THE APPROXIMATION OF SOLUTIONS TO INITIAL BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS IN ONE SPACE VARIABLE*

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1. Introduction. In a recent paper [3] the author has constructed integral operators which map solutions of the heat equation

$$h_{xx} = h_t \tag{1.1}$$

onto solutions of the parabolic equation

$$u_{xx} + q(x, t)u = u_t \tag{1.2}$$

and used these operators to obtain reflection principles for Eq. (1.2) which are analogous to the Schwarz reflection principle for analytic functions of a complex variable. (We note that the more general equation

$$v_{xx} + a(x, t)v_x + b(x, t)v = v_t \tag{1.3}$$

can be reduced to an equation of the form (1.2) by the change of variables

$$v(x, t) = u(x, t) \exp\left\{-\frac{1}{2}\int_0^x a(s, t)\, ds\right\}\Big).$$

In this paper we will show how these operators can be used to obtain approximate solutions to the first initial boundary value problem for Eq. (1.2) (or (1.3)) in a rectangle and quarter plane. More specifically, our approach provides an analogue for Eqs. (1.2) and (1.3) of the method of separation of variables and the "method of images" for the heat equation, and is an extension of the use of integral operator methods for approximating solutions of boundary value problems for elliptic equations (cf. [1, 2, 6, 10]) to the case of initial boundary value problems for parabolic equations. Numerical examples using the methods described in this paper will be published elsewhere.

2. The first initial boundary value problem in a rectangle. Let $u(x, t)$ be a (strong) solution of Eq. (1.2) in the rectangle $R = \{(x, t): -1 < x < 1, O < t < T\}$ such that $u(x, t)$ continuously assumes the initial-boundary data

$$u(-1, t) = f(t), \quad u(1, t) = g(t); \quad 0 \le t < T, \quad u(x, 0) = h(x); \quad -1 \le x \le 1. \tag{2.1}$$

(A strong, or classical, solution of Eq. (1.2) is a solution of Eq. (1.2) which is twice

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continuously differentiable with respect to x, continuously differentiable with respect to t, and satisfies Eq. (1.2) pointwise.) Let $\bar{R}$ denote the closure of R and assume that $q(x, t) \in C^1(\bar{R})$, and that for each fixed x, $-1 \leq x \leq 1$, $q(x, t)$ is an analytic function of t for $|t - \frac{1}{2}T| \leq \frac{1}{2}T$. (This domain of regularity is chosen in order to guarantee the global existence of the integral operators used in this paper—cf. [3].) Our aim is to construct a function $w(x, t)$ which is a solution of Eq. (1.2) in R and approximates $u(x, t)$ arbitrarily closely in the maximum norm on compact subsets of R. This will be accomplished by constructing a complete family of solutions to Eq. (1.2) in the maximum norm and then minimizing the L_2 norm of a finite linear combination of these solutions over the base and vertical sides of R.

We first consider Eq. (1.1). In [9] Rosenbloom and Widder have constructed a set of polynomial solutions to Eq. (1.1) which are defined by

$$h_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k} t^k}{(n-2k)!\, k!} = (-t)^{n/2} H_n\left(\frac{x}{(-4t)^{1/2}}\right), \tag{2.2}$$

where $H_n(z)$ denotes the Hermite polynomials. In [12] Widder showed that the set $\{h_n(x, t)\}$ was complete in the space of solutions to Eq. (1.1) which are analytic in a neighborhood of the origin, i.e. if $h(x, t)$ is a solution of Eq. (1.1) which is analytic for $|x| \leq x_0$, $|t| \leq t_0$ (where x and t are complex variables) then on the rectangle $-x_0 \leq x \leq x_0$, $-t_0 \leq t \leq t_0$, $h(x, t)$ can be approximated in the maximum norm by a finite linear combination of members of the set $\{h_n(x, t)\}$. The lemma below shows that the set $\{h_n(x, t)\}$ is in fact complete for the space of strong solutions of Eq. (1.1) which are defined in R and continuous in $\bar{R}$.

