THE ARCHIMEDEAN PROPERTY IN AN ORDERED SEMIGROUP

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Introduction

By an ordered semigroup we mean a semigroup with a simple order which is compatible with the semigroup operation. Several authors, for example Alimov [1], Clifford [2], Conrad [4] and Hion [7], studied the archimedean property in some special kinds of ordered semigroups. For a general ordered semigroup, Fuchs [6] defined the archimedean equivalence as follows:

$a \sim b$ if and only if one of the four conditions

$$a \leqq b \leqq a^n, \quad b \leqq a \leqq b^n, \quad a^n \leqq b \leqq a, \quad b^n \leqq a \leqq b$$

holds for some positive integer n.

Then he mentioned that this relation is an equivalence relation. But this is not correct. In fact, let $S = \{0, a, b\}$ with the product $xy = 0$ for every $x, y \in S$ and with the order $a < 0 < b$. Then it is easily checked that S is an ordered semigroup and that $a \sim 0$ and $b \sim 0$. However, $a \sim b$ does not hold. It seems to be troublesome to define the archimedean equivalence suitably in a general ordered semigroup. In the present note, we restrict our attention to nonnegatively ordered semigroups in the sense defined in § 1. We define the archimedean equivalence in natural way. Even in these semigroups, the archimedean equivalence is not always a congruence relation. The main purpose of § 2 is to give necessary and sufficient conditions in order that the archimedean equivalence is a congruence relation. Such a nonnegatively ordered semigroup is called a-regular. Many ordered semigroups, for example all nonnegatively ordered commutative semigroups and the nonnegative cones of all ordered inverse semigroups are a-regular. In § 3, we study the structure of a-regular nonnegatively ordered semigroups P. The quotient semigroup of P modulo the archimedean equivalence is an ordered idempotent semigroup, whose structure was completely determined in our previous paper [8]. By the aid of this knowledge, we show, in this note, the structure of P is known to some extent.

1. Preliminaries

By an *ordered semigroup*, we mean a semigroup S with a simple order which satisfies

$$a \leqq b \text{ implies } ac \leqq bc \text{ and } ca \leqq cb \text{ for every } c \in S.$$

An element c of S is said to *lie between a and b* if either $a \leqq c \leqq b$ or $b \leqq c \leqq a$. A subset T of S is called *convex* if T contains with two of its elements all elements of S which lie between them. An element p of S is called *positive* if $p^2 > p$, while q is called *negative* if $q^2 < q$. Since the order is simple, an element p of S is nonnegative if and only if $p^2 \geqq p$. An element p of S is called *positive* (*nonnegative*) *in the strict sense* if $ps > s$ and $sp > s$ ($ps \geqq s$ and $sp \geqq s$) for every $s \in S$. Clearly if p is positive (nonnegative) in the strict sense, then p is positive (nonnegative). An ordered semigroup S is called *positively* (*nonnegatively*) *ordered* (*in the strict sense*), if every element of S is positive (nonnegative) (in the strict sense). The number of distinct powers of an element a of an ordered semigroup S is called the *order of a*. A mapping of an ordered semigroup S into an ordered semigroup T is called an *o-isomorphism*, if it is a semigroup-isomorphism and an order-isomorphism at the same time. If there is an o-isomorphism of S onto T, then we say that S is *o-isomorphic* to T.

Now we give some lemmas which we need in the following sections.

LEMMA 1.1 ([9] Lemma 1 and its Corollary). *The set P of nonnegative elements of an ordered semigroup S, if it is nonvoid, is a subsemigroup of S. The set E of idempotents of S, if it is nonvoid, is a subsemigroup of S.*

The set P of nonnegative elements of S is called the *non-negative cone* of S. If the set E of idempotents of S is nonvoid, we denote by $\mathscr{D}_E$ the $\mathscr{D}$-equivalence in the semigroup E, in order to distinguish it from that in the original semigroup S.

LEMMA 1.2 ([9] Lemma 2). *In an ordered semigroup S, if p is nonnegative and q is nonpositive and if $p \leqq q$, then both pq and qp are idempotents which lie between p and q.*

LEMMA 1.3. *An idempotent semigroup S is a semilattice of rectangular bands. Every rectangular band which is a constituent of the decomposition is a $\mathscr{D}$-class of S.*

The first half of the above Lemma was given in [2] Exercise 1 for § 4.2. Then the second half can be shown easily.

