THE APPROACH TO NORMALITY OF THE SOLUTIONS OF RANDOM BOUNDARY AND EIGENVALUE PROBLEMS WITH WEAKLY CORRELATED COEFFICIENTS*

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Abstract. A general class of linear self-adjoint random boundary value problems with weakly correlated coefficients is considered. The earlier result that the distribution function of the solution approaches the normal as the correlation length ε tends to zero is generalized somewhat. Correction terms are derived that yield estimates for the distribution function when ε is small but nonzero. The results are also applied to the eigenvalues and eigenfunctions of a corresponding class of random eigenvalue problems. The discussion is given in terms of second-order equations, but extensions to higher-order problems are readily apparent.

1. Introduction. For many years it has been of interest to find conditions under which the distribution of the solution of a random differential equation tends to the normal. In 1930, while studying Brownian motion, Uhlenbeck and Ornstein [8] established that the solution $y(t)$ of certain initial-value problems has approximately a normal distribution. They accomplished this by showing that

$$\begin{aligned} \langle y^n \rangle &\cong 0, && n \text{ odd} \\ &\cong 1 \cdot 3 \cdot 5 \cdots (n-1)\langle y^2 \rangle^{n/2}, && n \text{ even} \end{aligned} \tag{1.1}$$

where $\langle \cdot \rangle$ denotes the mathematical expectation.

In 1966 Boyce [1] established a similar result for a class of linear self-adjoint boundary value problems

$$L[y] = f(x), \qquad 0 < x < 1 \tag{1.2}$$

with boundary conditions

$$U_i[y] = 0 \tag{1.3}$$

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at the end points. The operator L has the form

$$L[y] = \sum_{j=0}^{m} (-1)^j [r_j(x) y^{(j)}]^{(j)}. \tag{1.4}$$

Randomness entered the problem (1.2), (1.3) only through the forcing function f, which was assumed to be weakly correlated with correlation length $\varepsilon \ll 1$. The solution $y(x)$ was shown to satisfy equations similar to (1.1), with the approximation becoming exact as $\varepsilon \to 0$. This means that the distribution of $y(x)/\sqrt{\varepsilon}$ approaches a normal distribution as $\varepsilon \to 0$. Combining the methods of [7] and [1] with a perturbation expansion, Purkert and vom Scheidt [4, 7] generalized this result to the case in which $f, r_0, \ldots, r_{m-1}$ are independent and weakly correlated processes. In Sec. 2 we extend this result still further so as to permit r_m also to be a weakly correlated stochastic process.

In many problems arising in applications the correlation length ε is small, but not vanishingly so. This raises the question of how far the solution $y(x)$ departs from normality when ε is a small positive number. To answer this question we calculate the second nonzero term in the Hermite-Chebyshev expansion of the distribution function of $y(x)$; this is the most important contribution of the present paper. It turns out that two cases must be considered, depending on whether the correction term is of order $\sqrt{\varepsilon}$ or of order ε, and these are discussed in Secs. 3 and 4 respectively. Sec. 5 is devoted to an example that gives some feeling for the orders of magnitude of the terms. Purkert and vom Scheidt [3, 5, 6, 7] have also considered self-adjoint eigenvalue problems

$$L[u] = \lambda u, \qquad 0 < x < 1, \tag{1.5}$$

$$U_i[u] = 0, \tag{1.6}$$

where the coefficients in L are weakly correlated processes with common correlation length ε. They showed in [5] that $(\lambda_j - \mu_j)/\sqrt{\varepsilon}$ approaches a normal random variable as $\varepsilon \to 0$. Here μ_j is the jth eigenvalue of the mean problem obtained by replacing each random coefficient in L by its mean. At the same time they established the asymptotic normality of the eigenfunctions of (1.5), (1.6). By proceeding much as in Secs. 3 and 4 it is possible to determine correction terms for the distributions of the eigenvalues and eigenfunctions for small positive values of ε. These matters are discussed briefly in Sec. 6.

In [3] and [5], in particular, Purkert and vom Scheidt have also developed much necessary background material in a detailed and formal manner. We will use their definitions and nomenclature in this paper without explicit reference.

2. Boundary-value problems with weakly correlated coefficients. Although the same methods can be applied to higher-order problems, in this section we consider the nonhomogeneous two-point boundary-value problem consisting of the second-order differential equation

$$L[y] = -[p(x, \omega) y']' + q(x, \omega) y = f(x, \omega), \qquad 0 < x < 1, \tag{2.1}$$

and the boundary conditions

$$U_i[y] = 0; \qquad i = 1, 2. \tag{2.2}$$

The coefficients in the differential equation (2.1) are stochastic processes defined on an underlying probability space $(\Omega, \mathscr{F}, P)$. Except possibly on an ω-set of probability zero, the sample functions of p, q, and f satisfy the standard conditions: f and q are continuous,

p is continuously differentiable, and p is never zero in $0 \le x \le 1$. The emphasis here will be on the consequences of assuming that the leading coefficient $p(x, \omega)$ is random, since this extends the results of [4]. This section also provides a framework for the following sections.

The boundary conditions (2.2) are deterministic and are such that the problem is self-adjoint. For example, for the problem (2.1), (2.2) it is sufficient that the boundary conditions be separated, in which case

$$U_1[y] = a_1 y(0) + a_2 y'(0) = 0, \tag{2.3a}$$

$$U_2[y] = b_1 y(1) + b_2 y'(1) = 0, \tag{2.3b}$$

where a_1, a_2, b_1, and b_2 are given constants.

We can express p, q, and f as a sum of their respective means and a random fluctuation:

$$p(x, \omega) = p_0(x) + \eta p_1(x, \omega), \qquad p_0(x) = \langle p(x, \omega)\rangle, \tag{2.4a}$$

$$q(x, \omega) = q_0(x) + \eta q_1(x, \omega), \qquad q_0(x) = \langle q(x, \omega)\rangle, \tag{2.4b}$$

$$f(x, \omega) = f_0(x) + \eta f_1(x, \omega), \qquad f_0(x) = \langle f(x, \omega)\rangle, \tag{2.4c}$$

where

$$\langle p_1(x, \omega)\rangle = \langle q_1(x, \omega)\rangle = \langle f_1(x, \omega)\rangle = 0 \tag{2.5}$$

and η is an indexing parameter; sometimes $\eta = 1$. In addition, we assume that p_1, q_1, and f_1 are pairwise independent wide-sense stationary processes.

We are interested in the case in which p_1, q_1, and f_1 are also weakly correlated, a term that has been defined by Purkert and vom Scheidt [4, 5] in the following way. Let $S = (x_1, \ldots, x_n)$ be an n-tuple of real numbers and let $\varepsilon > 0$ be a positive constant. Let $S_1 = (x_{i_1}, \ldots, x_{i_k})$ be a subset of S, and suppose that $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_k}$; this ordering can always be attained by relabeling the elements of S_1 if necessary. Then S_1 is said to be ε-neighboring if

$$|x_{i_1} - x_{i_2}| \le \varepsilon, \qquad |x_{i_2} - x_{i_3}| \le \varepsilon, \ldots, \qquad |x_{i_{k-1}} - x_{i_k}| \le \varepsilon.$$

A single element subset is always ε-neighboring. The subset S_1 is maximally ε-neighboring, with respect to S, if S_1 is ε-neighboring, but is not contained in any larger ε-neighboring subset of S. It can be shown [5] that S can be separated into disjoint maximally ε-neighboring subsets in a unique way. Then a stationary process $h(x, \omega)$ is said to be weakly correlated with correlation length ε if, for each n,

$$\langle h(x_1, \omega) \cdots h(x_n, \omega)\rangle = \langle h(x_{11}) \cdots h(x_{1p_1})\rangle \cdots \langle h(x_{k1}) \cdots h(x_{kp_k})\rangle, \tag{2.6}$$

where the n-tuple S has been separated into the maximally ε-neighboring subsets $(x_{11}, \ldots, x_{1p_1}), \ldots, (x_{k1}, \ldots x_{kp_k})$ and $\sum_{i=1}^{k} p_i = n$.

Thus we assume that p_1, q_1, and f_1 have the property (2.6). In the particular case where $n = 2$ this reduces to

$$\begin{aligned} \langle p_1(x_1)p_1(x_2)\rangle &= 0, && |x_2 - x_1| > \varepsilon; \\ &= \sigma_p^2 \rho_p(x_2 - x_1), && |x_2 - x_1| \le \varepsilon; \end{aligned} \tag{2.7}$$

$$\begin{aligned} \langle q_1(x_1)q_1(x_2)\rangle &= 0, && |x_2 - x_1| > \varepsilon; \\ &= \sigma_q^2 \rho_q(x_2 - x_1), && |x_2 - x_1| \le \varepsilon; \end{aligned} \tag{2.8}$$

$$\begin{aligned} \langle f_1(x_1)f_1(x_2)\rangle &= 0, && |x_2 - x_1| > \varepsilon; \\ &= \sigma_f^2 \rho_f(x_2 - x_1), && |x_2 - x_1| \le \varepsilon. \end{aligned} \tag{2.9}$$

In (2.7) through (2.9) σ_p^2, σ_q^2, and σ_f^2 are the (constant) variances and ρ_p, ρ_q, and ρ_f are the autocorrelation functions of p_1, q_1, and f_1 respectively. Eq. (2.7) says that $p_1(x_1)$ and $p_1(x_2)$ are uncorrelated except in a strip of width $\sqrt{2}\varepsilon$ about the line $x_1 = x_2$, and similarly for (2.8) and (2.9).

Finally, we assume that the distributions of p_1, q_1, and f_1 are such as to allow all of the analytical procedures that we will use without specific justification. This includes the interchange of the order of various types of limiting processes.

It is convenient to write $y(x, \omega)$ as a perturbation series in η,

$$y(x, \omega) = y_0(x) + \sum_{k=1}^{\infty} y_k(x, \omega)\eta^k. \tag{2.10}$$

Substituting this expression for y in (2.1) and (2.2) and equating coefficients of like powers of η, we find that $y_0(x)$ satisfies

$$L_0[y_0] \equiv -[p_0(x)y_0']' + q_0(x)y_0 = f_0(x), \qquad 0 < x < 1, \tag{2.11}$$

$$U_i[y_0] = 0; \qquad i = 1, 2. \tag{2.12}$$

Further, from the linear terms in η, we have

$$L_0[y_1] = f_1(x, \omega) - q_1(x, \omega)y_0(x) + [p_1(x, \omega)y_0'(x)]', \qquad 0 < x < 1 \tag{2.13}$$

$$U_i[y_1] = 0; \qquad i = 1, 2. \tag{2.14}$$

In general, for $k \ge 2$:

$$L_0[y_k] = -q_1(x, \omega)y_{k-1}(x, \omega) + [p_1(x, \omega)y_{k-1}'(x, \omega)]', \qquad 0 < x < 1 \tag{2.15}$$

$$U_i[y_k] = 0; \qquad i = 1, 2. \tag{2.16}$$

If zero is not an eigenvalue of L_0 subject to the boundary conditions (2.12), then $y_0(x)$ is given by

$$y_0(x) = \int_0^1 G(x, x_1)f_0(x_1)\, dx_1, \tag{2.17}$$

where $G(x, x_1)$ is the (deterministic) Green's function associated with L_0 and the given boundary conditions. In the same way

$$y_1(x, \omega) = \int_0^1 G(x, x_1)g_1(x_1, \omega)\, dx_1 \tag{2.18}$$

where g_1 is the expression on the right-hand side of (2.13):

$$g_1(x, \omega) = f_1(x, \omega) - q_1(x, \omega)y_0(x) + [p_1(x, \omega)y_0'(x)]'. \tag{2.19}$$

Similar formulas can be written down for $y_2(x, \omega)$, $y_3(x, \omega)$, $\ldots$.

