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THE APPROXIMATION OF FRACTIONAL POWERS OF A CLOSED OPERATOR

BY

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Abstract

This paper provides a unified approach to some perturbation problems recently considered by the author in joint work with S. Kantorovitz. The key result is that the semigroup of unbounded operators $\{J^\alpha\}$ formed by the fractional powers of a closed operator J can be approximated in a canonical way by a certain family of bounded semigroups.

1. Introduction

In [5] a general technique for establishing similarity of certain singular perturbations of unbounded operators was developed. For closed operators M and J acting in a Banach space X, perturbations of the form $M + i\eta J$, $n \in R$, were shown, under suitable conditions, to be similar to M (cf. [5, Theorem 3.3]). The proof in [5] involves embedding J in a semigroup $\{J^\alpha\}$ of unbounded operators which possesses a boundary group (of bounded operators); these boundary values then implement the similarity.

More precisely, similarity results are obtained when $J = J^1$, where $\{J^\alpha\}$ is a *regular* semigroup of unbounded operators; that is, there exists a sequence (or net) of semigroups of bounded operators $\{J_N^\alpha\}_{\alpha \in C^+}$, $N \in Z^+$, such that for each $\alpha \in C^+$,

$$\text{Domain}(J^\alpha) = \{x \in X \mid \lim_{N \to \infty} J_N^\alpha x \text{ exists in } X\};$$

for each $N \in Z^+$, $\{J_N^\alpha\}$ is holomorphic on C^+, of class (C_0) on $(0, \infty)$, and has a boundary group $\{J_N^{i\eta}\}_{\eta \in R}$; and certain other technical conditions are satisfied. Then a boundary group $\{J^{i\eta}\}_{\eta \in R}$ is obtained as the limit, in the strong operator topology, of the groups $\{J_N^{i\eta}\}_{\eta \in R}$ as $N \to \infty$ (cf. [5, Theorem 2.2]).

In order to apply this theory to explicit examples, ad hoc methods were used to establish appropriate approximating semigroups. For example, in the case where M is the operation of multiplication by x, and J is the Volterra operator

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acting in $L^p(0, \infty)$, $1 < p < \infty$ (with maximal domains), $\{J^\alpha\}$ is the Riemann-Liouville semigroup

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t)\, dt;$$

the restrictions of that semigroup to $L^p(0, N)$, $N \in Z^+$, provide the approximating semigroups $\{J_N^\alpha\}$, where $J_N^\alpha = P_N J^\alpha$, and $P_N f(x) = \chi_{[0,N]}(x) f(x)$. The boundary group $\{J^{i\eta}\}$ is then the strong limit, as $N \to \infty$, of the boundary groups $\{J_N^{i\eta}\}$.

On the other hand, when $Jf(x) = \int_x^\infty f(t)\, dt$ in $L^p(0, \infty)$, the holomorphic semigroups $\{W_\varepsilon^\alpha\}_{\alpha \in C^+}$, $\varepsilon > 0$, where

$$W_\varepsilon^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty e^{\varepsilon(x-t)} (t - x)^{\alpha-1} f(t)\, dt,$$

were used to approximate the Weyl fractional integrals

$$J^\alpha f(x) = \int_x^\infty (t - x)^{\alpha-1} f(t) \frac{dt}{\Gamma(\alpha)}.$$

Thus the boundary group in this case is obtained as the limit, in the strong operator topology, of the boundary groups $\{W_\varepsilon^{i\eta}\}_{\eta \in R}$.

In [2] Fisher shows that the boundary group for the Riemann-Liouville semigroup acting in $L^p(0, \infty)$ is also the strong limit of the boundary groups of the semigroups $\{R_\varepsilon^\alpha\}$, as $\varepsilon \to 0^+$, where

$$R_\varepsilon^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x e^{\varepsilon(t-x)} (x - t)^{\alpha-1} f(t)\, dt.$$

This result, and the key observation that $R_\varepsilon^1 = R(\varepsilon; -D)$, where $D = J^{-1}$, suggest a unified approach to the approximation problem. It is the purpose of this paper to show that in a suitable general setting there is a canonical choice for $\{J^\alpha\}$ and the approximating semigroups: for certain closed operators J (which are one-to-one with inverse D), we take J^α to be the αth power of J as defined by Balakrishnan [1], and R_ε^α to be the abstract Bessel potential $R(\varepsilon; -D)^\alpha$ (the αth power of R_ε^1). We then have (in Section 2):

THEOREM A. *Let* $R_\varepsilon = R(\varepsilon; -D)$ *for* $\varepsilon > 0$. *Then* $\{R_\varepsilon^\alpha\}_{\alpha \in C^+}$, $\varepsilon > 0$, *is an approximating family of semigroups for* $\{J^\alpha\}_{\alpha \in C^+}$.

