

THE APPROXIMATE DISTRIBUTIONS OF THE MEAN AND VARIANCE OF A SAMPLE OF INDEPENDENT VARIABLES

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1. Introduction. In this paper we shall study the mean and variance of a large number, n (a sample of size n) of mutually independent random variables:

$$(1) \quad \xi_1, \xi_2, \dots, \xi_n,$$

having the same probability distribution represented by a (cumulative) distribution function $P(x)$. The r th moment, absolute moment, and semi-invariant of $P(x)$ are denoted by α_r , β_r , and γ_r respectively. It is assumed that for a certain integer $k \geq 3$, $\beta_k < \infty$ and that $\alpha_2 > 0$. Hence there is no loss of generality in assuming that

$$(2) \quad \alpha_1 = 0, \quad \alpha_2 = 1.$$

The characteristic function corresponding to $P(x)$ is denoted by $p(t)$.

We put

$$(3) \quad \bar{\xi} = \frac{1}{n} \sum_{r=1}^n \xi_r, \quad \eta = \frac{1}{n} \sum_{r=1}^n (\xi_r - \bar{\xi})^2$$

$$(4) \quad F(x) = \Pr\{\sqrt{n}\bar{\xi} \leq x\}, \quad G(x) = \Pr\left\{\frac{\sqrt{n}(\eta - 1)}{\sqrt{\alpha_4 - 1}} \leq x\right\}.$$

The definition of $G(x)$ implies that $\alpha_4 < \infty$ and $\alpha_4 - 1 > 0$. The case $\alpha_4 - 1 = 0$ provides an easy degenerated case which will be treated separately (section 4).

Cramér's theorem of asymptotic expansion¹ reads as follows:

THEOREM 1. *If $P(x)$ is non-singular and if $\beta_k < \infty$ for some integer $k \geq 3$, then*

$$(5) \quad F(x) = \Phi(x) + \Psi(x) + R(x)$$

where

$$(6) \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}y^2} dy.$$

$\Psi(x)$ is a certain linear combination of successive derivatives $\Phi^{(3)}(x), \dots, \Phi^{(3(k-3))}(x)$ with each coefficient of the form $n^{-\frac{1}{2}\nu}$ times a quantity depending only on $k, \alpha_3, \dots, \alpha_{k-1}$ ($1 \leq \nu \leq k-3$) and

$$(7) \quad |R(x)| \leq Q/n^{\frac{1}{2}(k-2)}$$

where Q is a constant depending only on k and $P(x)$.

¹ H. CRAMÉR: *Random Variables and Probability Distributions* (1937), Ch. 7. This book will be referred to as (C).

In particular, putting $k = 3$ we get that $|F(x) - \Phi(x)| \leq Qn^{-1}$ provided $P(x)$ is non-singular and $\beta_3 < \infty$. If the condition of non-singularity of $P(x)$ be removed, then Liapounoff's theorem² furnishes the weaker result: $|F(x) - \Phi(x)| \leq A\beta_3 n^{-1} \log n$ where A is a numerical constant.

Very recently Berry³ succeeded in removing the factor $\log n$ from Liapounoff's theorem under no other condition than that $\beta_3 < \infty$. We state here Berry's theorem:

THEOREM 2. If $\beta_3 < \infty$, then

$$(8) \quad |F(x) - \Phi(x)| \leq \frac{A\beta_3}{\sqrt{n}}$$

where A is a numerical constant.

An essential step in the proof of these results is the selection of a weighting function $w(x)$ and the appraisal of the integral

$$(9) \quad \int_{-\infty}^{\infty} w(u) \{F(u+x) - \Phi(u+x) - \Psi(u+x)\} du$$

($\Psi \equiv 0$ when $k = 3$). In his book¹ Cramér proves Theorem 1 by taking $w(u) = \frac{1}{\Gamma(\omega)} (-u)^{\omega-1}$ when $u < 0$ and $w(u) = 0$ when

$$(10) \quad u \geq 0 \quad (0 < \omega < 1)$$

and proves Liapounoff's theorem by taking

$$(11) \quad w(u) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-u^2/2\epsilon^2}.$$

On the other hand, Berry uses the following weighting function in his proof of Theorem 2:

$$(12) \quad w(u) = \frac{1 - \cos Tu}{u^2}.$$

The unfortunate selection of the function (11) accounts for the presence of the factor $\log n$ in Liapounoff's theorem.

Now Cramér's proof of Theorem 1, based on the integral (9) with $w(u)$ defined in (10), makes use of a result on that integral due to M. Riesz. A more elementary proof than this can be devised. In fact, one has only to use, with Berry, the function (12) and to adopt his elementary appraisal⁴ of the integral

² (C), Ch. 7.

³ A. C. BERRY: "The accuracy of the Gaussian approximation to the sum of independent variates." *Trans. Amer. Math. Soc.*, Vol. 49 (1941), pp. 122-136. This paper will be referred to as (B).

⁴ Berry proves the inequality (in our notation):

$$\left| \int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x+a) - \Phi(x+a)\} dx \right| \leq \int_0^T \frac{(T-t) |f(t) - e^{-t^2}|}{t} dt$$

(9) in order to obtain the proof of Theorem 1. One of our purposes is therefore to give an elementary proof of Theorem 1, without reference to the above-mentioned result due to M. Riesz. Section 2 is devoted to this work.

We ought to add that Cramér's theorem and Berry's theorem correspond to Theorems 1 and 2 for the case in which the random variables (1) do not follow the same distribution. The proof given in Section 2 is adaptable to these more general theorems when subjected to appropriate modifications; the assumption of a common distribution function for (1) is only made for the sake of convenience.

So much for the known results for the approximate distribution of $\bar{\xi}$. By a purely formal operational method Cornish and Fisher⁵ obtain terms of successive approximation to the distribution function of any random variable X with the help of its semi-invariants. It is hardly necessary to emphasize the importance of turning Cornish and Fisher's formal result (asymptotic expansion without appraisal of the remainder) into a mathematical theorem of asymptotic expansion which gives the order of magnitude of the remainder. In this paper we achieve this for the simplest function of (1) next to $\bar{\xi}$, viz. the η in (3). We do not seek to remove the assumption of a common distribution for (1), as there will be no practical significance (e.g. in statistics) of η if the variables (1) do not have the same probability distribution. Section 3 is devoted to the proof of the following theorems:

THEOREM 3. *If $\alpha_3 < \infty$ and $\alpha_4 - 1 - \alpha_3^2 \neq 0$ (it cannot be negative), then*

$$(13) \quad |G(x) - \Phi(x)| \leq \frac{A}{\sqrt{n}} \left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2} \right)^{3/2}$$

where A is a numerical constant.

THEOREM 4. *Let $P(x)$ be non-singular and let $\alpha_{2k} < \infty$ for some integer $k > 3$. Then*

$$(14) \quad G(x) = \Phi(x) + \chi(x) + R_1(x),$$

where $\Phi(x)$ is the function (6), $\chi(x)$ is a linear combination of the derivatives $\Phi'(x)$, \dots , $\Phi^{(3(k-3))}(x)$ with each coefficient of the form $n^{-1/2}$ times a quantity depending only on k and $\alpha_3, \alpha_4, \dots, \alpha_{2k-2}$, and

(B), p. 128. The "appraisal" mentioned here refers to (50) which is contained in B, p. 128. But Berry's appraisal of the integral in the right-hand side of the above inequality is in default. He writes

$$\frac{\epsilon}{6} \int_0^{c/\epsilon} \left(\frac{1.1}{\epsilon} - t \right) t^2 e^{-t^2} dt = \frac{1.1}{6} \sqrt{\frac{\pi}{2}} - \frac{\epsilon}{3} - \frac{1}{6} \int_{c/\epsilon}^{\infty} \left\{ (1.1 - c)t^2 + c - \frac{2c}{t^2} \right\} e^{-t^2/2} dt$$

(B, p. 132, line 3) whilst the last integral ought to be

$$\int_{c/\epsilon}^{\infty} \{ (1.1 - c)t^2 + c - 2\epsilon t \} e^{-t^2/2} dt.$$

⁵ E. A. Cornish and R. A. Fisher: "Moments and cumulants in the specification of distributions." (Revue de l'Institut International de Statistique (1937), pp. 1-14.)

$$(15) \quad |R_1(x)| \leq \frac{Q_k}{n^{\frac{1}{3}(k-2)}} \quad \text{if } k = 4, 5 \text{ or } 6$$

$$(16) \quad |R_1(x)| \leq \frac{Q'_k}{n^{k(k-1)/(2k+3)}} \quad \text{if } k \geq 7$$

where Q_k and Q'_k are constants depending only on k and $P(x)$.

It may be noticed that Theorem 3 is a "Berryian" theorem about $G(x)$, its characteristic feature being the absence of any condition on the distribution function except the two on its moments, and that Theorem 4 is a "Cramerian" theorem about $G(x)$, the characteristic feature being the assumption of non-singularity of $P(x)$ besides that $\alpha_{2k} < \infty$.

In proving these theorems we have devised a method which is applicable to getting similar results about functions other than η , such as functions commonly used in applied statistics: the higher moments about the means, the moment ratios (e.g. K. Pearson's b_1 and b_2), the covariance, the coefficient of correlation, and "Student's" t -statistic. Works on such functions are being done by my university colleagues, and the results will be published shortly.

If ξ is any of the random variables (1), then

$$0 \leq \epsilon\{a(\xi^2 - 1) + b\xi\} = a^2(\alpha_4 - 1) + 2ab\alpha_3 + b^2$$

for all real (a, b) . Hence $\alpha_4 - 1 - \alpha_3^2 \geq 0$, and $\alpha_4 - 1 - \alpha_3^2 = 0$ means that there is unit probability that ξ assumes exactly two values. This easily degenerated case is first eliminated in Theorem 3 by the assumption $\alpha_4 - 1 - \alpha_3^2 \neq 0$ and then considered in section 4. In Theorem 4 the condition $\alpha_4 - 1 - \alpha_3^2 \neq 0$ is implied since ξ cannot be a random variable of the nature just described owing to the non-singularity of $P(x)$.

2. Lemmas. Throughout this paper A, B, C , etc. will denote positive numerical constants; A_k, B_k (A_{km}, B_{km}), etc., will denote positive constants depending only on some integer k (integers k and m), and Q_k (Q_{km}) will denote a positive constant depending only on k (k and m) and the distribution function $P(x)$. $\vartheta, \Theta, \Theta_k, (\Theta_{km}), \Lambda_k$ (Λ_{km}) will denote respectively quantities such that $|\vartheta| \leq 1$, $|\Theta| \leq A$, $|\Theta_k| \leq A_k$ ($|\Theta_{km}| \leq A_{km}$), $|\Lambda_k| \leq Q_k$ ($|\Lambda_{km}| \leq Q_{km}$). These symbols do not necessarily stand for the same quantity at each occurrence. Thus $2\vartheta = \Theta$, $k\Theta_k = \Theta_k$ etc. In particular any positive functions of $k, \alpha_3, \dots, \alpha_k$ is a Q_k .

1.1. Cramér obtains the asymptotic expansion of the characteristic function of the distribution of $\sqrt{n}\xi$, viz. $\epsilon(e^{it\sqrt{n}\xi})$, when (1) do not have the same distribution, valid for $|t| \leq Q_k n^{1/6}$. Since we assume a common distribution for (1), so that the characteristic function is $\left\{p\left(\frac{t}{\sqrt{n}}\right)\right\}^n$, we are able to derive an asymptotic expansion valid for $|t| \leq Q_k \sqrt{n}$. The extension to $\left\{p\left(\frac{t_1}{\sqrt{n}}\right), \right.$

$\dots, \frac{t_m}{\sqrt{n}} \Big) \Big\}^n$ presents no difficulty. This is done in the following three lemmas, of which Lemma 3 contains the final result.