Our primary interest is the calculation of moments of $y_1(x, \omega)$. These are given by the expressions

$$\langle y_1(x, \omega)\rangle = \int_0^1 G(x, x_1)\langle g_1(x_1, \omega)\rangle\, dx_1, \tag{2.20}$$

$$\langle y_1^2(x, \omega)\rangle = \int_0^1 \int_0^1 G(x, x_1)G(x, x_2)\langle g_1(x_1, \omega)g_1(x_2, \omega)\rangle\, dx_1\, dx_2, \tag{2.21}$$

and in general by

$$\langle y_1^n(x, \omega)\rangle = \int_0^1 \cdots \int_0^1 G(x, x_1) \cdots G(x, x_n)\langle g_1(x_1, \omega) \cdots g_1(x_n, \omega)\rangle\, dx_1 \cdots dx_n. \tag{2.22}$$

To evaluate these expressions it is necessary to consider some properties of $g_1(x, \omega)$. Hereafter, for the sake of brevity, we will usually omit explicit indication of ω as an independent variable.

In the first place, we have

$$\begin{aligned}\langle p_1'(x)\rangle &= \left\langle \lim_{\Delta\to 0} \frac{p_1(x+\Delta) - p_1(x)}{\Delta}\right\rangle \\ &= \lim_{\Delta\to 0} \frac{1}{\Delta} \langle p_1(x+\Delta) - p_1(x)\rangle = 0 \end{aligned}\tag{2.23}$$

because of (2.5). Then

$$\begin{aligned}\langle g_1(x)\rangle &= \langle f_1(x) - q_1(x)y_0(x) + p_1'(x)y_0'(x) + p_1(x)y_0''(x)\rangle \\ &= \langle f_1(x)\rangle - \langle q_1(x)\rangle y_0(x) + \langle p_1'(x)\rangle y_0'(x) + \langle p_1(x)\rangle y_0''(x) \\ &= 0 \end{aligned}\tag{2.24}$$

by (2.5) and (2.23). Consequently, from (2.20),

$$\langle y_1(x, \omega)\rangle = I_1(x) = 0. \tag{2.25}$$

Using (2.19) to form the product $g_1(x_1)g_1(x_2)$ and then taking the mean, we obtain

$$\begin{aligned}\langle g_1(x_1)g_1(x_2)\rangle = \langle f_1(x_1)f_1(x_2)\rangle &+ \langle q_1(x_1)q_1(x_2)\rangle y_0(x_1)y_0(x_2) \\ &+ \langle [p_1(x_1)y_0'(x_1)]'[p_1(x_2)y_0'(x_2)]'\rangle; \end{aligned}\tag{2.26}$$

we have used (2.5) and the independence of f_1, p_1, and q_1 to eliminate cross-product terms such as $\langle f_1(x_1)q_1(x_2)\rangle y_0(x_2)$. The last term in (2.26) involves the quantities $\langle p_1(x_1)p_1(x_2)\rangle$, $\langle p_1(x_1)p_1'(x_2)\rangle$, $\langle p_1'(x_1)p_1(x_2)\rangle$, and $\langle p_1'(x_1)p_1'(x_2)\rangle$. The first of these is given by (2.7), so let us consider the others. We will need to assume that the correlation function ρ_p is twice differentiable. First we have

$$\begin{aligned}\langle p_1(x_1)p_1'(x_2)\rangle &= \left\langle p_1(x_1) \lim_{\Delta\to 0} \frac{p_1(x_2+\Delta) - p_1(x_2)}{\Delta}\right\rangle \\ &= \lim_{\Delta\to 0} \frac{1}{\Delta} [\langle p_1(x_1)p_1(x_2+\Delta)\rangle - \langle p_1(x_1)p_1(x_2)\rangle]. \end{aligned}\tag{2.27}$$

If $|x_2 - x_1| \le \varepsilon$ and if Δ is such that $|x_2 - x_1 + \Delta| \le \varepsilon$ also, then from (2.7) we have

$$\begin{aligned}\langle p_1(x_1)p_1'(x_2)\rangle &= \sigma_p^2 \lim_{\Delta\to 0} \frac{1}{\Delta} [\rho_p(x_2 - x_1 + \Delta) - \rho_p(x_2 - x_1)] \\ &= \sigma_p^2 \rho_p'(x_2 - x_1), \qquad |x_2 - x_1| \le \varepsilon. \end{aligned} \tag{2.28a}$$

At the endpoints ρ_p' is interpreted as a one-sided derivative. For $|x_2 - x_1| > \varepsilon$ and for Δ small enough so that $|x_2 - x_1 + \Delta| > \varepsilon$ also, then (2.27) and (2.7) yield

$$\langle p_1(x_1)p_1'(x_2)\rangle = 0, \qquad |x_2 - x_1| > \varepsilon. \tag{2.28b}$$

In the same way we find that

$$\begin{aligned}\langle p_1'(x_1)p_1(x_2)\rangle &= 0, & |x_2 - x_1| > \varepsilon; \\ &= -\sigma_p^2 \rho_p'(x_2 - x_1), & |x_2 - x_1| \le \varepsilon.\end{aligned} \tag{2.29}$$

Proceeding analogously with the remaining term, we obtain

$$\begin{aligned}\langle p_1'(x_1)p_1'(x_2)\rangle &= 0, & |x_2 - x_1| > \varepsilon; \\ &= -\sigma_p^2 \rho_p''(x_2 - x_1), & |x_2 - x_1| \le \varepsilon.\end{aligned} \tag{2.30}$$

Finally, substituting from (2.7), (2.8), (2.9), (2.28), (2.29), and (2.30) into (2.26), we have

$$\begin{aligned}\langle g_1(x_1)g_1(x_2)\rangle &= \sigma_f^2 \rho_f(x_2 - x_1) + \sigma_q^2 \rho_q(x_2 - x_1) y_0(x_1) y_0(x_2) \\ &\quad + \sigma_p^2 \rho_p(x_2 - x_1) y_0''(x_1) y_0''(x_2) - \sigma_p^2 \rho_p''(x_2 - x_1) y_0'(x_1) y_0'(x_2) \\ &\quad + \sigma_p^2 \rho_p'(x_2 - x_1)[y_0''(x_1) y_0'(x_2) - y_0'(x_1) y_0''(x_2)], \qquad |x_2 - x_1| \le \varepsilon,\end{aligned} \tag{2.31a}$$

and

$$\langle g_1(x_1)g_1(x_2)\rangle = 0, \qquad |x_2 - x_1| > \varepsilon. \tag{2.31b}$$

This, in turn, gives $\langle y_1^2(x)\rangle$ from (2.21). To simplify the latter expression it is convenient to introduce the new coordinates (Fig. 1)

$$s = x_2 - x_1, \qquad z = (x_1 + x_2)/2 \tag{2.32}$$

or

$$x_1 = z - s/2, \qquad x_2 = z + s/2. \tag{2.33}$$

Then, since the integrand is even in s, we can write

$$\langle y_1^2(x)\rangle = 2 \int_0^1 \int_0^\varepsilon G\left(x, z - \frac{s}{2}\right) G\left(x, z + \frac{s}{2}\right) \left\langle g_1\left(z - \frac{s}{2}\right) g_1\left(z + \frac{s}{2}\right)\right\rangle ds\, dz - 2J_1 - 2J_2, \tag{2.34}$$

where J_1 and J_2 are the integrals of the same integrand over the triangular regions T_1 and T_2 respectively. Note that J_1 and J_2 are of order ε^2. By expanding the integrand in powers of s and keeping only the first terms, we obtain the contribution to $\langle y_1^2(x)\rangle$ that is linear in ε, namely,

$$\langle y_1^2(x)\rangle = I_2(x) = A_1(x)\varepsilon + O(\varepsilon^2), \tag{2.35a}$$

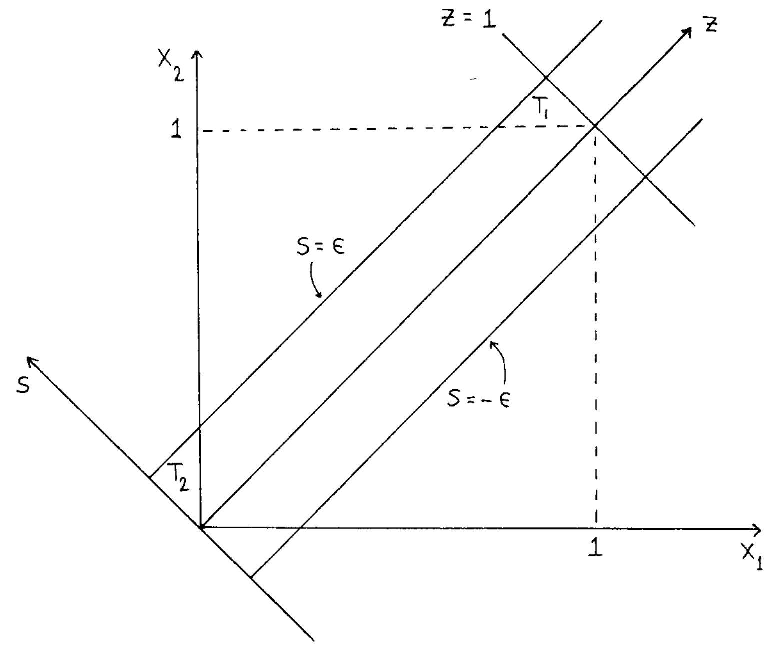

FIG. 1.

where

$$A_1(x) = 2\int_0^1 G^2(x, z)[\sigma_f^2 + \sigma_q^2 y_0^2(z) + \sigma_p^2\{y_0''^2(z) - \rho_p''(0)y_0'^2(z)\}]\,dz. \tag{2.35b}$$

To evaluate higher moments it is necessary to use (2.22). It is possible, although somewhat tedious, to show that even though $g_1(x, \omega)$ is not stationary, it nevertheless has the property (2.6). It is also helpful to define the set R_k by

$$R_k = \{(x_1, \ldots, x_k) \mid 0 \le x_1, \ldots, x_k \le 1 \quad \text{and} \quad (x_1, \ldots, x_k) \quad \text{is } \varepsilon\text{-neighboring}\};$$

it is clear that the volume of R_k is of order ε^{k-1}. Examining (2.22) first for $n = 3$, we note that the only contribution to $\langle y_1^3(x)\rangle$ comes from points in R_3; thus

$$\begin{aligned}\langle y_1^3(x)\rangle &= \iiint_{R_3} G(x, x_1)G(x, x_2)G(x, x_3)\langle g_1(x_1)g_1(x_2)g_1(x_3)\rangle\,dx_1\,dx_2\,dx_3 \\ &= I_3(x) = A_3(x)\varepsilon^2 + O(\varepsilon^3).\end{aligned} \tag{2.36}$$