LEMMA 2.1. Let $h(x, t)$ be a (strong) solution of Eq. (1.1) in R which is continouus in $\bar{R}$. Then, given $\epsilon > 0$, there exist constants $a_1, \cdots, a_N$ such that

$$\max_{(x,t)\in\bar{R}} \left| h(x, t) - \sum_{n=0}^{N} a_n h_n(x, t) \right| < \epsilon.$$

Proof: By the Weierstrass approximation theorem and the maximum principle for the heat equation [7], there exists a solution $w_1(x, t)$ of Eq. (1.1) in R which assumes polynomial initial and boundary data such that

$$\max_{(x,t)\in\bar{R}} |h(x, t) - w_1(x, t)| < \epsilon/3. \tag{2.3}$$

Let

$$w_1(-1, t) = \sum_{m=0}^{M} b_m t^m, \qquad w_1(1, t) = \sum_{m=0}^{M} c_m t^m,$$

and look for a solution of Eq. (1.1) in the form

$$v(x, t) = \sum_{m=0}^{M} v_m(x) t^m \tag{2.4}$$

where $v(-1, t) = w_1(-1, t)$, $v(1, t) = w_1(1, t)$. Substituting Eq. (2.4) into Eq. (1.1) leads to the following recursion scheme for the $v_m(x)$:

$$\begin{aligned} v_M'' &= 0; & v_M(-1) &= b_M, & v_M(1) &= c_M, \\ v_{M-1}'' &= Mv_M; & v_{M-1}(-1) &= b_{M-1}, & v_{M-1}(1) &= c_{M-1}, \\ &\vdots & & & &\vdots \\ v_0'' &= v_1; & v_0(-1) &= b_0, & v_0(1) &= c_0. \end{aligned} \tag{2.5}$$

Eq. (2.5) shows that each $v_m(x)$ is a polynomial in x and is uniquely determined. Now consider $w_2(x, t) = w_1(x, t) - v(x, t)$. By the method of separation of variables it is seen that there exist constants $d_1, \cdots, d_L$ such that

$$\max_{(x,t)\in\bar{R}} \left| w_2(x, t) - \sum_{l=0}^{L} d_l \sin \frac{l\pi}{2}(x+1) \exp\left(-\frac{l^2\pi^2 t}{4}\right)\right| < \frac{\epsilon}{3}. \tag{2.6}$$

Hence there exists a solution $w_3(x, t)$ of Eq. (1.1) which is an entire function of the complex variables x and t such that

$$\max_{(x,t)\in\bar{R}} |h(x, t) - w_3(x, t)| < \frac{2\epsilon}{3}. \tag{2.7}$$

From the previously mentioned results of [12] there exist positive constants $a_1, \cdots, a_n$ such that

$$\max_{(x,t)\in\bar{R}} \left| w_3(x, t) - \sum_{n=0}^{N} a_n h_n(x, t)\right| < \frac{\epsilon}{3}, \tag{2.8}$$

and the proof of the lemma now follows immediately from the triangle inequality.

We now want to construct a complete family of solutions to Eq. (1.2) which is analogous to the family $\{h_n(x, t)\}$ for the heat equation. To accomplish this we make use of the integral operators constructed in [3]. Let $u(x, t) \in C^0(\bar{R})$ be a (strong) solution of Eq. (1.2) in R such that $u(O, t) = 0$. Then from [3] we have that $u(x, t)$ can be represented in the form

$$u(x, t) = h(x, t) + \int_0^x K(s, x, t)h(s, t)\, ds \tag{2.9}$$

where $h(x, t)$ is a solution of Eq. (1.1) in R satisfying $h(0, t) = 0$ and $K(s, x, t)$ is defined by

$$K(s, x, t) = \tfrac{1}{2}[E(s, x, t) - E(-s, x, t)] \tag{2.10}$$

where $\tilde{E}(\xi, \eta, t) = E(\xi - \eta, \xi + \eta, t)$ can be constructed by the recursion scheme

$$\tilde{E}(\xi, \eta, t) = \lim_{n\to\infty} \tilde{E}_n(\xi, \eta, t)$$

$$\tilde{E}_1(\xi, \eta, t) = -\frac{1}{2}\int_0^\xi q(s, t)\, ds + \frac{1}{2}\int_0^\eta q(s, t)\, ds,$$