LEMMA 1.

$$(17) \quad \log p(t) = \sum_{r=2}^{k-1} \frac{\gamma_r(it)^r}{r!} + \Theta_k \beta_k |t|^k, \quad \text{for } |t| \leq \beta_k^{1/k}.$$

PROOF: Since $p(t) = 1 + \sum_{r=1}^{k-1} \frac{\alpha_r(it)^r}{r!} + \frac{\vartheta \beta_k |t|^k}{k!} = 1 + q(t)$ say, we have, for $\beta_k^{1/k} |t| \leq 1$,

$$q(t) \leq \sum_{r=2}^k \frac{\beta_r |t|^r}{r!} \leq \sum_{r=2}^k \frac{(\beta_k^{1/k} |t|)^r}{r!} < \sum_{r=2}^{\infty} \frac{1}{r!} = e - 2 < \frac{3}{4}.$$

Hence

$$(18) \quad \log p(t) = \sum_{1 \leq j \leq [\frac{1}{2}(k-1)]} (-1)^{j+1} \frac{\{q(t)\}^j}{j} + \Theta |q(t)|^{[\frac{1}{2}(k+1)]}.$$

For $1 \leq j \leq [\frac{1}{2}(k-1)]$ let us expand each $(-1)^{j+1} j^{-1} \{q(t)\}^j$ to get a polynomial $q_j(t)$ of degree $k-1$ and a remainder $r_j(t)$. In doing this we regard $q(t)$ formally as a polynomial of degree k in t . For this polynomial we have the majorating relation

$$q(t) \ll e^{\beta_k^{1/k} |t|},$$

whence

$$\frac{(-1)^j}{j} \{q(t)\}^j \ll e^{j \beta_k^{1/k} |t|},$$

which gives

$$(19) \quad |r_j(t)| \leq \sum_{r=k}^{\infty} \frac{j^r \beta_k^{r/k} |t|^r}{r!} \leq j^k \beta_k |t|^k e^{j \beta_k^{1/k} |t|} \leq j^k e^j \beta_k |t|^k \leq A_k \beta_k |t|^k.$$

Similarly,

$$(20) \quad |q(t)|^{[\frac{1}{2}(k+1)]} \leq A_k \beta_k |t|^k.$$

From (18), (19), (20) we obtain

$$(21) \quad \log p(t) = \sum_{1 \leq j \leq [\frac{1}{2}(k-1)]} q_j(t) + \Theta_k \beta_k |t|^k.$$

Since the sum in (21) must equal the sum in (17), the Lemma is proved.

LEMMA 2. Let $(\xi_1, \xi_2, \dots, \xi_m)$ be a random point with $\epsilon(\xi_i) = 0$ and $\epsilon(|\xi_i|^k) = \beta_{ki} < \infty$ for some integer $k \geq 3$ ($i = 1, \dots, m$). Let $p(t_1, \dots, t_m)$ be the characteristic function. Then for $|t_i| \leq m^{-2+1/k} \beta_{ki}^{-1/k} \sqrt{n}$ ($i = 1, \dots, m$) we have

$$(22) \quad n \log p\left(\frac{t_1}{\sqrt{n}}, \dots, \frac{t_m}{\sqrt{n}}\right) = \sum_{r=2}^{k-1} \frac{i^r U_r}{r! n^{\frac{1}{2}(r-2)}} + \frac{\Theta_k V_k}{n^{\frac{1}{2}(k-2)}}$$

where U_r and V_r are the r th semi-invariant and the absolute moment respectively of $\Sigma t_i \zeta_i$

PROOF: If $|t_i| \leq m^{-2+1/k} \beta_{ki}^{-1/k} \sqrt{n}$, then $V_k^{1/k} \leq m^{(k-1)/k} (\Sigma \beta_{ki} |t_i|^k)^{1/k} \leq m^{(k-1)/k} (\Sigma \beta_{ki}^{1/k} |t_i|) \leq \sqrt{n}$. Since $p\left(\frac{t_1}{\sqrt{n}}, \dots, \frac{t_m}{\sqrt{n}}\right)$ is the value at $t = \frac{1}{\sqrt{n}}$ of the characteristic function of $\Sigma t_i \zeta_i$, it follows from Lemma 1 that for $\sqrt{n} \geq V_k^{1/k}$ we have (22).

LEMMA 3. Let $(\zeta_1, \dots, \zeta_m)$ be a random point with $\epsilon(\zeta_i) = 0$, $\epsilon(\zeta_i^2) = 1$ and $\epsilon(|\zeta_i|^k) = \beta_{ki} < \infty$ for some integer $k \geq 3$. Let $\rho_{ij} = \epsilon(\zeta_i \zeta_j)$ ($\rho_{ii} = 1$; $i, j = 1, \dots, m$) and the matrix $\|\rho_{ij}\|$ be positive definite. Let

$$(23) \quad \Delta = \det. |\rho_{ij}|, \quad \varphi(t_1, \dots, t_m) = e^{-\frac{1}{4} \sum_{i,j=1}^m \rho_{ij} t_i t_j}.$$

Let $p(t_1, \dots, t_m)$ be the characteristic function. Then there exists a B_{km} such that for $|t_i| \leq \frac{B_{km} \Delta \sqrt{n}}{\beta_{ki}^{3/k}}$ ($i = 1, \dots, m$) we have

$$(24) \quad \left\{ p\left(\frac{t_1}{\sqrt{n}}, \dots, \frac{t_m}{\sqrt{n}}\right) \right\}^n = \varphi(t_1, \dots, t_m) \{1 + \psi(it_1, \dots, it_m)\} \\ + \frac{\Theta_{km}}{n^{3(k-2)/4}} \left\{ \sum_{i=1}^m \beta_{ki}^{3(k-2)/k} (|t_i|^k + |t_i|^{k+1} + \dots + |t_i|^{3(k-2)}) \right\} e^{-\Delta/4m^{m-1} \sum_{i=1}^m t_i^2}$$

where $\psi(it_1, \dots, it_m)$ is a polynomial each of whose terms has the form

$$\frac{1}{n^{\nu/2}} a_{\nu_1 \dots \nu_m} (it_1)^{\nu_1} \dots (it_m)^{\nu_m},$$

with $1 \leq \nu \leq k-3$, $3 \leq \nu_1 + \dots + \nu_m \leq 3(k-3)$, and $a_{\nu_1 \dots \nu_m}$ depending only on k and the moments $\epsilon(\zeta_1^{\mu_1} \dots \zeta_m^{\mu_m})$, $3 \leq \mu_1 + \dots + \mu_m \leq k-1$. If $k = 3$, then $\psi = 0$.

PROOF. If $|t_i| \leq m^{-2+(1/k)} \beta_{ki}^{-3/k} \Delta \sqrt{n}$, then $|t_i| \leq m^{-2+(1/k)} \beta_{ki}^{-1/k} \sqrt{n}$ since $\Delta \leq 1$ and $\beta_{ki} \geq 1$. It follows from Lemma 2 and the fact $U_2 = \Sigma \rho_{ij} t_i t_j$ that

$$(25) \quad \left\{ p\left(\frac{t_1}{\sqrt{n}}, \dots, \frac{t_m}{\sqrt{n}}\right) \right\}^n = \varphi(t_1, \dots, t_m) e^s \\ = \varphi(t_1, \dots, t_m) \left\{ 1 + \sum_{j=1}^{k-3} \frac{s^j}{j!} + \frac{\vartheta |s|^{k-2} e^{|s|}}{(k-2)!} \right\}$$

where

$$(26) \quad s = \frac{i^3}{\sqrt{n}} \sum_{r=0}^{k-3} \frac{i^r U_{r+3}}{(r+3)! n^{r/2}} + \frac{\Theta_k V_k}{n^{3(k-2)/4}}.$$

Regarding s formally as a polynomial in n^{-1} let us expand each $(j!)^{-1}s^j$ ($1 \leq j \leq k-3$) to get a polynomial s_j of degree $k-3$ in n^{-1} and a remainder r_j . For the formal polynomial s we have the majorating relation

$$(27) \quad s \ll \frac{A_k}{\sqrt{n}} \sum_{r=0}^{k-3} \frac{V_{r+3}}{r! n^{r/2}} \ll \frac{A_k}{\sqrt{n}} \sum_{r=0}^{k-3} \frac{V_k^{(r+3)/k}}{r! n^{r/2}} \ll \frac{A_k V_k^{3/k}}{\sqrt{n}} e^{V_k^{1/k} n^{-1/2}},$$

whence

$$\frac{1}{j!} s^j \ll A_k \frac{V_k^{3j/k}}{n^{j/2}} e^{j V_k^{1/k} n^{-1/2}},$$

which gives

$$|r_j| \leq \frac{A_k V_k^{3j/k}}{n^{j/2}} \sum_{v=k-2-j}^{\infty} \frac{j^v V_k^{v/k}}{v! n^{v/2}} \leq \frac{A_k V_k^{(k-2+2j)/k}}{n^{\frac{1}{2}(k-2)}} e^{j(V_k^{1/k}/\sqrt{n})}.$$

Since $V_k^{1/k} n^{-1/2} \leq 1$ as shown in the proof of Lemma 2, we have

$$\begin{aligned} |r_j| &\leq \frac{A_k V_k^{(k-2+2j)/k}}{n^{\frac{1}{2}(k-2)}} \leq \frac{A_{km} (\sum_i \beta_{ki} |t_i|^k)^{(k-2+2j)/k}}{n^{\frac{1}{2}(k-2)}} \\ &\leq \frac{A_{km} (\sum_i \beta_{ki}^{1/k} |t_i|)^{k-2+2j}}{n^{\frac{1}{2}(k-2)}} \leq \frac{A_{km} \sum_i \beta_{ki}^{(k-2+2j)/k} |t_i|^{k-2+2j}}{n^{\frac{1}{2}(k-2)}}. \end{aligned}$$

Since $\beta_{ki} \geq 1$ we have $\beta_{ki}^{(k-2+2j)/k} \leq \beta_{ki}^{3(k-2)/k}$. Hence

$$(28) \quad |r_j| \leq \frac{A_{km} \sum_i \beta_{ki}^{3(k-2)/k} |t_i|^{k-2+2j}}{n^{\frac{1}{2}(k-2)}}.$$

Similarly

$$(29) \quad \frac{|s|^{k-2}}{(k-2)!} \leq \frac{A_{km} \sum_i \beta_{ki}^{3(k-2)/k} |t_i|^{3(k-2)}}{n^{\frac{1}{2}(k-2)}}.$$