Next, consider $\langle y_1^4(x)\rangle$. There is a contribution to this quantity from points in R_4, and there are other contributions from points where x_1, x_2, x_3, and x_4 are ε-neighboring in pairs. Thus

$$\begin{aligned}\langle y_1^4(x)\rangle = 3I_2^2(x) &+ \iiiint_{R_4} G(x, x_1)\cdots G(x, x_4)\langle g_1(x_1)\cdots g_1(x_4)\rangle\,dx_1\cdots dx_4 \\ &- 3\iiiint_{R_4} G(x, x_1)\cdots G(x, x_4)\langle g_1(x_1)g_1(x_2)\rangle\langle g_1(x_3)g_1(x_4)\rangle\,dx_1\cdots dx_4.\end{aligned} \tag{2.37}$$

The coefficient 3 in the first and last terms on the right side of (2.37) arises from the number of ways that four objects can be separated into two parts, while the last term is needed to compensate for the duplication occurring in the first two terms. In the last term, note that the integrand is zero in the part of R_4 that is not in $R_2 \times R_2$. More briefly, we can rewrite (2.37) as

$$\langle y_1^4(x) \rangle = 3I_2^2(x) + I_4(x), \tag{2.38a}$$

where $I_4(x)$ refers to the last two terms in (2.37), which are of order ε^3. The quantity $I_4(x)$ will be important later, but for the present we need to keep only the ε^2 term. Thus we obtain

$$\langle y_1^4(x) \rangle = 3A_1^2(x)\varepsilon^2 + O(\varepsilon^3). \tag{2.38b}$$

Similarly,

$$\langle y_1^5(x) \rangle = 10I_2(x)I_3(x) + O(\varepsilon^4) = 10A_1(x)A_3(x)\varepsilon^3 + O(\varepsilon^4). \tag{2.39}$$

In general, we find that

$$\langle y_1^{2n}(x) \rangle = \frac{(2n)!}{2^n n!} I_2^n(x) + O(\varepsilon^{n+1}) = \frac{(2n)!}{2^n n!} A_1^n(x)\varepsilon^n + O(\varepsilon^{n+1}); \tag{2.40}$$

$$\begin{aligned}\langle y_1^{2n+1}(x) \rangle &= \frac{(2n+1)!}{3!2^{n-1}(n-1)!} I_2^{n-1}(x)I_3(x) + O(\varepsilon^{n+2}) \\ &= \frac{(2n+1)!}{3!2^{n-1}(n-1)!} A_1^{n-1}(x)A_3(x)\varepsilon^{n+1} + O(\varepsilon^{n+2}).\end{aligned} \tag{2.41}$$

We now define the normalized random variable

$$\xi(x, \omega) = \frac{y_1(x, \omega)}{\sqrt{A_1(x)\varepsilon}}. \tag{2.42}$$

Then

$$\langle \xi^{2n} \rangle = \frac{\langle y_1^{2n} \rangle}{A_1^n \varepsilon^n} = \frac{(2n)!}{2^n n!} + O(\varepsilon), \tag{2.43}$$

$$\langle \xi^{2n+1} \rangle = \frac{\langle y_1^{2n+1} \rangle}{(A_1\varepsilon)^{n+1/2}} = \frac{(2n+1)!}{3!2^{n-1}(n-1)!} \frac{A_3}{A_1^{3/2}} \sqrt{\varepsilon} + O(\varepsilon^{3/2}). \tag{2.44}$$

Thus, as $\varepsilon \to 0$, the distribution F_ξ of ξ approaches the standard normal distribution Φ. In other words,

$$\lim_{\varepsilon \to 0} F_\xi(u) = \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt. \tag{2.45}$$

To the extent that the second- and higher-order terms in (2.10) can be neglected, we have shown that the distribution of

$$\frac{y(x, \omega) - y_0(x)}{\sqrt{A_1(x)\varepsilon}} \tag{2.46}$$

approaches Φ as $\varepsilon \to 0$.

This generalizes the results in [1], [4], and [7] to the case in which the leading coefficient p in (2.1) is random. The further extension to higher-order self-adjoint boundary-value problems is straightforward.

3. The distribution function; first correction term. In the preceding section we showed that the distribution function F_ξ of the random variable $\xi = y_1/\sqrt{A_1 \varepsilon}$ approaches the standard normal distribution Φ as the correlation length $\varepsilon \to 0$. We can look upon this result as generating the first term in an asymptotic expansion of F_ξ in terms of ε. In order to investigate more carefully the behavior of F_ξ for small but nonzero ε, it is natural to try to determine at least the second term in this expansion. We will do this by means of an expansion in terms of Chebyshev-Hermite polynomials, as outlined by Gnedenko and Kolmogorov [2].

Let $p_\xi(u)$ be the density function of ξ, and consider the representation

$$p_\xi(u) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{c_k}{k!} e^{-u^2/2} H_k(u), \tag{3.1}$$

where H_k is the Chebyshev-Hermite polynomial of degree k. The polynomials $\{H_k\}$ are defined by

$$H_k(u) = (-1)^k e^{u^2/2} \frac{d^k}{du^k} (e^{-u^2/2}) \tag{3.2}$$

and satisfy the orthogonality relation

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-u^2/2} H_j(u) H_k(u)\, du &= 0, \qquad \text{if } j \neq k; \\ &= \sqrt{2\pi}\, k!, \qquad \text{if } j = k. \end{aligned} \tag{3.3}$$

The first few Chebyshev-Hermite polynomials are

$$\begin{gathered} H_0(u) = 1, \qquad H_1(u) = u, \qquad H_2(u) = u^2 - 1, \qquad H_3(u) = u^3 - 3u, \\ H^4(u) = u^4 - 6u^2 + 3. \end{gathered} \tag{3.4}$$

From (3.1) and (3.3) it follows that

$$c_k = (-1)^k \int_{-\infty}^{\infty} p_\xi(u) H_k(u)\, du, \qquad k = 0, 1, 2, \dots. \tag{3.5}$$

In particular,

$$c_0 = \int_{-\infty}^{\infty} p_\xi(u)\, du = 1, \tag{3.6a}$$

$$c_1 = -\int_{-\infty}^{\infty} u p_\xi(u)\, du = -\langle \xi \rangle = -\frac{\langle y_1 \rangle}{\sqrt{A_1 \varepsilon}} = 0; \tag{3.6b}$$

consequently

$$p_\xi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \left[1 + \sum_{k=2}^{\infty} \frac{(-1)^k c_k}{k!} H_k(u) \right]. \tag{3.7}$$

The corresponding expression for the distribution function $F_\xi(u)$ is

$$F_\xi(u) = \Phi(u) + \sum_{k=2}^{\infty} \frac{c_k}{k!} \Phi_k(u), \tag{3.8}$$

where

$$\Phi_k(u) = \frac{(-1)^k}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2} H_k(t)\, dt. \tag{3.9}$$

By using (3.2) it follows that

$$\Phi_k(u) = \frac{(-1)^{k-1}}{\sqrt{2\pi}} e^{-u^2/2} H_{k-1}(u), \tag{3.10}$$

so

$$F_\xi(u) = \Phi(u) + \frac{1}{\sqrt{2\pi}} \sum_{k=2}^{\infty} \frac{(-1)^{k-1} c_k}{k!} e^{-u^2/2} H_{k-1}(u). \tag{3.11}$$

Our goal is to determine the contribution to the series (3.11) that is of lowest order in ε. This can be done formally by relating the coefficients c_k to the moments of ξ, again using some results given in [2].

The characteristic function $\psi_\xi(t)$ of ξ is defined by

$$\psi_\xi(t) = \int_{-\infty}^{\infty} e^{itu}\, dF_\xi(u). \tag{3.12}$$

Hence, from (3.8),

$$\begin{aligned} \psi_\xi(t) &= \int_{-\infty}^{\infty} e^{itu}\, d\Phi(u) + \sum_{k=2}^{\infty} \frac{c_k}{k!} \int_{-\infty}^{\infty} e^{itu}\, d\Phi_k(u) \\ &= e^{-t^2/2} \left[1 + \sum_{k=2}^{\infty} \frac{c_k}{k!} (-it)^k \right]. \end{aligned} \tag{3.13}$$

On the other hand, we also have the expansions

$$\psi_\xi(t) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} (it)^k \tag{3.14}$$

where $\alpha_k = \langle \xi^k \rangle$, and

$$\log \psi_\xi(t) = \sum_{k=1}^{\infty} \frac{\beta_k}{k!} (it)^k, \tag{3.15}$$

where β_k is the kth semi-invariant of ξ. Starting from the expressions (2.43) and (2.44) for α_{2n} and α_{2n+1}, we will first determine β_k and then find c_k. Throughout this process we will keep terms of order $\sqrt{\varepsilon}$ and neglect terms of order ε.

To relate $\{\beta_k\}$ with $\{\alpha_k\}$ we need to calculate the logarithm of $\psi_\xi(t)$ from (3.14). We first substitute for α_k from (2.43) and (2.44), letting $B = A_3/A_1^{3/2}$ and $w = it$. Then

$$\begin{aligned} \log \psi_\xi(t) &= \log \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} w^k \\ &= \log \left[1 + \sum_{n=1}^{\infty} \frac{w^{2n}}{2^n n!} + \sum_{n=1}^{\infty} \frac{B\sqrt{\varepsilon}\, w^{2n+1}}{3!\,2^{n-1}(n-1)!} + O(\varepsilon) \right] \end{aligned}$$

$$= \log\left[\sum_{n=0}^{\infty}\frac{w^{2n}}{2^n n!} + \frac{B\sqrt{\varepsilon}\,w^3}{3!}\sum_{n=1}^{\infty}\frac{w^{2n-2}}{2^{n-1}(n-1)!} + O(\varepsilon)\right]$$

$$= \log\left[\left(1 + \frac{B\sqrt{\varepsilon}\,w^3}{3!}\right)e^{w^2/2} + O(\varepsilon)\right]$$

$$= \frac{w^2}{2} + \log\left(1 + \frac{B\sqrt{\varepsilon}\,w^3}{3!}\right) + O(\varepsilon)$$

$$= \frac{w^2}{2} + \frac{B\sqrt{\varepsilon}}{3!}\,w^3 + O(\varepsilon). \tag{3.16}$$

Comparing (3.15) with (3.16), and neglecting terms of order ε, we obtain

$$\beta_1 = 0, \qquad \beta_2 = 1, \qquad \beta_3 = B\sqrt{\varepsilon}, \quad \text{and} \quad \beta_k = 0 \quad \text{for} \quad k \geq 4. \tag{3.17}$$

Next we write

$$\psi_\xi(t) = \exp\,[\log \psi_\xi(t)] = \exp\left[\sum_{k=1}^{\infty}\frac{\beta_k}{k!}(it)^k\right]$$

$$= \exp\left[-\frac{t^2}{2} + \frac{B\sqrt{\varepsilon}}{3!}(it)^3 + O(\varepsilon)\right]$$

$$= \exp\,(-t^2/2)\exp\left[\frac{B\sqrt{\varepsilon}}{3!}(it)^3\right] + O(\varepsilon). \tag{3.18}$$

By comparing (3.13) with (3.18) we obtain

$$1 + \sum_{k=2}^{\infty}\frac{c_k}{k!}(-it)^k = \exp\left[\frac{B\sqrt{\varepsilon}}{3!}(it)^3\right] + O(\varepsilon)$$

$$= 1 + \frac{B\sqrt{\varepsilon}}{3!}(it)^3 + O(\varepsilon). \tag{3.19}$$

Again neglecting terms of order ε, we find that

$$c_2 = 0, \qquad c_3 = -B\sqrt{\varepsilon}, \qquad \text{and} \qquad c_k = 0 \qquad \text{for} \qquad k \geq 4. \tag{3.20}$$

Substituting these coefficients in (3.11) gives the desired result:

$$F_\xi(u) = \Phi(u) - \frac{1}{\sqrt{2\pi}}\frac{B\sqrt{\varepsilon}}{3!}e^{-u^2/2}H_2(u) + O(\varepsilon)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-t^2/2}\,dt - \frac{A_3\sqrt{\varepsilon}}{\sqrt{2\pi}\;3!A_1^{3/2}}(u^2 - 1)e^{-u^2/2} + O(\varepsilon), \tag{3.21}$$

where A_1 and A_3 are defined by (2.35) and (2.36) respectively. The corresponding formula for the density function is

$$p_\xi(u) = \phi(u)\left[1 + \frac{A_3\sqrt{\varepsilon}}{3!A_1^{3/2}}(u^3 - 3u) + O(\varepsilon)\right], \tag{3.22}$$

where

$$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \tag{3.23}$$

is the normal density function.

