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Report

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THE ASYMPTOTIC BEHAVIOR OF A FINITE ENERGY PLANE

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Abstract. Given a compact 3-manifold M equipped with the contact form λ we consider smooth maps $u : \mathbb{C} \to \mathbb{R} \times M$ solving the Cauchy-Riemann equations $Tu \circ i = J(u) \circ Tu$, for a distinguished class of almost complex structures J on $\mathbb{R} \times M$ which are $\mathbb{R}$-invariant and related to λ. If the map is non constant and of finite energy, the projection into M necessarily approaches as $|z| \to \infty$ a periodic solution of the Reeb vector field associated with the contact form.

Assuming the periodic solution to be non degenerate we shall describe the asymptotic behavior of the map u. The paper is a revised version of [5] and includes also [6].

Contents

1. Introduction, Notations, Results

We consider a compact oriented 3-manifold M and choose a contact form λ. Its existence is guaranteed by J. Martinet [13]. By definition, a contact form λ is a 1-form on M such that $\lambda \wedge d\lambda$ defines a volume-form on M. We assume that the orientation of M agrees with the orientation induced by this volume-form. Since the functional $\lambda_m \colon T_mM \to \mathbb{R}$ does not vanish, with the contact-form λ there is associated a 2-dimensional vector bundle $\xi \to M$ over M, whose fibre $\xi_m \subset T_mM$ is defined by

$$\xi_m = \ker(\lambda_m), \quad m \in M.$$

This plane bundle is a so called contact structure of M. The skew symmetric form $\omega = d\lambda \mid \xi \oplus \xi$ is nondegenerate on each fibre and hence defines a symplectic form on

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the vector spaces $\xi_m \subset T_mM$. We denote by (ξ, ω) this symplectic vector bundle. In addition, again in view of the fact that $\lambda \wedge d\lambda$ is a volume-form, the kernel $\ker d\lambda \subset TM$ is 1-dimensional and defines the line-bundle l transversal to ξ having the fibres

$$l_m = \big\{h \in T_mM \mid d\lambda(h,k) = 0, \quad \text{for all} \quad k \in T_mM\big\}.$$

There is a unique nonvanishing vector field $X = X_\lambda$ defined by

$$i_X d\lambda = 0 \qquad \text{and} \qquad i_X\lambda = 1. \tag{1}$$

It is called the Reeb vector field of λ. Thus the tangent bundle TM of M splits into the line-bundle $l \to M$ having the preferred section X and the symplectic plane bundle $\xi \to M$ having the preferred symplectic form $d\lambda$,

$$TM = \mathbb{R}\,X \oplus \xi.$$

If φ_t denotes the flow of X satisfying by definition $\frac{d}{dt}\varphi_t(m) = X\big(\varphi_t(m)\big)$ and $\varphi_0(m) = m \in M$, we conclude from (1) that $\frac{d}{dt}(\varphi_t^*\lambda) = 0$ and $\frac{d}{dt}(\varphi_t^* d\lambda) = 0$. Consequently, $d\varphi_t$ leaves ξ invariant,

$$d\varphi_t(\xi_m) = \xi_{\varphi_t(m)}, \quad m \in M.$$

Moreover, since X is time-independent, $\varphi_t \circ \varphi_s = \varphi_{t+s}$, from which we conclude

$$d\varphi_t X(m) = X\big(\varphi_t(m)\big).$$

Thus $d\varphi_t$ leaves the splitting $\mathbb{R}X \oplus \xi$ of TM invariant. With

$$\pi : \mathbb{R}X \oplus \xi \to \xi$$

we denote the projection along X. The symplectic vector bundle $(\xi, d\lambda) \to M$ has a distinguished class of almost complex structures $J\colon \xi \to \xi$ satisfying $J(m) \in \mathcal{L}(\xi_m, \xi_m)$ and $J(m)^2 = -\operatorname{Id}$, which are compatible with $d\lambda$ in the sense that

$$g_J(a,b) = d\lambda(a, J(m)b) \tag{2}$$

defines a positive definite inner product on each fibre ξ_m. This space of complex structures is contractible, as is well known, see f.e. [1, 3, 11].