From (25), (28), (29) we get

$$\begin{aligned} \left\{ p\left(\frac{t_1}{\sqrt{n}}, \dots, \frac{t_m}{\sqrt{n}}\right) \right\}^n &= \varphi(t_1, \dots, t_m) \left\{ 1 + \sum_{j=1}^{k-3} s_j + \sum_{j=1}^{k-3} r_j + \frac{\vartheta |s|^{k-2}}{(k-2)!} e^{|s|} \right\} \\ &= \varphi(t_1, \dots, t_m) \{1 + \psi(it_1, \dots, it_m)\} \\ &\quad + \frac{\Theta_{km}}{n^{\frac{1}{2}(k-2)}} \{ \sum_i \beta_{ki}^{3(k-2)/k} (|t_i|^k + |t_i|^{k+1} + \dots + |t_i|^{3(k-2)}) \} \varphi(t_1, \dots, t_m) e^{|s|} \end{aligned}$$

where $\psi(it_1, \dots, it_m)$ stands for $\sum s_j$. The assertion about $\psi(it_1, \dots, it_m)$ announced in the lemma can now be seen without difficulty. It remains to show that with suitable B_{km} in the lemma, we have

$$\varphi(t_1, \dots, t_m) e^{|s|} \leq e^{-\Delta/4m^{m-1} \sum_{i=1}^m t_i^2}$$

i.e.

$$(30) \quad -\frac{1}{2} \sum_{i,j=1}^m \rho_{ij} t_i t_j + |s| \leq -\frac{\Delta}{4m^{m-1}} \sum_{i=1}^m t_i^2.$$

From (27) we have

$$(31) \quad |s| \leq \frac{A_k}{\sqrt{n}} V_k^{3/k} \leq \frac{A_{km}}{\sqrt{n}} \left(\sum_i \beta_{ki} |t_i|^k \right)^{3/k} \\ \leq \frac{A_{km}}{\sqrt{n}} \left(\sum_i \beta_{ki}^{1/k} |t_i| \right)^3 \leq \frac{A_{km}}{\sqrt{n}} \sum_i \beta_{ki}^{3/k} |t_i|^3.$$

If we choose $B_{km} \leq (4m^{m-1} A_{km})^{-1}$ (and $B_{km} \leq m^{-2+(1/k)}$ in order that the earlier results may not be affected), the A_{km} here coinciding with the last written A_{km} in (31), we have, for $|t_i| \leq B_{km} \beta_{ki}^{-3/k} \Delta \sqrt{n}$,

$$(32) \quad |s| \leq \frac{\Delta}{4m^{m-1}} \sum_{i=1}^m t_i^2.$$

On the other hand, if $\lambda_1, \lambda_2, \dots, \lambda_m$ are the latent roots of $\|\rho_{ij}\|$ then each $\lambda_i \leq m$ since their sum is m . Letting λ_1 be the smallest one we have

$$(33) \quad \frac{1}{2} \sum_{i,j} \rho_{ij} t_i t_j \geq \frac{1}{2} \lambda_1 \sum t_i^2 = \frac{\lambda_1 \lambda_2 \dots \lambda_m}{2\lambda_2 \dots \lambda_m} \sum t_i^2 \geq \frac{\Delta}{2m^{m-1}} \sum t_i^2.$$

(32) and (33) imply (30). Hence the lemma is proved.

Let us write down the particular cases $m = 1$ and $m = 2$ of (24):

$$(34) \quad \left\{ p \left(\frac{t}{\sqrt{n}} \right) \right\}^n = e^{-\frac{1}{2}t^2} (1 + \psi(it)) \\ + \frac{\Theta_k}{n^{\frac{1}{2}(k-2)}} \beta_k^{2(k-2)/k} \{ |t|^k + |t|^{k+1} + \dots + |t|^{3(k-2)} \} e^{-t^2/4}, \left(|t| \leq \frac{A_k \sqrt{n}}{\beta_k^{3/k}} \right) \\ \left\{ p \left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}} \right) \right\}^n = e^{-\frac{1}{2}(t_1^2 + t_2^2 + 2\rho t_1 t_2)} \{ 1 + \psi(it_1, it_2) \} \\ (35) \quad + \frac{\Theta_k}{n^{\frac{1}{2}(k-2)}} \left\{ \sum_{i=1}^2 \beta_{ki}^{2(k-2)/k} (|t_i|^k + |t_i|^{k+1} + \dots + |t_i|^{3(k-2)}) \right\} e^{-(1-\rho^2)(t_1^2 + t_2^2)/8} \\ \left(|t_i| \leq \frac{A_k(1-\rho^2)\sqrt{n}}{\beta_{ki}^{3/k}}, \quad \rho = \epsilon(\xi_1 \xi_2) \right).$$

More specially let us rewrite (34) and (35) with $k = 3$:

$$(36) \quad \left\{ p \left(\frac{t}{\sqrt{n}} \right) \right\}^n = e^{-\frac{1}{2}t^2} + \frac{\Theta}{\sqrt{n}} \beta_3 |t|^3 e^{-\frac{1}{2}t^2}, \quad \left(|t| \leq \frac{A\sqrt{n}}{\beta_3} \right); \\ (37) \quad \left\{ p \left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}} \right) \right\}^n = e^{-\frac{1}{2}(t_1^2 + t_2^2 + 2\rho t_1 t_2)} \\ + \frac{\Theta}{\sqrt{n}} (\beta_{31} |t_1|^3 + \beta_{32} |t_2|^3) e^{-(1-\rho^2)(t_1^2 + t_2^2)/8}, \quad \left(|t_i| \leq \frac{A(1-\rho^2)\sqrt{n}}{\beta_{3i}} \right).$$

In this paper only these last four formulae are needed; they are used in the proofs of Theorems 2, 1, 3, 4 respectively. Cases of $m > 2$ of (24) will be needed for the works on other functions alluded to in the introduction.

1.2. In the following group of lemmas, which culminate in Lemma 7, one finds a generalization of the Riemann-Lebesgue theorem, viz. Lemma 6.

LEMMA 4. Let $f(x)$ be a polynomial of degree $m > 0$, with real coefficients:

$$(38) \quad f(x) = \sum_{i=0}^m a_i x^{m-i} \quad (a_0 \neq 0)$$

Then

$$(38) \quad \left| \int_0^1 e^{if(x)} dx \right| \leq \frac{A_m}{|a_0|^{1/m}}.$$

PROOF: It is sufficient to prove the inequality for $\int_0^1 \cos f(x) dx$. Divide the interval into A_m sub-intervals in each of whose interior none of the derivatives $f^{(i)}(x)$ ($i = 1, \dots, m$) vanishes. It is sufficient to consider one of these sub-intervals, say (a, b) . Consequently each of the polynomials $f^{(i)}(x)$ are monotonic in (a, b) . Let

$$(39) \quad I = \int_a^b \cos f(x) dx.$$

Suppose first that $f'(x)$ is positive and increasing for $a < x \leq b$. Then

$$\begin{aligned} |I| &\leq \epsilon + \left| \int_{a+\epsilon}^b \frac{f'(x) \cos f(x) dx}{f'(x)} \right| \\ &= \epsilon + \frac{1}{f'(a+\epsilon)} \left| \int_{a+\epsilon}^{b_1} f'(x) \cos f(x) dx \right|, \quad (a + \epsilon \leq b_1 \leq b), \end{aligned}$$

by the second mean-value theorem. Hence

$$(40) \quad |I| \leq \epsilon + \frac{2}{f'(a+\epsilon)}.$$

Now $0 < f'(a + \frac{1}{2}\epsilon) = f'(a + \epsilon) - \epsilon f'''(a + \theta\epsilon)/2$, $\frac{1}{2} \leq \theta \leq 1$. Hence $f'(a + \epsilon) > \frac{1}{2}\epsilon f'''(a + \theta\epsilon)$. Since $f''(x)$ is monotonic, we have either $f'(a + \epsilon) > \frac{1}{2}\epsilon f'''(a + \epsilon)$ or $f'(a + \epsilon) > \frac{1}{2}\epsilon f'''(a + \frac{1}{2}\epsilon)$. In other words, there exists a constant C_2 , independent of a or ϵ , such that $\frac{1}{2} \leq C_2 \leq 1$ and $f'(a + \epsilon) > \frac{1}{2}\epsilon f'''(a + C_2\epsilon)$.

If $f'''(x) \geq 0$, we have, as before $f''(a + C_2\epsilon) > \frac{1}{2}C_2\epsilon f'''(a + C_3\epsilon)$, where C_3 is independent of a or ϵ and $\frac{1}{2} \leq C_3 \leq 1$. If $f'''(x) < 0$, then, since $0 < f''(a + 2C_2\epsilon) = f''(a + C_2\epsilon) + C_2\epsilon f'''(a + \theta_1 C_2\epsilon)$, $\frac{1}{2} \leq \theta_1 \leq 1$, we have $f''(a + C_2\epsilon) > -C_2\epsilon f'''(a + 2\theta_1 C_2\epsilon)$. As $f'''(x)$ is monotonic, either $f''(a + C_2\epsilon) > -C_2\epsilon f'''(a + C_2\epsilon)$ or $f''(a + C_2\epsilon) > -C_2\epsilon f'''(a + 2C_2\epsilon)$. In all cases we obtain $f''(a + C_2\epsilon) > B_3 \epsilon |f'''(a + C_3\epsilon)|$, where B_3 and C_3 are independent of a or ϵ , and $\frac{1}{2} \leq C_3 \leq 2$. Hence $f'(a + \epsilon) > \frac{1}{2}B_3\epsilon^2 |f'''(a + C_3\epsilon)|$. Arguing with $\pm f'''(a + C_3\epsilon)$ as we did with $f''(a + C_2\epsilon)$, and so on until we come to $f^{(m)}$,

we obtain $f'(a + \epsilon) > B_m \epsilon^{m-1} |f^{(m)}(a + C_m \epsilon)| = B_m \epsilon^{m-1} |a_0|$. Substituting in (40) and putting $\epsilon = |a_0|^{-1/m}$ we obtain $|I| \leq A_m |a_0|^{-1/m}$. The proof presupposes that $C_m \epsilon < b - a$. If the reverse inequality is true, then $|I| \leq b - a < C_m |a_0|^{-1/m}$. Hence the lemma is true for $f'(x)$ positive and increasing in (a, b) .

If $f'(x)$ is positive and decreasing in (a, b) , then $I = \int_0^{b-a} \cos(-f(b-y)) dy$, $-f(b-y)$ being a polynomial with the leading coefficient $\pm a_0$ and the first derivative $f'(b-y)$, which is positive and increasing. This case reduces therefore to the preceding one. Finally, if $f'(x)$ is negative, we have only to notice that $I = \int_a^b \cos(-f(x)) dx$. Hence the lemma is proved.

LEMMA 5. Let $f(x)$ be the polynomial (38a), and let $a_r \neq 0$ for some r , $0 \leq r < m$. Then

$$(41) \quad \left| \int_0^1 e^{if(x)} dx \right| \leq \frac{A_m}{|a_r|^{A_m}}.$$

PROOF: We may assume that $|a_r| \geq 1$, (41) being trivial if $|a_r| < 1$. If $r = 0$ this reduces to Lemma 4. Suppose that the lemma is true for a_0, a_1, \dots, a_{r-1} . Let $f_1(x) = a_0 x^m + \dots + a_{r-1} x^{m-r+1}$, $f_2(x) = f(x) - f_1(x)$ and divide $(0, 1)$ into A_m sub-intervals in each of which $f_1(x)$ is monotonic. It is sufficient to consider one of these sub-intervals, say, (a, b) . We have

$$\begin{aligned} I &= \int_a^b \cos \{f_1(x) + f_2(x)\} dx \\ &= \int_a^b \cos f_1(x) \cos f_2(x) dx - \int_a^b \sin f_1(x) \sin f_2(x) dx. \end{aligned}$$

We have only to consider the integral of cosines, say J . Divide (a, b) into sub-intervals in each of whose interior $\cos f_1(x)$ is monotonic and does not vanish. The number of such intervals does not exceed $(\frac{1}{2}\pi)^{-1} |f_1(b) - f_1(a)| \leq (\frac{1}{2}\pi)^{-1} (|f_1(b)| + |f_1(a)|) < 2(|a_0| + \dots + |a_{r-1}|)$. Then, by the second mean-value theorem,

$$|J| \leq 2(|a_0| + \dots + |a_{r-1}|) \left| \int_a^{b_1} \cos f_2(x) dx \right| \quad (a \leq b_1 \leq b).$$

Hence, applying Lemma 4 to $f_2(x)$, we get

$$(42) \quad |I| \leq \frac{A_m(|a_0| + \dots + |a_{r-1}|)}{|a_r|^{1/(m-r)}} \leq \frac{A_m(|a_0| + \dots + |a_{r-1}|)}{|a_r|^{1/m}}.$$

On the hypothesis of induction we have $|I| \leq A_m |a_i|^{-B_m}$ ($i = 0, \dots, r-1$). If $|a_i| \geq |a_r|^{1/2m}$ for some $i < r$, then $|I| \leq A_m |a_r|^{-B_m/2m}$; if $|a_i| < |a_r|^{1/2m}$, then by (42), $|I| \leq A_m |a_r|^{-1/2m}$. The proof is therefore complete.