$$\begin{aligned} \tilde{E}_{n+1}(\xi, \eta, t) = {} & -\frac{1}{2}\int_0^\xi q(s, t)\, ds + \frac{1}{2}\int_0^\eta q(s, t)\, ds, \\ & + \int_0^\eta\int_0^\xi \left(\frac{\partial}{\partial t}\tilde{E}_n(\xi, \eta, t) - q(\xi + \eta, t)\tilde{E}_n(\xi, \eta, t)\right) d\xi\, d\eta; \qquad n \geq 1. \end{aligned} \tag{2.11}$$

The sequence $\{\tilde{E}_n\}$ converges uniformly for $(x, t) \in \bar{R}$, $-1 \le s \le 1$. The convergence of the sequence $\{\tilde{E}_n\}$ is quite rapid and good approximations can be found by terminating the recursion process after several iterations. Error estimates for such an approximating procedure can be found in [3]. If instead of the condition $u(0, t) = 0$ we have that $u(x, t)$ satisfies $u_x(0, t) = 0$, then we can represent $u(x, t)$ in the form

$$u(x, t) = h(x, t) + \int_0^x M(s, x, t)h(s, t)\, ds \tag{2.12}$$

where $h(x, t)$ is a solution of Eq. (1.1) in R satisfying $h_x(0, t) = 0$ and $M(s, x, t)$ is defined by

$$M(s, x, t) = \tfrac{1}{2}[G(s, x, t) + G(-s, x, t)] \tag{2.13}$$

where $\tilde{G}(\xi, \eta, t) = G(\xi - \eta, \xi + \eta, t)$ can be constructed via the recursion scheme

$$\begin{aligned} \tilde{G}(\xi, \eta, t) &= \lim_{n\to\infty} \tilde{G}_n(\xi, \eta, t), \\ \tilde{G}_{n+1}(\xi, \eta, t) &= -\frac{1}{2}\int_0^\xi q(s, t)\, ds - \frac{1}{2}\int_0^\eta q(s, t)\, ds \\ &\quad + \int_0^\eta \int_0^\xi \left(\frac{\partial}{\partial t}\tilde{G}_n(\xi, \eta, t) - q(\xi, \eta, t)\tilde{G}_n(\xi, \eta, t)\right) d\xi\, d\eta; \qquad n \ge 1. \end{aligned} \tag{2.14}$$

The sequence $\{\tilde{G}_n\}$ again converges rapidly and uniformly for $(x, t) \in \bar{R}$, $-1 \le s \le 1$. Observing that for $n \ge 0$, $h_{2n}(x, t)$ is an even function of x and that for $n \ge 0$, $h_{2n+1}(x, t)$ is an odd function of x, we now define the particular solutions $u_n(x, t)$ of Eq. (1.2) by

$$\begin{aligned} u_{2n}(x, t) &= h_{2n}(x, t) + \int_0^x M(s, x, t)h_{2n}(s, t)\, ds; \qquad n \ge 0, \\ u_{2n+1}(x, t) &= h_{2n+1}(x, t) + \int_0^x K(s, x, t)h_{2n+1}(s, t)\, ds; \qquad n \ge 0. \end{aligned} \tag{2.15}$$

LEMMA 2.2: Let $u(x, t)$ be a (strong) solution of Eq. (1.2) in R which is continuous in $\bar{R}$. Then, given $\epsilon > 0$, there exist constants $a_1, \cdots, a_N$ such that

$$\max_{(x,t)\in\bar{R}} \left| u(x, t) - \sum_{n=0}^{N} a_n u_n(x, t) \right| < \epsilon.$$

