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THE AR-PROPERTY FOR ROBERTS' EXAMPLE OF A COMPACT CONVEX SET WITH NO EXTREME POINTS PART 1: GENERAL RESULT

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Abstract. We prove that the original compact convex set with no extreme points, constructed by Roberts (1977) is an absolute retract, therefore is homeomorphic to the Hilbert cube. Our proof consists of two parts. In this first part, we give a sufficient condition for a Roberts space to be an AR. In the second part of the paper, we shall apply this to show that the example of Roberts is an AR.

1. Introduction

In 1975 Roberts [R1] constructed a striking example of a compact convex set without any extreme points, giving a counter-example to the Krein-Milman theorem [KM] for non-locally convex linear metric spaces. After 1975 it was hoped that Roberts' example could be used as a counter-example to the following question (we call it *the AR-problem*): Is every convex set in a linear metric space an AR?[1] See [BD], [G]. In fact, for about fifteen years, Roberts' example was the main target for attacking the AR-problem.

In [NT1], see also [N2], it was shown that every needle point space contains a compact convex AR-set with no extreme points. In particular, the spaces L_p, $0 \leq p < 1$, as well as the linear metric space constructed originally by Roberts in [R1], contain compact convex AR-sets with no extreme points. Let us observe, however, that the results of [NT1] and [N2] do not apply to the original compact convex set, constructed in [R1]. In fact, the compact convex set, constructed originally by Roberts in [R1], has been distorted by the arguments in [NT1], [N2].

On the other hand, it was shown in [NT2] that all Roberts spaces have the fixed point property. (By *a Roberts space* we mean any compact convex set with no extreme points constructed by Roberts' method of needle point spaces; see [R1], [R2], [KP], [KPR]). However, the result of [NT2] does not say anything about the AR-property of Roberts spaces. Therefore, the AR-problem about Roberts' example [R1], posed in [BD], [G], has not yet been answered.

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[1] In the non-compact case the AR-problem has been answered negatively by Cauty who recently constructed a σ-compact linear metric space which is not an AR; see Fund. Math. **146**(1994), 85–99. For compact convex sets, the AR-problem, however, is still open.

The aim of this paper is to solve this problem. As a consequence of our result we obtain an affirmative answer to a problem of Dobrowolski and Mogilski; see [DM]. Our result shows that Roberts' example, while rather pathological in functional analysis, has a nice topological structure: In fact, it has the topology of the Hilbert cube, the simplest infinite dimensional compact object in topology. Our proof consists of two parts: In the first part (Sections 2-4), we establish a sufficient condition for a Roberts space to be an AR; see Theorem 2. Then, we apply our sufficient condition to prove in the second part (Sections 5-6) that the compact convex set with no extreme points, constructed by Roberts [R1], is an AR.

The fixed point property for *all* Roberts spaces was completely established. We have tried to obtain a similar answer for the AR-property of Roberts spaces. However, there are still some difficulties we have not yet found a way to overcome. It seems to the authors that the AR-property is somewhat harder than the fixed point property even for compact convex sets. In fact, our result shows that the AR-property of Roberts spaces is not easy, even in the very special case of Roberts' example [R1]. The AR-problem for Roberts spaces is still an important question. Further investigation to the AR-problem for Roberts spaces should be followed up.

Notation and conventions. By a linear metric space we mean a topological linear space X which is metrizable. By Kakutani's theorem (see, for instance [Re]) there is an invariant metric ρ on X. We denote $\|x\| = \rho(x, \theta)$, where θ is the zero element of X, which is called *an F-norm*.

By [Re, Theorem 1.2.2] we may assume that $\|.\|$ is *monotone* (*or non-decreasing*), that is,

$$\|\lambda x\| \leq ([|\lambda|] + 1)\, \|x\| \text{ for every } x \in X \text{ and } \lambda \in \mathbb{R}, \tag{1}$$

where $[\alpha]$ denotes the greatest integer that is smaller than α. In particular

$$\|\lambda x\| \leq \|x\| \text{ for every } x \in X \text{ and } \lambda \in \mathbb{R} \text{ with } |\lambda| \leq 1. \tag{2}$$

In this paper *all linear metric spaces are assumed to be equipped with monotone F-norms*. In particular, (2) will be used frequently throughout the paper.

Let A be a subset of a linear metric space X. By conv A we denote the convex hull of A in X and span A denotes the linear subspace of X spanned by A. We also use the following notation:

$$A^+ = \operatorname{conv}(A \cup \{\theta\})\,; \;\; \hat{A} = \operatorname{conv}\left(A^+ \cup (-A^+)\right) = \operatorname{conv}(A \cup (-A) \cup \{\theta\})\,; \tag{3}$$

and if $x, y \in X$ and $B \subset X$, we write

$$\begin{aligned} \|x - A\| &= \inf\{\|x-y\| \,:\, y \in A\}\,; \\ \operatorname{dis}(A,B) &= \inf\{\|x-y\| \,:\, x \in A,\ y \in B\}\,; \\ \operatorname{diam} A &= \sup\{\|x-y\| \ x, y \in A\}; \\ [x,y] &= \{tx + (1-t)y \,:\, t \in [0,1]\}\,. \end{aligned}$$

If A is a finite set, then card A denotes the cardinality of A. Let $f : \mathbb{R} \to \mathbb{R}$ be a function; we denote $\operatorname{supp} f = \{t \in \mathbb{R} \,:\, f(t) \neq 0\}$.

2. The AR-property for Roberts spaces

The following theorem was established by Roberts in [R1]; see also [R2], [KPR, Proof of Theorem 9.4].