LEMMA 6. Let $f(x)$ be the polynomial (38a) and $g(x)$ be summable over $(-\infty, \infty)$. Then for every r we have

$$(43) \quad \lim_{|a_r| \rightarrow \infty} \int_{-\infty}^{\infty} e^{if(x)} g(x) dx = 0, \quad \text{uniformly in } a_i (i \neq r).$$

PROOF: By Lemma 5 We have

$$\lim_{|a_r| \rightarrow \infty} \int_0^1 e^{if(x)} dx = 0, \quad \text{uniformly in } a_i (i \neq r).$$

Hence

$$(44) \quad \lim_{|a_r| \rightarrow \infty} \int_a^b e^{if(x)} dx = 0, \quad \text{uniformly in } a_i (i \neq r)$$

for if $a \neq 0$ and $b \neq 0$, then (a, b) is the sum or the difference of two intervals of the form $(0, c)$ or $(c, 0)$, and for the latter intervals the transformation $x = \pm cy$ reduces the interval of integration to $(0, 1)$.

Let G be any open set of finite measure. Then G is the sum of a sequence $\{I_n\}$ of non-overlapping intervals. Since $\sum mI_n = mG < \infty$, we have

$$\sum_{n \geq N} mI_n < \epsilon, \quad n \geq N.$$

Hence

$$\left| \int_G e^{if(x)} dx \right| < \epsilon + \sum_{n=1}^N \left| \int_{I_n} e^{if(x)} dx \right|$$

which, together with (44), implies

$$(45) \quad \lim_{|a_r| \rightarrow \infty} \int_G e^{if(x)} dx = 0 \quad \text{uniformly in } a_i (i \neq r).$$

Let S be any set of finite measure. Then there is an open set G such that $G \supset S$ and $m(G - S) < \epsilon$. Hence

$$\left| \int_S e^{if(x)} dx \right| < \epsilon + \left| \int_G e^{if(x)} dx \right|.$$

Hence, by (45),

$$(46) \quad \lim_{|a_r| \rightarrow \infty} \int_S e^{if(x)} dx = 0 \quad \text{uniformly in } a_i (i \neq r).$$

Now let $h(x)$ be any positive "simple" summable function, i.e. $h(x) = a_\nu > 0$ for $x \in S$ ($\nu = 1, 2, \dots, n$) and $h(x) = 0$ otherwise. Since $h(x)$ is summable, each S_ν must be of finite measure. Hence

$$\left| \int_{-\infty}^{\infty} e^{if(x)} h(x) dx \right| \leq \sum_{\nu=1}^n a_\nu \left| \int_{S_\nu} e^{if(x)} dx \right|$$

which, together with (46), implies

$$\lim_{|a_r| \rightarrow \infty} \int_{-\infty}^{\infty} e^{if(x)} h(x) dx = 0 \quad \text{uniformly in } a_i (i \neq r).$$

Finally, let $g(x)$ be any summable function ≥ 0 . Then by a well-known theorem⁶ we have $g(x) = \lim h_n(x)$, where $\{h_n(x)\}$ is an ascending sequence of positive summable simple functions. Hence

$$\left| \int_{-\infty}^{\infty} e^{if(x)} g(x) dx \right| \leq \left| \int_{-\infty}^{\infty} e^{if(x)} h_n(x) dx \right| + \int_{-\infty}^{\infty} (g(x) - h_n(x)) dx.$$

By monotonic convergence the last integral tends to 0 as $n \rightarrow \infty$. Hence

$$\left| \int_{-\infty}^{\infty} e^{if(x)} g(x) dx \right| \leq \epsilon + \left| \int_{-\infty}^{\infty} e^{if(x)} h_n(x) dx \right|,$$

which implies (43). If $g(x)$ is any summable function, we have only to consider the customary expression of $g(x)$ as the difference of two non-negative functions. This completes the proof.

LEMMA 7. Let $P(x)$ be a non-singular distribution function of a random variable X , and let

$$(47) \quad p(t_1, t_2, \dots, t_m) = \int_{-\infty}^{\infty} e^{i \sum_{r=1}^m t_r x^r} dP.$$

Then for every r and every positive constant c we have

$$(48) \quad \text{l.u.b.}_{|t_r| \geq c} |p(t_1, \dots, t_m)| < 1.$$

PROOF: We have $P(x) = a_1 P_1(x) + a_2 P_2(x)$, where $P_1(x)$ is absolutely continuous, P_2 is singular, $a_1 > 0$, $a_1 + a_2 = 1$. Hence

$$|p(t_1, t_2, \dots, t_m)| \leq a_1 \left| \int_{-\infty}^{\infty} e^{i \sum_{r=1}^m t_r x^r} P_1'(x) dx \right| + a_2.$$

By Lemma 6 we may find $C > 0$ such that

$$|p(t_1, t_2, \dots, t_m)| \leq \frac{1}{2}a_1 + a_2 < 1, \quad \text{if any } |t_i| > C.$$

Suppose that

$$\text{l.u.b.}_{|t_r| \geq c} p(t_1, \dots, t_m) = 1,$$

then $c < C$ and we must have

$$(49) \quad \text{l.u.b.}_{c \leq |t_r| \leq C, |t_i| \leq C(i \neq r)} |p(t_1, \dots, t_m)| = 1.$$

Since $p(t_1, \dots, t_m)$ is a continuous function, it must attain its least upper bound in any bounded closed set. It follows that there is a point (t_1^0, \dots, t_m^0) such that⁷ $t_r^0 \neq 0$ ($|t_r^0| \geq c$) and $p(t_1^0, \dots, t_m^0) = 1$. But this implies that the distribution of $\sum t_i^0 X^i$ is discrete, i.e. that the distribution of X itself is discrete,

⁶ H. Kestelman: *Modern Theories of Integration* (1937), p. 108.

⁷ Cf. (C), p. 26.

which contradicts the non-singularity of $P(x)$. Hence (49) is false and (48) is true.

1.3. In his cited work Berry⁸ shows that if $F(x)$ is any distribution function and if $\Phi(x)$ is the function (6), then there is a constant a such that

$$(50) \quad \left| \int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x+a) - \Phi(x+a)\} dx \right| \geq \sqrt{\frac{2}{\pi}} T\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} dx - \pi \right\}$$

where $\delta = \sqrt{\frac{\pi}{2}}$ l.u.b. $|F(x) - \Phi(x)|$. This is easily extended to the following lemma, which needs no further proof.

LEMMA 8. Let $F(x)$ be a distribution function and $F_1(x)$ be a function having the following properties: (i) $F_1(x)$ is bounded for all x , (ii) $F_1(x) \rightarrow 1$ as $x \rightarrow \infty$, $F_1(x) \rightarrow 0$ as $x \rightarrow -\infty$, (iii) $F_1(x)$ has a bounded derivative, $|F_1'(x)| \leq M$. Let

$$\delta = \frac{1}{2M} \text{l.u.b.} |F(x) - F_1(x)|.$$

Then there exists a constant a such that

$$(51) \quad \left| \int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x+a) - F_1(x+a)\} dx \right| \geq 2MT\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} dx - \pi \right\}.$$

1.4. In section 3 we define, for given ϵ, k, λ and z , a function

$$(52) \quad G(x, y) = e^{-\epsilon y^{2k}} \text{ if } z < x \leq z + \lambda y^2, \quad G(x, y) = 0 \text{ otherwise.}$$

The introduction of $G(x, y)$ and the appraisal of its Fourier transform constitute the essence of our method of solving the problem of the asymptotic expansion of the distribution function $G(x)$. The solution of the same problem about other functions of (1) alluded to in section 3 is based on the introduction of functions playing the role of $G(x, y)$. We now prove the following lemma:

LEMMA 9. Let $G(x, y)$ be defined by (52) and let

$$(53) \quad g(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1x - it_2y} G(x, y) dx dy.$$

Then

$$(i) \quad |g(t_1, t_2)| \leq \frac{\lambda A_k}{\epsilon^{3/2k}}$$

$$(ii) \quad |g(t_1, t_2)| \leq \frac{A}{|t_2|^3} \left(\lambda + \frac{\lambda^2 |t_1|}{\epsilon^{1/3}} + \frac{\lambda^3 |t_1|^2}{\epsilon^{2/3}} \right) \text{ if } k = 3,$$

$$(iii) \quad |g(t_1, t_2)| \leq \frac{A_k}{|t_2|^2} \left(\frac{\lambda}{\epsilon^{1/2k}} + \frac{\lambda^2 |t_1|}{\epsilon^{3/2k}} \right).$$

⁸ (B), p. 128.

PROOF:

$$(i) \quad |g(t_1, t_2)| \leq \int_{R_2} G(x, y) dx dy = \lambda \int_{-\infty}^{\infty} y^2 e^{-\epsilon y^{2k}} dy = \frac{A_k \lambda}{\epsilon^{3/2k}}$$

(ii) Putting $k = 3$ we have

$$g(t_1, t_2) = \frac{e^{-it_1 x}}{it_1} \int_{-\infty}^{\infty} e^{-\epsilon y^6 - it_2 y} (1 - e^{-it_1 \lambda y^2}) dy,$$

$$|g(t_1, t_2)| \leq \frac{1}{|t_1| |t_2|^3} \left| \int_{-\infty}^{\infty} u(y) v'''(y) dy \right|,$$

where $u(y) = e^{-\epsilon y^6} (1 - e^{-it_1 \lambda y^2})$, $v(y) = e^{-it_2 y}$. On integrating by parts we obtain

$$(54) \quad |g(t_1, t_2)| \leq \frac{1}{|t_1| |t_2|^3} \left| \int_{-\infty}^{\infty} v(y) u'''(y) dy \right| \leq \frac{1}{|t_1| |t_2|^3} \int_{-\infty}^{\infty} |u'''(y)| dy.$$

Elementary calculation establishes that

$$\begin{aligned} \frac{|u'''(y)|}{|t_1|} &\leq e^{-\epsilon y^6} (216\lambda \epsilon^3 |y|^{17} + 756\lambda \epsilon^2 |y|^{11} \\ &\quad + 336\lambda \epsilon |y|^5 + 8\lambda^3 |t_1|^2 |y|^3 + 12\lambda^2 |t_1| |y|). \end{aligned}$$

Substituting in (54) and making the transformation $y = \epsilon^{-1/6} x$ we get the result.