LEMMA 1.4 ([8] Theorem 1). *In an ordered idempotent semigroup S, each $\mathscr{D}$-class consists of either only one $\mathscr{L}$-class or only one $\mathscr{R}$-class.*

A $\mathscr{D}$-class of an ordered idempotent semigroup S which consists of only one $\mathscr{L}$-class ($\mathscr{R}$-class) is called a $\mathscr{D}$-class of $\mathscr{L}$-type ($\mathscr{R}$-type). By Lemma 1.3, the set of $\mathscr{D}$-classes of an ordered idempotent semigroup S forms a semilattice, which is called the *associated semilattice* of S. In the associated semilattice, we denote the partial order by $\preccurlyeq$ and the semilattice operation by $\circ$.

LEMMA 1.5 ([8] Theorem 3). *The associated semilattice S^* of an ordered idempotent semigroup S is a tree semilattice, i.e. a semilattice in which $\{\xi; \xi \preccurlyeq \alpha\}$ forms a simply ordered set for every $\alpha \in S^*$.*

In the tree semilattice S^*, $\alpha \in S^*$ is called a *branching element* of S^*, if there exist β and γ such that $\alpha \prec \beta$, $\alpha \prec \gamma$ and $\alpha = \beta \circ \gamma$.

Finally we give the following well-known lemma, which is implicitly included in [5] Théorème 3 in p. 179.

LEMMA 1.6. *Let S be an ordered semigroup and let ρ be a congruence relation on S such that every ρ-class is convex. For ρ-classes A and B, we define $A \leqq B$ if and only if $a \leqq b$ for some $a \in A$ and $b \in B$. Then the quotient semigroup S/ρ is an ordered semigroup. Moreover, if $A < B$, then $a < b$ for every $a \in A$ and $b \in B$.*

2. The archimedean equivalence

In what follows, we always denote by P a nonnegatively ordered semigroup and by E the set of idempotents of P. For $x, y \in P$, we define the *archimedean equivalence* $\sim$ as follows:

$x \sim y$ if and only if $x \leqq y \leqq x^n$ or $y \leqq x \leqq y^n$ for some positive integer n.

LEMMA 2.1. *The archimedean equivalence in P is an equivalence relation.*

PROOF. It suffices to prove only the transitivity. Let $a \sim b$ and $b \sim c$. Then

(1) if $a \leqq b \leqq a^n$ and $b \leqq c \leqq b^m$, then $a \leqq b \leqq c \leqq b^m \leqq a^{mn}$;

(2) if $a \leqq b \leqq a^n$ and $c \leqq b \leqq c^m$, then, according as $a \leqq c$ or $c \leqq a$, we have $a \leqq c \leqq b \leqq a^n$ or $c \leqq a \leqq b \leqq c^m$;

(3) if $b \leqq a \leqq b^n$ and $b \leqq c \leqq b^m$, then, according as $a \leqq c$ or $c \leqq a$, we have $a \leqq c \leqq b^m \leqq a^m$ or $c \leqq a \leqq b^n \leqq c^n$;

(4) if $b \leqq a \leqq b^n$ and $c \leqq b \leqq c^m$, then $c \leqq b \leqq a \leqq b^n \leqq c^{mn}$.

Thus, in all cases, we have $a \sim c$.

An equivalence class of P modulo the archimedean equivalence $\sim$ is called an *archimedean class*.

LEMMA 2.2. *Each archimedean class of P is a convex subsemigroup of P which is nonnegatively ordered in the strict sense.*