Theorem 1. *Let $\{A_n\}$ be a sequence of finite sets of non-zero points in a complete linear metric space X with the following properties:*

$$A_0 \text{ consists of only one single point}; \tag{4}$$

$$\begin{aligned} &A_{n+1} = \cup \{A(a, \varepsilon_{n+1}) : a \in A_n\}, \text{ where } \varepsilon_{n+1} = m(n)^{-1}2^{-n-1}, \\ &m(n) = \operatorname{card} A_n, \text{ and } A(a, \varepsilon_{n+1}) \text{ is a finite set, which is called} \\ &\quad \text{an } \varepsilon_{n+1}\text{-needle set for } a, \text{ that is,} \end{aligned} \tag{5}$$

$$\|b\| < \varepsilon_{n+1} \text{ for every } b \in A(a, \varepsilon_{n+1}); \tag{5-a}$$

$$a \in \operatorname{conv} A(a, \varepsilon_{n+1}); \tag{5-b}$$

$$\begin{aligned} &\text{For each } b \in A(a, \varepsilon_{n+1})^+, \text{ there exists an } \alpha \in [0,1] \\ &\text{such that } \|b - \alpha a\| < \varepsilon_{n+1}. \end{aligned} \tag{5-c}$$

Denote $A = \overline{\bigcup_{n=o}^{\infty} (A_n)^+} \subset X$; $C = \operatorname{conv}(A \cup (-A)) \subset X$. Then, C is a compact convex set with no extreme points.

Remark 1. By (5-b), $\{(A_n)^+\}$ and $\{\hat{A}_n\}$, see (3), are increasing sequences. Therefore, the set A is convex, and by (5), A, and hence C, are compact. It is easy to see that

$$C = \overline{\bigcup_{n=o}^{\infty} \hat{A}_n} \subset X. \tag{6}$$

For every $a \in A_n$, let $A_{n+1}(a) = A(a, \varepsilon_{n+1})$. For $k \geq n+2$, we define $A_k(a)$ by induction:

$$A_k(a) = \cup \{A(b, \varepsilon_k) : b \in A_{k-1}(a)\}; \tag{7}$$

$$C_n(a) = \overline{\bigcup_{k=1}^{\infty} \hat{A}_{n+k}(a)} \subset X. \tag{8}$$

Observe that

$$A_{n+k} = \cup \{A_{n+k}(a) : a \in A_n\}. \tag{9}$$

The following notation will be used frequently throughout the paper:

$$A_n = \{a_1^n, \ldots, a_{m(n)}^n\}, \text{ where } m(n) = \operatorname{card} A_n. \tag{10}$$

In this part, we give a sufficient condition under which the compact convex set C in Theorem 1 is an AR.

Theorem 2. *C is an AR if the following conditions hold:*

(i) *For every $n \in \mathbb{N}$, if $x_i \in \operatorname{span} C_n(a_i^n) \backslash \{\theta\}$, $i = 1, \ldots, m(n)$, then the set $\{x_1, \ldots, x_{m(n)}\}$ is linearly independent in X (see (8) and (10)).*

(ii) *For every $k \in \mathbb{N}$, the sets $A_{k+1}(a) = A(a, \varepsilon_{k+1})$, $a \in A_k$, have the same cardinality (see (5)).*

Remark 2. It follows from (i) that the union of (9) is disjoint. Therefore, Condition (ii) implies that $\operatorname{card} A_{k+1}(a) = m(k)^{-1} m(k+1)$ for every $a \in A_k$.

At first we establish the following facts, which will be used in the proof of Theorem 2.

Lemma 1 [NT2]. *Let X be a linear metric space and let $a \in X$ be a non-zero point. Then, there is a retraction $r_a : X \to [-a, a]$ such that*

$$\|x - r_a(x)\| \leq 4\|x - [-a, a]\| \quad \textit{for every } x \in X.$$

Lemma 2. $\|x - [-a_i^n, a_i^n]\| \leq m(n)^{-1}2^{-n+1}$ *for every* $x \in C_n(a_i^n)$ *and* $i = 1, \ldots, m(n)$ (*see* (8), (10)).

Proof. Since $A_{n+1}(a_i^n)^+ = A(a_i^n \varepsilon_{n+1})^+$, by (5-c) for every $x \in A_{n+1}(a_i^n)^+$ there exists an $\alpha \in [0, 1]$ such that

$$\|x - \alpha a_i^n\| < \varepsilon_{n+1} = m(n)^{-1}2^{-n-1}.$$

Therefore Lemma 2 holds for $x \in A_{n+1}(a_i^n)^+$. We shall prove by induction that, for every $x \in A_{n+k}(a_i^n)^+$,

$$\|x - [\theta, a_i^n]\| < m(n)^{-1}(2^{-n-1} + 2^{-n-2} + \cdots + 2^{-n-k}). \tag{11}$$

In fact, let $A_{n+k}(a_i^n) = \{a_1, \ldots, a_p\}$. By Condition (ii) of Theorem 2, p does not depend on $i \in \{1, \ldots, m(n)\}$. From (5) and (7) we get

$$A_{n+k+1}(a_i^n) = \cup_{j=1}^p A(a_j, \varepsilon_{n+k+1}).$$

So, for every $x \in A_{n+k+1}(a_i^n)^+$ we have $x = \sum_{j=1}^p \lambda_j x_j$, where $x_j \in A(a_j, \varepsilon_{n+k+1})^+$, $\lambda_j \geq 0,\ j = 1, \ldots, p$, and $\sum_{j=1}^p \lambda_j \leq 1$; see (3). By (5-c), for every $j = 1, \ldots, p$, there exists an $\alpha_j \in [0, 1]$, such that

$$\|x_j - \alpha_j a_j\| < \varepsilon_{n+k+1} = m(n+k)^{-1}2^{-n-k-1}. \tag{12}$$

By Remark 2,

$$m(n+k) = \operatorname{card} A_{n+k} = m(n) \operatorname{card} A_{n+k}(a_i^n) = pm(n). \tag{13}$$

Let $y = \sum_{j=1}^p \lambda_j \alpha_j a_j \in A_{n+k}(a_i^n)^+$. Then from (12) and (13) we get

$$\begin{aligned} \|x - y\| &= \Big\| \sum_{j=1}^p \lambda_j (x_j - \alpha_j a_j) \Big\| < p\varepsilon_{n+k+1} \\ &= pm(n+k)^{-1}2^{-n-k-1} \\ &= p(m(n)p)^{-1}2^{-n-k-1} = m(n)^{-1}2^{-n-k-1}. \end{aligned} \tag{14}$$