Proof: We first show that $u(x, t)$ can be represented in the form

$$u(x, t) = h(x, t) + \frac{1}{2}\int_{-x}^x [K(s, x, t) + M(s, x, t)]h(s, t)\, ds \tag{2.16}$$

where $h(x, t)$ is a solution of Eq. (1.1) in R. Eq. (2.16) is a Volterra integral equation of the second kind for $h(x, t)$ and can be uniquely solved for $h(x, t)$ where $h(x, t)$ is defined in R and continuous in $\bar{R}$ [11]. It remains to be shown that $h(x, t)$ is a solution of Eq. (1.1). From Eqs. (2.10) and (2.13) we have that $K(s, x, t) = -K(-s, x, t)$ and $M(s, x, t) = M(-s, x, t)$ and hence we can rewrite Eq. (2.16) in the form

$$\begin{aligned} u(x, t) &= \tfrac{1}{2}(h(x, t) - h(-x, t)) + \frac{1}{2}\int_0^x K(s, x, t)[h(s, t) - h(-s, t)]\, ds \\ &\quad + \tfrac{1}{2}(h(x, t) + h(-x, t)) + \frac{1}{2}\int_0^x M(s, x, t)[h(s, t) + h(-s, t)]\, ds. \end{aligned} \tag{2.17}$$

Applying the differential operator (1.2) to both sides of Eq. (2.17), using the fact that $K(s, x, t)$ and $M(s, x, t)$ are solutions of the following initial boundary value problems [3]

$$K_{xx} - K_{ss} + q(x, t)K = K_t \,, \tag{2.18a}$$

$$K(x, x, t) = -\frac{1}{2}\int_0^x q(s, t)\, ds, \qquad K(0, x, t) = 0, \tag{2.18b}$$

$$M_{xx} - M_{ss} + q(x, t)M = M_t \,, \tag{2.19a}$$

$$M(x, x, t) = -\frac{1}{2}\int_0^x q(s, t)\, ds, \qquad M_x(0, x, t) = 0, \tag{2.19b}$$

and rewriting the resulting expression in the form of Eq. (2.16), gives

$$0 = (h_{xx} - h_t) + \frac{1}{2}\int_{-x}^{x} [K(s, x, t) + M(s, x, t)](h_{ss}(s, t) - h_t(s, t))\, ds. \tag{2.20}$$

Since solutions of Volterra integral equations of the second kind are unique we can conclude that $h(x, t)$ is a solution of Eq. (1.1) in R.

Using Lemma 2.1, we now approximate $h(x, t)$ by a linear combination of the polynomials defined in Eq. (2.2) such that

$$\max_{(x,t)\in\bar{R}} \left| h(x, t) - \sum_{n=0}^{N} a_n h_n(x, t) \right| < \frac{\epsilon}{1 + C} \tag{2.21}$$

where

$$C = \max_{\substack{(x,t)\in\bar{R} \\ -1\le s\le 1}} |K(s, x, t) + M(s, x, t)|.$$

Eqs. (2.16), (2.17) and the fact that $h_{2n}(x, t)$ is an even function of x and $h_{2n+1}(x, t)$ is an odd function of x for $n \geq 0$ now show that

$$\max_{(x,t)\in\bar{R}} \left| u(x, t) - \sum_{n=0}^{N} a_n u_n(x, t) \right| < \epsilon. \tag{2.22}$$

THEOREM 2.1: Let $u(x, t)$ be a (strong) solution of Eq. (1.2) in R which is continuous in $\bar{R}$ and satisfies the initial-boundary data (2.1). Let R_0 be a compact subset of R. Let N be a positive integer and define a_{kn} and b_k, $k = 0, 1, \cdots, N$, $n = 0, 1, \cdots, N$, by the formulas

$$a_{kn} = \int_0^T u_n(-1, t)u_k(-1, t)\, dt + \int_{-1}^{1} u_n(x, 0)u_k(x, 0)\, dx + \int_0^T u_n(1, t)u_k(1, t)\, dt,$$

$$b_k = \int_0^T f(t)u_k(-1, t)\, dt + \int_{-1}^{1} h(x)u_k(x, 0)\, dx + \int_0^T g(t)u_k(1, t)\, dt. \tag{2.23}$$

Then there exists a unique solution $c_1, \cdots, c_N$ of the linear algebraic system

$$\sum_{n=0}^{N} a_{kn}c_n = b_k\,; \qquad k = 0, 1, \cdots, N, \tag{2.24}$$

and given $\epsilon > 0$ we have

$$\max_{(x,t)\in R_0} \left| u(x, t) - \sum_{n=0}^{N} c_n u_n(x, t) \right| < \epsilon \tag{2.25}$$

for N sufficiently large.