Fixing an almost complex structure J compatible with $d\lambda$ we are interested in smooth maps

$$\widetilde{u} := (a,u)\colon \mathbb{C} \to \mathbb{R} \times M,$$

solving the equations

$$\begin{gathered} \pi\frac{\partial u}{\partial s} + J(u)\pi\frac{\partial u}{\partial t} = 0 \\ (u^*\lambda) \circ i = da, \end{gathered} \tag{3}$$

where $z = s + it \in \mathbb{C}$. In order to reformulate this equation we introduce the special almost complex structure $\widetilde{J}$ on the 4-manifold $\mathbb{R} \times M$ as follows:

$$\widetilde{J}(a,m)(h,k) = \big(-\lambda_m(k), J(m)\pi k + hX(m)\big)$$

for $(h,k) \in T_{(a,m)}(\mathbb{R} \times M)$. It is $\mathbb{R}$-invariant. One verifies immediately that $\widetilde{J}^2(h,k) = -(h,k)$. The equation (3) is equivalent to

$$\widetilde{u}_s + \widetilde{J}(\widetilde{u})\widetilde{u}_t = 0. \tag{4}$$

There are plenty of solutions of (4) which are not interesting to us. For example, if $x\colon \mathbb{R} \to M$ is a solution of the Reeb field $\dot{x} = X(x)$ on M, then

$$\widetilde{u}(s+it) := \big(s, x(t)\big) \in \mathbb{R} \times M \tag{5}$$

is a solution, as is readily verified. As was shown in Hofer [4], there is an interesting class of solutions singled out by an "energy requirement". Introduce the class of functions

$$\Sigma = \big\{ f \in C^\infty(\mathbb{R}, [0,1]) \mid \quad f' \geq 0 \big\}$$

and define for $f \in \Sigma$ the 1-form λ_f on $\mathbb{R} \times M$ by

$$\lambda_f(a,m)(h,k) = f(a)\lambda_m(k).$$

For a solution $\widetilde{u} = (a,u)$ of (4) one computes

$$\begin{aligned} \widetilde{u}^* d\lambda_f = \frac{1}{2}\big[& f'(a)\big(a_s^2 + a_t^2 + \lambda(u_s)^2 + \lambda(u_t)^2\big) \\ & + f(a)\big(|\pi u_s|_J^2 + |\pi u_t|_J^2\big)\big] ds \wedge dt, \end{aligned} \tag{6}$$

which is a nonnegative integrand. We used the norm $|h|_J^2 := g_J(h,h)$ for $h \in \xi$, where g_J is defined in (2). Therefore, if $\widetilde{u}$ is a solution of (4), then

$$0 \leqslant \int_{\mathbb{C}} \widetilde{u}^* d\lambda_f \leqslant \infty,$$

and we define the energy $E(\widetilde{u}) \in [0, \infty]$ of a solution by

$$E(\widetilde{u}) = \sup_{f \in \Sigma} \int_{\mathbb{C}} \widetilde{u}^* d\lambda_f.$$

Definition 1.1. *A finite energy plane is a solution $\widetilde{u} = (a,u)$ of (4) satisfying, in addition,*

$$0 < E(\widetilde{u}) < \infty.$$

For the trivial solutions $\widetilde{u}$ defined in (5) we have $E(\widetilde{u}) = \infty$. Indeed, taking a function $f \in \Sigma$ satisfying $f' \neq 0$ we compute

$$\int_{\mathbb{C}} \widetilde{u}^* d\lambda_f = \big[f(\infty) - f(-\infty)\big] \int_{\mathbb{R}} dt = \infty.$$

The significance of the concept of "finite energy plane" lies in the following result relating finite energy planes to periodic orbits of the Reeb vector field X.

Theorem 1.2. *Assume $\widetilde{u} = (a,u)\colon \mathbb{C} \to \mathbb{R} \times M$ is a finite energy plane. Then*

$$T := \int_{\mathbb{C}} u^* d\lambda > 0$$

and there exists a sequence $R_k \to \infty$ such that $\lim_{k\to\infty} u\big(R_k e^{2\pi i t}\big) = x(Tt)$ in $C^\infty(\mathbb{R})$ for a T-periodic solution $x(t)$ of the Reeb vector field $\dot{x}(t) = X\big(x(t)\big)$. If this solution is nondegenerate, then

$$\lim_{R\to\infty} u\big(Re^{2\pi i t}\big) = x(Tt),$$

with convergence in $C^\infty(\mathbb{R})$.