We point out that Theorem A also follows from a result of Hirsch (cf. [3, Theorem 10]); our proof uses different techniques and is elementary in that it involves only properties of fractional powers.

Under suitable conditions we can apply Theorem 2.2 in [5] to obtain the boundary group $\{J^{i\eta}\}$; in the setting of Theorem B, the hypotheses of [5, Theorem 2.2] may be weakened.

THEOREM B. *Let* $-D$ *be the infinitesimal generator of a semigroup* $\{T_t\}_{t>0}$ *of uniformly bounded operators. If for each* $\varepsilon > 0$, $\{R_\varepsilon^\alpha\}_{\alpha \in C^+}$ *has a boundary group* $\{R_\varepsilon^{i\eta}\}_{\eta \in R}$, *and* $\|R_\varepsilon^{i\eta}\| \leq Me^{\nu|\eta|}$, *where* M *and* ν *are independent of* $\varepsilon > 0$, *then there exists a strongly continuous group* $\{J^{i\eta}\}_{\eta \in R}$ *of bounded operators satisfying* (i)–(iv) *of* [5, *Theorem* 2.2]. *In addition,* $J^\alpha J^{i\eta} = J^{i\eta} J^\alpha$.

Proofs of the existence and uniform boundedness of the approximating boundary groups for the cases discussed above may be found in [5, Theorem 4.2] and [2, Corollary 3.4]; in the latter we see that Muckenhoupt's singular integrals provide a useful tool for verifying the hypotheses of Theorem B in explicit examples. The proof in [5] employs different techniques. Section 2 closes with a brief discussion of the infinitesimal generator of the boundary group $\{J^{i\eta}\}_{\eta \in R}$.

In Section 3 we discuss perturbations of the form $M + \alpha J$, where M is a certain closed operator and J satisfies the hypotheses of Theorem B. We have the following:

THEOREM C. $M + \alpha J$ *and* $M + \beta J$ *are similar if* $\alpha, \beta \in C\backslash\{0\}$ *and* $\operatorname{Re} \alpha = \operatorname{Re} \beta$; *the similarity is implemented by* $J^{i\,\mathrm{m}(\alpha-\beta)}$. *If* $D(M) \subset D(J)$, *then* M *and* $M + i\eta J$ *are similar for* $\eta \in R$.

Perturbations of Heisenberg-Volterra type (cf. Kantorovitz [6]) also arise rather naturally in this setting; a preliminary result is discussed in Theorem D.

Throughout this paper, X will denote a Banach space, $D(J)$ the domain of the operator J, and $R(\varepsilon; J)$ its resolvent. Theorems 2.2, 3.3 and 3.4 of [5] are required, but their contents will be made clear in the present discussion.

2. The approximating semigroups

Let J be a closed, densely-defined linear operator in X, with dense range. Suppose that $R^+ \subset \rho(-J)$, the resolvent set of $-J$, and that the resolvent of $-J$ satisfies

$$\|\lambda R(\lambda; -J)\| \leq M \quad \text{for } \lambda > 0. \tag{1}$$

Since J is closed and satisfies (1), we may embed J in a one-parameter family of closed operators $\{J^\alpha\}_{\alpha \in C^+}$, where J^α is defined by Balakrishnan's fractional powers of closed operators (cf. [1]): for $\alpha \in C$ with $0 < \operatorname{Re} \alpha < 1$ and $x \in D(J)$,

$$J^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} R(\lambda; -J) Jx \, d\lambda; \tag{2}$$

for $n - 1 < \operatorname{Re} \alpha < n$ and $x \in D(J^n)$, $J^\alpha x = J^{\alpha-n+1}(J)^{n-1}x$, and for $n - 1 < \operatorname{Re} \alpha \leq n$ and $x \in D(J^{n+1})$, $J^\alpha x = J^{\alpha-n+1}(J)^{n-1}x$. Then the operators J^α are closable, and we will also denote their closures by J^α. Moreover, in light of our hypotheses on J,

$$D(J^\infty) = \bigcap_{n \in Z^+} D(J^n) \subset D$$

$$= \left\{ x \in \bigcap_{\alpha,\beta > 0} D(J^\alpha J^\beta) \;\middle|\; \begin{array}{l} J^\alpha J^\beta x = J^{\alpha+\beta} x \\ J^\alpha x \text{ strongly continuous for } \alpha > 0 \\ J^\alpha x \to x \text{ as } \alpha \to 0+ \end{array} \right\}$$

by Lemmas 2.2, 2.4 and 2.5 in [1]. Thus $\{J^\alpha\}_{\alpha>0}$ is a semigroup of closed operators in the sense of the definition in [4], since $\overline{D(J^\infty)} = X$ (by [1, Lemma 3.1]).