The second term on the right side of (3.21) gives the dominant contribution to $F_\xi(u) - \Phi(u)$ when ε is nonzero but small; at least this is so when $A_3 \neq 0$. Unfortunately, there are important cases in which $A_3 = 0$. For example, this will occur when the distribution of ξ is symmetric about $\xi = 0$. To deal with this situation we must calculate the $O(\varepsilon)$ term in (3.21), and this is done in the next section.

Before turning to that problem, let us briefly consider the calculation of $\langle y_1^3(x)\rangle$, which leads to the coefficient A_3 in (3.21) and (3.22). The only nonzero contribution to the integral

$$\langle y_1^3(x)\rangle = \int_0^1 \int_0^1 \int_0^1 G(x, x_1)G(x, x_2)G(x, x_3)\langle g_1(x_1)g_1(x_2)g_1(x_3)\rangle \, dx_1 \, dx_2 \, dx_3 \tag{3.24}$$

comes from the region where at least one of the following sets of inequalities is satisfied:

$$|x_1 - x_2| \le \varepsilon \quad \text{and} \quad |x_2 - x_3| \le \varepsilon; \tag{3.25a}$$

$$\text{or} \quad |x_2 - x_3| \le \varepsilon \quad \text{and} \quad |x_3 - x_1| \le \varepsilon; \tag{3.25b}$$

$$\text{or} \quad |x_3 - x_1| \le \varepsilon \quad \text{and} \quad |x_1 - x_2| \le \varepsilon. \tag{3.25c}$$

This is a right cylinder whose axis is the line $x_1 = x_2 = x_3$ and whose cross-section in a plane perpendicular to the axis is a regular six-pointed star.

It is helpful to introduce a new orthogonal coordinate system z, s_1, s_2 by the transformation

$$z = (x_1 + x_2 + x_3)/3, \qquad s_1 = (x_1 - 2x_2 + x_3)/\sqrt{2}, \qquad s_2 = (x_1 - x_3)/\sqrt{2}. \tag{3.26}$$

The coordinate z lies along the axis of the cylinder, while s_1 and s_2 are in a transverse plane. The Jacobian of this transformation is one, and its inverse is

$$x_1 = z + \frac{\sqrt{2}}{6} s_1 + \frac{\sqrt{2}}{2} s_2, \qquad x_2 = z - \frac{\sqrt{2}}{3} s_1,$$

$$x_3 = z + \frac{\sqrt{2}}{6} s_1 - \frac{\sqrt{2}}{2} s_2. \tag{3.27}$$

In a plane $z = c$ the cross-section of the cylinder is the star S shown in Fig. 2. The s_1, s_2 coordinates of the points of S are $(\pm 3\, \varepsilon/\sqrt{2}, \pm\varepsilon/\sqrt{2})$ and $(0, \pm \sqrt{2}\, \varepsilon)$, where all combinations of plus and minus signs occur. If we introduce the new coordinates in (3.24), expand the integrand in the transverse variables s_1, s_2, and keep only the lowest terms, we find that

$$\langle y_1^3(x)\rangle = \iint_S ds_1 \, ds_2 \int_0^1 G^3(x, z)\langle g_1^3(z)\rangle \, dz + O(\varepsilon^3)$$

$$= 6\, \varepsilon^2 \int_0^1 G^3(x, z)\langle g_1^3(z)\rangle \, dz + O(\varepsilon^3), \tag{3.28}$$

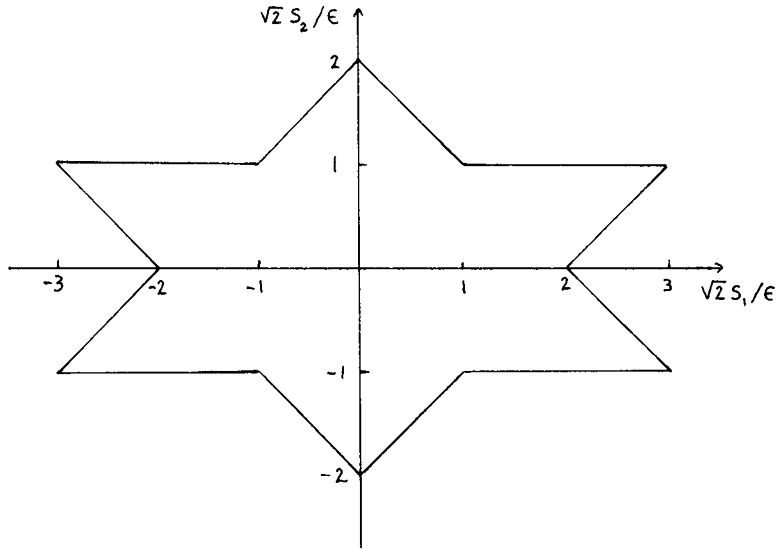

FIG. 2.

where $6\,\varepsilon^2$ is the area of S. The calculation of $\langle g_1^3(z)\rangle$ is greatly simplified by the assumption that p_1, q_1, and f_1 are independent and the fact that $\langle p_1\rangle$, $\langle p_1'\rangle$, $\langle q_1\rangle$, and $\langle f_1\rangle$ are all zero. Thus most of the terms in $\langle g_1^3(z)\rangle$ are zero, and

$$\langle g_1^3(z)\rangle = \langle f_1^3\rangle - \langle q_1^3\rangle y_0^3 + \langle\{[p_1 y_0']'\}^3\rangle. \tag{3.29}$$

Since A_3 is the coefficient of ε^2 in (3.28), we have

$$A_3(x) = 6\int_0^1 G^3(x, z)\langle g_1^3(z)\rangle\, dz, \tag{3.30}$$

where $\langle g_1^3(z)\rangle$ is given by (3.29). A further determination of A_3 requires some specific hypotheses about the distributons of p_1, q_1, and f_1.

4. The distribution function; second correction term. In order to deal with the case in which $A_3(x) = 0$, and hence $\langle y_1^3(x)\rangle = O(\varepsilon^3)$, we must determine the $O(\varepsilon)$ term on the right side of (3.21). In turn, this requires the determination of higher-order terms in $I_2(x)$ and also a consideration of $I_4(x)$. We write

$$I_2(x) = A_1(x)\varepsilon + A_2(x)\varepsilon^2 + O(\varepsilon^3), \tag{4.1}$$

$$I_3(x) = 0 + O(\varepsilon^3), \tag{4.2}$$

$$I_4(x) = A_4(x)\varepsilon^3 + O(\varepsilon^4), \tag{4.3}$$

where $A_1(x)$ is the same as before, $A_3(x)$ has been set equal to zero, and the calculation of $A_2(x)$ and $A_4(x)$ will be discussed later. We will follow the same line of argument as in Sec. 3. We will first determine the moments $\langle y_1^k(x)\rangle$ or $\langle \xi^k(x)\rangle$, which give the coefficients α_k in the series (3.14) for the characteristic function $\psi_\xi(t)$. Next we find the semi-invariants β_k, and finally the coefficients c_k in the series (3.11) for $F_\xi(u)$. Throughout the derivation we will keep the terms that contribute to order ε in $F_\xi(u)$.

By keeping the $O(\varepsilon^{n+1})$ term in $\langle y_1^{2n}\rangle$ we obtain, for $n \geq 2$,

$$\langle y_1^{2n}(x)\rangle = \frac{(2n)!}{2^n n!} I_2^n(x) + \frac{(2n)!}{2^{n-2}(n-2)!4!} I_2^{n-2}(x) I_4(x) + O(\varepsilon^{n+2}). \tag{4.4}$$

The first term on the right side was obtained in (2.40) and is $O(\varepsilon^n)$. The second term is the contribution of order ε^{n+1}; its coefficient comes from a consideration of the number of ways that one set of four and $n-2$ pairs can be chosen from a set of $2n$ objects. Note that if $A_3(x)$ were not zero, then there would be another contribution of order ε^{n+1} of the form $I_2^{n-3}(x) I_3^2(x)$. From (4.1) and the binomial theorem it follows that

$$I_2^n = (A_1\varepsilon)^n \left[1 + n\frac{A_2}{A_1}\varepsilon\right] + O(\varepsilon^{n+2}). \tag{4.5}$$

Then

$$\langle y_1^{2n}\rangle = \frac{(2n)!}{2^n n!}(A_1\varepsilon)^n \left[1 + n\frac{A_2}{A_1}\varepsilon\right] + \frac{(2n)!}{2^{n-2}(n-2)!4!}(A_1\varepsilon)^{n-2} A_4 \varepsilon^3 + O(\varepsilon^{n+2}). \tag{4.6}$$

Since $\xi(x) = y_1(x)/\sqrt{A_1(x)\varepsilon}$, we have

$$\begin{aligned}\alpha_{2n} &= \langle \xi^{2n}\rangle = \langle y_1^{2n}\rangle/(A_1\varepsilon)^n \\ &= (2n)!\left\{\frac{1}{2^n n!} + \left[\frac{A_2/A_1}{2^n(n-1)!} + \frac{A_4/A_1^2}{2^{n-2}(n-2)!4!}\right]\varepsilon\right\} + O(\varepsilon^2),\ n \geq 2.\end{aligned} \tag{4.7}$$

Of course,

$$\alpha_0 = 1, \qquad \alpha_2 = 1 + (A_2/A_1)\varepsilon + O(\varepsilon^2). \tag{4.8}$$