Since $y \in A_{n+k}(a_i^n)^+$, by (11) there is a $z \in [\theta, a_i^n]$ such that

$$\|y - z\| < m(n)^{-1}(2^{-n-1} + 2^{-n-2} + \cdots + 2^{-n-k}).$$

Hence from (11) and (14) we get, for every $x \in A_{n+k+1}(a_i^n)^+$,

$$\begin{aligned} \|x - [\theta, a_i^n]\| &\leq \|x - z\| \leq \|x - y\| + \|y - z\| \\ &< m(n)^{-1}2^{-n-k-1} + m(n)^{-1}(2^{-n-1} + \cdots + 2^{-n-k}) \\ &= m(n)^{-1}(2^{-n-1} + \cdots + 2^{-n-k-1}). \end{aligned}$$

Consequently, (11) has been proved by induction. Therefore

$$\|x - [\theta, a_i^n]\| < m(n)^{-1}2^{-n} \text{ for every } x \in \cup_{k=1}^\infty A_{n+k}(a_i^n)^+.$$

Now, for every $x \in \hat{A}_{n+k}(a_i^n)$ and $k \in \mathbb{N}$, we have $x = \lambda_1 x_1 - \lambda_2 x_2$, where $x_j \in A_{n+k}(a_i^n)^+$ and $\lambda_j \in [0, 1],\ j = 1, 2$, with $\lambda_1 + \lambda_2 = 1$; see (3). Take $a_j \in [\theta, a_i^n]$ such that $\|x_j - a_j\| < m(n)^{-1}2^{-n},\ j = 1, 2$. Then we have

$$a = \lambda_1 a_1 - \lambda_2 a_2 \in \lambda_1[\theta, a_i^n] - \lambda_2[\theta, a_i^n] \subset [-a_i^n, a_i^n],$$

and

$$\begin{aligned}\|x-a\| &= \|\lambda_1 x_1 - \lambda_2 x_2 - \lambda_1 a_1 + \lambda_2 a_2\| \\ &\le \|x_1 - a_1\| + \|x_2 - a_2\| < 2m(n)^{-1}2^{-n} = m(n)^{-1}2^{-n+1}.\end{aligned}$$

Therefore

$$\|x - [-a_i^n, a_i^n]\| < m(n)^{-1}2^{n+1} \text{ for } i = 1, \dots, m(n).$$

Consequently, we get the assertion by (8). The lemma is proved.

Claim 1. Using notation (8) and (10), for every $n \in \mathbb{N}$ and for every $x \in C$, there exist $x_i \in C_n(a_i^n)$, see (8), $i = 1, \dots, m(n)$, such that $x = \sum_{i=1}^{m(n)} x_i$.

Proof. Let $x \in C$. Take a sequence $\{x_k\} \subset \cup_{i=0}^{\infty} \hat{A}_i$ such that $x_k \to x$; see (6). By Remark 1, $\{\hat{A}_i\}$ is an increasing sequence. Therefore $\{x_k\} \subset \cup_{i=n+1}^{\infty} \hat{A}_i$ for some $n \in \mathbb{N}$. We may assume that $x_k \in \hat{A}_{n+n(k)}$ for every $k \in \mathbb{N}$. From (9) we get $A_{n+n(k)} = \cup_{i=1}^{m(n)} A_{n+n(k)}(a_i^n)$. Therefore, there exist $x_i^k \in \hat{A}_{n+n(k)}(a_i^n)$ and $\lambda_i^k \in [0,1]$, $i = 1, \dots, m(n)$, such that

$$\sum_{i=1}^{m(n)} \lambda_i^k \le 1, \text{ and } x_k = \sum_{i=1}^{m(n)} \lambda_i^k x_i^k. \tag{15}$$

From (8) we get $x_i^k \in \hat{A}_{n+n(k)}(a_i^n) \subset C_n(a_i^n)$ for every $k \in \mathbb{N}$ and $i = 1, \dots, m(n)$. By the compactness of $C_n(a_i^n)$, by passing into subsequences if necessary, we may assume that $\lambda_i^k \to \lambda_i$ and $x_i^k \to y_i \in C_n(a_i^n)$ for every $i = 1, \dots, m(n)$. Therefore from (15) we get

$$x = \sum_{i=1}^{m(n)} \lambda_i y_i, \text{ where } \lambda_i \in [0,1],\ i = 1, \dots, m(n), \text{ and } \sum_{i=1}^{m(n)} \lambda_i \le 1. \tag{16}$$

Consequently, letting $x_i = \lambda_i y_i$ we get $x_i \in C_n(a_i^n)$ for every $i = 1, \dots, m(n)$. The claim is proved.

From Claim 1 and from (16) it follows that

$$C \subset \sum \{C_n(a) : a \in A_n\} \text{ and } C = \operatorname{conv}(\cup\{C_n(a) : a \in A_n\}) \text{ for every } n \in \mathbb{N}.$$

(However, $C \ne \sum\{C_n(a) : a \in A_n\}$.)

Definition 1. Let $a \in A_n$. We say that a non-zero point $x \in C_n(a)$, see (8), is *maximal*, if $\lambda x \notin C_n(a)$ for any $|\lambda| > 1$.

Definition 2. We say that the expression of $x \in C$ by (16) is *standard* if y_i is maximal for every $y_i \in C_n(a_i^n)$ with $y_i \ne \theta$.

Remark 3. It is easy to see that every $x \in C$ has a standard expression, and under Condition (i) of Theorem 2, any standard expression is unique.

Observe that, under Condition (i) of Theorem 2, the expression of x in Claim 1 is unique. Therefore, we can define $P_i^n : C \to C_n(a_i^n)$, $i = 1, \dots, m(n)$, by

$$P_i^n(x) = x_i \text{ for every } x = \sum_{i=1}^{m(n)} x_i \in C.$$

Claim 2. $P_i^n : C \to C_n(a_i^n)$ is continuous for every $i = 1, \ldots, m(n)$.