We now consider the family $\{J^\alpha\}_{\alpha \in C^+}$, in order to determine a canonical approximating family of semigroups. By (1) and the fact that Ran (J) is dense, we have that for all $x \in X$, $\lim_{\lambda \to 0+} \lambda R(\lambda; -J)x = 0$, and also that J is one-to-one. Let D denote the inverse of J; of course, D is closed, densely-defined with dense range, $R^+ \subset \rho(-D)$ and (1) holds with J replaced by D. Moreover, if for $\varepsilon > 0$, $R_\varepsilon = R(\varepsilon; -D)$, then $R^+ \subset \rho(-R_\varepsilon)$ and, for $\lambda > 0$,

$$R(\lambda; -R_\varepsilon) = \frac{1}{\lambda} - \frac{1}{\lambda^2} R\left(\frac{1 + \lambda\varepsilon}{\lambda}; -D\right).$$

Therefore R_ε satisfies (1), so we may define, for $\alpha \in C^+$, $R_\varepsilon^\alpha = R(\varepsilon; -D)^\alpha$, again using Balakrishnan's definition. Now R_ε^α is bounded, and by the above-mentioned lemmas in [1], $\{R_\varepsilon^\alpha\}_{\alpha \in C^+}$ is a holomorphic semigroup of class (C_0) on $(0, \infty)$. In fact, since

$$R(\lambda; -R_\varepsilon)R_\varepsilon x = \frac{1}{\lambda} R\left(\frac{1 + \lambda\varepsilon}{\lambda}; -D\right) x \quad \text{for } \lambda > 0,\ x \in X,$$

the change of variables $\mu = (1 + \lambda\varepsilon)/\lambda$ yields

$$R_\varepsilon^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_\varepsilon^\infty (\mu - \varepsilon)^{-\alpha} R(\mu; -D)x \, d\mu, \quad 0 < \operatorname{Re} \alpha < 1 \tag{3}$$

In order to prove that $\{R_\varepsilon^\alpha\}_{\alpha \in C^+}$ is an approximating family of semigroups (as $\varepsilon \to 0^+$) for $\{J^\alpha\}_{\alpha \in C^+}$, we shall need the following three lemmas.

LEMMA 1. *For each $\alpha \in C^+$, $D(J^\infty)$ is a core for J^α; that is, $J^\alpha = \overline{J^\alpha | D(J^\infty)}$.*

Proof. We may assume that $0 < \operatorname{Re} \alpha < 1$. Since $J^\alpha = \overline{J^\alpha | D(J)}$, we must show that

$$J^\alpha | D(J) \subset \overline{J^\alpha | D(J^\infty)}.$$

Let $x \in D(J)$; then $x = R(\lambda; -J)y$ for some $y \in X$ and $\lambda > 0$. Since $\overline{D(J^\infty)} = X$, $y = \lim_{n \to \infty} y_n$, where $\{y_n\} \subset D(J^\infty)$. Thus

$$J^\alpha x = J^\alpha R(\lambda; -J)y = \lim_{n \to \infty} J^\alpha R(\lambda; -J)y_n,$$

since $J^\alpha R(\lambda; -J)$ is bounded. But $\{R(\lambda; -J)y_n\} \subset D(J^\infty)$, and $R(\lambda; -J)y_n \to x$ as $n \to \infty$. Therefore $x \in D(\overline{J^\alpha | D(J^\infty)})$, and $J^\alpha x = \overline{J^\alpha | D(J^\infty)}x$.

Next we note that if $\lambda > 0$, then $\lambda \in \rho(-(I - \varepsilon R_\varepsilon))$, since

$$(4)\quad R(\lambda; -(I - \varepsilon R_\varepsilon)) = \frac{1}{\varepsilon} R\left(\frac{\lambda + 1}{\varepsilon}; -R_\varepsilon\right) = \frac{1}{\lambda + 1} + \frac{\varepsilon}{(\lambda + 1)^2} R\left(\frac{\lambda\varepsilon}{\lambda + 1}; -D\right).$$

Therefore $I - \varepsilon R_\varepsilon$ satisfies (1), because D does, and so we may define $(I - \varepsilon R_\varepsilon)^\alpha$ using Balakrishnan's definition. We now prove:

LEMMA 2. *Let $\alpha \in C^+$, $\varepsilon > 0$ and $x \in D(J^\alpha)$. Then*

$$(5)\qquad R_\varepsilon^\alpha x = (I - \varepsilon R_\varepsilon)^\alpha J^\alpha x.$$

