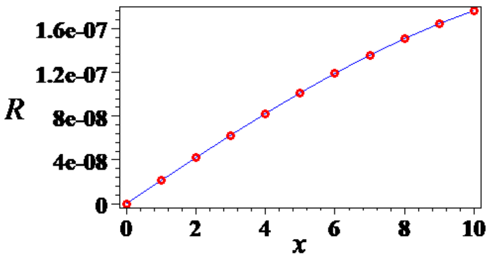

Fig. 14: The application of the splines of the seventh order of approximation ($n = 10$)

Example 4. Now consider the integral equation:

$$u(x) + \int_0^1 \exp(x+s)\, u(s) ds = f(x),$$

where $x \in [0,1]$.

The exact solution of the integral equation is the next: $u(x) = \exp(-x)$. The right side of the integral equation was constructed according to the exact solution. Applying spline approximations of the fifth order to the solution of the equation, [19], we obtain the error which is shown in Fig. 15. When applying spline approximations of the seventh order to the solution of the equation, we obtain the error which is shown in Fig. 16. A program was developed in the Maple environment. A uniform grid of nodes was built on [0,1], consisting of 16 nodes ($n = 16$).

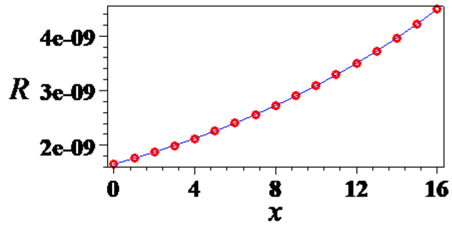

Fig. 15: The plot of the error obtained using spline approximations of the fifth order (Digits=20, $n = 16$)

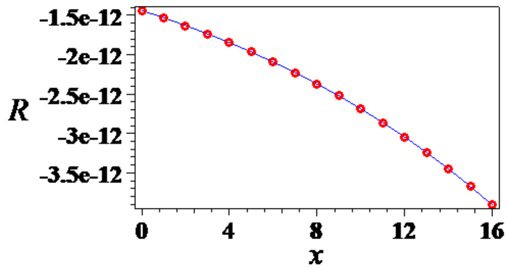

Fig. 16: The plot of the error obtained using spline approximations of the seventh order ($n = 16$)

5 Conclusion

In this paper, we consider the solution of integral equations of the second kind using splines of the seventh order of approximation. The results of solving the same integral equations using splines of the order of approximation less than 7 are given also. It should be noted that with the same number of grid nodes, polynomial splines of the seventh order of approximation provide a smaller error compared to splines of a lower degree. In the future, numerical schemes for integro-differential equations will be constructed. It is supposed to develop numerical methods for solving integral equations with a weak singularity based on the use of splines of the seventh order of approximation.

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G. O. Alcybeev executed the numerical experiments, and I. G. Burova developed the theoretical part.

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Conflict of Interest

The authors have no conflict of interest to declare.