(iii) We have

$$|g(t_1, t_2)| \leq \frac{1}{|t_1|} \left| \int_{-\infty}^{\infty} e^{-\epsilon y^{2k} - it_2 y} (1 - e^{-it_1 \lambda y^2}) dy \right|.$$

Integrating by parts twice we obtain

$$|g(t_1, t_2)| \leq \frac{1}{|t_1| |t_2|^2} \int_{-\infty}^{\infty} \left| \frac{d^2}{dy^2} \{e^{-\epsilon y^{2k}} (1 - e^{-it_1 \lambda y^2})\} \right| dy.$$

By elementary calculations we get

$$|g(t_1, t_2)| \leq \frac{1}{|t_2|^2} \int_{-\infty}^{\infty} (4k^2 \lambda \epsilon y^{4k} + 2k(k+3) \lambda \epsilon y^{2k} + 4\lambda^2 |t_1| y^2 + 2\lambda) e^{-\epsilon y^{2k}} dy$$

which, on the transformation $y = \epsilon^{-1/2k} x$, gives the result.

1.5. We prove a few additional lemmas used in the proof of Theorems 3 and 4.

LEMMA⁹ 10. Let $u(x_1, \dots, x_m) \geq 0$ be summable in the m -dimensional space and let

$$(55) \quad v(t_1, \dots, t_m) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-it_1 x_1 - \dots - it_m x_m} u(x_1, \dots, x_m) dx_1 \dots dx_m.$$

⁹ Although the author believes that this lemma is almost classical, a proof is given owing to lack of reference.

If $v(t_1, \dots, t_m)$ is summable in the m -dimensional space, then

$$(56) \quad u(x_1, \dots, x_m) = \frac{1}{(2\pi)^m} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{it_1 x_1 + \dots + it_m x_m} v(t_1, \dots, t_m) dt_1 \dots dt_m.$$

PROOF: Except for a constant factor the function $u(x_1, \dots, x_m)$ may be regarded as a probability density function. Hence by the well-known inversion formula of (55),

$$(57) \quad \int \dots \int_{a_i \leq x_i \leq b_i \quad (i=1, \dots, m)} u(x_1, \dots, x_m) dx_1 \dots dx_m \\ = \frac{1}{(2\pi)^m} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\prod_{j=1}^m \frac{e^{it_j b_j} - e^{it_j a_j}}{it_j} \right) v(t_1, \dots, t_m) dt_1 \dots dt_m.$$

Now $u(x_1, \dots, x_m)$ is almost everywhere the symmetric derivative of the interval function in the left-hand side of (57):

$$u(x_1, \dots, x_m) = \lim_{\epsilon \rightarrow 0} \frac{1}{(2\epsilon)^m} \int \dots \int_{x_i - \epsilon \leq y_i \leq x_i + \epsilon \quad (i=1, 2, \dots, m)} u(y_1, \dots, y_m) dy_1 \dots dy_m.$$

Hence

$$(58) \quad u(x_1, \dots, x_m) = \frac{1}{(2\pi)^m} \lim_{\epsilon \rightarrow 0} \frac{1}{(2\epsilon)^m} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\prod_{j=1}^m \frac{e^{it_j \epsilon} - e^{-it_j \epsilon}}{it_j} \right) e^{it_1 x_1 + \dots + it_m x_m} v(t_1, \dots, t_m) dt_1 \dots dt_m.$$

Owing to dominated convergence the order of the limit sign and the integration sign in (58) may be inverted: Hence (56) is true.

LEMMA 11. We have

$$(59) \quad \int_{-\infty}^{\infty} e^{-itu} \frac{1 - \cos Tu}{u^2} du = \begin{cases} \pi(T - |t|) & \text{if } |t| \leq T, \\ 0 & \text{if } |t| > T. \end{cases}$$

PROOF: The Fourier transform of the function in the right-hand side of (59) is

$$\pi \int_{-T}^T e^{itu} (T - |t|) dt = \frac{2\pi}{u^2} (1 - \cos Tu).$$

Hence (59) follows from (56).

LEMMA 12.

$$(60) \quad |\epsilon(\xi_1 + \dots + \xi_n)^k| \leq A_k n^{k/2} \beta_k$$

PROOF. As (60) is true for $k = 1$, let us assume, for induction, that it is true for $1, 2, \dots, k$. Then, by symmetry,

$$\epsilon(\xi_1 + \dots + \xi_n)^{k+1} = n \epsilon\{\xi_1(\xi_1 + \dots + \xi_n)^k\} = n \sum_{r=0}^k \binom{k}{r} \epsilon(\xi_1^{r+1} U^{k-r})$$

where $U = \xi_2 + \cdots + \xi_k$. Since $\epsilon(\xi_1) = 0$, we have

$$\epsilon(\xi_1 + \cdots + \xi_n)^{k+1} = n \sum_{r=1}^k \binom{k}{r} \epsilon(\xi_1^{r+1} U^{k-r}).$$

On the hypotheses of induction we have $|\epsilon(U^{k-r})| \leq A_k(n-1)^{\frac{1}{2}(k-r)} \beta_{k-r} < A_k n^{\frac{1}{2}(k-1)} \beta_{k-r}$. Hence

$$|\epsilon(\xi_1 + \cdots + \xi_n)^{k+1}| \leq k! A_k n^{\frac{1}{2}(k+1)} \Sigma \beta_{r+1} \beta_{k-r} \leq A_{k+1} n^{\frac{1}{2}(k+1)} \beta_{k+1}.$$

Therefore the induction is complete.

3. Elementary Proof of Theorem 1. 2.1 We have defined

$$(61) \quad F(x) = Pr\{\sqrt{n}\bar{\xi} \leq x\}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}y^2} dy$$

with the characteristic functions

$$(62) \quad f(t) = \left\{ p \left(\frac{t}{\sqrt{n}} \right) \right\}^n, \quad \varphi(t) = e^{-\frac{1}{2}t^2}.$$

Following Berry¹⁰ we use the equation

$$(63) \quad \int_{-\infty}^{\infty} \{F(x) - \Phi(x)\} e^{itz} dx = \frac{f(t) - \varphi(t)}{-it}.$$

Let $\psi(it)$ be the polynomial in (34), and let us define $\Psi(x)$ as the function obtained from $\psi(it)$ through the replacement of each power $(it)^r$ by $(-1)^r \Phi^{(r)}(x)$.

Integration by parts shows $(-1)^{r-1} \int_{-\infty}^{\infty} e^{itz} \Phi^{(r)}(x) dx = (it)^{r-1} \varphi(t)$, whence

$$(64) \quad \int_{-\infty}^{\infty} \Psi(x) e^{itz} dx = \frac{\psi(it)\varphi(t)}{-it}.$$

From (63) and (64) we obtain

$$(65) \quad \int_{-\infty}^{\infty} \{F(x) - \Phi(x) - \Psi(x)\} e^{itz} dx = \frac{f(t) - \varphi(t)\{1 + \psi(it)\}}{-it}.$$

The function $\Psi(x)$ defined here is precisely the $\Psi(x)$ appearing in (5) under Theorem 1. Our task is to prove that

$$(66) \quad |F(x) - \Phi(x) - \Psi(x)| \leq \frac{Q_k}{n^{(k-2)/2}}.$$

Following Berry¹¹ we replace x by $x + a$ in (65), getting

$$(67) \quad \begin{aligned} \int_{-\infty}^{\infty} \{F(x+a) - \Phi(x+a) - \Psi(x+a)\} e^{itz} dx \\ = \frac{e^{-\frac{1}{2}ia^2} [f(t) - \varphi(t)\{1 + \psi(it)\}]}{-it} \end{aligned}$$

¹⁰ (B), p. 127, Equation (23).

¹¹ (B), p. 127.

multiply both sides of (67) by $T - |t|$ and integrate with respect to t in $(-T, T)$:

$$\begin{aligned} 2 \int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x+a) - \Phi(x+a) - \Psi(x+a)\} dx \\ = \int_{-T}^T \frac{(T - |t|) e^{-it} [f(t) - \varphi(t)\{1 + \psi(it)\}]}{-it} dt \end{aligned}$$

the reversion of order of integration involved is obviously justifiable. Hence

$$(68) \quad \left| \int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x+a) - \Phi(x+a) - \Psi(x+a)\} dx \right| \leq T \int_0^T \frac{|f(t) - \varphi(t)\{1 + \psi(it)\}|}{t} dt.$$

2.2. When in particular $k = 3$, (68) becomes

$$(69) \quad \left| \int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x+a) - \Phi(x+a)\} dx \right| \leq T \int_0^T \frac{|f(t) - \varphi(t)|}{t} dt.$$

If we choose a to be the a in (50), the left-hand side of (69) is not less than

$$\sqrt{\frac{2}{\pi}} T\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} dx - \pi \right\}, \quad \delta = \sqrt{\frac{\pi}{2}} \text{l.u.b. } |F(x) - \Phi(x)|.$$

On the other hand, taking $T = \frac{A\sqrt{n}}{\beta_3}$ as in (36) the right-hand side of (69) is not greater than

$$A \int_0^{\infty} t^2 e^{-t^2} dt = A.$$

Hence

$$(70) \quad T\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} dx - \pi \right\} \leq A.$$

Now the left-hand side of (70), as a function of $T\delta$, is positive and increasing for sufficiently large $T\delta$, and becomes infinite as $T\delta \rightarrow \infty$. Hence (70) implies that $T\delta \leq A$, i.e.

$$\text{l.u.b. } |F(x) - \Phi(x)| \leq \frac{A}{T} = \frac{A\beta_3}{\sqrt{n}},$$

giving Theorem 2.

2.3. Coming back to the general case, we see that the function $\Phi(x) + \Psi(x)$ has a bounded derivative: $|\Phi'(x) + \Psi'(x)| \leq Q_k$, and also has all the properties of the function $F_1(x)$ in Lemma 8. On choosing a in (69) to be the a in (51) we obtain

$$(71) \quad Q_k T \delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} dx - \pi \right\} \leq T \int_0^T \frac{|f(t) - \varphi(t)\{1 + \psi(it)\}|}{t} dt,$$

where

$$\delta = Q_k \text{ l.u.b. } |F(x) - \Phi(x) - \Psi(x)|.$$

Let us take $T = (A_k \beta_k^{-3/k} \sqrt{n})^{k-2}$ with A_k in accordance with (34). Then

$$(72) \quad \begin{aligned} T \int_0^T \frac{|f(t) - \varphi(t)\{1 + \psi(it)\}|}{t} dt \\ = Q_k n^{\frac{1}{4}(k-2)} \int_0^{T^{1/(k-2)}} + Q_k n^{\frac{1}{4}(k-2)} \int_{Q_k \sqrt{n}}^T = J_1 + J_2 \quad \text{say.} \end{aligned}$$

By (34) we have

$$(73) \quad J_1 \leq Q_k \int_0^\infty (t^{k-1} + \dots + t^{3k-7}) e^{-\frac{1}{4}t^2} dt = Q_k.$$

Also,

$$(74) \quad J_2 \leq Q_k n^{\frac{1}{4}(k-2)} \int_{Q_k \sqrt{n}}^T \frac{|p(t/\sqrt{n})|^n}{t} dt + Q_k n^{\frac{1}{4}(k-2)} \int_{Q_k \sqrt{n}}^T \frac{\varphi(t)|1 + \psi(it)|}{t} dt.$$

The second term in the right-hand side of (74) is evidently $\leq Q_k$. The first term does not exceed

$$(75) \quad Q_k n^{\frac{1}{4}(k-3)} T \text{ l.u.b. }_{t \geq Q_k} |p(t)|^n.$$

At this step we make use of the non-singularity of $P(x)$ and apply Lemma 7 for $m = 1$. We have

$$\text{l.u.b. }_{t \geq Q_k} |p(t)| = e^{-Q_k}.$$

Hence (75) does not exceed $Q_k n^{\frac{1}{4}(2k-5)} e^{-Q_k n} \leq Q_k$. We have therefore

$$(76) \quad T \delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} dx - \pi \right\} \leq Q_k, \quad T = Q_k n^{\frac{1}{4}(k-2)}.$$

Arguing with (76) as we did with (70) we conclude that

$$\text{l.u.b. } |F(x) - \Phi(x) - \Psi(x)| \leq \frac{Q_k}{T} = \frac{Q_k}{n^{\frac{1}{4}(k-2)}}.$$

(72) is valid for $T \geq 1$. If $T < 1$, we have only to suppress the term J_2 . Hence Theorem 1 is proved.