Proof. Assume that $\{x_k\} \subset C$ and $x_k \to x$. Write

$$x_k = \sum_{i=1}^{m(n)} x_i^k,\ x = \sum_{i=1}^{m(n)} x_i,\ \text{where } x_i^k,\ x_i \in C_n(a_i^n) \text{ for } i = 1, \ldots, m(n).$$

We shall show that $P_i^n(x_k) = x_i^k \to x_i = P_i^n(x)$. If it is not the case, then by the compactness of $C_n(a_i^n)$ there exists a sequence $\{k(\ell)\} \subset \mathbb{N}$ such that $x_i^{k(\ell)} \to y_i \in C_n(a_i^n)$ for every $i = 1, \ldots, m(n)$ and that $y_i \neq x_i$ for some $i \in \{1, \ldots, m(n)\}$. Then

$$x_{k(\ell)} = \sum_{i=1}^{m(n)} x_i^{k(\ell)} \to x = \sum_{i=1}^{m(n)} y_i.$$

From the uniqueness of the expression of x in Claim 1, we have $x_i = y_i$ for every $i = 1, \ldots, m(n)$, a contradiction. The claim is proved.

Remark 4. By Claim 2, the map

$$P_n = \sum_{i=1}^{m(n)} r_{a_i^n} P_i^n : C \to D_n = \sum_{i=1}^{m(n)} [-a_i^n, a_i^n]$$

is continuous for every $n \in \mathbb{N}$, where $r_{a_i^n},\ i = 1, \ldots, m(n)$, were defined in Lemma 1. Moreover, by Lemmas 1 and 2 we have

$$\begin{aligned}\|P_n(x) - x\| &\leq \sum_{i=1}^{m(n)} \|r_{a_i^n}(x_i) - x_i\| \leq \sum_{i=1}^{m(n)} 4\|x_i - [-a_i^n, a_i^n]\| \\ &< 4m(n)m(n)^{-1}2^{-n+1} = 2^{n+3}.\end{aligned}$$

So, if $P_n(C) \subset C$, then C is admissible, therefore, it is an AR; see [K1], [K2]. This problem seems to be easy, however we are unable to get it done. (Observe that D_n is not contained in C.)

Thus, we must take a long way to the proof of Theorem 2 which involves the following characterization of ANR-spaces, established by the first author in [N1]; see also [NS].

3. A characterization of ANR-spaces

Let X be a metric space. For an open cover $\mathcal{U}$ of X, let $\mathcal{N}(\mathcal{U})$ denote the nerve of $\mathcal{U}$, equipped with *the Whitehead topology*.

Let $\{\mathcal{U}_n\}$ be a sequence of open covers of X. We say that $\{\mathcal{U}_n\}$ is *a zero sequence* iff $\sup\{\operatorname{diam} U : U \in \mathcal{U}_n\} \to 0$ as $n \to \infty$.

We denote $\mathcal{U} = \cup_{n=1}^{\infty} \mathcal{U}_n$ and $\mathcal{K}(\mathcal{U}) = \cup_{n=1}^{\infty} \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})$. For every $\sigma \in \mathcal{K}(\mathcal{U})$, we write

$$n(\sigma) = \sup\{n \in \mathbb{N} : \sigma \in \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})\}.$$

We say that a map $f : \mathcal{U} \to X$ is *a selection* if $f(U) \in U$ for every $U \in \mathcal{U}$. The proof of Theorem 2 is based on the following.

Theorem 3 [N1]. *A metric space X is an ANR if and only if there exists a zero sequence of open covers $\{\mathcal{U}_n\}$ of X with the following property: For any selection $g : \mathcal{U} \to X$, there exists a map $f : \mathcal{K}(\mathcal{U}) \to X$ such that if $\{\sigma_n\}$ is a sequence of* *simplices of* $\mathcal{K}(\mathcal{U})$ *for which* $n(\sigma_k) \to \infty$, *then* $\operatorname{diam}\{f(\sigma_k) \cup g(\sigma_k^o)\} \to 0$, *where* σ^o *denotes the set of all vertices of a simplex* $\sigma \in \mathcal{K}(\mathcal{U})$.

We are going to describe a sequence $\{\mathcal{U}_n\}$ of open covers of C satisfying the conditions of Theorem 3. Using notation (10), for every $i = 1, \dots, m(n)$, we subdivide each interval $[-a_i^n, a_i^n]$ into $2k(n)$ subintervals by $c_0^i = -a_i^n < c_1^i < \cdots < c_{2k(n)}^i = a_i^n$ such that

$$\|c_j^i - c_{j+2}^i\| < m(n)^{-1}2^{-n-1} \text{ for every } j = 0, \dots, 2(k(n)-1). \tag{17}$$

By Lemma 1 for every $i = 1, \dots, m(n)$, there exists a retraction $r_i^n : C_n(a_i^n) \to [-a_i^n, a_i^n]$ such that

$$\|r_i^n(x) - x\| \le 4\|x - [-a_i^n, a_i^n]\| \text{ for every } x \in C_n(a_i^n). \tag{18}$$

For every $i = 1, \dots, m(n)$ and $j = 0, \dots, 2(k(n)-1)$ we put

$$S_0^i(n) = \{x \in [-a_i^n, a_i^n] \ : \ -a_i^n \le x < c_2^i\}; \tag{19-a}$$

$$S_{2(k(n)-1)}^i(n) = \{x \in [-a_i^n, a_i^n] \ : \ c_{2(k(n)-1)}^i < x \le a_i^n; \tag{19-b}$$

$$S_j^i(n) = \{x \in [-a_i^n, a_i^n] \ : \ c_j^i < x < c_{j+2}^i\} \text{ for } j = 1, \dots, 2k(n)-3. \tag{19-c}$$

$$U_j^i(n) = (r_i^n)^{-1}(S_j^i(n)) \subset C_n(a_i^n). \tag{20}$$

Denote

$$\mathbf{J}(n) = \{0, \dots, 2(k(n)-1)\}^{m(n)}. \tag{21}$$

For $\mathbf{j} = (j(1), \dots, j(m(n))) \in \mathbf{J}(n)$, write

$$U_{\mathbf{j}} = U_{j(1)}^1(n) + \cdots + U_{j(m(n))}^{m(n)}(n) \text{ and } \mathcal{U}_n = \{U = U_{\mathbf{j}} \cap C : \ \mathbf{j} \in \mathbf{J}(n)\}. \tag{22}$$

Our aim is to check that the sequence $\{\mathcal{U}_n\}$, defined by (22), satisfies the conditions of Theorem 3. First, we shall show that $\{\mathcal{U}_n\}$ is a zero-sequence of open covers of C.