Also, because $A_3(x) = 0$, we have $\langle y_1^{2n+1}(x)\rangle = O(\varepsilon^{n+2})$, and

$$\alpha_{2n+1} = \langle \xi^{2n+1}\rangle = O(\varepsilon^{3/2}). \tag{4.9}$$

Next we consider

$$\log \psi_\xi(t) = \log\left[\sum_{k=0}^{\infty} \frac{\alpha_k}{k!} w^k\right], \tag{4.10}$$

where $w = it$. Since to order ε only the even-powered terms contribute, we can write

$$\begin{aligned}\log \psi_\xi(t) &= \log\left\{1 + \left(1 + \frac{A_2}{A_1}\varepsilon\right)\frac{w^2}{2} + \sum_{n=2}^{\infty}\left[\frac{1}{2^n n!} + \varepsilon\left(\frac{A_2/A_1}{2^n(n-1)!}\right.\right.\right. \\ &\qquad \left.\left.\left. + \frac{A_4/A_1^2}{2^{n-2}(n-2)!4!}\right)\right] w^{2n} + O(\varepsilon^{3/2})\right\} \\ &= \log\left\{e^{w^2/2}\left[1 + e^{-w^2/2}\varepsilon\left\{\sum_{n=1}^{\infty}\frac{A_2/A_1}{2^n(n-1)!}w^{2n}\right.\right.\right. \\ &\qquad \left.\left.\left. + \sum_{n=2}^{\infty}\frac{A_4/A_1^2}{2^{n-2}(n-2)!4!}w^{2n}\right\} + O(\varepsilon^{3/2})\right]\right\} \\ &= \frac{w^2}{2} + e^{-w^2/2}\varepsilon\left[\sum_{n=0}^{\infty}\frac{A_2/A_1}{2^{n+1}n!}w^{2n+2} + \sum_{n=0}^{\infty}\frac{A_4/A_1^2}{2^n n!4!}w^{2n+4}\right] + O(\varepsilon^{3/2})\end{aligned}$$

$$= \frac{w^2}{2} + \varepsilon\left(\frac{A_2}{2A_1} w^2 + \frac{A_4}{A_1^2}\frac{w^4}{4!}\right) + O(\varepsilon^{3/2})$$

$$= \left(1 + \varepsilon\frac{A_2}{A_1}\right)\frac{w^2}{2} + \varepsilon\frac{A_4}{A_1^2}\frac{w^4}{4!} + O(\varepsilon^{3/2}). \tag{4.11}$$

Thus, referring to (3.15), we obtain the semi-invariants β_k:

$$\beta_1 = 0, \qquad \beta_2 = 1 + \varepsilon\frac{A_2}{A_1}, \qquad \beta_3 = 0, \qquad \beta_4 = \varepsilon\frac{A_4}{A_1^2}, \qquad \beta_k = 0 \quad \text{for} \quad k \geq 5, \tag{4.12}$$

where terms of order $\varepsilon^{3/2}$ have been neglected. Next we write

$$\psi_\xi(t) = \exp\left[\sum_{k=1}^{\infty}\frac{\beta_k}{k!}(it)^k\right]$$

$$= \exp\left\{-\frac{t^2}{2} + \varepsilon\left[-\frac{A_2}{A_1}\frac{t^2}{2} + \frac{A_4}{A_1^2}\frac{t^4}{4!}\right] + O(\varepsilon^{3/2})\right\}$$

$$= e^{-t^2/2}\left\{1 + \varepsilon\left[-\frac{A_2}{A_1}\frac{t^2}{2} + \frac{A_4}{A_1^2}\frac{t^4}{4!}\right]\right\} + O(\varepsilon^{3/2}). \tag{4.13}$$

Then by comparing (4.13) with (3.13) we determine the coefficients c_k:

$$c_2 = \frac{A_2}{A_1}\varepsilon, \qquad c_3 = 0, \qquad c_4 = \frac{A_4}{A_1^2}\varepsilon, \qquad c_k = 0 \qquad \text{for} \qquad k \geq 5, \tag{4.14}$$

where again terms of order $\varepsilon^{3/2}$ have been neglected. Finally, from (3.11) we have

$$F_\xi(u) = \Phi(u) - \frac{\varepsilon}{\sqrt{2\pi}} e^{-u^2/2}\left\{\frac{A_2/A_1}{2!} H_1(u) + \frac{A_4/A_1^2}{4!} H_3(u)\right\} + O(\varepsilon^{3/2})$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-t^2/2}\,dt - \frac{\varepsilon}{\sqrt{2\pi}} e^{-u^2/2}\left\{\frac{A_2/A_1}{2!} u + \frac{A_4/A_1^2}{4!}(u^3 - 3u)\right\} + O(\varepsilon^{3/2}). \tag{4.15}$$

By differentiation we obtain the density function

$$p_\xi(u) = \phi(u)\left\{1 + \varepsilon\left[\frac{A_2/A_1}{2!}(u^2 - 1) + \frac{A_4/A_1^2}{4!}(u^4 - 6u^2 + 3)\right] + O(\varepsilon^{3/2})\right\}, \tag{4.16}$$

where $\phi(u)$ is given by (3.23).

Let us now consider the calculation of the coefficient A_2 in (4.1). This involves a determination of the contributions of order ε^2 in (2.34). First consider the integral J_1, which is given by

$$J_1 = \iint_{T_1} G\left(x, z - \frac{s}{2}\right) G\left(x, z + \frac{s}{2}\right)\left\langle g_1\left(z - \frac{s}{2}\right) g_1\left(z + \frac{s}{2}\right)\right\rangle ds\,dz, \tag{4.17}$$

where the triangle T_1 is indicated in Fig. 1; note that the area of T_1 is $\varepsilon^2/4$. Thus, if we expand the integrand in powers of s and z about the point $z = 1$, $s = 0$, we need keep only the first, or constant, term in the expansion in order to obtain the ε^2 contribution to J_1.

The result is

$$J_1 = \frac{\varepsilon^2}{4} G^2(x, 1)\langle g_1^2(1)\rangle + O(\varepsilon^3), \tag{4.18}$$

where $\langle g_1^2(1)\rangle$ is found by evaluating the expression

$$\langle g_1^2(z)\rangle = \sigma_f^2 + \sigma_q^2 y_0^2(z) + \sigma_p^2 y_0''^2(z) - \sigma_p^2 \rho_p''(0) y_0'^2(z) \tag{4.19}$$

for $z = 1$. Eq. (4.19) follows from (2.31a) by introducing the variables s and z, and then setting $s = 0$. The integral J_2 over the triangle T_2 is calculated in a similar way:

$$J_2 = \frac{\varepsilon^2}{4} G^2(x, 0)\langle g_1^2(0)\rangle + O(\varepsilon^3), \tag{4.20}$$

where $\langle g_1^2(0)\rangle$ is also obtained from (4.19).

There may also be a contribution of order ε^2 arising from the main integral in (2.34). This contribution can be found by expanding the integrand in (2.34) in powers of s and keeping the linear, as well as the constant, term. The calculation is rendered somewhat more complicated by the fact that the derivative of one of the Green's function factors is discontinuous when $z \pm (s/2) = x$. Thus we must consider the interval $|z - x| \le \varepsilon/2$ separately, so that

$$\int_0^1 \int_0^\varepsilon G\left(x, z - \frac{s}{2}\right) G\left(x, z + \frac{s}{2}\right)\left\langle g_1\left(z - \frac{s}{2}\right) g_1\left(z + \frac{s}{2}\right)\right\rangle ds\, dz$$
$$= \int_{\substack{0 \\ |z-x| > \varepsilon/2}}^1 \int_0^\varepsilon \cdots ds\, dz + \int_{x-\varepsilon/2}^{x+\varepsilon/2} \int_0^\varepsilon \ldots ds\, dz. \tag{4.21}$$

For $|z - x| > \varepsilon/2$, we can write

$$G\left(x, z + \frac{s}{2}\right) = G(x, z) + G'(x, z)\frac{s}{2} + O(s^2),$$
$$G\left(x, z - \frac{s}{2}\right) = G(x, z) - G'(x, z)\frac{s}{2} + O(s^2) \tag{4.22}$$

where G' refers to the derivative with respect to the second argument of G. Then

$$G\left(x, z + \frac{s}{2}\right) G\left(x, z - \frac{s}{2}\right) = G^2(x, z) + O(s^2). \tag{4.23}$$

By starting from (2.31a) and proceeding in a similar way it is possible to show that

$$\left\langle g_1\left(z + \frac{s}{2}\right) g_1\left(z - \frac{s}{2}\right)\right\rangle = \sigma_f^2 + \sigma_q^2 y_0^2(z) + \sigma_p^2 y_0''^2(z) - \sigma_p^2 \rho_p''(0) y_0'^2(z) + O(s^2). \tag{4.24}$$

For example, consider the second term on the right side of (2.31a):

$$\sigma_q^2 \rho_q(x_2 - x_1) y_0(x_1) y_0(x_2) = \sigma_q^2 \rho_q(s) y_0\left(z - \frac{s}{2}\right) y_0\left(z + \frac{s}{2}\right)$$
$$= \sigma_q^2 \left[1 + O(s^2)\right]\left[y_0(z) - \frac{s}{2} y_0'(z) + O(s^2)\right]\left[y_0(z) + \frac{s}{2} y_0'(z) + O(s^2)\right]$$
$$= \sigma_q^2 y_0^2(z) + O(s^2). \tag{4.25}$$

The other terms on the right side of (4.24) are obtained in a like manner. Thus

$$\underset{|z-x|>\varepsilon/2}{\int_0^1 \int_0^\varepsilon} G\left(x, z - \frac{s}{2}\right) G\left(x, z + \frac{s}{2}\right)\left\langle g_1\left(z - \frac{s}{2}\right) g_1\left(z + \frac{s}{2}\right)\right\rangle ds\, dz$$

$$= \underset{|z-x|>\varepsilon/2}{\int_0^1 \int_0^\varepsilon} [G^2(x, z)\langle g_1^2(z)\rangle + O(s^2)]\, ds\, dz$$

$$= \varepsilon\left(\int_0^{x-\varepsilon/2} + \int_{x+\varepsilon/2}^1\right) G^2(x, z)\langle g_1^2(z)\rangle\, dz + O(\varepsilon^3)$$

$$= \varepsilon \int_0^1 G^2(x, z)\langle g_1^2(z)\rangle\, dz - \varepsilon \int_{x-\varepsilon/2}^{x+\varepsilon/2} G^2(x, z)\langle g_1^2(z)\rangle\, dz + O(\varepsilon^3). \tag{4.26}$$

Upon substituting (4.26) into (4.21) we are led to consider the combination

$$\int_{x-\varepsilon/2}^{x+\varepsilon/2} \int_0^\varepsilon G\left(x, z - \frac{s}{2}\right) G\left(x, z + \frac{s}{2}\right)\left\langle g_1\left(z - \ + \frac{s}{2}\right) g_1\left(z + \frac{s}{2}\right)\right\rangle ds\, dz$$
$$- \varepsilon \int_{x-\varepsilon/2}^{x+\varepsilon/2} G^2(x, z)\langle g_1^2(z)\rangle\, dz \tag{4.27}$$