The first part of Theorem 1.2 has been proved in [4]. The strengthening for a non-degenerate asymptotic limit will be proved in the present paper. The distinguished periodic orbit x associated to a suitable sequence $R_k \to \infty$ will be called, in the following, an asymptotic limit. As stated, the asymptotic limit is unique provided there exist a non-degenerate one. In general this should not be case. However we do not know an explicit counter example.

As for the existence question, we recall that if M has a non-vanishing $\pi_2(M)$ then for every contact form λ and every compatible almost complex structure J there exists a finite energy plane. One can also show that for the three-sphere S^3 there exists a finite energy plane for every choice of contact form and compatible J. All these results, with the exception of the case where λ is a tight contact form on S^3 have been proved in [4]. Theorem 1.2 then guarantees a periodic solution for the associated Reeb vector fields X.

We are not concerned in the following with the existence question. Rather we assume the existence of a finite energy plane and determine its asymptotic behavior as $|z| \to \infty$. We shall assume that the T-periodic solution $x(t)$ guaranteed by the first part of Theorem 1.2 is nondegenerate. This requires that it has only one Floquet multiplier equal to 1, and hence is isolated in the set of periodic solutions of the Reeb vector field having their periods close to T. We reformulate the second part of Theorem 1.2 as follows

Theorem 1.3. *Let $\widetilde{u} = (a, u) \colon \mathbb{C} \to \mathbb{R} \times M$ be a nonconstant, finite energy plane as in Theorem 1.2, with an asymptotic T-periodic orbit $x(t)$ which is nondegenerate, then*

$$\lim_{R \to \infty} u\big(Re^{2\pi i t}\big) = x(T \cdot t),$$

moreover the convergence is in $C^\infty(\mathbb{R})$.

The theorem allows us to study, for R large, the finite energy plane in a tubular neighborhood of its limit $x(t)$. It is convenient to consider the holomorphic cylinder $\widetilde{v} = \widetilde{u} \circ \varphi = (a, v)$, with the biholomorphic map $\varphi \colon \mathbb{R} \times S^1 \to \mathbb{C} \setminus \{0\}$ defined by $\varphi(s, t) = e^{2\pi(s+it)}$. Then $v(s, t) \to x(Tt)$ as $s \to \infty$ in $C^\infty(S^1)$. We shall construct local coordinates $\mathbb{R} \times \mathbb{R}^2$ in a tubular neighborhood of $x(t)$. In these coordinates the map $\widetilde{v}$ is represented by $(a, v) = \big(a(s,t), \vartheta(s,t), z(s,t)\big) \colon [s_0, \infty) \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$. If $T = k\tau$, $k \geq 1$, where τ is the minimal period of the periodic solution $x(t)$, then $\vartheta(s, t+1) = \vartheta(s,t) + k$, while the other functions a, z are 1-periodic in t. The main contents of this paper is the proof of the following asymptotic description of a non degenerate finite energy plane.

Theorem 1.4. *There exist constants a_0, $\vartheta_0 \in \mathbb{R}$ and $d > 0$ such that*

$$\big|\partial^\beta[a(s,t) - Ts - a_0]\big| \leqslant Me^{-ds}$$

$$\big|\partial^\beta[\vartheta(s,t) - kt - \vartheta_0]\big| \leqslant Me^{-ds}$$

for all multi-indices β, with constants $M = M_\beta$. Moreover, we have the asymptotic formula for the transversal approach to $x(t)$:

$$z(s,t) = \mathrm{e}^{\int_{s_0}^{s} \alpha(\tau)d\tau}[e(t) + \widehat{r}(s,t)] \in \mathbb{R}^2$$

where $\partial^\beta \widehat{r}(s,t) \to 0$ as $s \to \infty$ uniformly in t for all derivatives. Here $\alpha \colon [s_0, \infty) \to \mathbb{R}$ is a smooth function satisfying $\alpha(s) \to \mu < 0$ as $s \to \infty$. The number μ is an eigenvalue of a self-adjoint operator A in $L^2(S^1, \mathbb{R}^2)$ related to the linearized

Reeb vector field X along the limit orbit $x(t)$. The operator is defined by $A = -J_0 \frac{d}{dt} - S_\infty(t)$, with $S_\infty(t) = S_\infty(t+1)$ a symmetric, 1-periodic, smooth 2×2 matrix function defined by $S_\infty(t) = -TJ_0\pi_m dX(m)\pi_m$, where $m = (kt, 0) \in \mathbb{R}\times\mathbb{R}^2$. Moreover,

$$e(t) = e(t+1) \neq 0,$$

is an eigenvector of A belonging to the eigenvalue $\mu < 0$.

