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THE APPROXIMATION PROPERTY DOES NOT IMPLY THE BOUNDED APPROXIMATION PROPERTY

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ABSTRACT. There is a Banach space which has the approximation property but fails the bounded approximation property. The space can be chosen to have separable conjugate, hence there is a nonnuclear operator on the space which has nuclear adjoint. This latter result solves a problem of Grothendieck [2].

I. **Introduction.** Let $(X, \|\cdot\|)$ be a Banach space. We show that if there is a constant λ so that $(X, |\cdot|)$ has the λ-metric approximation property (λ-m.a.p., in short) for each equivalent norm $|\cdot|$ on X, then X^* has the bounded approximation property (b.a.p., in short). This result is used to construct an example of a Banach space which possesses the approximation property (a.p.) but fails the b.a.p.

For ε, λ positive constants, we say that X has the (ε, λ)-m.a.p. provided that, for each finite dimensional subspace Z of X and each $\delta>0$, there is a finite rank operator T on X so that $\|T\| \leqq \lambda+\delta$ and $\|Tz-z\| \leqq (\varepsilon+\delta)\|z\|$ for each $z \in Z$. An intermediate step in our construction is that if X has the (ε, λ)-m.a.p. for some ε, $0<\varepsilon<1$, then X has the $\lambda(1-\varepsilon)^{-1}$-m.a.p.

We use the standard notation in Banach space theory. Let us only recall the types of approximation conditions a Banach space X may satisfy. X has the a.p. if for each compact subset K of X and $\varepsilon>0$, there is a finite rank operator (=bounded, linear operator) T on X so that $\|Tk-k\| \leqq \varepsilon$ for each $k \in K$. If always T can be chosen with $\|T\| \leqq \lambda$ then X is said to have the λ-m.a.p. A space which has the λ-m.a.p. for some λ is said to have the b.a.p. For equivalent formulations of these definitions (which we use without further reference) the reader is referred to [2] and [4].

We wish to thank Professor A. Pełczyński for a revision of an earlier incorrect proof of the main result. Pełczyński's description [6] led us to the proof presented here.

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II. **Implications among approximation conditions.** Given a Banach space $(X, \|\cdot\|)$, let $\mathscr{A}$ be the family of equivalent norms, $|\cdot|$, on X whose dual norms on X^* are of the form $|x^*|=\|x^*\|+M\,d(x^*, Z)$. Here M ranges over positive constants, Z ranges over finite dimensional subspaces of X^*, and $d(x^*, Z)\equiv\inf\{\|x^*-z\|:z\in Z\}$ is the $\|\cdot\|$-distance of x^* to Z. Since finite dimensional subspaces of X^* are weak* closed, it is evident that each such norm on X^* is the dual of an equivalent norm on X.

PROPOSITION 1. *Suppose that $(X, |\cdot|)$ has the λ-m.a.p. for each norm, $|\cdot|$, in $\mathscr{A}$. Let $0<\varepsilon<1$. Then $(X^*, \|\cdot\|)$ has the $(\varepsilon, \lambda[1+2\varepsilon^{-1}\lambda])$-m.a.p.*

PROOF. Suppose that Z is a finite dimensional subspace of X^*. Let $\beta>\lambda$ and $\delta>0$. Define $|\cdot|$ on X^* by $|x^*|=\|x^*\|+2\varepsilon^{-1}\beta\,d(x^*, Z)$.

Pick a finite dimensional subspace Y of X such that for each $z\in Z$, $\|z\|\leqq(1+\delta)\sup\{z(y):y\in Y, \|y\|\leqq 1\}$. Since $(X, |\cdot|)$ has the λ-*m.a.p.*, there is a finite rank operator T on X so that $Ty=y$ for $y\in Y$ and $|T|\leqq\beta$.

We have, for $x^*\in X^*$,

$$\|T^*x^*\| + 2\varepsilon^{-1}\beta\,d(T^*x^*, Z) \leqq \beta[\|x^*\| + 2\varepsilon^{-1}\beta\,d(x^*, Z)].$$

Hence $\|T^*x^*\|\leqq\beta(1+2\varepsilon^{-1}\beta)\|x^*\|$ whence $\|T\|\leqq\beta(1+2\varepsilon^{-1}\beta)$. Now for $z\in Z$, $2\varepsilon^{-1}\beta\,d(T^*z, Z)\leqq\beta\|z\|$, so there exists $w\in Z$ satisfying $\|T^*z-w\|\leqq\frac{1}{2}\varepsilon\|z\|$. But for $y\in Y$, $(T^*z)y=z(Ty)=z(y)$, and thus $\sup\{|z(y)-w(y)|: y\in Y, \|y\|\leqq 1\}\leqq\frac{1}{2}\varepsilon\|z\|$. Therefore $\|z-w\|\leqq\frac{1}{2}\varepsilon(1+\delta)\|z\|$, from which it follows that

$$\|T^*z - z\| \leqq [\tfrac{1}{2}\varepsilon(1 + \delta) + \tfrac{1}{2}\varepsilon]\,\|z\| \leqq (1 + \delta)\varepsilon\,\|z\|.$$

Since $\delta>0$, $\beta>\lambda$ are arbitrary, the conclusion follows.

PROPOSITION 2. *Suppose $(X, \|\cdot\|)$ has the (ε, λ)-m.a.p. with $\varepsilon<1$. Then X has the $(1-\varepsilon)^{-1}\lambda$-m.a.p.*

PROOF. We thank Professor W. J. Davis for the proof given here. Davis' proof is rather more revealing than proofs discovered by us.

Suppose Z is a finite dimensional subspace of X. Let $0<\varepsilon<\delta<1$ and $\beta>\lambda$.