By the mean-value theorem, this combination can be expressed as

$$\varepsilon \int_{x-\varepsilon/2}^{x+\varepsilon/2} \left[G\left(x, z - \frac{\hat{s}}{2}\right) G\left(x, z + \frac{\hat{s}}{2}\right)\left\langle g_1\left(z - \frac{\hat{s}}{2}\right) g_1\left(z + \frac{\hat{s}}{2}\right)\right\rangle - G^2(x, z)\langle g_1^2(z)\rangle\right] dz \tag{4.28}$$

where $0 < \hat{s} < \varepsilon$. Even though $G'(x, t)$ is not continuous when $t = x$, the one-sided derivatives of G are bounded throughout the region under consideration, and therefore

$$\left|G\left(x, z \pm \frac{\hat{s}}{2}\right) - G(x, z)\right| \le \bar{G}\varepsilon \tag{4.29}$$

where

$$\bar{G} = \sup |G'(x, z)|, \qquad 0 \le x,\ z \le 1; \tag{4.30}$$

at a discontinuity point, G' refers to a one-sided derivative. Further, from (4.19) and (4.24) it follows that

$$\left|\left\langle g_1\left(z - \frac{\hat{s}}{2}\right) g_1\left(z + \frac{\hat{s}}{2}\right)\right\rangle - \langle g_1^2(z)\rangle\right| = O(\varepsilon^2). \tag{4.31}$$

By adding and subtracting suitable quantities to the integrand in (4.28), using the triangle inequality, and referring to (4.29) and (4.31), we find that the integrand in (4.28) is of order ε. Consequently, the expression in (4.28) is of order ε^3. Thus, from (4.21) and (4.26),

$$\int_0^1 \int_0^\varepsilon G\left(x, z - \frac{s}{2}\right) G\left(x, z + \frac{s}{2}\right)\left\langle g_1\left(z - \frac{s}{2}\right) g_1\left(z + \frac{s}{2}\right)\right\rangle ds\, dz$$
$$= \varepsilon \int_0^1 G^2(x, z)\langle g_1^2(z)\rangle\, dz + O(\varepsilon^3); \tag{4.32}$$

that is, this integral has no terms of order ε^2. Finally, going back to (2.34), we have

$$\langle y_1^2(x)\rangle = I_2(x) = 2\varepsilon \int_0^1 G^2(x, z)\langle g_1^2(z)\rangle\, dz - \frac{\varepsilon^2}{2}\left[G^2(x, 1)\langle g_1^2(1)\rangle + G^2(x, 0)\langle g_1^2(0)\rangle\right] + O(\varepsilon^3), \quad (4.33)$$

where $\langle g_1^2(z)\rangle$ is given by (4.18). Thus $A_1(x)$ and $A_2(x)$ in (4.1) are the coefficients of ε and ε^2 respectively in (4.33); of course, $A_1(x)$ is the same as given by (2.35). The preceding derivation assumes tacitly that x is not within a distance $\varepsilon/2$ of either endpoint. If this is not so, then the argument must be modified slightly, but the result is the same.

Now we turn to the calculation of the coefficient A_4 in (4.3). From (2.37) and (2.38) we have

$$I_4(x) = \iiiint_{R_4} G(x, x_1) \cdots G(x, x_4)[\langle g_1(x_1) \cdots g_1(x_4)\rangle - 3\langle g_1(x_1)g_1(x_2)\rangle\langle g_1(x_3)g_1(x_4)\rangle]\, dx_1 \cdots dx_4, \quad (4.34)$$

where the region of integration is that portion of the four-dimensional cube $0 \le x_1, x_2, x_3, x_4 \le 1$ satisfying

$$|x_1 - x_2| \le \varepsilon, \qquad |x_2 - x_3| \le \varepsilon, \qquad \text{and} \qquad |x_3 - x_4| \le \varepsilon, \quad (4.35)$$

or one of eleven other sets of inequalities obtained from (4.35) by permuting the variables. If we remove the absolute value bars, then we can replace (4.35) by twenty-four sets of inequalities, of which

$$0 \le x_1 - x_2 \le \varepsilon, \qquad 0 \le x_2 - x_3 \le \varepsilon, \qquad 0 \le x_3 - x_4 \le \varepsilon \quad (4.36)$$

is typical. The integration region is a portion of a cylinder whose axis is the line $x_1 = x_2 = x_3 = x_4$. It is convenient to introduce new coordinates s_1, s_2, s_3, s_4 by the transformation

$$s_1 = (x_1 + x_2 + x_3 + x_4)/2, \qquad s_2 = (x_1 - x_2 + x_3 - x_4)/2,$$
$$s_3 = (x_1 + x_2 - x_3 - x_4)/2, \qquad s_4 = (x_1 - x_2 - x_3 + x_4)/2, \quad (4.37a)$$

or its inverse

$$x_1 = (s_1 + s_2 + s_3 + s_4)/2, \qquad x_2 = (s_1 - s_2 + s_3 - s_4)/2,$$
$$x_3 = (s_1 + s_2 - s_3 - s_4)/2, \qquad x_4 = (s_1 - s_2 - s_3 + s_4)/2. \quad (4.37b)$$

The $s_1 s_2 s_3 s_4$-coordinate system is an orthogonal system, and is oriented so that the s_1-axis lies along the axis of the cylinder. A straightforward calculation shows that

$$\frac{\partial(s_1, s_2, s_3, s_4)}{\partial(x_1, x_2, x_3, x_4)} = \frac{\partial(x_1, x_2, x_3, x_4)}{\partial(s_1, s_2, s_3, s_4)} = 1. \quad (4.38)$$

To find the part of I_4 that is of lowest order in ε we can expand the integrand in the transverse variables and then keep only the first term. This amounts to evaluating the integrand on the s_1-axis, so that

$$I_4(x) = V \int_0^2 G^4\left(x, \frac{s_1}{2}\right)\left\langle g_1^4\left(\frac{s_1}{2}\right)\right\rangle ds_1 - 3\bar{V} \int_0^2 G^4\left(x, \frac{s_1}{2}\right)\left\langle g_1^2\left(\frac{s_1}{2}\right)\right\rangle^2 ds_1 + O(\varepsilon^4), \quad (4.39)$$

where V is the volume of the three-dimensional cross-section of the cylindrical integration region R_4, and $\bar{V}$ is the volume of the cross-section of the smaller cylinder $(R_2 \times R_2) \cap R_4$. The cross-section of R_4 is a star-shaped region with twenty-four points, as shown in Fig. 3. It is bounded by twelve identical plane faces, one of which is shaded in the figure. The inequality sets typified by (4.36) divide the region into twenty-four congruent subregions, each of which is a parallelepiped. For example, in an $s_2 s_3 s_4$-subspace corresponding to $s_1 =$ constant, the inequalities (4.36) yield the parallelepiped with vertices

$$(0, 0, 0), \quad (0, \varepsilon, 0), \quad (\varepsilon, \varepsilon, 0), \quad (\varepsilon, 2\varepsilon, 0), \quad (\varepsilon/2, \varepsilon/2, \varepsilon/2),$$
$$(\varepsilon/2, \varepsilon/2, -\varepsilon/2), \quad (\varepsilon/2, 3\varepsilon/2, \varepsilon/2), \quad (\varepsilon/2, 3\varepsilon/2, -\varepsilon/2).$$

The volume of this parallelepiped is readily found to be $\varepsilon^3/2$, so the volume of the entire cross-section is $12\varepsilon^3$. A further examination reveals that $\bar{V} = V/2$. Substituting these values in (4.39) and letting $z = s_1/2$ to normalize the interval of integration, we finally obtain

$$I_4(x) = 24\varepsilon^3 \int_0^1 G^4(x, z)[\langle g_1^4(z)\rangle - \tfrac{3}{2}\langle g_1^2(z)\rangle^2]\, dz + O(\varepsilon^4), \quad (4.40)$$

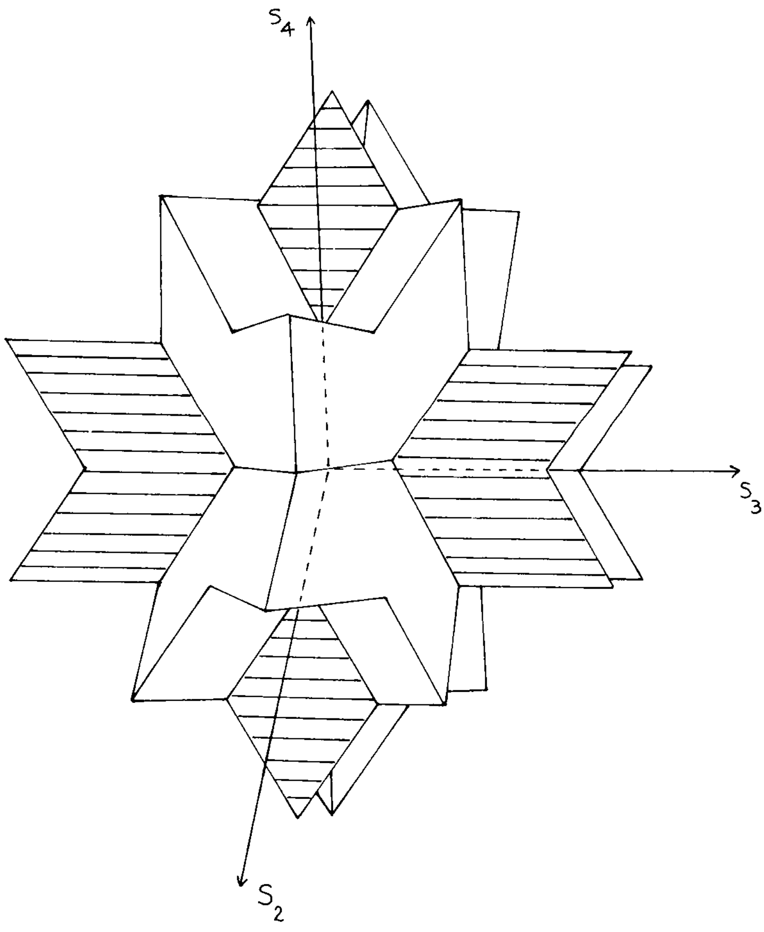

FIG. 3.

so

$$A_4(x) = 24 \int_0^1 G^4(x, z)[\langle g_1^4(z)\rangle - \tfrac{3}{2}\langle g_1^2(z)\rangle^2]\, dz. \tag{4.41}$$

5. Example. To illustrate the application of the results derived in the preceding sections, consider the example

$$-y'' + \eta q_1(x, \omega)y = x + \eta f_1(x, \omega), \qquad y(0) = y(1) = 0. \tag{5.1a,b}$$

In the notation of Sec. 2 we have

$$p_0(x) = 1, \qquad p_1(x, \omega) = 0, \qquad q_0(x) = 0, \qquad f_0(x) = x; \tag{5.2}$$

the statistical properties of q_1 and f_1 will be specified later. The coefficient $p(x, \omega)$ was chosen to be nonrandom in order to simplify the necessary calculations. As in Sec. 2 we assume that

$$y(x, \omega) = y_0(x) + \eta y_1(x, \omega) + O(\eta^2), \tag{5.3}$$

whereupon it follows that y_0 satisfies

$$L_0[y_0] \equiv -y_0'' = x, \qquad y(0) = y(1) = 0. \tag{5.4}$$

The Green's function for L_0 with the given boundary conditions is

$$\begin{aligned} G(x, x_1) &= x_1(1 - x), \qquad 0 \le x_1 \le x \\ &= x(1 - x_1), \qquad x \le x_1 \le 1 \end{aligned} \tag{5.5}$$

and

$$y_0(x) = \int_0^1 G(x, x_1) f_0(x_1)\, dx_1 = \tfrac{1}{6}(x - x^3). \tag{5.6}$$

Further,

$$y_1(x, \omega) = \int_0^1 G(x, x_1) g_1(x_1, \omega)\, dx_1, \tag{5.7}$$

where

$$g_1(x, \omega) = f_1(x, \omega) - q_1(x, \omega) y_0(x). \tag{5.8}$$

To estimate the distribution function of $y_1(x, \omega)$ the first step is to normalize y_1 by calculating $A_1(x)$ from (2.35b). Since $p_1(x, \omega) \equiv 0$, we have

$$\begin{aligned} A_1(x) &= 2 \int_0^1 G^2(x, z)[\sigma_f^2 + \sigma_q^2\, y_0^2(z)]\, dz \\ &= \tfrac{2}{3}\sigma_f^2 x^2(1 - x)^2 + \tfrac{1}{18}\sigma_q^2 x^2 \left(\frac{23}{1260} - \frac{2x^3}{15} + \frac{x^4}{10} + \frac{4x^5}{35} - \frac{2x^6}{21} - \frac{2x^7}{63} + \frac{x^8}{36}\right). \end{aligned} \tag{5.9}$$

Next we use (3.29) and (3.30) to determine $A_3(x)$; this requires information, or assumptions, about $\langle q_1^3\rangle$ and $\langle f_1^3\rangle$. If it turns out that $A_3(x) \neq 0$, then (3.22) gives the density function p_ξ up to terms of order $\sqrt{\varepsilon}$. Let us suppose, however, that $\langle q_1^3 \rangle = \langle f_1^3 \rangle = 0$ so that $A_3(x) = 0$. In this case we wish to use (4.16) to estimate p_ξ, and this requires the evaluation of $A_2(x)$ and $A_4(x)$ from (4.33) and (4.41) respectively. Since $G(x, 0) = G(x, 1) = 0$ from (5.5), it follows at once from (4.33) that $A_2(x) = 0$.