From this asymptotic description of $\widetilde{u}$ we shall deduce, using the similarity principle, the following global consequences

Theorem 1.5. *Let $\widetilde{u} = (a, u)\colon \mathbb{C} \to \mathbb{R}\times M$ be a nonconstant finite energy plane with nondegenerate asymptotic periodic orbit $x(t)$. Let $P = \{x(t) \mid t \in \mathbb{R}\} \subset M$. Then the sets*

$$\big\{z \in \mathbb{C} \mid u(z) \in P\big\}$$
$$\big\{z \in \mathbb{C} \mid \pi\circ Tu(z) = 0\big\}$$

consist of finitely many points.

This means that the map $u\colon \mathbb{C} \to M$ intersects its limit $x(t)$ in at most finitely many points. Moreover, the tangent map Tu has maximal rank except at finitely many points, using that $\pi \circ Tu(z)\colon T_z\mathbb{C} \to \xi_{u(z)}$ is complex linear, in view of the identity $\pi\circ Tu\circ i = J\circ\pi\circ Tu$.

These results are important in a series of applications of holomorphic curves methods to problems in low-dimensional topology and Hamiltonian dynamics, see [7, 8, 9, 10]. There we use holomorphic curve methods in symplectisations to construct open book decompositions for certain three-manifolds, [9], as well as global surfaces of sections for Hamiltonian flows on three-dimensional energy surfaces, [10]. One concludes, in particular, that a Hamiltonian flow on a strictly convex energy surface in $\mathbb{R}^4$ has either precisely 2 or infinitely many periodic orbits, see [10].

There are three technical ingredients to any application. The first is a complete description of the behavior of finite energy planes at infinity, which is the same as the behavior of a finite energy surface near a non removable singularity. This is the contents of the present paper. The second ingredient is the study of embedding properties of finite energy surfaces and their projections into the contact manifold. Here methods from algebraic topology like intersection theory, Maslov indices and winding numbers combined with the asymptotic analysis from the present paper play a crucial role, see [7]. The third ingredient is a Fredholm theory and implicit function type techniques in order to describe families of finite energy planes, see [8].

2. Periodic orbits of X and local coordinates near the ends

We consider a T-periodic solution $x(t)$ for the Reeb vector field $\dot{x} = X_\lambda(x)$. Then $x(0) = x(T)$ and for the linearization of the flow φ_t we have

$$d\varphi_T X\big(x(0)\big) = X\big(\varphi_T(x(0)\big) = X\big(x(0)\big).$$

Hence 1 is an eigenvalue of $d\varphi_T\big(x(0)\big) \in \mathcal{L}(T_{x(0)}M, T_{x(0)}M)$. The periodic solution in called nondegenerate if this is the only eigenvalue equal to 1 of the linear map

$d\varphi_T\big(x(0)\big)$. Since $d\varphi_T\big(x(0)\big)$ leaves the splitting $X\big(x(0)\big) \oplus \xi_{x(0)}$ invariant this is equivalent to the requirement that

$$d\varphi_T\big(x(0)\big) \colon \xi_{x(0)} \to \xi_{x(0)}$$

has no eigenvalue equal to 1. Dynamically a nondegenerate T-periodic solution is isolated on M in the set of periodic solutions having periods close to T. In order to study the asymptotic behavior it is convenient in the following to consider a cylinder instead of a plane. Let $\varphi \colon \mathbb{R} \times S^1 \to \mathbb{C} \setminus \{0\}$ be the biholomorphic map defined by

$$\varphi(s,t) = e^{2\pi(s+it)},$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. If $\widetilde{u} = (a,u)$ is a finite energy plane, we define the finite energy cylinder

$$\widetilde{v} \colon \mathbb{R} \times S^1 \to \mathbb{R} \times M$$

by the composition

$$\widetilde{v} = \widetilde{u} \circ \varphi.$$

In what follows we will use the same letter a to denote a map $\mathbb{C} \to \mathbb{R}$ and also the map $\mathbb{R} \times S^1 \to \mathbb{R}$ obtained by composing a with φ. The map $\widetilde{v} = (a,v)$ satisfies

$$\begin{gathered}\widetilde{v}_s + \widetilde{J}(\widetilde{v})\widetilde{v}_t = 0 \quad \text{on } \mathbb{R} \times S^1 \\ \int_{\mathbb{R}\times S^1} v^* d\lambda = \int_{\mathbb{C}} u^* d\lambda > 0 \\ 0 < E(\widetilde{v}) = E(\widetilde{u}) < \infty.\end{gathered} \tag{7}$$