Proof. Fix $x \in D(J^\infty)$, and suppose $0 < \operatorname{Re} \alpha < 1$. Then

$$J^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \mu^{-\alpha} R(\mu; -D)x \, d\mu;$$

using (4) and the first resolvent equation, we have

$$(I - \varepsilon R_\varepsilon)^\alpha J^\alpha x = \left(\frac{\sin \pi\alpha}{\pi}\right)^2 \int_0^\infty \lambda^{\alpha - 1} R(\lambda; -(I - \varepsilon R_\varepsilon)) \times (I - \varepsilon R_\varepsilon) \int_0^\infty \mu^{-\alpha} R(\mu; -D)x \, d\mu \, d\lambda$$

$$(6)\quad = J^\alpha x - \varepsilon \left(\frac{\sin \pi\alpha}{\pi}\right)^2 \int_0^\infty \frac{\lambda^\alpha}{(\lambda + 1)^2} R\left(\frac{\lambda\varepsilon}{\lambda + 1}; -D\right) \int_0^\infty \mu^{-\alpha} R(\mu; -D)x \, d\mu \, d\lambda$$

$$= J^\alpha x - \varepsilon \left(\frac{\sin \pi\alpha}{\pi}\right)^2 \int_0^\infty \frac{\lambda^\alpha}{\lambda + 1} \int_0^\infty \frac{\mu^{-\alpha}}{\mu(\lambda + 1) - \lambda\varepsilon} \times \left[R\left(\frac{\lambda\varepsilon}{\lambda + 1}; -D\right) - R(\mu; -D)\right] x \, d\mu \, d\lambda.$$

We now use an argument similar to that in [8, Proposition 4.9], to which we refer for notation. Let $\varepsilon > 0$, $R > 0$ and ϕ be an angle such that $\tan \phi < 1/M$, where M is the constant in (1). Let C be the closed contour formed by the straight lines from 0 to R, R to $R + i\varepsilon$, $R + i\varepsilon$ to $(\varepsilon/\tan \phi) + i\varepsilon$, and $(\varepsilon/\tan \phi) + i\varepsilon$ to 0. Since $\rho(-D)$ contains the sector $|\arg \lambda| < \operatorname{Tan}^{-1}(1/M)$ (cf. [1, Lemma 6.1]), the integral in (6) with respect to λ, taken around C, is zero. Using the fact that

$$\sup_{|\arg \lambda| = \theta} \|\lambda R(\lambda; -D)\| < \infty \quad \text{for } |\theta| < \operatorname{Tan}^{-1}(1/M)$$

[1, Lemma 6.1], it follows that the integral with respect to λ in (6) is the limit, as $\varepsilon \to 0$, of integrals in which the path for λ is a line parallel to and height ε above the x-axis, from $\operatorname{Re} \lambda = \varepsilon/\tan \phi$ to ∞. Similarly, the integral with respect to μ is the limit of integrals along paths parallel to and slightly below the x-axis. Calculating residues, we have

$$\begin{aligned}(I - \varepsilon R_\varepsilon)^\alpha J^\alpha x &= J^\alpha x - \varepsilon \left(\frac{\sin \pi\alpha}{\pi}\right) \int_0^\infty \frac{\lambda^\alpha}{\lambda + 1} \left(\frac{(-\lambda - i0)\varepsilon}{\lambda + i0 + 1}\right)^{-\alpha} R\left(\frac{\lambda\varepsilon}{\lambda + 1}; -D\right) x \, d\lambda \\ &+ \frac{\sin \pi\alpha}{\pi} \int_0^\infty \mu^{-\alpha} \left[-1 + \left(\frac{\mu - i0}{\mu - i0 - \varepsilon}\right)^\alpha\right] R(\mu; -D)x \, d\mu \\ &= -\varepsilon \left(\frac{\sin \pi\alpha}{\pi}\right) e^{\pi i\alpha} \int_0^\infty \frac{\lambda^\alpha}{\lambda + 1} \left(\frac{\lambda\varepsilon}{\lambda + 1}\right)^{-\alpha} R\left(\frac{\lambda\varepsilon}{\lambda + 1}; -D\right) x \, d\lambda \\ &+ \left(\frac{\sin \pi\alpha}{\pi}\right) e^{\pi i\alpha} \int_0^\varepsilon (\varepsilon - \mu)^{-\alpha} R(\mu; -D)x \, d\mu \\ &+ \frac{\sin \pi\alpha}{\pi} \int_\varepsilon^\infty (\mu - \varepsilon)^{-\alpha} R(\mu; -D)x \, d\mu.\end{aligned}$$

We make the change of variables $\mu = \lambda\varepsilon/(\lambda + 1)$ in the first integral on the right-hand side, and obtain

$$(I - \varepsilon R_\varepsilon)^\alpha J^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_\varepsilon^\infty (\mu - \varepsilon)^{-\alpha} R(\mu; -D)x \, d\mu = R_\varepsilon^\alpha x,$$

by (3).