Construct by induction finite rank operators T_n on X so that

$$\|T_1z - z\| \leqq \delta\,\|z\| \quad \text{for } z\in Z, \qquad \|T_{n+1}x - x\| \leqq \delta\,\|x\|$$

for $x\in\operatorname{span} Z\cup T_nX\cup T_{n-1}X\cup\cdots\cup T_1X$, and $\|T_n\|\leqq\beta$.

Define S_n by $I-S_n=(I-T_n)(I-T_{n-1})\cdots(I-T_1)$. Then for $z\in Z$, $\|(I-S_n)z\|\leqq\delta^n\|z\|$. Also,

$$S_n = (I - T_n)(I - T_{n-1})\cdots(I - T_2)T_1 + (I - T_n)(I - T_{n-1})\cdots(I - T_3)T_2 + \cdots + (I - T_n)T_{n-1} + T_n,$$

so that $\|S_n\|\leqq\delta^{n-1}\beta+\cdots+\delta\beta+\beta<(1-\delta)^{-1}\beta$.

Hence X has the $(\tau, (1-\delta)^{-1}\beta)$-m.a.p. for each $\tau>0$, $\delta>\varepsilon$ and $\beta>\lambda$, whence X has the $(1-\varepsilon)^{-1}\lambda$-m.a.p.

Setting $\varepsilon=\frac{1}{2}$ in the above two propositions yields:

THEOREM 1. *If, for each* $|\cdot|$ *in* $\mathscr{A}$, $(X, |\cdot|)$ *has the* λ*-m.a.p., then* X^* *has the* $2\lambda(1+4\lambda)$*-m.a.p.*

REMARK 1. If X^* has the λ-m.a.p. then for each $|\cdot|$ in $\mathscr{A}$, $(X^*, |\cdot|)$ has the λ-m.a.p. (hence also $(X, |\cdot|)$ has the λ-m.a.p.). For if $|x^*|=\|x^*\|+M\,d(x^*, Z)$, Y is a finite dimensional subspace of X^*, and $\varepsilon>0$, then there is a finite rank operator T on X^* so that $\|T\|\leqq\lambda+\varepsilon$ and $Tx^*=x^*$ for $x^*\in\text{span } Y\cup Z$. Since T is the identity on Z, $|T|\leqq\|T\|\leqq\lambda+\varepsilon$.

REMARK 2. It is known [**3**, Theorem 4] that if $(X, |\cdot|)$ has the 1-m.a.p. for each $|\cdot|$ in $\mathscr{A}$, then X^* has the 1-m.a.p. We do not know whether a similar result is true with "1" replaced by "λ". It may even be true that if X^* has the b.a.p., then X^* also has the 1-m.a.p. This is the case when X^* is separable [**4**, Remark 4.11].

EXAMPLE. *There is a Banach space which has the a.p. but fails the b.a.p.*

PROOF. Of course, we need the important result of Enflo [**1**] that there is a Banach space which fails the a.p. Lindenstrauss [**5**] (see [**3**] for a specific example) had shown that a consequence of this is the existence of a Banach space X which possesses the 1-m.a.p., but whose conjugate fails the a.p. By Theorem 1, there is a sequence $(|\cdot|_n)$ of equivalent norms on this X so that $(X, |\cdot|_n)$ fails the n-m.a.p. Thus $(\Sigma(X, |\cdot|_n))_{l_2}$ fails the b.a.p. but possesses the a.p.

Note that $(\Sigma(X, |\cdot|_n))_{l_2}$ can be chosen to have separable conjugate, since Lindenstrauss' construction can yield an X with X^* separable.

III. **Nonnuclear operators with nuclear adjoints.** The example constructed in §II justifies the following proposition.

PROPOSITION 3. *If a Banach space* X *has the a.p. but fails the b.a.p., and* X^* *is separable, then there is a nonnuclear operator* T *on* X *such that* T^* *is nuclear.*

Since the proof is an almost immediate consequence of results of [**2**], we give only some indications.

Let $N(X)$ denote the space of nuclear operators on X [**2**, Definition 4] and $L_0(X)$ the space of finite rank operators on X. Since the weak*-continuous nuclear operators form a closed subset of $N(X^*)$, it is enough to show that $\{T^*: T\in N(X)\}$ is not closed in $N(X^*)$.

Consider the natural mappings $X^*\hat{\otimes}X\xrightarrow{\varphi}N(X)\xrightarrow{\chi}[L_0(X)]^*\xleftarrow{\psi}N(X^*)$. Here $\chi(T)S=\text{trace } ST$ and $\psi(T)S=\text{trace}(TS^*)$.

Observe that

(i) φ is an isometry onto, because X has the a.p. (cf. [**2**, Proposition 35, $A \Rightarrow B_1$]).

(ii) $\chi\varphi$ is not an isomorphic embedding, for otherwise (cf. [**2**, Proposition 39, proof of $B_1 \Rightarrow A_1$]) X would have the b.a.p.

(iii) ψ is an isometry onto. For given $F \in L_0(X)^*$, consider the factorization (cf. [**2**, Proposition 27, (a)$\Rightarrow$(d)]) $X^* \to L_\infty \to L_1 \to X^*$ of the operator induced on X^* by F. X^* is separable, so the Dunford-Pettis theorem yields (cf. [**2**, Lemma 9]) that $L_\infty \to L_1 \to X^*$ is nuclear.

Since (i), (ii), and (iii) imply that the range of $\psi^{-1}\chi$ is not closed, it only remains to observe that $\psi^{-1}\chi(T) = T^*$ for each $T \in N(X)$.

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