A solution $\widetilde{v}$ of (7) satisfies the estimate

$$\sup_{\mathbb{R}\times S^1} \big|\nabla\widetilde{v}(s,t)\big| < \infty,$$

from which one derives estimates for all derivatives

$$\sup_{\mathbb{R}\times S^1} \big|\partial^\alpha\widetilde{v}(s,t)\big| < \infty, \quad |\alpha| \geq 1. \tag{8}$$

For a proof of these estimates, based on a “bubbling off” analysis and elliptic estimates, we refer to Hofer [4].

In order to prove Theorem 1.3 we start with

Proposition 2.1. *Let $\widetilde{u}$ be a finite energy plane and assume there exists a sequence $R_k \to \infty$ such that $u\big(R_k e^{2\pi it}\big) \to x(Tt)$ in $C^\infty(S^1, M)$. Assume further that x is a non-degenerate T-periodic solution of the Reeb vector field $\dot{x} = X(x)$ associated with the contact form λ. Then given any S^1-invariant C^∞ neighborhood W of the loop $x(T\cdot)$ in $C^\infty(S^1, M)$ there exists an $R_0 > 0$ such that $u\big(Re^{2\pi i\cdot}\big) \in W$ for all $R \geq R_0$.*

Proof. We view M as being embedded in some $\mathbb{R}^n$ and equip the Frechet space $C^\infty(S^1, \mathbb{R}^n)$ with a translation invariant and S^1-invariant metric which we restrict to the subspace $C^\infty(S^1, M)$. The S^1-action on the loop space is the one induced by S^1 itself. Let $\mathcal{T} \subset C^\infty(S^1, M)$ be the collection of all loops corresponding to periodic solutions of $\dot{x} = X(x)$. With $x_T(\cdot) = x(T\cdot) \in \mathcal{T}$ we denote the loop corresponding to the distinguished T-periodic solution x of the proposition. Since x is non degenerate we find two disjoint and S^1-invariant open sets V_1 and V_2 in

$C^\infty(S^1, M)$ having the properties that $\mathcal{T} \subset (V_1 \cup V_2)$ and $V_1 \cap \mathcal{T} = S^1 * x_T$. In the holomorphic polar coordinates φ the finite energy plane $\widetilde{u}$ becomes the finite energy cylinder $\widetilde{v} = \widetilde{u} \circ \varphi = (b, v)\colon \mathbb{R} \times S^1 \to \mathbb{R} \times M$ and by hypotheses there exists a sequence $s_k \to \infty$ such that

$$v(s_k, \cdot) \to x_T \qquad \text{in } C^\infty(S^1, M).$$

Hence $v(s_k, \cdot) \in V_1$ for k large. Recall from the proof of Theorem 1.2 that every sequence $\sigma_k \to \infty$ possesses a subsequence ${\sigma'}_k$ such that $v({\sigma'}_k, \cdot)$ converges in $C^\infty(S^1, M)$ to an element of $\mathcal{T}$. Using this remark we prove proposition 2.1 indirectly. Assuming that $v(s,t)$ does not converge to $S^1 * x_T$ as $s \to \infty$ we find a sequence $\sigma_k \to \infty$ satisfying $v(\sigma_k, \cdot) \in V_2$ for k large, and passing to subsequences, we may assume that $s_k < \sigma_k < s_{k+1}$ for all k.

Since $s \mapsto v(s, \cdot)$ is a continuous path in $C^\infty(S^1, M)$ there is a sequence ${s'}_k \in (s_k, \sigma_k)$ satisfying $v({s'}_k, \cdot) \notin V_1 \cup V_2$. By theorem 2.1 again we deduce a subsequence ${s''}_k$ of ${s'}_k$ such that $v({s''}_k, \cdot)$ converges to an element $y \in \mathcal{T}$ satisfying $y \notin V_1 \cup V_2$ and hence contradicting $\mathcal{T} \subset (V_1 \cup V_2)$. This finishes the proof of Proposition 2.1. $\square$