We have shown that (5) holds for all $x \in D(J^\infty)$, $\alpha \in C^+$ with $0 < \operatorname{Re} \alpha < 1$. Since both sides of (5) are holomorphic functions of $\alpha \in C^+$ for $x \in D(J^\infty)$, the equality holds for all $\alpha \in C^+$ and $x \in D(J^\infty)$. Now if $x \in D(J^\alpha)$ for $\alpha \in C^+$, we use Lemma 1 and the fact that R_ε^α and $(I - \varepsilon R_\varepsilon)^\alpha$ are bounded operators to obtain the desired result.

LEMMA 3. *For each* $\alpha \in C^+$ *and* $\varepsilon > 0$, $R_\varepsilon^\alpha = J^\alpha(I - \varepsilon R_\varepsilon)^\alpha \supset (I - \varepsilon R_\varepsilon)^\alpha J^\alpha$.

Proof. Let $x \in D(J^\infty)$, and suppose that $0 < \operatorname{Re} \alpha < 1$. Then

$$\begin{aligned}(7)\quad (I - \varepsilon R_\varepsilon)^\alpha x &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha - 1} R(\lambda; -(I - \varepsilon R_\varepsilon))(I - \varepsilon R_\varepsilon)x \, d\lambda \\ &= x - \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\lambda^{\alpha - 1}}{\lambda + 1} R\left(\frac{\lambda + 1}{\lambda\varepsilon}; -J\right) Jx \, d\lambda,\end{aligned}$$

by the first resolvent equation. The right-hand side of (7) certainly belongs to $D(J^\infty)$, and

$$(8)\qquad J^\alpha(I - \varepsilon R_\varepsilon)^\alpha x = (I - \varepsilon R_\varepsilon)^\alpha J^\alpha x,$$

because J^α commutes with the integrand in (7). Since, for $x \in D(J^\infty)$, both sides of (8) are holomorphic functions of $\alpha \in C^+$, (8) holds for all $\alpha \in C^+$.

Now let $x \in D(J^\alpha)$, for $\alpha \in C^+$. By Lemma 1, there exists a sequence $\{x_n\} \subset D(J^\infty)$ such that $x_n \to x$, and $J^\alpha x_n \to J^\alpha x$. Thus $(I - \varepsilon R_\varepsilon)^\alpha x_n \to (I - \varepsilon R_\varepsilon)^\alpha x$, $\{(I - \varepsilon R_\varepsilon)^\alpha x_n\} \subset D(J^\infty)$, and

$$J^\alpha(I - \varepsilon R_\varepsilon)^\alpha x_n = (I - \varepsilon R_\varepsilon)^\alpha J^\alpha x_n \to (I - \varepsilon R_\varepsilon)^\alpha J^\alpha x.$$

Because J^α is closed, the inclusion in the statement of the lemma follows. That $R_\varepsilon^\alpha = J^\alpha(I - \varepsilon R_\varepsilon)^\alpha$ follows immediately from Lemma 2, since $D(J^\alpha)$ is dense and both operators are bounded.

We now have:

THEOREM A. *For each $\alpha \in C^+$,*

$$\text{Domain}(J^\alpha) = \{x \in X \mid \lim_{\varepsilon \to 0+} R_\varepsilon^\alpha x \textit{ exists in } X\};$$

for $x \in D(J^\alpha)$, $J^\alpha x = \lim_{\varepsilon \to 0+} R_\varepsilon^\alpha x$.

Proof. Fix $\alpha \in C^+$, and let $x \in D(J^\alpha)$; then $(I - \varepsilon R_\varepsilon)^\alpha x \to x$ as $\varepsilon \to 0^+$, for all $x \in X$. Indeed, $\varepsilon R_\varepsilon \to 0$ as $\varepsilon \to 0^+$ in the strong operator topology, so we see from (4) that

$$R(\lambda;\ -(I - \varepsilon R_\varepsilon))x \to \frac{x}{\lambda + 1} \text{ as } \varepsilon \to 0^+ \quad \text{for all } x \in X.$$

Moreover, $\|I - \varepsilon R_\varepsilon\| \le K$, where K is a constant which does not depend on ε. Therefore, for each $x \in X$,

$$R(\lambda;\ -(I - \varepsilon R_\varepsilon))(I - \varepsilon R_\varepsilon)x \to \frac{1}{\lambda + 1} x \quad \text{as } \varepsilon \to 0^+.$$

In addition, the integrand in the first integral in (7) is $O(\lambda^{\operatorname{Re}\alpha - 1})$ as $\lambda \to 0^+$, and is $O(\lambda^{\operatorname{Re}\alpha - 2})$ as $\lambda \to \infty$. Therefore it follows from the dominated convergence theorem that

$$(I - \varepsilon R_\varepsilon)^\alpha x \to \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\lambda^{\alpha - 1}}{\lambda + 1} x \, d\lambda = x,$$

as claimed.