The analytical calculation of $A_4(x)$ from (4.41), while not difficult in principle, requires a rather lengthy integration process, which ultimately yields a polynomial of degree twenty whose coefficients depend on $\langle f_1^4 \rangle$, $\sigma_f^2 \sigma_q^2$, and $\langle q_1^4 \rangle$. In most cases it is probably better to evaluate $A_4(x)$ numerically for those values of x that are of interest.

Since $A_2(x) = 0$ in this example, the expression (4.16) for $p_\xi(u)$ reduces to

$$\begin{aligned} p_\xi(u) &= \phi(u) + \varepsilon C e^{-u^2/2}(u^4 - 6u^2 + 3) + O(\varepsilon^{3/2}) \\ &= \phi(u) + \varepsilon \phi_1(u) + O(\varepsilon^{3/2}), \end{aligned} \tag{5.10}$$

where $C(x) = A_4(x)/24\sqrt{2\pi}\, A_1^2(x)$. In the accompanying plots we have chosen $x = 0.5$ and have assumed that

$$\sigma_f^2 = \sigma_q^2 = 1.0; \qquad \langle f_1^4 \rangle = 1.75; \qquad \langle q_1^4 \rangle = 2.0. \tag{5.11}$$

The solid and dashed curves in Fig. 4 show the normal density function $\phi(u)$ and the correction term $\phi_1(u)$ respectively. The curves in Fig. 5 are plots of $p_\xi(u)$ from (5.10) when $\varepsilon = 0$, 0.1, and 0.5 respectively. In this example, at least, the actual density function is close to the normal even for fairly large values of ε.

6. The corresponding eigenvalue problem. Purkert and vom Scheidt [5, 7] have established properties similar to those in Sec. 2 for a large class of eigenvalue problems.

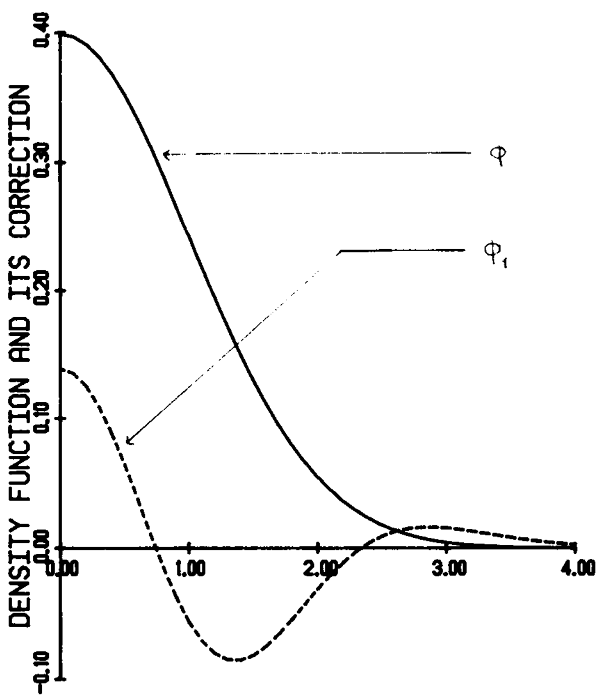

Fig. 4.

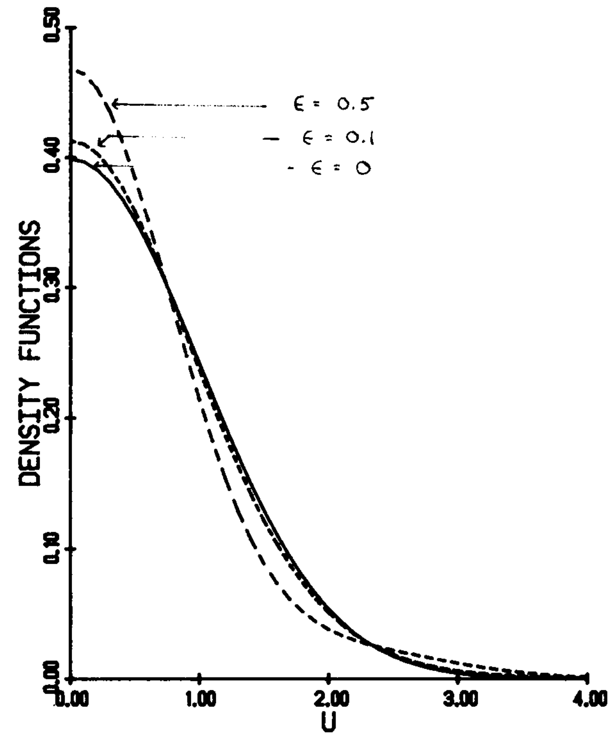

FIG. 5.

Here we describe briefly the extension of the results in Secs. 3 and 4 to this kind of problem. For simplicity we consider an eigenvalue problem analogous to the boundary-value problem of Sec. 2; it will be apparent that the derivation can be extended to a much larger class of problems, such as that investigated in [5]. Suppose that

$$L[y] = -[p(x, \omega)y']' + q(x, \omega)y = \lambda r(x, \omega)y, \qquad 0 < x < 1 \tag{6.1}$$

$$U_i[y] = 0; \qquad i = 1, 2. \tag{6.2}$$

We assume that the coefficients $p(x, \omega)$ and $q(x, \omega)$ satisfy the conditions given in Sec. 2, and that the boundary conditions again have the form (2.3). The coefficient $r(x, \omega)$ is continuous and positive on $0 \le x \le 1$ except possibly for an ω-set of probability zero. We assume that $p(x, \omega)$ and $q(x, \omega)$ again have the form (2.4a, b), and that

$$r(x, \omega) = r_0(x) + \eta r_1(x, \omega), \qquad r_0(x) = \langle r(x, \omega)\rangle, \tag{6.3}$$

where $r_0(x) > 0$ on $0 \le x \le 1$. The random perturbations p_1, q_1, and r_1 have mean zero, and we assume them to be pairwise independent and wide-sense stationary with correlation length ε. Thus p_1 and q_1 satisfy (2.7) and (2.8) respectively, while

$$\begin{aligned} \langle r_1(x_1)r_1(x_2)\rangle &= 0, & |x_2 - x_1| &> \varepsilon \\ &= \sigma_r^2\, \rho_r(x_2 - x_1), & |x_2 - x_1| &\le \varepsilon. \end{aligned} \tag{6.4}$$

Under the given assumptions the problem (6.1), (6.2) has a sequence of eigenvalues

$\{\lambda_i(\omega)\}$ and corresponding eigenfunctions $\{y_i(x, \omega)\}$. To determine them we assume that the nth eigenvalue and eigenfunction have the form

$$\lambda_n = \sum_{j=0}^{\infty} \lambda_{nj}\eta^j, \qquad y_n(x) = \sum_{j=0}^{\infty} y_{nj}(x)\eta^j. \tag{6.5}$$

A straightforward perturbation analysis leads to the problems

$$-(p_0 y'_{n0})' + q_0 y_{n0} = \lambda_{n0} r_0 y_{n0}, \qquad U_i[y_{n0}] = 0 \tag{6.6}$$

and

$$-(p_0 y'_{n1})' + q_0 y_{n1} - \lambda_{n0} r_0 y_{n1} = (p_1 y'_{n0})' - q_1 y_{n0} + \lambda_{n0} r_1 y_{n0} + \lambda_{n1} r_0 y_{n0}, \qquad U_i[y_{n1}] = 0 \tag{6.7}$$

for y_{n0}, λ_{n0} and y_{n1}, λ_{n1} respectively. Let us denote the nth eigenvalue of (6.6) by μ_n and the corresponding eigenfunction, normalized with respect to the weight function r_0, by w_n. Then (6.7) has a solution if and only if the right side of the differential equation is orthogonal to w_n. From this condition we find that

$$\lambda_{n1}(\omega) = \int_0^1 h_n(x, \omega) w_n(x)\, dx, \tag{6.8}$$

where

$$h_n(x, \omega) = -[p_1(x, \omega) w'_n(x)]' + q_1(x, \omega) w_n(x) - \mu_n r_1(x, \omega) w_n(x). \tag{6.9}$$

If (6.8) is satisfied, then y_{n1} can be found from (6.7) by means of an eigenfunction expansion

$$y_{n1}(x, \omega) = \sum_{\substack{i=1 \\ i\neq n}}^{\infty} \frac{a_{ni}(\omega)}{\mu_i - \mu_n} w_i(x), \tag{6.10}$$

where

$$a_{ni}(\omega) = \int_0^1 h_n(x, \omega) w_i(x)\, dx, \qquad i \neq n. \tag{6.11}$$

An analysis similar to that in Secs. 2, 3, and 4 can be based on (6.8) through (6.11).