We shall study the finite energy cylinder $\widetilde{v} = (a, v)\colon \mathbb{R} \times S^1 \to \mathbb{R} \times M$ as introduced above. We know by the previous discussion that given any S^1-invariant neighborhood W of $x(T\cdot)$ in $C^\infty(S^1, M)$ we have $v(s, \cdot) \in W$ for all s large enough. Hence we can study the solution $v\colon \mathbb{R} \times S^1 \to M$ for large s locally in a tubular neighborhood of the periodic solution x. For this purpose we shall first introduce convenient local coordinates in M near the periodic solution $x(t)$. The coordinates and the contact form in the coordinates will be given by

$$S^1 \times \mathbb{R}^2, \quad f \cdot \lambda_0$$

where the periodic solution lies on $S^1 \times \{0\}$, f is a positive function, λ_0 is the standard contact form

$$\lambda_0 = d\vartheta + x dy$$

on $S^1 \times \mathbb{R}^2$. Since $S^1 = \mathbb{R}/\mathbb{Z}$ we work in the covering space and denote by $(\vartheta, x, y) \in \mathbb{R}^3$ the coordinates, ϑ mod 1. Recall first that if a diffeomorphism $\varphi\colon (N, \mu) \to (M, \lambda)$ between two contact manifolds satisfies $\varphi^*\lambda = \mu$, then the corresponding Reeb vector fields are transformed into each other by

$$X_\mu = (d\varphi)^{-1} \cdot X_\lambda \circ \varphi,$$

as is easily verified. Hence φ maps the solutions of X_μ onto the solutions of X_λ. Indeed, for the flows we conclude $\varphi_t^\lambda \circ \varphi = \varphi \circ \varphi_t^\mu$, for all $t \in \mathbb{R}$. This will be used in the proof of

Lemma 2.2. *Let (M, λ) be a 3-dimensional contact manifold, and let $x(t)$ be a T-periodic solution of the corresponding Reeb vector field $\dot{x} = X_\lambda(x)$ on M. Let τ be the minimal period such that $T = k\tau$ for some positive integer k. Then there is an open neighborhood $U \subset S^1 \times \mathbb{R}^2$ of $S^1 \times \{0\}$ and an open neighborhood $V \subset M$ of $P = \{x(t) \mid t \in \mathbb{R}\}$ and a diffeomorphism $\varphi\colon U \to V$ mapping $S^1 \times \{0\}$ onto P such that*

$$\varphi^*\lambda = f \cdot \lambda_0, \tag{9}$$

with a positive smooth function $f\colon U \to \mathbb{R}$ *satisfying*

$$f(\vartheta, 0, 0) = \tau \quad \text{and} \quad df(\vartheta, 0, 0) = 0 \tag{10}$$

for all $\vartheta \in S^1$.

Proof. Let $\varphi_0\colon U \to V$ be a local diffeomorphism mapping $S^1 \times \{0\}$ onto P such that the contact structure $\ker(\varphi_0^* \lambda)$ is transversal to $S^1 \times \{0\}$. By J. Martinet [13], we find a local diffeomorphism $\varphi_1\colon U \to U'$ in the local coordinates, where U and U' are open neighborhoods of $S^1 \times \{0\} \subset S^1 \times \mathbb{R}^2$, satisfying $\varphi_1(S^1 \times \{0\}) = S^1 \times \{0\}$ and

$$\varphi_1^*(\varphi_0^* \lambda) = g\lambda_0$$

with a nonvanishing smooth function $g\colon U \to \mathbb{R}$. Denoting in the covering space $(\vartheta, x, y) \in \mathbb{R}^3$ the coordinates, the function g is periodic in ϑ of period 1. The Reeb vector field $X_{g\lambda_0}$ associated with the contact form $g\lambda_0$ on $S^1 \times \mathbb{R}^2$ is computed to be

$$X_{g\lambda_0}(\vartheta, x, y) = \left(\frac{1}{g} + \frac{x}{g^2} g_x\right)\frac{\partial}{\partial \vartheta} + \frac{1}{g^2}(g_y - x g_\vartheta)\frac{\partial}{\partial x} - \left(\frac{1}{g^2} g_x\right)\frac{\partial}{\partial y}.$$