It now follows immediately from (5) that $J^\alpha x = \lim_{\varepsilon \to 0+} R_\varepsilon^\alpha x$. On the other hand, suppose that $\lim_{\varepsilon \to 0+} R_\varepsilon^\alpha x$ exists. Then by Lemma 3, $\lim_{\varepsilon \to 0+} J^\alpha(I - \varepsilon R_\varepsilon)^\alpha x$ exists. Since $(I - \varepsilon R_\varepsilon)^\alpha x \to x$, and $(I - \varepsilon R_\varepsilon)^\alpha x \in D(J^\alpha)$ by Lemma 3, we have that $x \in D(J^\alpha)$, and $J^\alpha x = \lim_{\varepsilon \to 0+} R_\varepsilon^\alpha x$, since J^α is closed.

In special cases we can now apply Theorem 2.2 in [5] to obtain the boundary group $\{J^{i\eta}\}_{\eta \in R}$. Note that if $-D$ is the infinitesimal generator of a strongly continuous semigroup $\{T_t\}_{t>0}$ such that $\|T_t\| \le L$ for all $t > 0$, then we have the representation

$$(9) \qquad R_\varepsilon^\alpha x = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\varepsilon t} t^{\alpha-1} T_t x\, dt, \quad x \in X, \alpha \in C^+,$$

which may be obtained by a direct computation (cf. [2, p. 426]); the regularity requirement of Theorem 2.2 in [5] may now be weakened.

THEOREM B. *Let* $-D$ *be the infinitesimal generator of a strongly continuous semigroup* $\{T_t\}_{t>0}$ *of uniformly bounded operators, and let* $\{R_\varepsilon^\alpha\}_{\alpha \in C^+}$, $\varepsilon > 0$, *be the approximating family of semigroups obtained in Theorem A. If for each* $\varepsilon > 0$ $\{R_\varepsilon^\alpha\}_{\alpha \in C^+}$ *has a boundary group* $\{R_\varepsilon^{i\eta}\}_{\eta \in R}$, *and* $\|R_\varepsilon^{i\eta}\| \le Me^{\nu|\eta|}$, *where* M *and* ν *are constants independent of* $\varepsilon > 0$, *then there exists a strongly continuous group of bounded linear operators* $\{J^{i\eta}\}_{\eta \in R}$ *on* X *such that:*

(i) $J^{i\eta}x = \lim_{\varepsilon \to 0^+} R_\varepsilon^{i\eta}x$, $x \in X$, $\eta \in R$;
(ii) $\|J^{i\eta}\| \le Me^{\nu|\eta|}$;
(iii) $J^{i\eta}J^\alpha = J^\alpha J^{i\eta} = J^{\alpha+i\eta}$, $\alpha > 0$, $\eta \in R$, *as operators in* X*; and*
(iv) *if* $x \in D(J^\infty)$, *then* $J^{i\eta}x = \lim_{\xi \to 0^+} J^{\xi+i\eta}x$.

Proof. First we show that $\{J^\alpha\}_{\alpha \in C^+}$ is a regular semigroup in the sense of Definition 2.1 in [5]. Since each of the approximating semigroups $\{R_\varepsilon^\alpha\}_{\alpha \in C^+}$ has boundary values on the imaginary axis, we need only check that the following holds: if $\gamma_\varepsilon(s)$ is the Nörlund function of $\{R_\varepsilon^\alpha\}_{\alpha \in C^+}$, and $(\alpha_{0,\varepsilon}, \alpha_{1,\varepsilon})$ is the largest interval such that the equation $\gamma_\varepsilon(s) = \pi/2\alpha$ has a unique solution $s_\varepsilon = s_{0,\varepsilon}(\alpha)$ when $0 \le \alpha_{0,\varepsilon} < \alpha < \alpha_{1,\varepsilon} \le \infty$, then $\alpha_{1,\varepsilon} > 1$. However, it follows from Stirling's formula that $\gamma_\varepsilon(s) \le \pi/2$ for each $\varepsilon > 0$. Therefore $\alpha_{1,\varepsilon} > 1$.