Lemma 3. $\mathcal{U}_n$ *is an open cover of* C *for every* $n \in \mathbb{N}$.

Proof. First every $U \in \mathcal{U}_n$ is open in C. In fact, let $U = U_{\mathbf{j}} \cap C \in \mathcal{U}_n$, where $U_{\mathbf{j}} = U_{j(1)}^1(n) + \cdots + U_{j(m(n))}^{m(n)}(n)$. Observe that $U_{\mathbf{j}} \cap C = \cap_{i=1}^{m(n)} (P_i^n)^{-1}(U_{j(i)}^i(n))$. Since $U_{j(i)}^i(n)$ is open in $C_n(a_i^n)$, see (20), and since P_i^n is continuous for $i = 1, \dots, m(n)$ by Claim 2, it follows that $U = U_{\mathbf{j}} \cap C$ is open in C.

Now, we prove that $\mathcal{U}_n$ covers C. By Claim 1, for every $x \in C$, there exist $x_i \in C_n(a_i^n)$, $i = 1, \dots, m(n)$, such that $x = \sum_{i=1}^{m(n)} x_i$. Then $r_i^n(x_i) \in [-a_i^n, a_i^n]$, $i = 1, \dots, m(n)$. Hence, there exists $S_{j(i)}^i(n)$ so that $r_i^n(x_i) \in S_{j(i)}^i(n)$, whence $x_i \in U_{j(i)}^i(n)$; see (19), (20). Let $\mathbf{j} = (j(1), \dots, j(m(n))) \in \mathbf{J}(n)$; see (21). Then $x \in U_{\mathbf{j}} = U_{j(1)}^1(n) + \cdots + U_{j(m(n))}^{m(n)}(n)$; see (22). Consequently, $\mathcal{U}_n$ covers C, and the lemma is proved.

Lemma 4. $\{\mathcal{U}_n\}$ *is a zero sequence: In fact,* $\operatorname{diam} U < 2^{-n+5}$ *for every* $U \in \mathcal{U}_n$.

Proof. Let

$$U = U_{\mathbf{j}} \cap C = (U^1_{j(1)}(n) + \cdots + U^{m(n)}_{j(m(n))}(n)) \cap C \in \mathcal{U}_n; \text{ see (22)}.$$

Let $x \in U^i_{j(i)}(n),\ i = 1, \ldots, m(n)$. Then, from (18) and from Lemma 2, we get

$$\|x - r^n_i(x)\| \leq 4\|x - [-a^n_i, a^n_i]\| \leq 4m(n)^{-1}2^{-n+1} = m(n)^{-1}2^{-n+3}.$$

From (17) and (19,a-c) we obtain

$$\operatorname{diam} S^i_{j(i)}(n) = \|c^i_{j(i)} - c^i_{j(i)+2}\| < m(n)^{-1}2^{-n-1}.$$

Observe that, if $x \in U^i_{j(i)}(n)$, then $r^n_i(x) \in S^i_{j(i)}(n)$, see (20). For every $x, y \in U^i_{j(i)}(n)$, since $r^n_i(x), r^n_i(y) \in S^i_{j(i)}(n)$, see (20), we have

$$\begin{aligned} \|x - y\| &\leq \|x - r^n_i(x)\| + \|r^n_i(x) - r^n_i(y)\| + \|r^n_i(y) - y\| \\ &\leq m(n)^{-1}2^{-n+3} + \operatorname{diam} S^i_{j(i)}(n) + m(n)^{-1}2^{-n+3} \\ &< m(n)^{-1}2^{-n+5}. \end{aligned}$$

Consequently

$$\operatorname{diam} U_{\mathbf{j}} \leq \sum_{i=1}^{m(n)} \operatorname{diam} U^i_{j(i)}(n) \leq m(n)m(n)^{-1}2^{-n+5} = 2^{-n+5}.$$

The lemma is proved.

The following fact is essential in our proof of Theorem 2.

Lemma 5. *Under Condition (i) of Theorem 2, if* $\mathbf{j} = (j(1), \ldots, j(m(n)))$, $\mathbf{i} = (i(1), \ldots, i(m(n))) \in \mathbf{J}(n)$, *see (21), such that* $U_{\mathbf{j}} \cap U_{\mathbf{i}} \neq \emptyset$, *then*

$$|j(k) - i(k)| \leq 1 \text{ for every } k = 1, \ldots, m(n).$$

Proof. Assume that $x \in U_{\mathbf{j}} \cap U_{\mathbf{i}}$. Then we have

$$x = x^1_{j(1)} + \cdots + x^{m(n)}_{j(m(n))} = x^1_{i(1)} + \cdots + x^{m(n)}_{i(m(n))},$$

where

$$x^k_{j(k)} \in U^k_{j(k)}(n) \subset C_n(a^n_k) \text{ and } x^k_{i(k)} \in U^k_{i(k)}(n) \subset C_n(a^n_k) \text{ for every } k = 1, \ldots, m(n).$$

From the uniqueness of the expression of x in Claim 1, we get $x^k_{j(k)} = x^k_{i(k)}$ for every $k = 1, \ldots, m(n)$. Hence, from (20) we get

$$r^n_k(x^k_{j(k)}) \in S^k_{j(k)}(n) \cap S^k_{i(k)}(n) \text{ for every } k = 1, \ldots, m(n).$$

Consequently,

$$|j(k) - i(k)| \leq 1 \text{ for every } k = 1, \ldots, m(n).$$

The lemma is proved.

