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# THE ARENS PRODUCT AND DUALITY IN $B^*$-ALGEBRAS. II

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ABSTRACT. Let $A$ be a commutative $B^*$-algebra, $\Phi$ its carrier space and $A^*$ the conjugate space of $A$. Let $A'$ be the closed subspace of $A^*$ spanned by $\Phi$. We show that $A$ is a dual algebra if and only if $A'=A^*$ and for each $x\in A$, the mapping $T_x:f\to f*x$ is a weakly completely continuous operator on $A^*$. This improves an early result by B. J. Tomiuk and the author. A similar result holds for general $B^*$-algebras.

**1. Notation and preliminaries.** Notation and definitions not explicitly given are taken from [7].

Let $A$ be a Banach algebra and $A^*$ its conjugate space. For each $x\in A$ and $f\in A^*$, define

$$(f*x)(y) = f(xy) \qquad (y \in A).$$

Then $f*x\in A^*$. Let $T_x$ be the operator from $A^*$ into itself given by

$$T_x(f) = f*x \qquad (f \in A^*).$$

The mapping $T_x$ is called weakly completely continuous on $A^*$ if for every bounded net $\{f_t\}\subset A^*$, there exist a subnet $\{f_k\}\subset\{f_t\}$ and an element $f\in A^*$ such that $T_x(f_k)\to T_x(f)$ weakly; i.e.,

$$F(f*x) = \lim_k F(f_k*x)$$

for all $F\in A^{**}$, where $A^{**}$ denotes the second conjugate space of $A$.

In this paper, all algebras and spaces under consideration are over the complex field $C$.

2. **Lemmas.** Let $A$ be a $B^*$-algebra. It is well known that $A$ is Arens regular and $A^{**}$ is a $B^*$-algebra under the Arens product $*$ (see [1, p. 869, Theorem 7.1]).

LEMMA 2.1. *Let $A$ be a $B^*$-algebra and $\pi$ the canonical mapping of $A$ into $A^{**}$. Then $A$ is a dual algebra if and only if $\pi(A)$ is a closed two-sided ideal of $A^{**}$.*

PROOF. This is Theorem 5.1 in [7].

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LEMMA 2.2. *Let $A$ be a dual $B^*$-algebra. For each $x \in A$, the mapping $T_x: f \to f * x$ is weakly completely continuous on $A^*$.*

PROOF. Let $\{f_t\}$ be a net in $A^*$ such that $\|f_t\| \leqq 1$. By Alaoglu's theorem [2, p. 424, Theorem V.4.2], there exist a subnet $\{f_k\}$ of $\{f_t\}$ and a linear functional $f \in A^*$ such that

$$f(y) = \lim_k f_k(y) \qquad (y \in A).$$

Let $x \in A$ and $F \in A^{**}$. Since $A$ is a dual algebra, by Lemma 2.1, $\pi(x) * F \in \pi(A)$ and therefore

$$\begin{aligned} F(f * x) &= (\pi(x) * F)(f) = f(\pi(x) * F) \\ &= \lim_k f_k(\pi(x) * F) \\ &= \lim_k F(f_k * x). \end{aligned}$$

Hence $T_x$ is a weakly completely continuous operator on $A^*$. This completes the proof.

3. **A characterization of commutative dual $B^*$-algebras.** In this section, $A$ will denote a commutative $B^*$-algebra and $\Phi$ its carrier space. Let $A'$ be the closed subspace of $A^*$ spanned by $\Phi$.

LEMMA 3.1. *If $A$ is a dual commutative $B^*$-algebra, then we have $A' = A^*$.*

PROOF. Let $x \in A$ and $f \in A^*$. Since $A$ is a dual algebra, the carrier space $\Phi$ of $A$ is discrete. For each $\varphi \in \Phi$, let $e_\varphi$ be the element of $A$ corresponding to the characteristic function of $\varphi$. By the proof of [4, p. 21, Theorem 6], $\{e_\varphi : \varphi \in \Phi\}$ is a maximal orthogonal family of selfadjoint minimal idempotents in $A$ such that

$$x = \sum_\varphi e_\varphi x = \sum_\varphi k_\varphi e_\varphi,$$

where $k_\varphi \in C$. Hence there exists only a countable number of $k_\varphi \neq 0$; say $k_{\varphi 1}, k_{\varphi 2}, \cdots$. Let

$$x_n = \sum_{i=1}^{n} k_{\varphi i} e_{\varphi i} \qquad (n = 1, 2, \cdots).$$

Then we can write $x = \lim_{n\to\infty} x_n$. Since

$$e_\varphi y = e_\varphi y e_\varphi = \varphi(y) e_\varphi$$

for all $y \in A$, we have $f * e_\varphi = f(e_\varphi)\varphi\,(\varphi \in \Phi)$. It now follows that $f * x_n \in A'$. Since $f * x = \lim_{n\to\infty} f * x_n$, $f * x \in A'$.