Proof: Let $G(x, t, \xi, \tau)$ be the Green's function for Eq. (1.2) in R. Then $u(x, t)$ can be represented in the form

$$u(x, t) = \int_0^t \frac{\partial}{\partial \xi} G(x, t, -1, \tau) f(\tau)\, d\tau - \int_0^t \frac{\partial}{\partial \xi} G(x, t, 1, \tau) g(\tau)\, d\tau + \int_{-1}^1 G(x, t, \xi, 0) h(\xi)\, d\xi \qquad (2.26)$$

where $G(x, t, \xi, \tau)$ is continuous for $(x, t, \xi, \tau) \in R_0 \times \partial R$. Hence for $(x, t) \in R_0$ we have by Schwarz's inequality

$$\max_{(x,t)\in R_0} |u(x, t)|^2 \leq C\left[\int_0^T |f(\tau)|^2\, d\tau + \int_0^T |g(\tau)|^2\, d\tau + \int_{-1}^1 |h(\xi)|^2\, d\xi\right] \qquad (2.27)$$

where

$$C = \max_{(x,t)\in R_0} \left\{\int_0^T \left|\frac{\partial}{\partial \xi} G(x, t, -1, \tau)\right|^2 d\tau + \int_0^T \left|\frac{\partial}{\partial \xi} G(x, t, 1, \tau)\right|^2 d\tau + \int_{-1}^1 |G(x, t, \xi, 0)|^2\, d\xi. \qquad (2.28)$$

From Lemma 2.2 we can conclude that for N sufficiently large there exist constants $C_1, \cdots, C_N$ such that

$$\max_{(x,t)\in R} \left|u(x, t) - \sum_{n=0}^N c_n u_n(x, t)\right|^2 < \frac{\epsilon^2}{2C(T + 1)} \qquad (2.29)$$

and Eq. (2.27) (applied to $u(x, t) - \sum_{n=0}^N c_n u_n(x, t)$ instead of $u(x, t)$) shows that a suitable choice of the constants $c_1, \cdots, c_N$ can be determined by minimizing the quadratic functional

$$Q(c_1, \cdots, c_N) = \int_0^T \left|f(\tau) - \sum_{n=0}^N c_n u_n(-1, \tau)\right|^2 d\tau + \int_0^T \left|g(\tau) - \sum_{n=0}^N c_n u_n(1, \tau)\right|^2 d\tau + \int_{-1}^1 \left|h(\xi) - \sum_{n=0}^N c_n u_n(\xi, 0)\right|^2 d\xi. \qquad (2.30)$$

We note that $Q(c_1, \cdots, c_N)$ is always positive or zero and hence its only stationary point represents a minimum. This minimum can be found by solving the set of equations $\partial Q/\partial c_k = 0$ and this leads to the system (2.23), (2.24). Since the set $\{u_n(x, t)\}_{n=0}^N$ is linearly independent (this follows from the fact that the set $\{v_n(x, t)\}_{n=0}^N$ is linearly independent) the coefficient matrix (a_{kn}) is nonsingular and hence the system (2.23), (2.24) has a unique solution. (Here use has been made of the fact that if a solution of Eq. (1.2) vanishes on the base and vertical sides of R it must be identically zero throughout R [7]). If $c_1, \cdots, c_N$ is the solution of the system (2.24) then Eqs. (2.27) and (2.29) imply the validity of Eq. (2.25).

We note in passing that error estimates for the above approximation procedure can be found if one can estimate the maximum of $|u(x, t) - \sum_{n=0}^N c_n u_n(x, t)|$ on the base and vertical sides of R. The maximum principle for parabolic equations [7] then immediately gives estimates for $|u(x, t) - \sum_{n=0}^\infty c_n u_n(x, t)|$ in the interior of R.