PROOF. Let A be an archimedean class of P and let $a, b \in A$ and $a \leqq c \leqq b$. Since $a \sim b$, we have $b \leqq a \leqq b^n$ or $a \leqq b \leqq a^n$. If $b \leqq a \leqq b^n$, then $a = b = c$, and if $a \leqq b \leqq a^n$, then $a \leqq c \leqq b \leqq a^n$. Thus, in both cases, we have $a \sim c$ and so A is convex. Next we suppose that $a, b \in A$. Then, since $a \sim b$, we have $a \leqq b \leqq a^n$ or $b \leqq a \leqq b^n$. If $a \leqq b \leqq a^n$, then $a \leqq a^2 \leqq ab \leqq a^{n+1}$ and so $a \sim ab$. If $b \leqq a \leqq b^n$, then $b \leqq b^2 \leqq ab \leqq b^{n+1}$ and so $b \sim ab$. Thus, in both cases, we have $ab \in A$ and so A is a subsemigroup. Finally, by way of contradiction, we suppose that $ab < a$ for some $a, b \in A$. Then we have $ab^2 \leqq ab$. On the other hand, since $b \leqq b^2$, we have $ab \leqq ab^2$. Hence $ab = ab^2$ and so $ab = ab^n$ for every positive integer n. Since $ab < a \leqq a^2$, we have $b < a$. But $a \sim b$ and so $b < a \leqq b^m$ for some positive integer m. Hence $a \leqq a^2 \leqq ab^m = ab < a$, which is a contradiction. Thus $a \leqq ab$ for every $a, b \in A$. Similarly we can prove $a \leqq ba$. Thus A is nonnegative in the strict sense.

LEMMA 2.3. *For an archimedean class A of P, the following conditions are equivalent to one another*:

(1) *A contains an idempotent,*

(2) *A has the greatest element,*

(3) *A has the zero element,*

(4) *every element of A is an element of finite order,*

(5) *A contains an element of finite order.*

Moreover, under these conditions, an idempotent of A is the greatest element and also the zero element of A.

PROOF. (1) implies (2). In fact, let e be an idempotent of A and let $a \in A$. Then we wave $a \leqq e \leqq a^n$ or $e \leqq a \leqq e^n = e$. Thus, in both cases, we have $a \leqq e$. Incidentally we have shown that an idempotent of A is the greatest element of A. (2) implies (3). In fact, let g be the greatest element of A and let $a \in A$. By Lemma 2.2, we have $g \leqq ga$ and $g \leqq ag$, and also $ag \in A$ and $ga \in A$ and so $ga \leqq g$ and $ag \leqq g$. Thus $ga = ag = g$. Incidentally we have shown that the greatest element of A is the zero element of A. (3) implies (4). In fact, let A have the zero element 0 and let $a \in A$. Then $0 \leqq a \leqq 0^n = 0$ or $a \leqq 0 \leqq a^n$. In the former case, we have $a = 0$ and $a = a^2$. In the latter case, we have $0 \leqq a^n \leqq 0^n = 0$ and so $a^n = 0$ and $a^n = a^{n+1}$. (4) implies (5) trivially. Finally (5) implies (1). In fact, let a be an element of finite order in A. Then $a^n = a^{n+1}$ for some positive integer n, and a^n is an idempotent of A.

COROLLARY 2.4. *Every archimedean class of P contains at most one idempotent.*

If an archimedean class A satisfies any one of the conditions in Lemma

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2.3, then A is called a *periodic archimedean class*. Otherwise A is called a *nonperiodic archimedean class*.

LEMMA 2.5. *In P, each nonperiodic archimedean class A is positively ordered in the strict sense.*

PROOF. By Lemma 2.2, we have $a \leqq ab$ for every $a, b \in A$. Now, by way of contradiction, we assume that $a = ab$. Then we have $a = ab^m$ for every positive integer m. Since $a \sim b$, we have either $a \leqq b \leqq a^n$ or $b \leqq a \leqq b^n$. If $b \leqq a \leqq b^n$, then $a^2 \leqq ab^n = a \leqq a^2$ and so $a = a^2$. If $a \leqq b \leqq a^n$, then $a^n \leqq b^n$ and so $a \leqq a^2 \leqq a^{n+1} \leqq ab^n = a$ and $a = a^2$. Hence, in both cases, a is an idempotent of A, which contradicts that A is non-periodic. Thus we have $a < ab$. We can prove $a < ba$ in a similar way.

EXAMPLE 2.6. Let $K_1 = \{e, f, a, g\}$ be a system with the multiplication table

	e	f	a	g
e	e	e	e	e
f	f	f	f	f
a	f	g	g	g
g	g	g	g	g

and with the order $e < f < a < g$. It is easily checked that K_1 is an ordered semigroup.