Suppose $A' \neq A^*$. Then there exists a nonzero linear functional $F \in A^{**}$ such that $F(A') = (0)$. Therefore

$$(\#) \qquad (\pi(x) * F)(f) = F(f * x) = 0,$$

for all $f \in A^*$ and $x \in A$. Let $G \in A^{**}$. By Goldstine's theorem [2, p. 424, Theorem V.4.5], there exists a net $\{x_t\} \subset A$ such that $\pi(x_t) \to G$ weakly in $A^{**}$. Thus $\pi(x_t) * F \to G * F$ weakly in $A^{**}$. Since by (#), $\pi(x_t) * F = 0$ for all $t$, we have $G * F = 0$. Hence $A^{**} * F = (0)$. Since $A^{**}$ is a $B^*$-algebra, it now follows that $F = 0$. This is a contradiction. Therefore $A' = A^*$.

THEOREM 3.2. *Let $A$ be a commutative $B^*$-algebra. Then the following statements are equivalent*:

(i) *$A$ is a dual algebra.*

(ii) *$A' = A^*$ and for each $x \in A$, the mapping $T_x: f \to f * x$ is weakly completely continuous on $A^*$.*

PROOF. (i)$\Rightarrow$(ii). This follows from Lemma 2.2 and Lemma 3.1. (ii)$\Rightarrow$(i). Suppose (ii) holds. Let $F \in A^{**}$ and $x \in A$. By Goldstine's theorem, there exists a net $\{x_t\} \subset A$ such that $\pi(x_t) \to F$ weakly. Let $\varphi \in \Phi$ and let $\{\varphi_p\} \subset \Phi$ be a net converging to $\varphi$ in $\Phi$. Since $T_x$ is a weakly completely continuous operator and since $\{\varphi_p\}$ is bounded, there exist a subnet $\{\varphi_k\} \subset \{\varphi_p\}$ and an element $g \in A^*$ such that

$$G(g * x) = \lim_k G(\varphi_k * x) \qquad (G \in A^{**}).$$

Therefore we have

$$\begin{aligned}(\pi(x) * F)(\varphi) &= \lim_t \varphi(xx_t) = \lim_t \lim_k \varphi_k(xx_t) \\ &= \lim_t \lim_k \pi(x_t)(\varphi_k * x) = \lim_t \pi(x_t)(g * x) \\ &= F(g * x) = \lim_k F(\varphi_k * x) = \lim_k (\pi(x) * F)(\varphi_k).\end{aligned}$$

This shows that $(\pi(x) * F) | \Phi$ is a continuous function on $\Phi$, where $(\pi(x) * F) | \Phi$ denotes the restriction of $\pi(x) * F$ to $\Phi$. Clearly $(\pi(x) * F) | \Phi$ vanishes at infinity on $\Phi$. If $\pi(x) * F \neq 0$, it follows from $A' = A^*$ that $(\pi(x) * F) | \Phi \neq 0$. Since $A$ is a commutative $B^*$-algebra, we conclude that $\pi(x) * F \in \pi(A)$ for all $x \in A$ and $F \in A^{**}$. Therefore $\pi(A)$ is an ideal of $A^{**}$ and so by Lemma 2.1, $A$ is a dual algebra. This completes the proof of the theorem.

REMARK. Let $A$ be a dual commutative $B^*$-algebra. Then by the preceding theorem, $A'' = A^{**}$, where $A'' = A'^*$. Therefore the statements (5) and (6) of Theorem 4.2 in [7] coincide. In this case, the products $\circ$ and $*$ of Theorem 4.2 in [7] also coincide on $A^{**}$.

## 4. Duality in general $B^*$-algebras.

LEMMA 4.1. *Let $A$ be a Banach algebra and $B$ a closed subalgebra of $A$. Let $x \in B$, $f \in A^*$ and $g \in B^*$. If the mapping $f \to f*x$ is weakly completely continuous on $A^*$, then the mapping $g \to g*x$ is weakly completely continuous on $B^*$.*

PROOF. This is clear.

Let $A$ be a $B^*$-algebra. It is well known that $A$ is a dual algebra if and only if every maximal commutative *-subalgebra of $A$ is dual (see [5, p. 179, Theorem 1]). Now by using Lemma 2.2, Theorem 3.2 and Lemma 4.1, we can easily prove the following result:

THEOREM 4.2. *Let $A$ be a $B^*$-algebra. Then $A$ is a dual algebra if and only if the following conditions are satisfied:*

(a) *for every $x \in A$, the mapping $T_x: f \to f*x$ is weakly completely continuous on $A^*$;*

(b) *for every maximal commutative *-subalgebra $B$ of $A$, $B' = B^*$.*

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