3. The first initial boundary value problem in a quarter plane. In this section we will derive constructive methods for approximating solutions of Eq. (1.2) which satisfy the initial-boundary data

$$\begin{aligned} u(0, t) &= 0; \qquad 0 \le t < T, \\ u(x, 0) &= f(x); \qquad 0 \le x < \infty, \end{aligned} \tag{3.1}$$

where we assume $f(x)$ is continuous, $f(0) = 0$, and there exist positive constants M and A such that

$$|f(x)| \le M \exp Ax^2; \qquad 0 \le x < \infty. \tag{3.2}$$

We will look for a solution $u(x, t)$ of Eq. (1.2) in $0 < x < \infty, 0 < t < T < 1/4A$ which is continuous for $0 \le x < \infty$, $0 \le t < T$, satisfies the initial-boundary data (3.1), and satisfies a bound of the form

$$|u(x, t)| \le M_1 \exp A_1 x^2; \qquad 0 \le x < \infty, \qquad 0 \le t < T \tag{3.3}$$

for some positive constants M_1 and A_1 (cf. [7], Ch. 4). For the sake of simplicity we will only consider the case when $q(x, t) = q(x)$ is independent of t, and make the assumption that $q(x)$ is continuously differentiable for $0 \le x < \infty$ and is bounded in absolute value by a positive constant C for $0 \le x < \infty$. In order to exploit the construction of the kernel $K(s, x, t)$ already given in Eqs. (2.10), (2.11) we will assume without loss of generality that $q(x)$ has been extended to a continuously differentiable function defined for $-\infty < x < \infty$. The method we will use to solve the initial-boundary value problem (1.2), (3.1), is basically an application of the reflection principle (or "method of images") for parabolic equations derived in [3].

We look for a solution of Eqs. (1.2) and (3.1) in the form

$$u(x, t) = h(x, t) + \int_0^x K(s, x)h(s, t)\, ds \tag{3.9}$$

where $K(s, x)$ is defined by Eqs. (2.10) and (2.11) (noting that $q(x, t) = q(x)$ is independent of t and hence so is $K(s, x, t) = K(s, x)$) and $h(x, t)$ is a (strong) solution of Eq. (1.1) for $0 \le x < \infty$, $0 < t < T$, satisfying $h(0, t) = 0$. Note that by the reflection principle for the heat equation we can conclude that $h(x, t)$ is in fact a solution of the heat equation for $-\infty < x < \infty$, $0 < t < T$ and hence $u(x, t)$ is a strong solution of Eq. (1.2) in this region. Evaluating Eq. (3.9) at $t = 0$ leads to a Volterra integral equation of the second kind for the unknown function $h(x, 0)$ and from this data along with $h(0, t) = 0$ it is possible to construct $h(x, t)$ in the region $0 < x < \infty$, $0 < t < T$, provided we know that $h(x, 0)$ satisfies a bound of the form (3.2). However, the construction of $h(x, 0)$ and the estimation of its rate of growth is based on the construction and rate of growth of the resolvent kernel for Eq. (3.9). But the resolvent kernel is obtained by an iteration procedure involving the kernel $K(s, x)$ which in turn is constructed by the iteration procedure (2.11). Hence, in order to solve the initial boundary value problem (1.2), (3.1) by the use of the integral operator (3.9), it is important to provide a better method of constructing the resolvent kernel for Eq. (3.9). We will now show how this can be done by reducing the construction of the resolvent kernel to the problem of solving a Goursat problem for a hyperbolic equation.