EXAMPLE 2.7. Let $K_2 = \{e, f, a, g\}$ be an ordered semigroup with the product multiplicatively dual to that of K_1 and with the same order relation as K_1.

THEOREM 2.8. *In order that the archimedean equivalence in a nonnegatively ordered semigroup P is not a congruence relation, it is necessary and sufficient that P contains a subsemigroup o-isomorphic to either K_1 or K_2 in the above Examples.*

PROOF. Necessity. Let the archimedean equivalence $\sim$ in P be not a congruence relation. Then there exist elements $a, b, c \in P$ such that $a \sim b$ but either $ac \sim bc$ or $ca \sim cb$ does not hold. First we consider the case when $ac \sim bc$ does not hold and suppose without loss of generality that $a \leqq b \leqq a^n$. Then $ac \leqq bc \leqq a^n c$ and, since $ac \neq bc$, we have $n > 1$. Now we give a series of relations which hold for a, b and c.

(1) $(ac)^m < a$ *for every positive integer m.*
In fact, if $(ac)^m \geqq a$ for some m, then

$$a^n c = a^{n-1}(ac) \leqq (ac)^{m(n-1)}(ac) = (ac)^{m(n-1)+1}.$$

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Hence we have $ac \leqq bc \leqq a^n c \leqq (ac)^{m(n-1)+1}$ which contradicts that $ac \sim bc$ does not hold.

(2) $ac < a$.
The special case of (1) for $m = 1$.

(3) $ac^m = ac$ *for every positive integer* m.
In fact, by (2), we have $ac^2 \leqq ac$. On the other hand, since $c \leqq c^2$, we have $ac \leqq ac^2$. Hence $ac = ac^2$ and so $ac = ac^m$.

(4) $ca < ac$.
In fact, if $ac \leqq ca$, then, by (3), we have $a^n c = a^n c^n \leqq (ac)^n$. Hence $ac \leqq bc \leqq a^n c \leqq (ac)^n$, which contradicts that $ac \sim bc$ does not hold.

(5) $ca = cac$.
In fact, by (4), we have $ca \leqq c^2 a = c(ca) \leqq cac$. On the other hand, by (2), we have $cac \leqq ca$. Hence we have $ca = cac$.

(6) $a < a^2 c$.
In fact, if $a^2 c \leqq a$, then, by (3), we have $a^2 c = a^2 c^2 = (a^2 c)c \leqq ac$. On the other hand, since $a \leqq a^2$, we have $ac \leqq a^2 c$. Hence $ac = a^2 c$ and so $ac = a^n c$. Therefore $ac \leqq bc \leqq a^n c = ac$, which contradicts that $ac \sim bc$ does not hold.

(7) $aca < a$.
In fact, by (5) and (1), we have $aca = acac = (ac)^2 < a$.

(8) $(ac)^2 = ac$, $(ca)^2 = ca$.
In fact, by (7), we have $(ac)^2 = acac \leqq ac$ and $(ca)^2 = caca \leqq ca$. On the other hand, since ac and ca are nonnegative, these elements are idempotents.

(9) $(a^2 c)^2 = a^2 c = a^2$.
In fact, by (5) and (8), we have

$$(a^2 c)^2 = a^2 (ca)ac = a^2 (cac)ac = a(ac)^3 = a(ac) = a^2 c.$$

Hence, by (6) and (2), we have $a^2 \leqq (a^2 c)^2 = a^2 c \leqq a^2$ and so $(a^2 c)^2 = a^2 c = a^2$.

Now we put $ca = e$, $ac = f$, $a^2 = a^2 c = g$. Then, by (4), (2) and (6), we have $e < f < a < g$. Moreover

$e = e^2 \leqq ef \leqq ea \leqq eg = (ca)aa = (cac)aa = ca(cac)a = (ca)^3 = ca = e$ by (8) and (5),

$f = ac = acac = a(ca) = ac^2 a = fe \leqq f^2 \leqq fa \leqq fg = acaa = acaca = acacac = (ac)^3 = ac = f$ by (8), (5) and (3),

$f = ac = acac = aca = ae$ by (8) and (5),

$g = a^2 c = af \leqq a^2 \leqq ag = aa^2 = a^3 = g$ by (9),

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$g = a(ac) = a(ac)^2 = a^2cac = a^2(ca) = ge \leqq gf \leqq ga \leqq g^2 = (a^2c)^2 = a^2c = g$ by (8), (5) and (9).