Remark 5. Let $U_{\mathbf{j}} = U^1_{j(1)}(n) + \cdots + U^{m(n)}_{j(m(n))}(n)$, $\mathbf{j} = (j(1), \ldots, j(m(n))) \in \mathbf{J}(n)$, see (22). Then, we say that $U^i_{j(i)}(n)$ is *the i-th coordinate of* $U_{\mathbf{j}}$. Lemma 5 says that if $U_{\mathbf{j}}$ and $U_{\mathbf{i}}$ intersect, then the respective coordinates intersect.

4. Proof of Theorem 2

Since C is contractible, it suffices to prove that C is an ANR. Our aim is to show that the sequence $\{\mathcal{U}_n\}$, defined by (22), satisfies the conditions of Theorem 3.

Let $g : \mathcal{U} \to C$ be an arbitrary selection, where $\mathcal{U} = \cup_{n=1}^{\infty} \mathcal{U}_n$. Then, for every $U \in \mathcal{U}$, we have $U = U_{\mathbf{j}} \cap C = (U^1_{j(1)}(n) + \cdots + U^{m(n)}_{j(m(n))}(n)) \cap C$, where $\mathbf{j} = (j(1), \ldots, j(m(n))) \in \mathbf{J}(n)$, see (21) and (22). Let $g(U) = \sum_{i=1}^{m(n)} \lambda_{j(i)} x_{j(i)}$ denote the standard expression of $g(U)$, see Definition 2. Then $\lambda_{j(i)} x_{j(i)} \in U^i_{j(i)}(n)$ for each $i = 1, \ldots, m(n)$. We define $f_o(U)$ by the formula

$$f_o(U) = \sum_{i=1}^{m(n)} \lambda_{j(i)} r_i(x_{j(i)}) \in C, \tag{23}$$

where $r_i = r_{a_i^n},\ i = 1, \ldots, m(n)$, see (10), were defined by Lemma 1. By Remark 3 the standard expression is unique, hence f_o is well-defined. Thus, we get a map $f_o : \mathcal{U} \to C$. (Observe that Theorem 3 does not require f_o to be a selection.) From Lemmas 1 and 2 we get, for $U \in \mathcal{U}_n$,

$$\begin{aligned} \|g(U) - f_o(U)\| &= \Big\| \sum_{i=1}^{m(n)} \lambda_{j(i)} x_{j(i)} - \sum_{i=1}^{m(n)} \lambda_{j(i)} r_i(x_{j(i)}) \Big\| \\ &\le \sum_{i=1}^{m(n)} \|\lambda_{j(i)} x_{j(i)} - \lambda_{j(i)} r_i(x_{j(i)})\| \\ &\le \sum_{i=1}^{m(n)} \|x_{j(i)} - r_i(x_{j(i)})\| \le \sum_{i=1}^{m(n)} 4\|x_{j(i)} - [-a_i^n, a_i^n]\| \\ &\le 4m(n)m(n)^{-1}2^{-n+1} = 2^{-n+3}. \end{aligned} \tag{24}$$

Now, using the convexity of C we extend f_o to the map $f : \mathcal{K}(\mathcal{U}) \to C$ which is linear on each simplex of $\mathcal{K}(\mathcal{U})$. Let us check that f satisfies the required conditions. Let $\sigma = \langle U_1, \ldots, U_k\rangle \in \mathcal{K}(\mathcal{U})$, where $U_1, \ldots, U_p \in \mathcal{U}_{n(\sigma)}$ and $U_{p+1}, \ldots, U_k \in \mathcal{U}_{n(\sigma)+1}$, that is,

$$\begin{aligned} &\sigma = \langle \sigma_1, \sigma_2\rangle, \\ &\sigma_1 = \langle U_1, \ldots, U_p\rangle \in \mathcal{N}(\mathcal{U}_{n(\sigma)}) \text{ and } \sigma_2 = \langle U_{p+1}, \ldots, U_k\rangle \in \mathcal{N}(\mathcal{U}_{n(\sigma)+1}). \end{aligned} \tag{25}$$

The most important step in our proof of Theorem 2 is to estimate the diameters of $f(\sigma_i),\ i = 1, 2$. For the reader's convenience, we first outline the rough idea of our proof: By Remark 5, the respective coordinates of U_t intersect. Therefore, instead of working on U_t we can work on each coordinate of U_t. Then, using the retraction r_a of Lemma 1, we push every coordinate into a straight line in the space. Now, observe that, because the F-norm $\|.\|$ of the space is monotone, *the diameter of a finite set in a straight line in the space X does not increase when taking its convex hull*. So, if $\operatorname{diam} f_o(\sigma_i),\ i = 1, 2$, are really small, then $\operatorname{diam} f(\sigma_i),\ i = 1, 2$, are small too.

Now, we will present our arguments in details. Let $U_t = U_{\mathbf{j}_t} \cap C,\ t = 1, \ldots, p$, with $\mathbf{j}_t = (j_t(1), \ldots, j_t(m(n(\sigma)))) \in \mathbf{J}(n(\sigma))$, see (21) (22). Since $\cap_{t=1}^p U_t \neq \emptyset$, from Lemma 5 we get

$$|j_t(i) - j_{t'}(i)| \le 1 \text{ for every } t, t' \in \{1, \ldots, p\} \text{ and } i = 1, \ldots, m(n(\sigma)).$$

We may assume that $j_t(i) \le j_{t'}(i)$. Then we get

$$\|c^i_{j_t(i)} - c^i_{j_{t'}(i)}\| \le \|c^i_{j_t(i)} - c^i_{j_t(i)+2}\| < m(n(\sigma))^{-1}2^{-n(\sigma)-1}, \text{ see (17)}, \tag{26}$$

for every $t, t' \in \{1, \dots, p\}$ and $i = 1, \dots, m(n(\sigma))$. For every $i = 1, \dots, m(n(\sigma))$, we denote

$$B_i = \{\lambda_{j_t(i)} r_i(x_{j_t(i)}) : t = 1, \dots, p\} \subset [-a_i^{n(\sigma)}, a_i^{n(\sigma)}]. \tag{27}$$

Then we have

Claim 3. $\operatorname{diam}\operatorname{conv} B_i \le m(n(\sigma))^{-1}2^{-n(\sigma)+7}$ for every $i = 1, \dots, m(n(\sigma))$.