4. Proof of Theorem 3 and Theorem 4. 3.1. In connection with the random variables (1), we assume that $\beta_{2k} < \infty$ for some integer $k \geq 3$ and define

$$(77) \quad \eta = \frac{1}{n} \sum_{r=1}^n (\xi_r - \bar{\xi})^2, \quad G(z) = Pr \left\{ \frac{\sqrt{n}(\eta - 1)}{\sqrt{\alpha_4 - 1}} \leq z \right\}.$$

Now,

$$\eta = \frac{1}{n} \sum \xi_r^2 - \bar{\xi}^2 = 1 + \sqrt{\frac{\alpha_4 - 1}{n}} X - \frac{Y^2}{n}$$

where

$$(78) \quad X = \frac{1}{\sqrt{n}} \sum \frac{(\xi_r^2 - 1)}{\sqrt{\alpha_4 - 1}}, \quad Y = \sqrt{n} \bar{\xi}.$$

Hence

$$(79) \quad G(z) = Pr\{X - \lambda Y^2 \leq z\}$$

with

$$(80) \quad \lambda = \frac{1}{\sqrt{n(\alpha_4 - 1)}}.$$

Let W be the probability function of the distribution of the random point (X, Y) and $f(t_1, t_2)$ be the characteristic function:

$$(81) \quad W(S) = Pr\{(X, Y) \in S\} \text{ for every Borel set } S \text{ in } R_2,$$

$$(82) \quad f(t_1, t_2) = \epsilon(e^{it_1 X + it_2 Y}) = \left\{ p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right) \right\}^n$$

$$(83) \quad p(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_1(x^2-1)/(\sqrt{\alpha_4-1}) + it_2 x} dP.$$

Let $G_1(z)$ be the distribution function of X . Then

$$(84) \quad G(z) - G_1(z) = \int \int_{z < x \leq z + \lambda y^2} dW = K(z), \text{ say.}$$

Let

$$(85) \quad K_*(z) = \int \int_{z < x \leq z + \lambda y^2} e^{-iy^2 h} dW.$$

If we define (for fixed z) the function $G(x, y)$ by

$$(86) \quad G(x, y) = e^{-iy^2 h} \text{ if } z < x \leq z + \lambda y^2, \quad G(x, y) = 0 \text{ otherwise,}$$

then

$$(87) \quad K_*(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y) dW.$$

Letting

$$(88) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1 x - it_2 y} G(x, y) dx dy = g(t_1, t_2),$$

we replace x by $x - u$ in the integral and get

$$(89) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1x - it_2y} G(x - u, y) dx dy = e^{-it_1u} g(t_1, t_2).$$

Multiplying both sides by $\frac{1 - \cos Tu}{u^2}$ and integrating with respect to u we obtain, with the help of (59), Lemma 11,

$$(90) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1x - it_2y} dx dy \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} G(x - u, y) du = \begin{cases} \pi(T - |t_1|)g(t_1, t_2) & \text{if } |t_1| \leq T, \\ 0 & \text{if } |t_1| > T; \end{cases}$$

the reversion of order of integration in the left-hand side is obviously justifiable. By Lemma 9 the right-hand side of (90) is summable in the whole plane of (t_1, t_2) . Hence, by Lemma 10,

$$(91) \quad \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} G(x - u, y) du = \frac{1}{4\pi} \int \int_{|t_1| \leq T} (T - |t_1|) g(t_1, t_2) e^{it_1x + it_2y} dt_1 dt_2.$$

If we integrate both sides with respect to the probability function W , we obtain, on reversing the order of integration,

$$(92) \quad \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} du \int \int_{R_2} G(x - u, y) dW = \frac{1}{4\pi} \int \int_{|t_1| \leq T} (T - |t_1|) g(t_1, t_2) f(t_1, t_2) dt_1 dt_2.$$

By (86) and (87),

$$(93) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x - u, y) dW = K_*(u + z).$$

Hence

$$(94) \quad \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} K_*(u + z) du = \frac{1}{4\pi} \int \int_{|t_1| \leq T} (T - |t_1|) g(t_1, t_2) f(t_1, t_2) dt_1 dt_2.$$

We now take the functions

$$(95) \quad \varphi(t_1, t_2) = e^{-\frac{1}{2}(t_1^2 + t_2^2 + 2\rho t_1 t_2)}$$

and $\psi(it_1, it_2)$ as in (35), where

$$(96) \quad \rho = \int_{-\infty}^{\infty} \frac{(x^2 - 1)x}{\sqrt{\alpha_4 - 1}} dP = \frac{\alpha_3}{\sqrt{\alpha_4 - 1}}.$$

Since the condition $\alpha_4 - 1 - \alpha_3^2 \neq 0$ is assumed in Theorem 3 and implied in Theorem 4, we have $|\rho| < 1$. Let

$$(97) \quad w(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(1/2(1-\rho^2))(x^2+y^2-2\rho xy)}$$

and let $\gamma(x, y)$ be the function obtained from $\psi(it_1, it_2)$ through the replacement of each power $(it_1)^{r_1} (it_2)^{r_2}$ by $(-1)^{r_1+r_2} W_{r_1, r_2}(x, y) = (-1)^{r_1+r_2} \frac{\partial^{r_1+r_2} w(x, y)}{\partial x^{r_1} \partial y^{r_2}}$.

Since

$$(98) \quad w(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1 x - it_2 y} \varphi(t_1, t_2) dt_1 dt_2,$$

we have

$$(99) \quad w_{r_1, r_2}(x, y) = \frac{(-1)^{r_1+r_2}}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (it_1)^{r_1} (it_2)^{r_2} e^{-it_1 x - it_2 y} \varphi(t_1, t_2) dt_1 dt_2,$$

whence, by Fourier inversion,

$$(100) \quad (it_1)^{r_1} (it_2)^{r_2} \varphi(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1 x + it_2 y} w_{r_1, r_2}(x, y) dx dy.$$

From the definition of $\gamma(x, y)$ it follows therefore

$$(101) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1 x + it_2 y} \{w(x, y) + \gamma(x, y)\} dx dy = \varphi(t_1, t_2) \{1 + \psi(it_1, it_2)\}.$$

A comparison of (101) with $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1 x + it_2 y} dW = f(t_1, t_2)$ shows that (94) will remain true if $K_*(u)$ be replaced by

$$(102) \quad \int \int_{u < x \leq u+\lambda y^2} e^{-iy^2 x} (w(x, y) + \gamma(x, y)) dx dy = L_*(u), \text{ say,}$$

and $f(t_1, t_2)$ be replaced by $\varphi(t_1, t_2) \{1 + \psi(it_1, it_2)\}$. Hence

$$(103) \quad \begin{aligned} & \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{K_*(u+z) - L_*(u+z)\} du \\ &= \frac{1}{4\pi} \int \int_{|t_1| \leq T} (T - |t_1|) g(t_1, t_2) \{f(t_1, t_2) \\ & \quad - \varphi(t_1, t_2) [1 + \psi(it_1, it_2)]\} dt_1 dt_2. \end{aligned}$$

Let also

$$(104) \quad H(z) = \int \int_{x-\lambda y^2 \leq z} \{w(x, y) + \gamma(x, y)\} dx dy,$$

$$H_1(z) = \int \int_{x \leq z} \{w(x, y) + \gamma(x, y)\} dx dy,$$

$$(105) \quad L(z) = H(z) - H_1(z) = \int \int_{z < x \leq z + \lambda y^2} \{w(x, y) + \gamma(x, y)\} dx dy.$$

3.2. We now consider the particular case $k = 3$ and prove Theorem 3. For $k = 3$ we have $\psi \equiv \gamma \equiv 0$ and so

$$(106) \quad H(z) = \int \int_{x-\lambda y^2 \leq z} w(x, y) dx dy,$$

$$H_1(z) = \int \int_{x \leq z} w(x, y) dx dy = \Phi(z),$$

$$L(z) = H(z) - H_1(z),$$

$$(107) \quad L_*(z) = \int \int_{z < x \leq z + \lambda y^2} e^{-xy^2} w(x, y) dx dy,$$

$$(108) \quad \begin{aligned} & \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{K_*(u+x) - L_*(u+x)\} du \\ &= \frac{1}{4\pi} \int \int_{|t_1| \leq T} (T - |t_1|) g(t_1, t_2) \{f(t_1, t_2) - \varphi(t_1, t_2)\} dt_1 dt_2. \end{aligned}$$

Now

$$\begin{aligned} K_*(u) - L_*(u) &= \{G(u) - \Phi(u)\} - \{H(u) - \Phi(u)\} - \{G_1(u) - \Phi(u)\} \\ &\quad - \{K(u) - K_*(u)\} + \{L(u) - L_*(u)\}, \end{aligned}$$

$$\begin{aligned} 0 \leq H(u) - \Phi(u) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-iy^2} dy \int_u^{u+\lambda y^2} e^{-(1/2(1-\rho^2))(x-\rho y)^2} dx \\ &\leq \frac{\lambda}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} y^2 e^{-iy^2} dy = \frac{\lambda}{\sqrt{2\pi(1-\rho^2)}}, \end{aligned}$$

$$|G_1(u) - \Phi(u)| \leq \frac{A}{\sqrt{n}} \int_{-\infty}^{\infty} \left| \frac{x^2 - 1}{\sqrt{\alpha_4 - 1}} \right|^3 dP \leq \frac{A\alpha_6}{(\alpha_4 - 1)^{3/2}\sqrt{n}} \text{ by Theorem 2,}$$

$$0 \leq K(u) - K_*(u) \leq \epsilon(Y^6) \leq A\alpha_6\epsilon \text{ by Lemma 12,}$$

$$0 \leq L(u) - L_*(u) \leq A\epsilon.$$

Hence

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{G(u + \lambda) - \Phi(u + \lambda)\} du \\
 (109) \quad & = \Theta T \left\{ \alpha_6 \epsilon + \frac{\alpha_6}{(\alpha_4 - 1)^{3/2} \sqrt{n}} + \frac{1}{\sqrt{n} \sqrt{(\alpha_4 - 1)(1 - \rho^2)}} \right. \\
 & \quad \left. + \Theta T \int \int_{|t_1| \leq T} |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)| dt_1 dt_2. \right.
 \end{aligned}$$

It is easy to verify that

$$\frac{\alpha_6}{(\alpha_4 - 1)^{3/2}} + \frac{1}{\sqrt{(\alpha_4 - 1)(1 - \rho^2)}} \leq \left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2} \right)^{3/2}.$$