Next, we observe that the set

$$\tilde{D} = \left\{ x \in \bigcap_{\alpha,\beta \in C^+} D(J^\alpha J^\beta) \;\middle|\; \begin{array}{c} J^\alpha J^\beta x = J^{\alpha+\beta}x,\ \alpha, \beta > 0 \\ J^\alpha x \text{ strongly continuous}, \alpha > 0 \\ J^\alpha x \to x \text{ as } \alpha \to 0^+ \end{array} \right\}$$

contains $D(J^\infty)$, so $\tilde{D}$ is dense in X. Therefore, by Theorem 2.2 in [5], (i), (ii), (iv) and the fact that $J^{i\eta}J^\alpha = J^{\alpha+i\eta}$ hold. To complete the proof, we show that for each $\varepsilon > 0$, and $\alpha, \zeta \in C^+$,

$$(10) \qquad R_\varepsilon^\zeta J^\alpha \subset J^\alpha R_\varepsilon^\zeta;$$

in fact, equality holds in (10) if either α or ζ is purely imaginary. Now if $\alpha, \zeta \in C^+$, and $x \in D(J^\infty)$, it is easy to see that $R_\varepsilon^\zeta x \in D(J^\infty)$ and $J^\alpha R_\varepsilon^\zeta x = R_\varepsilon^\zeta J^\alpha x$. Using Lemma 1, we obtain (10) in the usual manner.

If Re $\alpha = 0$ (say $\alpha = i\eta$), then for $x \in D(J^\infty)$, $J^{i\eta}x = \lim_{\xi \to 0^+} J^{\xi+i\eta}x$, so that

$$R_\varepsilon^\zeta J^{i\eta}x = \lim_{\xi \to 0^+} R_\varepsilon^\zeta J^{\xi+i\eta}x = \lim_{\xi \to 0^+} J^{\xi+i\eta}R_\varepsilon^\zeta x = J^{i\eta}R_\varepsilon^\zeta x \quad \text{for } \zeta \in C^+,$$

by (10). Since both sides of (10) are bounded, and $D(J^\infty)$ is dense, equality holds in (10) if Re $\alpha = 0$, $\zeta \in \overline{C^+}$. A similar argument yields equality in (10) for Re $\zeta = 0, \alpha \in C^+$. Finally, we use (i) to obtain $J^{i\eta}J^\alpha = J^\alpha J^{i\eta}$ for $\alpha > 0, \eta \in R$. We omit the easy details.

In particular, if J is the Volterra operator acting in $L^p(0, \infty)$, $1 < p < \infty$, with maximal domain, then for $f \in L^p(0, \infty)$,

$$R_\varepsilon^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x e^{\varepsilon(t-x)}(x-t)^{\alpha-1} f(t)\, dt, \quad x \in (0, \infty).$$

Since we showed in [5, Theorem 4.2] that $\{R_\varepsilon^\alpha\}_{\alpha \in C^+}$ satisfies the requirements of Theorem B, we obtain as a special case the result of Fisher discussed in the introduction. (We emphasize that Balakrishnan's definition of fractional power gives precisely the Riemann-Liouville fractional integral

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt,$$

with maximal domain in $L^p(0, \infty)$. Indeed, if $f \in D(J)$ and $0 < \text{Re}\ \alpha < 1$, it is easy to see that the two coincide. Since each operator is equal to the closure of its restriction to $D(J)$, the rest follows by a familiar argument.)

In the event that $-D$ generates a semigroup as described above, we can obtain information about the infinitesimal generator of the group $\{J^{i\eta}\}_{\eta \in R}$.

COROLLARY. *Let A denote the infinitesimal generator of $\{J^{i\eta}\}_{\eta \in R}$. Then A is the limit, in the strong generalized sense* (cf. [7, VIII, Section 1]), *of the operators A_ε, where A_ε is the infinitesimal generator of $\{R_\varepsilon^{i\eta}\}_{\eta \in R}$. Moreover, $x \in D(A_\varepsilon)$ if and only if*

$$x^* = \int_0^\infty e^{-\varepsilon t} \log t T_t x\, dt$$

is in Domain(D), *and for $x \in D(A_\varepsilon)$,*

$$A_\varepsilon x = -i[Cx + (\varepsilon I + D)x^*],$$

where C is Euler's constant.

Proof. The statement regarding the A_ε's is proved in [3, Theorem 4 and Corollary 3.7]. That A is the strong generalized limit of the A_ε's follows from Theorem IX.2.16 in [7], because of (i) and (ii) in Theorem A.

3. Similarity

Now suppose that M is a closed linear operator acting in X with Domain $D(M)$, that J satisfies the hypotheses of Theorem B, and that for each $\varepsilon > 0$, the following hold:

(i) A is a non-zero bounded operator on X which commutes with M and R_ε;

(ii) $R_\varepsilon^{s+it}D(M) \subset D(M)$ for $s + it$ in some rectangle $0 \leq s \leq a$, $|t| \leq a$, where a is a constant which may depend on ε;

(iii) R_ε is M-Volterra with respect to A; that is, $R_\varepsilon D(M) \subset D(M)$ and $[R_\varepsilon, M] \subset AR_\varepsilon^2$.