Proof. From Lemmas 1 and 2 we get, for $t = 1, \dots, p,\ i = 1, \dots, m(n(\sigma))$,

$$\begin{aligned}
&\|\lambda_{j_t(i)} r_i(x_{j_t(i)}) - r_i(\lambda_{j_t(i)} x_{j_t(i)})\| \\
&\quad \le \|\lambda_{j_t(i)} r_i(x_{j_t(i)}) - \lambda_{j_t(i)} x_{j_t(i)}\| + \|\lambda_{j_t(i)} x_{j_t(i)} - r_i(\lambda_{j_t(i)} x_{j_t(i)})\| \\
&\quad \le 4\|x_{j_t(i)} - [-a_i^{n(\sigma)}, a_i^{n(\sigma)}]\| + 4\|\lambda_{j_t(i)} x_{j_t(i)} - [-a_i^{n(\sigma)}, a_i^{n(\sigma)}]\| \\
&\quad \le 4m(n(\sigma))^{-1}2^{-n(\sigma)+1} + 4m(n(\sigma))^{-1}2^{-n(\sigma)+1} \\
&\quad = m(n(\sigma))^{-1}2^{n(\sigma)+4}.
\end{aligned}$$

Since $\lambda_{j_t(i)} x_{j_t(i)} \in U^i_{j_t(i)}(n(\sigma))$, we have $r_i(\lambda_{j_t(i)} x_{j_t(i)}) \in S^i_{j_t(i)}(n(\sigma))$, see (20). Therefore, for every $t = 1, \dots, p,\ i = 1, \dots, m(n(\sigma))$,

$$\|c^i_{j_t(i)} - r_i(\lambda_{j_t(i)} x_{j_t(i)})\| \le \operatorname{diam} S^i_{j_t(i)}(n(\sigma)) \le m(n(\sigma))^{-1}2^{-n(\sigma)-1}.$$

It follows that

$$\begin{aligned}
&\|\lambda_{j_t(i)} r_i(x_{j_t(i)}) - c^i_{j_t(i)}\| \\
&\quad \le \|\lambda_{j_t(i)} r_i(x_{j_t(i)}) - r_i(\lambda_{j_t(i)} x_{j_t(i)})\| + \|r_i(\lambda_{j_t(i)} x_{j_t(i)}) - c^i_{j_t(i)}\| \\
&\quad \le m(n(\sigma))^{-1}2^{-n(\sigma)+4} + m(n(\sigma))^{-1}2^{-n(\sigma)-1} \\
&\quad < m(n(\sigma))^{-1}2^{-n(\sigma)+5}.
\end{aligned}$$

Consequently, from (26) we get, for $t, t' \in \{1, \dots, p\}$,

$$\begin{aligned}
&\|\lambda_{j_t(i)} r_i(x_{j_t(i)}) - \lambda_{j_{t'}(i)} r_i(x_{j_{t'}(i)})\| \\
&\quad \le \|\lambda_{j_t(i)} r_i(x_{j_t(i)}) - c^i_{j_t(i)}\| + \|c^i_{j_t(i)} - c^i_{j_{t'}(i)}\| + \|c^i_{j_{t'}(i)} - \lambda_{j_{t'}(i)} r_i(x_{j_{t'}(i)})\| \\
&\quad \le m(n(\sigma))^{-1}2^{-n(\sigma)+5} + m(n(\sigma))^{-1}2^{-n(\sigma)-1} + m(n(\sigma))^{-1}2^{-n(\sigma)+5} \\
&\quad < m(n(\sigma))^{-1}2^{-n(\sigma)+7}.
\end{aligned}$$

Therefore, $\operatorname{diam} B_i \le m(n(\sigma))^{-1}2^{-n(\sigma)+7}$. Since B_i is contained in a straight line in X, see (27), from (2) we get $\operatorname{diam}\operatorname{conv} B_i = \operatorname{diam} B_i$. The claim is proved.

By (25) for every $x \in \sigma$ there exist $x_i \in \sigma_i$ and $\lambda_i \in [0, 1]$, $i = 1, 2$ with $\lambda_1 + \lambda_2 = 1$, such that $x = \lambda_1 x_1 + \lambda_2 x_2$. Then we have

$$f(x) = \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

Using Claim 3 above we obtain the following fact which provides an estimation of $\operatorname{diam} f(\sigma_i)$, $i = 1, 2$.

Claim 4. (i) $\|f(x_1) - f_o(U_1)\| \le 2^{-n(\sigma)+7}$;
(ii) $\|f(x_2) - f_o(U_{p+1})\| \le 2^{-n(\sigma)+6}$.

(Observe that (ii) differs from (i) because $U_1 \in \mathcal{U}_{n(\sigma)}$; meanwhile $U_{p+1} \in \mathcal{U}_{n(\sigma)+1}$, see (25).)