If we compare (6.8) with (2.18) we see that they are of the same form with $h_n(x, \omega)$ corresponding to $g_1(x_1, \omega)$ and $w_n(x)$ to $G_1(x, x_1)$. Thus we can obtain estimates for the distribution function of λ_{n1} similar to (3.21) and (4.15), provided that we can determine the coefficients corresponding to A_1, A_2, A_3, and A_4. To do this, define I_2^*, I_3^*, and I_4^* analogously to I_2, I_3, and I_4, respectively. To evaluate I_2^* we again introduce the variables z and s by (2.32) so that

$$I_2^* = 2\int_0^1 \int_0^\varepsilon \left\langle h_n\left(z - \frac{s}{2}\right) h_n\left(z + \frac{s}{2}\right)\right\rangle w_n\left(z - \frac{s}{2}\right) w_n\left(z + \frac{s}{2}\right) ds\, dz - 2(J_1^* + J_2^*), \tag{6.12}$$

where J_1^* and J_2^* are the integrals of the same integrand over the triangles T_1 and T_2 in Fig. 1. To evaluate the main integral up to terms of order ε^2 we expand the integrand in powers of s and keep the first two terms. This requires that h_n be differentiable, and this in turn requires the additional hypothesis that p_1', q_1, and r_1 be differentiable with probability one. Assuming this to be true, we have

$$h_n\left(z-\frac{s}{2}\right)h_n\left(z+\frac{s}{2}\right)=\left[h_n(z)-\frac{s}{2}h_n'(z)+O(s^2)\right]\left[h_n(z)+\frac{s}{2}h_n'(z)+O(s^2)\right]$$
$$=h_n^2(z)+O(s^2) \tag{6.13}$$

and similarly for $w_n(z-s/2)w_n(z+s/2)$. Since there is no term of order s in (6.13), it follows that there is no ε^2 term in the main integral. To evaluate J_1^* and J_2^* we need keep only the first term in the expansion of the integrand in s, since the area of the triangular region is $\varepsilon^2/4$ in each case. The result is

$$I_2^*=2\varepsilon\int_0^1\langle h_n^2(z)\rangle w_n^2(z)\,dz-\frac{\varepsilon^2}{2}[\langle h_n^2(0)\rangle w_n^2(0)+\langle h_n^2(1)\rangle w_n^2(1)]+O(\varepsilon^3),$$
$$=A_1^*\varepsilon+A_2^*\varepsilon^2+O(\varepsilon^3). \tag{6.14}$$

To obtain the integrand in (6.14) more explicitly, we can start from (6.9) and use some of the results of Sec. 2, thereby obtaining

$$\langle h_n^2(z)\rangle=\sigma_p^2\,w_n''^2(z)-\sigma_p^2\,\rho_p''(0)w_n'^2(z)+\sigma_q^2\,w_n^2(z)+\mu_n^2\,\sigma_r^2\,w_n^2(z). \tag{6.15}$$

In the same way as in Secs. 3 and 4 it follows that

$$I_3^*=6\varepsilon^2\int_0^1\langle h_n^3(z)\rangle w_n^3(z)\,dz+O(\varepsilon^3)=A_3^*\varepsilon^2+O(\varepsilon^3),$$

and

$$I_4^*=24\varepsilon^3\int_0^1[\langle h_n^4(z)\rangle-\tfrac{3}{2}\langle h_n^2(z)\rangle^2]w_n^4(z)\,dz+O(\varepsilon^4)$$
$$=A_4^*\varepsilon^3+O(\varepsilon^4).$$

Finally, we define the normalized random variable ζ by

$$\zeta=\lambda_1/\sqrt{A_1^*\varepsilon}.$$

Then (3.21) and (4.15) can be used to give the distribution function for ζ simply by replacing A_1 by A_1^*, A_2 by A_2^*, and so on.

The analysis of (6.10) and (6.11) for the eigenfunctions is similar, but is rendered more complicated in practice by the summation in (6.10). Let

$$\hat{I}_2^{(i,j)}=\langle a_{ni}\,a_{nj}\rangle=\int_0^1\int_0^1\langle h_n(x_1)h_n(x_2)\rangle w_i(x_1)w_j(x_2)\,dx_1\,dx_2.$$

Then

$$\hat{I}_2^{(i,j)}=2\varepsilon\int_0^1\langle h_n^2(z)\rangle w_i(z)w_j(z)\,dz$$
$$-\frac{\varepsilon^2}{2}[h_n^2(0)w_i(0)w_j(0)+h_n^2(1)w_i(1)w_j(1)]+O(\varepsilon^3)$$
$$=\hat{A}_1^{(i,j)}\varepsilon+\hat{A}_2^{(i,j)}\varepsilon^2+O(\varepsilon^3),$$

where $\langle h_n^2(z)\rangle$ is given by (6.15). From (6.10) we have

$$\hat{I}_2(x) = \langle y_{n1}^2(x)\rangle = \sum_{\substack{i=1 \\ i,j\neq n}}^{\infty} \sum_{j=1}^{\infty} \frac{\langle a_{ni} a_{nj}\rangle w_i(x) w_j(x)}{(\mu_i - \mu_n)(\mu_j - \mu_n)}$$

$$= \hat{A}_1(x)\varepsilon + \hat{A}_2(x)\varepsilon^2 + O(\varepsilon^3),$$

where

$$\hat{A}_1(x) = \sum_{\substack{i=1 \\ i,j\neq n}}^{\infty} \sum_{j=1}^{\infty} \frac{\hat{A}_1^{(i,j)} w_i(x) w_j(x)}{(\mu_i - \mu_n)(\mu_j - \mu_n)},$$

and $\hat{A}_2(x)$ is given by a similar expression. In much the same way

$$\hat{I}_3(x) = \langle y_{n1}^3(x)\rangle = \hat{A}_3(x)\varepsilon^2 + O(\varepsilon^3),$$

where

$$\hat{A}_3(x) = \sum_{\substack{i=1 \\ i,k,j\neq n}}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\hat{A}_3^{(i,j,k)} w_i(x) w_j(x) w_k(x)}{(\mu_i - \mu_n)(\mu_j - \mu_n)(\mu_k - \mu_n)}$$

and

$$\hat{A}_3^{(i,j,k)} = 6 \int_0^1 \langle h_n^3(z)\rangle w_i(z) w_j(z) w_k(z)\, dz.$$

Also,

$$\hat{I}_4(x) = \hat{A}_4(x)\varepsilon^3 + O(\varepsilon^4),$$

where

$$\hat{A}_4(x) = \sum_{\substack{i=1 \\ i,j,k,l\neq n}}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\hat{A}_4^{(i,j,k,l)} w_i(x) w_j(x) w_k(x) w_l(x)}{(\mu_i - \mu_n)(\mu_j - \mu_n)(\mu_k - \mu_n)(\mu_l - \mu_n)}$$

and

$$\hat{A}_4^{(i,j,k,l)} = 24 \int_0^1 [\langle h_n^4(z)\rangle - \tfrac{3}{2}\langle h_n^2(z)\rangle^2] w_i(z) w_j(z) w_k(z) w_l(z)\, dz.$$

Finally we define the normalized random variable

$$v_n(x, \omega) = y_{n1}(x, \omega)/\sqrt{\hat{A}_1(x)\varepsilon}.$$

Then (3.21) and (4.15) can be used again to estimate the distribution function for $v_n(x)$ provided that A_1 is replaced by $\hat{A}_1$ and so forth.

Note added in proof: We mention here two extensions of the results in this paper that were obtained after the preparation of the original manuscript.

First, suppose that the coefficients p, q, and f are weakly correlated but not necessarily stationary. Then (2.7) through (2.9) are replaced by

$$\begin{aligned}\langle p_1(x_1)p_1(x_2)\rangle &= 0, && |x_2 - x_1| > \varepsilon;\\ &= R_p(x_1, x_2), && |x_2 - x_1| \le \varepsilon; \qquad \text{(A.1)}\\ \langle q_1(x_1)q_1(x_2)\rangle &= 0, && |x_2 - x_1| > \varepsilon;\\ &= R_q(x_1, x_2), && |x_2 - x_1| \le \varepsilon; \qquad \text{(A.2)}\\ \langle f_1(x_1)f_1(x_2)\rangle &= 0, && |x_2 - x_1| > \varepsilon;\\ &= R_f(x_1, x_2), && |x_2 - x_1| \le \varepsilon. \qquad \text{(A.3)}\end{aligned}$$

It follows that

$$\begin{aligned}\langle p_1(x_1)p_1'(x_2)\rangle &= 0, && |x_2 - x_1| > \varepsilon;\\ &= \partial_2 R_p(x_1, x_2), && |x_2 - x_1| \le \varepsilon; \qquad \text{(A.4)}\\ \langle p_1'(x_1)p_1(x_2)\rangle &= 0, && |x_2 - x_1| > \varepsilon;\\ &= \partial_1 R_p(x_1, x_2), && |x_2 - x_1| \le \varepsilon; \qquad \text{(A.5)}\\ \langle p_1'(x_1)p_1'(x_2)\rangle &= 0, && |x_2 - x_1| > \varepsilon;\\ &= \partial_{12}^2 R_p(x_1, x_2), && |x_2 - x_1| \le \varepsilon; \qquad \text{(A.6)}\end{aligned}$$

where ∂_i denotes partial differentiation with respect to the ith argument. Then, instead of (2.31a), in [1], we have

$$\begin{aligned}\langle g_1(x_1)g_1(x_2)\rangle = {} & R_f(x_1, x_2) + R_q(x_1, x_2)y_0(x_1)y_0(x_2)\\ & + R_p(x_1, x_2)y_0''(x_1)y_0''(x_2) + \partial_1 R_p(x_1, x_2)y_0'(x_1)y_0''(x_2)\\ & + \partial_2 R_p(x_1, x_2)y_0''(x_1)y_0'(x_2)\\ & + \partial_{12}^2 R_p(x_1, x_2)y_0'(x_1)y_0'(x_2), \qquad |x_2 - x_1| \le \varepsilon. \qquad \text{(A.7)}\end{aligned}$$

The quantity $A_1(x)$ is still given by

$$A_1(x) = 2\int_0^1 G^2(x, z)\langle g_1^2(z)\rangle\, dz, \qquad \text{(A.8)}$$

but now

$$\begin{aligned}\langle g_1^2(z)\rangle = {} & R_f(z, z) + R_q(z, z)y_0^2(z) + R_p(z, z)y_0''^2(z)\\ & + [\partial_1 R_p(z, z) + \partial_2 R_p(z, z)]y_0'(z)y_0''(z)\\ & + \partial_{12}^2 R_p(z, z)y_0'^2(z). \qquad \text{(A.9)}\end{aligned}$$

In a similar way

$$A_2(x) = -\tfrac{1}{2}[G^2(x, 0)\langle g_1^2(0)\rangle + G^2(x, 1)\langle g_1^2(1)\rangle] \qquad \text{(A.10)}$$

where $\langle g_1^2(0)\rangle$ and $\langle g_1^2(1)\rangle$ are found from (A.9). Eqs. (3.30) for $A_3(x)$ and (4.41) for $A_4(x)$ remain valid, but $\langle g_1^3(z)\rangle$ and $\langle g_1^4(z)\rangle$ must be interpreted a little differently.

With these modifications in mind, the expressions (3.21) and (4.15) for $F_\xi(u)$ and (3.22) and (4.16) for $p_\xi(u)$ remain valid in this more general setting, as well as the results in Sec. 6.

It is also possible to combine the results of Secs. 3 and 4 by calculating the term mentioned just above (4.5). This leads to the formula

$$p_\xi(u) = \phi(u)\left\{1 + \left[\frac{A_3/A_1^{3/2}}{3!}H_3(u)\right]\sqrt{\varepsilon} + \left[\frac{A_2/A_1}{2!}H_2(u) + \frac{A_4/A_1^2}{4!}H_4(u) + \frac{A_3^2/A_1^3}{72}H_6(u)\right]\varepsilon + O(\varepsilon^{3/2})\right\} \quad \text{(A.11)}$$

with a corresponding result for $F_\xi(u)$. Of course, (A.11) reduces to (4.16) when $A_3 = 0$ and to (3.22) when only the $O(\varepsilon^{1/2})$ term is retained.

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