If we define $T_\alpha = M + \alpha AJ$ for $\alpha \in C$, with Domain $D(T_\alpha) = D(M) \cap D(J)$, then an application of Theorems 3.3 and 3.4 in [5] gives:

THEOREM C. *If $\alpha, \beta \in C\backslash\{0\}$, and* Re α = Re β, *then T_α is similar to T_β, with $J^{i\,\mathrm{m}(\alpha-\beta)}$ implementing the similarity. Also, if $D(M) \subset D(J)$, then M and $T_{i\eta}$ are similar, for $\eta \in R$.*

We point out that in general the perturbations under consideration are not Kato perturbations (cf. [9, p. 190]). For example, if $Mf(x) = xf(x)$ and $Jf(x) = \int_x^\infty f(t)\,dt$ as discussed in the introduction, then $D(M) \subset D(J)$ (cf. [5, Lemma 4.9]). However, if $f \in D(M)$, then $\|Jf\|_p \leq |\eta|p\|Mf\|_p$, since $Jf = W_{1,0}^{-1}(Mf)$, where

$$W_{\alpha,\mu}^{(\nu)} f(x) = \frac{1}{\Gamma(\alpha)} x^{\mu-\nu-\alpha} \int_x^\infty t^\nu (t-x)^{\alpha-1} f(t)\,dt;$$

by Theorem 4.5.11 in [9], $\|W_{1,0}^{-1}\|_p \leq p$. Thus $M + i\eta J$ is a Kato perturbation if $|\eta| < 1/p$.

In [5] we restricted our attention to operators which satisfy the commutation relation in (iii) above. However, it is natural to assume that the operators M and R_ε satisfy a Heisenberg-Volterra commutation relation as discussed in [6]. That is, let M be a closed operator, and suppose that for each $\varepsilon > 0$,

$$R_\varepsilon D(M) \subset D(M) \quad \text{and} \quad [R_\varepsilon, M] \subset C_\varepsilon, \tag{11}$$

where C_ε is a bounded operator which commutes with R_ε. Then by the theorem in [6], $M + g'(R_\varepsilon)C_\varepsilon$ is similar to M, where g is any function holomorphic in a neighborhood of $\sigma(R_\varepsilon)$, and the similarity is implemented by $e^{g(R_\varepsilon)}$.

Now assume that $\lim_{\varepsilon\to0^+} C_\varepsilon x = Cx$ exists for all $x \in X$, and that $-D$ generates a strongly continuous semigroup $\{T_t\}_{t>0}$ such that $\|T_t\| \leq L$. Then for $\varepsilon > 0$, $\varepsilon \in \rho(-J)$, and $x \in$ Domain(D) if and only if $\lim_{\varepsilon\to0^+} R(\varepsilon; -J)x$ exists in X. Indeed, the operator D has all the properties required of J, so Theorem A holds with the roles of J and D reversed. If we set $S_\varepsilon = R(\varepsilon; -J)$, and assume that (11) holds with R_ε replaced by S_ε, we obtain from the theorem in [6] that

$$Me^{-tS_\varepsilon}x = e^{-tS_\varepsilon}(M - tC_\varepsilon)x \quad \text{for } x \in D(M). \tag{12}$$

But for $t > 0$, $e^{-tS_\varepsilon}x \to T_t x$, for all $x \in X$, by a classical result. If we let $\varepsilon \to 0^+$ in (12), we obtain $T_t x \in D(M)$, for $\|e^{-tS_\varepsilon}\| \leq K$ uniformly in $\varepsilon > 0$, and M is closed; also, $MT_t x = T_t(M - tC)x$. We have proved the following:

THEOREM D. *Let $-D$ be the infinitesimal generator of a strongly continuous semigroup $\{T_t\}_{t>0}$ of uniformly bounded operators, and let $S_\varepsilon = R(\varepsilon; -J)$, where $J = D^{-1}$ and $\varepsilon > 0$. Suppose that $S_\varepsilon D(M) \subset D(M)$ and $[S_\varepsilon, M] \subset C_\varepsilon$, where C_ε is bounded for each $\varepsilon > 0$, and that $\lim_{\varepsilon \to 0+} C_\varepsilon x = Cx$ for all $x \in X$. Then for each $t > 0$, $T_t(M - tC) \subset MT_t$.*

Remark. If D satisfies the hypotheses of Theorem D, and $R_\varepsilon = R(\varepsilon; -D)$ is M-Volterra with respect to A, then it is easy to see that $S_\varepsilon = R(\varepsilon; -J)$ satisfies the following:

$$S_\varepsilon D(M) \subset D(M) \text{ and } [S_\varepsilon, M] \subset -A(JS_\varepsilon)^2 \quad \text{for each } \varepsilon > 0.$$

Therefore, if we take $C_\varepsilon = JS_\varepsilon$ in Theorem D, we obtain $T_t(M + tA) \subset MT_t$, $t > 0$.

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