For the left-hand side of (109) we refer to (50) and take x to be the number a therein. Hence

$$\begin{aligned}
 & T\delta \left\{ 3 \int_0^{T\alpha} \frac{1 - \cos u}{u^2} du - \pi \right\} \leq AT \left\{ \alpha_6 \epsilon + \frac{1}{\sqrt{n}} \left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2} \right)^{3/2} \right\} \\
 (110) \quad & + AT \int \int_{|t_1| \leq T, |t_2| \leq T} |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)| dt_1 dt_2 \\
 & + AT \int \int_{|t_1| \leq T, |t_2| > T} |g(t_1, t_2)| dt_1 dt_2.
 \end{aligned}$$

By Lemma 9 (ii) we have

$$\begin{aligned}
 & T \int \int_{|t_1| \leq T, |t_2| > T} |g(t_1, t_2)| dt_1 dt_2 \\
 (111) \quad & \leq AT \int \int_{|t_1| \leq T, |t_2| > T} \frac{1}{|t_2|^3} \left(\lambda + \frac{\lambda^2 |t_1|}{\epsilon^4} + \frac{\lambda^3 |t_1|^2}{\epsilon^3} \right) dt_1 dt_2. \\
 & \leq A \left(\lambda + \frac{\lambda^2 T}{\epsilon^4} + \frac{\lambda^3 T^2}{\epsilon^3} \right).
 \end{aligned}$$

Hence

$$\begin{aligned}
 & T\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos u}{u^2} du - \pi \right\} \\
 (112) \quad & \leq A \left\{ \alpha_6 T\epsilon + \left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2} \right)^{3/2} \frac{T}{\sqrt{n}} + \lambda + \frac{\lambda^2 T}{\epsilon^4} + \frac{\lambda^3 T^2}{\epsilon^3} \right\} \\
 & + AT \int \int_{|t_1| \leq T, |t_2| \leq T} |g(t_1, t_2)| |f - \varphi| dt_1 dt_2.
 \end{aligned}$$

By Lemma 9 (i) with $k = 3$ we have

$$(113) \quad T \int \int_{|t_1| \leq T, |t_2| \leq T} |g| \cdot |f - \varphi| dt_1 dt_2 \leq \frac{AT\lambda}{\epsilon^{\frac{1}{2}}} \int \int_{|t_1| \leq T, |t_2| \leq T} |f - \varphi| dt_1 dt_2.$$

By (37) under Lemma 3,

$$(114) \quad |f - \varphi| \leq \frac{A}{\sqrt{n}} (\beta_{31} |t_1|^3 + \beta_{32} |t_2|^3) e^{-\frac{1}{2}(1-\rho^2)(t_1^2+t_2^2)} \quad \text{for } |t_i| \leq \frac{A(1-\rho^2)\sqrt{n}}{\beta_{3i}}$$

with

$$(115) \quad \begin{aligned} \beta_{31} &= \int_{-\infty}^{\infty} \left| \frac{x^2 - 1}{\sqrt{\alpha_4 - 1}} \right|^3 dP \leq \frac{4}{(\alpha_4 - 1)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (x^6 + 1) dP \\ &\leq \frac{8\alpha_6}{(\alpha_4 - 1)^{\frac{1}{2}}}, \quad \beta_{32} = \int_{-\infty}^{\infty} |x|^3 dP = \beta_3. \end{aligned}$$

We now take

$$(116) \quad T = \frac{A}{8} \left(\frac{\alpha_4 - 1 - \alpha_3^2}{\alpha_6} \right)^{\frac{1}{2}} \sqrt{n},$$

the A coinciding with that in (114). Then

$$(117) \quad \begin{aligned} \frac{A(1-\rho^2)\sqrt{n}}{\beta_{31}} &\geq \frac{A(1-\rho^2)(\alpha_4 - 1)^{\frac{1}{2}}\sqrt{n}}{8\alpha_6} \\ &= \frac{A(\alpha_4 - 1 - \alpha_3^2)\sqrt{\alpha_4 - 1}\sqrt{n}}{8\alpha_6} \geq \frac{A(\alpha_4 - 1 - \alpha_3^2)^{\frac{1}{2}}\sqrt{n}}{8\alpha_6^{3/2}} = T \end{aligned}$$

$$(118) \quad \begin{aligned} \frac{A(1-\rho^2)\sqrt{n}}{\beta_{32}} &= \frac{A(\alpha_4 - 1 - \alpha_3^2)\sqrt{n}}{(\alpha_4 - 1)\beta_3} \\ &\geq \frac{A(\alpha_4 - 1 - \alpha_3^2)^{\frac{1}{2}}\sqrt{n}}{\alpha_4^{3/2}\beta_3} \geq \frac{A(\alpha_4 - 1 - \alpha_3^2)^{\frac{1}{2}}\sqrt{n}}{\alpha_6^{3/2}} > T. \end{aligned}$$

Hence (114) is true for $|t_1| \leq T$ and $|t_2| \leq T$. Using this fact on (113) we obtain

$$(119) \quad \begin{aligned} T \int \int_{|t_1| \leq T, |t_2| \leq T} |g| |f - \varphi| dt_1 dt_2 &\leq \frac{AT\lambda}{\epsilon^{\frac{1}{2}}} \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{\alpha_6}{(\alpha_4 - 1)^{\frac{1}{2}}} |t_1|^3 + \beta_3 |t_2|^3 \right\} e^{-\frac{1}{2}(1-\rho^2)(t_1^2+t_2^2)} dt_1 dt_2 \\ &\leq \frac{AT\lambda}{\epsilon^{\frac{1}{2}}\sqrt{n}} \left(\frac{\alpha_6}{(\alpha_4 - 1)^{3/2}} + \beta_3 \right) \frac{1}{(1-\rho^2)^{5/2}} \\ &= \frac{AT\lambda}{\sqrt{n}\epsilon} (\alpha_6(\alpha_4 - 1) + \beta_3(\alpha_4 - 1)^{5/2}) \frac{1}{(\alpha_4 - 1 - \alpha_3^2)^{5/2}} \\ &= \frac{AT}{n\sqrt{\epsilon}} (\alpha_6\sqrt{\alpha_4 - 1} + \beta_3(\alpha_4 - 1)^{\frac{3}{2}}) \frac{1}{(\alpha_4 - 1 - \alpha_3^2)^{5/2}} \\ &\leq \frac{AT\alpha_6^{11/6}}{n\sqrt{\epsilon}(\alpha_4 - 1 - \alpha_3^2)^{5/2}}. \end{aligned}$$

Substituting in (112), setting $\epsilon = (\alpha_6 T)^{-1}$ and using (116) we obtain after some easy reduction

$$(120) \quad T\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos u}{u^2} du - \pi \right\} \leq A \left[1 + \frac{1}{\sqrt{n(\alpha_4 - 1)}} + \left(\frac{\alpha_6}{n(\alpha_4 - 1 - \alpha_3^2)} \right)^{\frac{1}{2}} + \left(\frac{\alpha_6}{n(\alpha_4 - 1 - \alpha_3^2)} \right)^{\frac{1}{2}} \right].$$

If $n \geq (\alpha_4 - 1 - \alpha_3^2)^{-1} \alpha_6$, then the right-hand side of (120) is $\leq A$, and so, arguing with (120), as we did with (70), we obtain

$$(121) \quad \text{l.u.b. } |G(u) - \Phi(u)| \leq \frac{A}{T} = \frac{A}{\sqrt{n}} \left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2} \right)^{\frac{1}{2}}.$$

For $n < (\alpha_4 - 1 - \alpha_3^2)^{-1} \alpha_6$, however, the right-hand side of (121) $\geq A(\alpha_4 - 1 - \alpha_3^2)^{-1} \alpha_6 \geq A$ and (121) becomes a triviality. Hence Theorem 3 is proved.

3.3. To prove Theorem 4, we start again with the identity (103). We have

$$(122) \quad K_*(u) - L_*(u) = \{G(u) - H(u)\} - \{G_1(u) - H_1(u)\} - \{K(u) - K_*(u)\} + \{L(u) - L_*(u)\},$$

$$(123) \quad 0 \leq K(u) - K_*(u) \leq \epsilon \epsilon(Y^{2k}) \leq Q_k \epsilon \quad \text{by Lemma 12,}$$

$$(124) \quad 0 \leq L(u) - L_*(u) \leq \epsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^{2k}(w(x, y) + |\gamma(x, y)|) dx dy \leq Q_k \epsilon.$$

Let us show that

$$(125) \quad |G_1(u) - H_1(u)| \leq Q_k/n^{\frac{1}{2}(k-1)}.$$

The function $X = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{\xi_i^2 - 1}{\sqrt{\alpha_4 - 1}} \right)$ has the same structure as $\sqrt{n} \bar{\xi}$ (with $(\alpha_4 - 1)^{-1}(\xi_i^2 - 1)$ playing the role of ξ_i); hence, by Theorem 1, there exists an asymptotic expansion of the distribution function $G_1(u)$. We shall see that the terms of this asymptotic expansion are precisely $H_1(u)$, whence (125) follows from Theorem 1.

It is obvious that for the polynomial $\psi(it_1, it_2)$ in (35) $\psi(it, 0)$ coincides with the polynomial $\psi(it)$ in (34). Hence the terms of the asymptotic expansion of $G_1(u)$ are the inversion of $e^{-\frac{1}{2}t^2} \{1 + \psi(it, 0)\}$ viz.

$$(126) \quad \Phi(u) + \frac{1}{2\pi} \int_{-\infty}^u dx \int_{-\infty}^{\infty} e^{-itx - \frac{1}{2}t^2} \psi(it, 0) dt.$$

On the other hand, by (104),

$$(127) \quad H_1(u) = \Phi(u) + \int_{-\infty}^u dx \int_{-\infty}^{\infty} \gamma(x, y) dy,$$

and by (101) with $t_2 = 0$,

$$(128) \quad \int_{-\infty}^{\infty} e^{itx} dx \int_{-\infty}^{\infty} \gamma(x, y) dy = e^{-\frac{1}{2}t^2} \psi(it, 0).$$

Inversion of (118) gives

$$(129) \quad \int_{-\infty}^{\infty} \gamma(x, y) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iz - \frac{1}{2}t^2} \psi(it, 0) dt$$

which establishes the equality of $H_1(u)$ and (126).