Thus the set consisting of four elements e, f, a and g forms a subsemigroup o-isomorphic to K_1. In the case when $ca \sim cb$ does not hold we can prove similarly that P contains a subsemigroup o-isomorphic to K_2.

Sufficiency. We suppose that P contains a subsemigroup o-isomorphic to K_1. Without loss of generality, we assume P contains the ordered semigroup K_1. Then, since $a^2 = g$, we have $a \sim g$. But $ae = f$, $ge = g$ and so $ae \sim ge$ does not hold. Thus the archimedean equivalence is not a congruence relation. In the case when P contains a subsemigroup o-isomorphic to K_2, we can obtain the same conclusion in a similar way.

A nonnegatively ordered semigroup P is called *a-regular* if the archimedean equivalence in P is a congruence relation.

COROLLARY 2.9. *A nonnegatively ordered semigroup P is a-regular if one of the following conditions is satisfied:*

(1) *P is commutative,*

(2) *P contains no elements of finite order except idempotents,*

(3) *P is the nonnegative cone of an ordered inverse semigroup.*

PROOF. In cases (1) and (2), it is trivial that P does not contain a subsemigroup o-isomorphic to K_1 or K_2. Since an ordered inverse semigroup contains no elements of finite order except idempotents ([9] Theorem 6), the case (3) is reduced to the case (2).

REMARK. When P is the nonnegative cone of an ordered regular semigroup which contains a non-idempotent element of finite order, then, by [9] Theorems 2 and 3, P contains a subsemigroup o-isomorphic to K_1 or K_2. Hence P is not a-regular.

THEOREM 2.10. *A nonnegatively ordered semigroup P is a-regular if and only if it satisfies the condition*

(α) $a \sim g = g^2$, $e = e^2 < g$ *and* $e\mathscr{D}_E g$ *imply either* $ea = g$ *or* $ae = g$.

PROOF. Let P be a-regular and let $a \sim g = g^2$, $e = e^2 < g$ and $e\mathscr{D}_E g$. Then, by Lemma 2.3, we have $a \leqq g$. Now we have also $e < a$. In fact, otherwise, $a \leqq e < g$ and so, by Lemma 2.2, we have $e \sim g$, which contradicts Corollary 2.4. First we suppose that the $\mathscr{D}_E$-class of E which contains e is of $\mathscr{L}$-type. Then $e = e^2 \leqq ea \leqq eg = e$. Hence we have $ea = e$. Therefore $(ae)^2 = aeae = ae$ and so ae is an idempotent. Since $\sim$ is a congruence relation, we have $ae \sim ge = g$. Hence, by Corollary 2.4, we have $ae = g$. If the $\mathscr{D}_E$-class which contains e is of $\mathscr{R}$-type, we can prove $ea = g$ in a similar way. Conversely we suppose that P is not a-regular. Then, by Theorem 2.8,

P contains a subsemigroup o-isomorphic to either K_1 or K_2. If P contains K_1, then three elements e, a and g of K_1 satisfy the assumption of the condition (α). But we have $ea = e \neq g$ and $ae = f \neq g$ and so the condition (α) does not hold. If P contains K_2, we can obtain the same conclusion in a similar way.

3. a-regular nonnegatively ordered semigroups

In this section, we denote by P an a-regular nonnegatively ordered semigroup and by $A(p)$ the archimedean class which contains an element $p \in P$. Since P is a-regular, the archimedean equivalence $\sim$ is a congruence relation and so, by Lemmas 2.2 and 1.6, the quotient semigroup $P/\sim$ is an ordered semigroup with the order defined in Lemma 1.6. We denote by $\bar{P}$ the ordered semigroup $P/\sim$.

THEOREM 3.1. *$\bar{P}$ is an ordered idempotent semigroup.*

PROOF. Let $A(p)$ be an element of $\bar{P}$. Then, since $p \sim p^2$, we have $(A(p))^2 = A(p^2) = A(p)$.