We look for a solution $h(x, t)$ of Eq. (1.1) in the form

$$h(x, t) = u(x, t) + \int_0^x \Gamma(s, x)u(s, t)\, ds \tag{3.10}$$

where $u(x, t)$ is a solution of Eq. (1.2) in $0 < x < \infty$, $0 < t < T$, is continuously differentiable for $0 \leq x < \infty$, $0 < t < T$, continuous for $0 \leq x < \infty$, $0 \leq t < T$, and satisfies the boundary condition $u(0, t) = 0$ for $0 \leq t < T$. Substituting Eq. (3.10) into Eq. (1.1) and integrating by parts shows that $h(x, t)$ will be a solution of Eq. (1.1) provided $\Gamma(s, x)$ satisfies the Goursat problem

$$\Gamma_{xx} - \Gamma_{ss} - q(s)\Gamma = 0 \tag{3.11a}$$

$$\Gamma(x, x) = \frac{1}{2}\int_0^x q(s)\, ds, \qquad \Gamma(0, x) = 0. \tag{3.11b}$$

From [5, p. 119], it is seen that the unique solution $\tilde{\Gamma}(\xi, \eta) = \Gamma(\xi - \eta, \xi + \eta)$ of Eqs. (3.11a), (3.11b) is given by the iterative scheme

$$\begin{aligned} \tilde{\Gamma}(\xi, \eta) &= \lim_{n\to\infty} \tilde{\Gamma}_n(\xi, \eta), \\ \tilde{\Gamma}_1(\xi, \eta) &= \frac{1}{2}\int_\eta^\xi q(s)\, ds, \\ \tilde{\Gamma}_{n+1}(\xi, \eta) &= \frac{1}{2}\int_\eta^\xi q(s)\, ds - \int_0^\eta \int_\eta^\xi q(\xi + \eta)\tilde{\Gamma}_n(\xi, \eta)\, d\xi\, d\eta; \qquad n \geq 1. \end{aligned} \tag{3.12}$$

Hence the existence of the operator (3.10) is established. From the initial conditions (3.11b) and (2.18b) satisfied by the kernels $\Gamma(s, x)$ and $K(s, x)$ respectively, it is seen that the operators (3.9) and (3.10) leave the Cauchy data assumed by $h(x, t)$ and $u(x, t)$ invariant. Hence from the uniqueness of the solution to Cauchy's problem for parabolic equations [8] we can conclude that the operators defined by Eqs. (3.9) and (3.10) are inverses of one another, i.e. $\Gamma(s, x)$ is the resolvent kernel of the operator (3.9).

We now want to obtain an estimate on the rate of growth of $\Gamma(s, x)$ for $0 \leq s \leq x$, $0 \leq x < \infty$. Since $x = \xi + \eta$, $s = \xi - \eta$, it is seen that under these restrictions on s and x we have $\xi \geq \eta$, $\eta \geq 0$. Since $|q(x)| < C$ for $0 \leq x < \infty$, it is seen from Eq. (3.12) that for $\xi \geq \eta$, $\eta \geq 0$, $|\tilde{\Gamma}(\xi, \eta)| \leq P(\xi, \eta)$ where $P(\xi, \eta)$ is defined by the recursion scheme

$$\begin{aligned} P(\xi, \eta) &= \lim_{n\to\infty} P_n(\xi, \eta), \\ P_1(\xi, \eta) &= C\xi, \\ P_{n+1}(\xi, \eta) &= C\xi + C\int_0^\eta \int_0^\xi P_n(\xi, \eta)\, d\xi\, d\eta. \end{aligned} \tag{3.13}$$

Hence

$$\begin{aligned} P(\xi, \eta) &= \sum_{k=0}^\infty \frac{C^{k+1}\xi^{k+1}\eta^k}{(k+1)!\, k!} \\ &\leq C\xi \sum_{k=0}^\infty \frac{C^k\xi^k\eta^k}{k!\, k!} \\ &= C\xi I_0(2(C\xi\eta)^{1/2}) \end{aligned} \tag{3.14}$$

where $I_0(z)$ denotes the modified Bessel function of the first kind. From the asymptotic expansion of $I_0(z)$ (cf. [4]) we can now conclude that there exists a positive constant C_1 such that

$$0 < P(\xi, \eta) < C_1\xi \exp (2(C\xi\eta)^{1/2}); \tag{3.15}$$

i.e. for $0 \leq s \leq x, 0 \leq x < \infty$,

$$|\Gamma(s, x)| \leq C_1 x \exp (\sqrt{C}\, x). \tag{3.16}$$

From the above analysis and the fact that $K(s, x)$ satisfies a Goursat problem of the same form as $\Gamma(s, x)$ (cf. Eqs. (2.18a), (2.18b)), it is seen that for $0 \leq s \leq x, 0 \leq x < \infty$, $K(s, x)$ also satisfies the inequality