Using (122), (123), (124), (125) on (103) we get

$$(130) \quad \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{G(u+z) - H(u+z)\} du = \Lambda_k T \left(\epsilon + \frac{1}{n^{\frac{1}{2}(k-2)}} \right) \\ + \Theta T \int \int |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)[1 + \psi(it_1, it_2)]| dt_1 dt_2.$$

If we expand

$$(131) \quad H(u) = \int \int_{-\lambda u^2 \leq u} \{w(x, y) + \gamma(x, y)\} dx dy$$

in powers of $n^{-\frac{1}{2}}$ up to and including the term $n^{-\frac{1}{2}(k-3)}$, the remainder is obviously $\Lambda_k n^{-\frac{1}{2}(k-2)}$. Hence

$$(132) \quad H(u) = \Phi(u) + \chi(u) + \Lambda_k/n^{\frac{1}{2}(k-2)},$$

where $\Phi(u) + \chi(u)$ is the group of terms of the Taylor expansion of (131) in powers of $n^{-\frac{1}{2}}$ up to and including the term $n^{-\frac{1}{2}(k-3)}$. From (130) and (132) we get

$$(133) \quad \left| \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{G(u+z) - \Phi(u+z) - \chi(u+z)\} du \right| \\ \leq Q_k T \left(\epsilon + \frac{1}{n^{\frac{1}{2}(k-2)}} \right) + AI,$$

where

$$(134) \quad I = T \int \int_{|t_1| \leq \tau} |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)\{1 + \psi(it_1, it_2)\}| dt_1 dt_2.$$

We are going to prove that the function $\chi(u)$ here defined satisfies all the requirements of the function $\chi(u)$ in Theorem 4. The structure of $\chi(u)$ announced in Theorem 4 is easily verifiable. It remains to prove the inequalities (15) and (16) satisfied by

$$|G(u) - \Phi(u) - \chi(u)|.$$

It is obvious that the function $\Phi(u) + \chi(u)$ has all the properties of the function $F_1(u)$ in Lemma 8, having a bounded derivative $|\Phi'(u) + \chi'(u)| \leq Q_k$. Hence, on taking z in (133) to be the number a in (51), the left-hand side of (133) does not exceed

$$Q_k T \delta \left(3 \int_0^{T\delta} \frac{1 - \cos u}{u^2} du - \pi \right), \quad \delta = Q_k \text{ l.u.b. } |G(u) - \Phi(u) - \chi(u)|.$$

Hence

$$(135) \quad T\delta \left(3 \int_0^{T\delta} \frac{1 - \cos u}{u^2} du - \pi \right) \leq Q_k T \left(\epsilon + \frac{1}{n^{\frac{1}{3}(k-2)}} \right) + Q_k I.$$

In order to appraise I we recall (35) under Lemma 3 (replacing therein each β_{ki} by the larger number $\beta_{k1}\beta_{k2}$, and merging the latter into Q_k)

$$(136) \quad |f(t_1, t_2) - \varphi(t_1, t_2) \{1 + \psi(it_1, it_2)\}| \leq \frac{Q_k}{n^{\frac{1}{3}(k-2)}} \{ \Sigma (|t_i|^k + \dots + |t_i|^{3(k-2)}) \} e^{-(1-\rho^2)(t_1^2+t_2^2)/8}$$

for

$$(137) \quad |t_i| \leq Q_k \sqrt{n}.$$

Put $T = (Q_k \sqrt{n})^l$, with Q_k here coinciding with that in (137) and then (136) is valid for $|t_1| \leq T^{1/l}$ and $|t_2| \leq T^{1/l}$. Write

$$I = T \int \int_{|t_1| \leq T^{1/l}, |t_2| \leq T^{1/l}} + T \int \int_{|t_1| \leq T, |t_2| > T^{1/l}} + T \int \int_{\substack{T^{1/l} < |t_1| \leq T \\ |t_2| \leq T^{1/l}}} = I_1 + I_2 + I_3.$$

By Lemma 9 (i),

$$(138) \quad I_1 \leq \frac{Q_k T}{n^{\frac{1}{3}\epsilon^{3/2k}}} \int \int |f - \varphi(1 + \psi)| dt_1 dt_2,$$

whence, by (136)

$$(139) \quad I_1 \leq \frac{Q_k T}{n^{\frac{1}{3}(k-1)} \epsilon^{3/2k}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\sum_{i=1}^2 (|t_i|^k + \dots + |t_i|^{3(k-2)}) \right) \cdot e^{-(1-\rho^2)(t_1^2+t_2^2)/8} dt_1 dt_2 \leq \frac{Q_k T}{n^{\frac{1}{3}(k-1)} \epsilon^{3/2k}}.$$

By Lemma 9 (iii) we have

$$I_2 \leq Q_k T \int \int_{|t_1| \leq T, |t_2| > T^{1/l}} \frac{1}{|t_2|^2} \left(\frac{1}{\sqrt{n\epsilon^{1/2k}}} + \frac{|t_1|}{n\epsilon^{3/2k}} \right) \{ |f(t_1, t_2)| + \varphi(t_1, t_2) |1 + \psi(it_1, it_2)| \} dt_1 dt_2.$$

Obviously,

$$(140) \quad \text{l.u.b.}_{t_2 > T^{1/k-2}} |\varphi(t_1, t_2) |1 + \psi(it_1, it_2)| = e^{-nQ_k}.$$

On the assumption of non-singularity of $P(x)$ we have, by Lemma 7,

$$(141) \quad \text{l.u.b.}_{|t_2| > T^{1/k-2}} |f(t_1, t_2)| = \text{l.u.b.}_{|t_2| > Q_k \sqrt{n}} \left| p \left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}} \right) \right|^n = \text{l.u.b.}_{|t_2| \geq Q_k} \left| p \left(\frac{t_1}{\sqrt{n}}, t_2 \right) \right|^n = e^{-nQ_k}.$$

Hence

$$\begin{aligned}
 I_2 &\leq Q_k T e^{-nQ_k} \int \int_{|t_1| \leq T, |t_2| > T^{1/l}} \frac{1}{|t_2|^2} \left(\frac{1}{\sqrt{n\epsilon^{1/2k}} + \frac{|t_1|}{n\epsilon^{3/2k}}} \right) dt_1 dt_2 \\
 (142) \qquad &= Q_k \left(\frac{n^{l-1}}{\epsilon^{1/2k}} + \frac{n^{(3/2)(l-1)}}{\epsilon^{3/2k}} \right) e^{-nQ_k}.
 \end{aligned}$$

For I_3 we have $|t_1| > T^{1/l} = Q_k \sqrt[n]{n}$, and so Lemma 7 is applicable to I_3 in the same manner as to I_2 . Using Lemma 9 (i) on the factor $|g(t_1, t_2)|$ we get

$$(143) \qquad I_3 \leq \frac{Q_k n^l e^{-nQ_k}}{\epsilon^{3/2k}}.$$

Combining (135), (138), (139), (142), (143) we obtain

$$\begin{aligned}
 T\delta \left(3 \int_0^{T\delta} \frac{1 - \cos u}{u^2} du - \pi \right) &\leq Q_k \left(n^{l/2} \epsilon + \frac{n^{l/2}}{n^{\frac{1}{4}(k-2)}} + \frac{n^{l/2}}{n^{\frac{1}{4}(k-1)} \epsilon^{3/2k}} \right. \\
 (144) \qquad &\quad \left. + Q_k \left(\frac{n^{l-1}}{\epsilon^{1/2k}} + \frac{n^{3/2(l-1)}}{\epsilon^{3/2k}} + \frac{n^l}{\epsilon^{3/2k}} \right) e^{-nQ_k} \right).
 \end{aligned}$$

Putting $\epsilon = \frac{1}{n^{k(k-1)/(2k+3)}}$ we get, as the last term in (144) is $\leq Q_k$,

$$T\delta \left(3 \int_0^{T\delta} \frac{1 - \cos u}{u^2} du - \pi \right) \leq Q_k + Q_k n^{l/2} \left(\frac{1}{n^{k(k-1)/(2k+3)}} + \frac{1}{n^{\frac{1}{4}(k-2)}} \right).$$

If $4 \leq k \leq 6$, we take $l = k - 2$ and get

$$T\delta \left(3 \int_0^{T\delta} \frac{1 - \cos u}{u^2} du - \pi \right) \leq Q_k + Q_k \left(\frac{1}{n^{(6-k)/(2(2k+3))}} + 1 \right) \leq Q_k.$$

Hence, by the argument following (70),

$$\text{l.u.b.} |G(u) - \Phi(u) - \chi(u)| \leq \frac{Q_k}{T} = \frac{Q_k}{n^{\frac{1}{4}(k-2)}},$$

giving (15). If $k \geq 7$, we take $l = \frac{2k(k-1)}{2k+3}$ and get

$$T\delta \left(3 \int_0^{T\delta} \frac{1 - \cos u}{u^2} du - \pi \right) \leq Q_k + Q_k \left(1 + \frac{1}{n^{(k-6)/(2(k+3))}} \right) \leq Q_k.$$

Hence

$$\text{l.u.b.} |G(u) - \Phi(u) - \chi(u)| \leq \frac{Q_k}{T} = \frac{Q_k}{n^{k(k-1)/2(k+3)}},$$

giving (16). Therefore Theorem 4 is proved.

5. When $\alpha_4 - 1 - \alpha_3^2 = 0$. If $\alpha_4 - 1 - \alpha_3^2 = 0$, then there is unit probability that ξ_i assumes exactly two values:

$$Pr\{\xi_i = a\} = p, \quad Pr\{\xi_i = b\} = q, \quad p + q = 1.$$

Let $\zeta_i = 1$ with probability p and $\zeta_i = 0$ with probability q . Then $\xi_i = b + (a - b)\zeta_i$, $\eta = (a - b)^2 \frac{1}{n} \sum (\zeta_i - \bar{\zeta})^2$. Hence it is sufficient to consider the variable $\frac{1}{n} \sum (\zeta_i - \bar{\zeta})^2 = \eta$. Letting $\sum \zeta_i = r = np + \sqrt{npq} X$ we have $\eta_1 = r - \frac{r^2}{n} = npq + (q - p)\sqrt{npq} X - pqX^2$. We now consider two distinct cases:

Case (i). $p \neq q$. Here

$$F(z) = \Pr \left\{ \frac{\eta_1 - n/pq}{|p - q|\sqrt{npq}} \leq z \right\} \\ = \Pr \{ (X + c\sqrt{n})^2 \geq c^2 n - 2|c|\sqrt{n}z \}, \quad c = \frac{p - q}{2\sqrt{pq}}.$$

Thus $F(z) = 1$ if $z \geq \frac{1}{2}|c|\sqrt{n}$. If $z < \frac{1}{2}|c|\sqrt{n}$, then

$$F(z) = \Pr \{ X \leq -cn - (c^2 n - 2|c|\sqrt{n}z)^{\frac{1}{2}} \} \\ + \Pr \{ X \geq -c\sqrt{n} + (c^2 n - 2|c|\sqrt{n}z)^{\frac{1}{2}} \} = F_1(z) + F_2(z).$$

To the random variable X Theorem 2 can be applied. Suppose that $c < 0$; then, by Tchebycheff's inequality,

$$F_2(z) \leq \Pr \{ X \geq -cn \} \leq \frac{1}{c^2 n} \leq \frac{1}{(p - q)^2 n}.$$

By Theorem 2,

$$F_1(z) = \Pr \{ X \leq -cn - (c^2 n - 2|c|\sqrt{n}z)^{\frac{1}{2}} \} \\ = \Phi(z) + \frac{\Theta z^2}{\sqrt{n}|p - q|} + \frac{\Theta(p^2 + q^2)}{\sqrt{npq}}.$$

Hence

$$(145) \quad |F(z) - \Phi(z)| \leq A \left\{ \frac{p^2 + q^2}{\sqrt{npq}} + \frac{z^2}{\sqrt{n}|p - q|} + \frac{1}{n(p - q)^2} \right\}.$$

The same inequality holds also for $c > 0$.

Case (ii). $p = q = 1/2$. Here $\eta_1 = \frac{1}{4}(n - X^2)$; hence

$$(146) \quad \Pr \left\{ \eta_1 \geq \frac{n - z}{4} \right\} = \Pr \{ X^2 \leq z \} = \frac{1}{\sqrt{2\pi}} \int_0^z x^{-\frac{1}{2}} e^{-x/2} dx + \frac{\Theta}{\sqrt{n}}.$$

There is no asymptotic expansion for the distribution function of η_1 . (See (C), p. 83.)