LEMMA 3.2. *The mapping φ which maps $e \in E$ to $A(e) \in \bar{P}$ is an o-isomorphism of E into $\bar{P}$.*

PROOF. By Corollary 2.4, φ is a one-to-one mapping. Then it is easily seen that φ is a semigroup-isomorphism and an order-isomorphism.

The image set of the o-isomorphism φ in the above Lemma 3.2 is denoted by $\bar{E}$. $\bar{E}$ is a subsemigroup of $\bar{P}$. For an archimedean class A, we have $A \in \bar{E}$ if and only if A contains an idempotent. Hence $\bar{E}$ is the set of periodic archimedean classes. The $\mathscr{D}$-equivalence in the ordered idempotent semigroup $\bar{P}$ is denoted by $\overline{\mathscr{D}}$. For $A \in \bar{P}$, the $\overline{\mathscr{D}}$-class which contains A is denoted by $\overline{\mathscr{D}}(A)$.

THEOREM 3.3. *If $A \in \bar{E}$, then $\overline{\mathscr{D}}(A) \subseteqq \bar{E}$.*

PROOF. Let $B \in \bar{P}$ such that $A\overline{\mathscr{D}}B$. First we suppose that $\overline{\mathscr{D}}(A)$ is a $\overline{\mathscr{D}}$-class of $\mathscr{L}$-type. Since $A \in \bar{E}$, A contains an element $e \in E$. We take $b \in B$ arbitrarily. If $b \leqq e$, then, by Lemma 1.2, be is an idempotent of P and $be \in BA = B$. If $e \leqq b$, then we have $e = e^2 \leqq eb \in AB = A$. Hence, by Lemma 2.3, we have $e = eb$ and so $(be)^2 = bebe = be$ and $be \in BA = B$. Hence be is an idempotent of B. Thus, in both cases, we obtain $B \in \bar{E}$. In the case when $\overline{\mathscr{D}}(A)$ is of $\mathscr{R}$-type, we can prove $B \in \bar{E}$ in a similar way.

By Theorem 3.3, each $\overline{\mathscr{D}}$-class $\bar{D}$ in $\bar{P}$ belongs to one and only one of the following two types:

(1) all archimedean classes in $\bar{D}$ are periodic,

(2) all archimedean classes in $\bar{D}$ are nonperiodic.

If a $\overline{\mathscr{D}}$-class $\bar{D}$ belongs to the type (1), then $\bar{D}$ is called a periodic $\overline{\mathscr{D}}$-class, while if $\bar{D}$ belongs to the type (2), it is called a nonperiodic $\overline{\mathscr{D}}$-class.

THEOREM 3.4. *If A is an archimedean class which belongs to a periodic $\overline{\mathscr{D}}$-class $\bar{D}$ and if A is not the least element of $\bar{D}$ with respect to the order in $\bar{P}$, then, in P, every element of A is at most of order 2.*

PROOF. Let $a \in A$. By assumption, there exists an archimedean class $B \in \bar{D}$ such that $B < A$. Since $\bar{D}$ is a periodic $\overline{\mathscr{D}}$-class, both A and B are periodic archimedean classes. Let e and f be idempotents of A and B, respectively. Then, since $B < A$, we have $f < a \leqq e$. First we suppose that the $\overline{\mathscr{D}}$-class $\bar{D}$ is of $\mathscr{L}$-type. Then $ef \in AB = A$ and $fe \in BA = B$. Since $ef \in E$ and $fe \in E$, we have $ef = e$ and $fe = f$ by Corollary 2.4, and so $e\mathscr{D}_E f$. In the case when $\bar{D}$ is of $\mathscr{R}$-type, we can prove $e\mathscr{D}_E f$ in a similar way. Hence, in both cases, by Theorem 2.10, we have $fa = e$ or $af = e$. On the other hand, since $f < a \leqq e$, we have $fa \leqq a^2 \leqq e^2 = e$ and $af \leqq a^2 \leqq e^2 = e$. Therefore we have $a^2 = e$.