$$|K(s, x)| \leq C_1 x \exp (\sqrt{C}\, x). \tag{3.17}$$

We now return to the initial boundary value problem (1.2), (3.1). From Eqs. (3.2), (3.10) and (3.16) we have

$$g(x) = h(x, 0) = f(x) + \int_0^x \Gamma(s, x)f(s)\, ds \tag{3.18}$$

and for $0 < x < \infty$

$$\begin{aligned} |g(x)| &\leq M \exp (Ax^2)[1 + C_1x^2 \exp \sqrt{C}\, x] \\ &\leq C_2 \exp [(A + \epsilon)x^2] \end{aligned} \tag{3.19}$$

for $\epsilon > 0$ fixed but arbitrarily small and C_2 a positive constant. Using the "method of images", we now define the solution $h(x, t)$ of Eq. (1.1) by

$$h(x, t) = \int_0^\infty \{s(x - y, t) - s(x + y, t)\}g(y)\, dy \tag{3.20}$$

where

$$s(x, t) = \frac{1}{(4\pi t)^{1/2}} \exp \left(-\frac{x^2}{4t}\right). \tag{3.21}$$

From [7] it is seen that $h(x, t)$ is a strong solution of Eq. (1.1) for $-\infty < x\ \infty, 0 < t < T$, is continuous for $-\infty < x < \infty, 0 \leq t \leq T$, assumes the initial-boundary data $h(0, t) = h(0, t) = 0, 0 \leq t < T, h(x, 0) = g(x), 0 \leq x \leq \infty$, and satisfies $|h(x, t)| \leq M_2 \exp A_2x^2$ for suitable constants M_2 and A_2 and $0 \leq x < \infty, 0 \leq t < T$. Since from our previous discussion we have

$$f(x) = g(x) + \int_0^x K(s, x)g(s)\, ds, \tag{3.22}$$

it is seen that Eqs. (3.18), (3.20) and (3.9) now define the solution of the initial boundary value problem (1.2), (3.1) for $0 \leq x < \infty, 0 \leq t \leq T$. From Eq. (3.17) and the bound on $h(x, t)$ we can conclude that the inequality (3.3) is valid.

For (x, t) restricted to compact subsets of $0 < x < \infty, 0 < t < T$, approximations of the solution to the initial boundary value problem (1.2), (3.1) can be obtained by using the recursion schemes (2.10)–(2.11) and (3.12) to approximate the kernels $K(s, x)$ and $\Gamma(s, x)$ respectively. Error estimates for such an approximation procedure can be found from estimates of the form (3.13), (3.14). For (x, t) again restricted to compact subsets of $0 < x < \infty, 0 < t < T$, the improper integral (3.20) can be accurately approximated by a proper integral by setting $s(x, t) = 0$ for $|x|$ sufficiently large. This is particularly useful if $f(x)$ satisfies a bound of the form $|f(x)| \leq M \exp Ax$ instead of the bound in Eq. (3.2), since in this case an estimate of the form (3.19) leads to a similar bound for $g(x)$, thus speeding up the convergence of the integral (3.20). The problem of dealing with the improper integral (3.20) is avoided completely if we make the assumption that $q(x)$ and $f(x)$ both vanish for $x \geq x_0$. In this case we have from Eq. (3.18) that

$$g(x) = \int_0^{x_0} \Gamma(s, x) f(s)\, ds \tag{3.23}$$

for $x \geq x_0$. But for $x \geq 3x_0$ we have $\xi \geq \eta = \frac{1}{2}(x - s) \geq x_0$, and hence from Eq. (3.12) it is seen that $\Gamma(s, x) \equiv 0$ for $0 \leq s \leq x_0$, $x > 3x_0$. Therefore from Eq. (3.23) it is seen that $g(x) \equiv 0$ for $x > 3x_0$ and the integral (3.20) reduces to a proper integral.

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