Proof of (i). Since $x_1 \in \sigma_1$, we have $x_1 = \sum_{t=1}^{p} \alpha_t U_t$, where $\alpha_t \in [0,1]$ and $\sum_{t=1}^{p} \alpha_t = 1$, see (25). By the definition of f and (23) we get

$$f(x_1) = \sum_{t=1}^{p} \alpha_t f_0(U_t) = \sum_{t=1}^{p} \alpha_t \sum_{i=1}^{m(n(\sigma))} \lambda_{j_t(i)} r_i(x_{j_t(i)}) = \sum_{i=1}^{m(n(\sigma))} \sum_{t=1}^{p} \alpha_t \lambda_{j_t(i)} r_i(x_{j_t(i)}) \tag{28}$$

Since $\sum_{t=1}^{p} \alpha_t \lambda_{j_t(i)} r_i(x_{j_t}(i)) \in \mathrm{conv} B_i$, see (27), it follows from Claim 3 that

$$\begin{aligned} \|f(x_1) - f_o(U_1)\| &\leq \sum_{i=1}^{m(n(\sigma))} \Big\| \sum_{t=1}^{p} \alpha_t \lambda_{j_t(i)} r_i(x_{j_t(i)}) - \lambda_{j_1(i)} r_i(x_{j_1(i)}) \Big\| \\ &\leq \sum_{i=1}^{m(n(\sigma))} \operatorname{diam} \operatorname{conv} B_i \\ &\leq m(n(\sigma)) m(n(\sigma))^{-1} 2^{-n(\sigma)+7} = 2^{-n(\sigma)+7}. \end{aligned}$$

Therefore (i) holds. Similarly, we get the proof of (ii).

Using Claim 4 we can compute $\operatorname{diam} f(\sigma)$

Claim 5. $\operatorname{diam} f(\sigma) \leq 2^{-n(\sigma)+10}$.

Proof. From Lemma 4 we get

$$\operatorname{diam} U_1 < 2^{-n(\sigma)+5} \text{ and } \operatorname{diam} U_{p+1} < 2^{-n(\sigma)+4}, \text{ see (25)}.$$

Since g is a selection and since $U_1 \cap U_{p+1} \neq \emptyset$ we have

$$\begin{aligned} \|g(U_1) - g(U_{p+1})\| &\leq \operatorname{diam} U_1 + \operatorname{diam} U_{p+1} \\ &\leq 2^{-n(\sigma)+5} + 2^{-n(\sigma)+4} < 2^{-n(\sigma)+6}. \end{aligned}$$

For each $x \in \sigma$, we write $x = \lambda_1 x_1 + \lambda_2 x_2$, where $x_i \in \sigma_i$ and $\lambda_i \in [0,1]$ with $\lambda_1 + \lambda_2 = 1$. Let $y = \lambda_1 U_1 + \lambda_2 U_{p+1} \in \sigma$. Then by Claim 4 we have

$$\begin{aligned} \|f(x) - f(y)\| &= \|\lambda_1 f(x_1) + \lambda_2 f(x_2) - \lambda_1 f_o(U_1) - \lambda_2 f_o(U_{p+1})\| \\ &\leq \|f(x_1) - f_o(U_1)\| + \|f(x_2) - f_o(U_{p+1})\| \\ &\leq 2^{-n(\sigma)+7} + 2^{-n(\sigma)+6} < 2^{-n(\sigma)+8}. \end{aligned} \tag{29}$$

Therefore, since $\lambda_1 + \lambda_2 = 1$, from (24) we get

$$\begin{aligned} \|f(y) - f_o(U_1)\| &= \|\lambda_1 f_o(U_1) + \lambda_2 f_o(U_{p+1}) - f_o(U_1)\| \\ &= \|\lambda_2 f_o(U_1) - \lambda_2 f_o(U_{p+1})\| \leq \|f_o(U_1) - f_o(U_{p+1})\| \\ &\leq \|f_o(U_1) - g(U_1)\| + \|g(U_1) - g(U_{p+1})\| + \|g(U_{p+1}) - f_o(U_{p+1})\| \\ &\leq 2^{-n(\sigma)+3} + 2^{-n(\sigma)+6} + 2^{-n(\sigma)+3} < 2^{-n(\sigma)+7}. \end{aligned}$$

Consequently, from (29) we obtain

$$\begin{aligned} \|f(x) - f(U_1)\| &\leq \|f(x) - f(y)\| + \|f(y) - f_o(U_1)\| \\ &\leq 2^{-n(\sigma)+8} + 2^{-n(\sigma)+7} < 2^{-n(\sigma)+9}. \end{aligned}$$

Therefore, $\operatorname{diam} f(\sigma) \leq 2^{-n(\sigma)+10}$. The claim is proved.

Now, we are able to complete our proof. Let $\sigma = \langle U_1, \ldots, U_k \rangle \in \mathcal{K}(\mathcal{U})$. Since g is a selection and since $\cap_{i=1}^{k} U_i \neq \emptyset$, from Lemma 4 we get

$$\begin{aligned}\operatorname{diam} g(\sigma^o) &\leq 2 \max\{\operatorname{diam} U_i : i = 1, \ldots, k\} \\ &= 2 \max\{2^{-n(\sigma)+5}, 2^{-n(\sigma)+4}\} = 2^{-n(\sigma)+6}.\end{aligned}$$

From (24) we obtain

$$\operatorname{dis}(f(\sigma), g(\sigma^o)) \leq \|f_o(U_1) - g(U_1)\| \leq 2^{-n(\sigma)+3}.$$

Therefore

$$\begin{aligned}\operatorname{diam}\{f(\sigma) \cup g(\sigma^o)\} &\leq \operatorname{diam} f(\sigma) + \operatorname{diam} g(\sigma^o) + \operatorname{dis}(f(\sigma), g(\sigma^o)) \\ &\leq 2^{-n(\sigma)+10} + 2^{-n(\sigma)+6} + 2^{-n(\sigma)+3} < 2^{-n(\sigma)+11}.\end{aligned}$$

Consequently, $\operatorname{diam}\{f(\sigma_k) \cup g(\sigma_k^o)\} \to 0$ as $n(\sigma_k) \to \infty$. Hence, C is an ANR by Theorem 3. Theorem 2 is proved.

Remark 6. Let us observe that Condition (ii) is *not essential* in the proof of Theorem 2. However, we prove Theorem 2 under this condition because it simplifies the proof and also because this condition is satisfied naturally in our application to the example of Roberts [R1].

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