THEOREM 3.5. *Suppose that, for $A \in \bar{P}$, there exists $B \in \bar{P}$ such that $A < B$* and *$\overline{\mathscr{D}}(A) \preccurlyeq \overline{\mathscr{D}}(B)$. Then A is a periodic archimedean class.*

PROOF. First we suppose that $\overline{\mathscr{D}}(A)$ is a $\overline{\mathscr{D}}$-class of $\mathscr{L}$-type. Then, since $\overline{\mathscr{D}}(A) = \overline{\mathscr{D}}(A) \circ \overline{\mathscr{D}}(B) = \overline{\mathscr{D}}(AB)$, we have $AB = A(AB) = A$. We take $a \in A$ and $b \in B$ arbitrarily. Then $ab \in AB = A$ and so $ab < b$. Hence we have $a^2b \leqq ab$. On the other hand, since $a \leqq a^2$, we have $ab \leqq a^2b$. Therefore $ab = a^2b = a(ab)$ with $a \in A$ and $ab \in A$. Hence, by Lemma 2.5, A is a periodic archimedean class. In the case when $\overline{\mathscr{D}}(A)$ is of $\mathscr{R}$-type, we can obtain the same conclusion in a similar way.

THEOREM 3.6. *Every nonperiodic $\overline{\mathscr{D}}$-class $\bar{D}$ consists of only one nonperiodic archimedean class.*

PROOF. By way of contradiction, we assume that $\bar{D}$ contains two distinct archimedean classes A and B. Without loss of generality, we suppose that $A < B$. Then $\overline{\mathscr{D}}(A) = \bar{D} = \overline{\mathscr{D}}(B)$ and A is a nonperiodic archimedean class, which contradicts Theorem 3.5.

COROLLARY 2.7. *Let A be a nonperiodic archimedean class and let B be an archimedean class such that $A < B$. Then there exists an archimedean class C such that $A < C$ and $\overline{\mathscr{D}}(A) \succ \overline{\mathscr{D}}(C)$.*

PROOF. We put $C = AB$. Then, by Lemma 1.2, we have $A \leqq C \leqq B$. If it were true that $A = C$, then $A = C < B$ and $\overline{\mathscr{D}}(A) = \overline{\mathscr{D}}(C) = \overline{\mathscr{D}}(AB) \preccurlyeq \overline{\mathscr{D}}(B)$, which contradicts Theorem 3.5. Hence we have $A < C$. Moreover $\overline{\mathscr{D}}(C) = \overline{\mathscr{D}}(AB) \preccurlyeq \overline{\mathscr{D}}(A)$ and the equality is excluded by Theorem 3.6. Thus we have $\overline{\mathscr{D}}(A) \succ \overline{\mathscr{D}}(C)$.

REMARK. Intuitively speaking, when we pursue the course on the associated semilattice of $\bar{P}$ according to the order, every nonperiodic archimedean class appears in the descending path. In particular, every branching element of the associated semilattice is a periodic $\overline{\mathscr{D}}$-class.

THEOREM 3.8. *Let A and B be archimedean classes such that $A < B$.*

(1) *If $AB < B$, then AB is a periodic archimedean class and, for every $a \in A$ and $b \in B$, the product ab is equal to the idempotent of AB.*

(2) *If $BA < B$, then BA is a periodic archimedean class and, for every $a \in A$ and $b \in B$, the product ba is equal to the idempotent of BA.*

PROOF. First we consider (1) and suppose that $AB < B$. Then $\overline{\mathscr{D}}(AB) = \overline{\mathscr{D}}(A) \circ \overline{\mathscr{D}}(B) \leqslant \overline{\mathscr{D}}(B)$ and so, by Theorem 3.5, AB is a periodic archimedean class. Let g be the idempotent of AB and let $a \in A$ and $b \in B$. Then, since $AB < B$, we have $g < b$ and so $ag \leqq ab$. On the other hand, by Lemma 2.3, g is the greatest element of AB and $A \leqq AB$. Hence we have $a \leqq g$. Therefore, by Lemma 1.2, ag is an idempotent and also $ag \in A(AB) = AB$. Hence we have $g = ag$. Since $ab \in AB$, we have $ab \leqq g = ag$ by Lemma 2.3 again. Thus $ab = ag = g$. The assertion (2) can be proved in a similar way.

REMARK. If $AB = B$, the product ab varies in general according to the choice of elements $a \in A$ and $b \in B$. For the study of the structure in this case, it needs to discuss beforehand the inner structure of archimedean classes.

Appendix

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