By construction, in view of the remark previous to the lemma this Reeb vector field is tangential to the periodic solution $(\alpha(t), 0, 0) = (\varphi_0 \circ \varphi_1)^{-1}\big(x(t)\big)$, where $\alpha(t + \tau) = \alpha(t) + 1$. Recall that τ is the minimal period of $x(t)$. As usual, we work in the covering space $\mathbb{R}$ of $S^1 = \mathbb{R}/\mathbb{Z}$. Therefore,

$$g_x(\vartheta, 0, 0) = g_y(\vartheta, 0, 0) = 0$$
$$X_{g\lambda_0}(\vartheta, 0, 0) = \frac{1}{g(\vartheta, 0, 0)}\frac{\partial}{\partial \vartheta}$$

and

$$\dot{\alpha}(t) = \frac{1}{g(\alpha(t), 0, 0)}. \tag{11}$$

Finally, we define a diffeomorphism $\varphi_2\colon S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}^2$ leaving $S^1 \times \{0\}$ invariant, by

$$\varphi_2(\vartheta, x, y) = (a(\vartheta), \dot{a}(\vartheta)x, y),$$

where $a(\vartheta) = \alpha(\tau\vartheta)$, so that $a(\vartheta + 1) = a(\vartheta) + 1$. Then the composition $\varphi = \varphi_0 \circ \varphi_1 \circ \varphi_2$ is a local diffeomorphism $S^1 \times \mathbb{R}^2 \to M$ mapping the periodic solution $S^1 \times \{0\}$ onto $x(t)$. It satisfies $\varphi^* \lambda = f\lambda_0$, with the function f defined by

$$f(\vartheta, x, y) = g\big(a(\vartheta), \dot{a}(\vartheta)x, y\big) \cdot \dot{a}(\vartheta).$$

The function f satisfies $f(\vartheta + 1, x, y) = f(\vartheta, x, y)$ and a computation, using (11) shows that $f = \tau$ and $f_\vartheta = f_x = f_y = 0$ at every point $(\vartheta, 0, 0) \in S^1 \times \{0\}$ as desired. This finishes the proof of the lemma. $\square$

From now on we shall work in the local coordinates $(\vartheta, x, y) \in \mathbb{R} \times \mathbb{R}^2$, ϑ mod 1 with the contact structure

$$\lambda = f \cdot \lambda_0, \quad \lambda_0 = d\vartheta + x dy$$

with a smooth and positive function $f\colon \mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$, defined near $\mathbb{R}\times\{0\}$ and periodic in ϑ: $f(\vartheta,x,y)=f(\vartheta+1,x,y)$ and satisfying (10). The Reeb vector field $X=X_\lambda(\vartheta,x,y)\in\mathbb{R}^3$ is periodic in ϑ, satisfies

$$X(\vartheta,0,0)=\frac{1}{\tau}(1,0,0),$$

and is given by

$$X(\vartheta,x,y)=\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}=\frac{1}{f^2}\begin{pmatrix}f+xf_x\\f_y-xf_\vartheta\\-f_x\end{pmatrix}.$$

The contact plane ξ_m at $m=(\vartheta,x,y)\in\mathbb{R}^3$, defined by $\xi_m=\big\{k\in\mathbb{R}^3\mid\lambda_m(k)=0\big\}$, is the two dimensional plane

$$\xi_m=\operatorname{span}\big\langle e_1,e_2\big\rangle,$$

where

$$\begin{aligned}e_1&=(\ \ \ 0,1,0)\\e_2&=(-x,0,1)\end{aligned}$$

at the point $m=(\vartheta,x,y)$. Since

$$d\lambda(e_1,e_2)=fd\lambda_0(e_1,e_2)=f$$

we find that the symplectic structure $d\lambda\mid\xi_m\oplus\xi_m$ is, in the basis e_1,e_2 of ξ_m given by the skew symmetric 2×2-matrix $\Omega=\Omega(\vartheta,x,y)$,

$$\Omega=fJ_0,\quad \text{where } J_0=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

is the standard symplectic structure of $\mathbb{R}^2$.

Given to us is an almost complex structure $j_m\colon\xi_m\to\xi_m$ compatible with $d\lambda\mid\xi_m$ and induced by the diffeomorphism $\varphi\colon\mathbb{R}^3\to M$ constructed in Lemma 2.2 via

$$j_m=(d\varphi_m)^{-1}\circ J_{\varphi(m)}\circ d\varphi_m,$$

where J is the almost complex structure chosen in Theorem 1.2. Since j_m is compatible with $d\lambda\mid\xi_m$ it is, in the basis e_1,e_2 of ξ_m, represented by a 2×2-matrix $J=J(m)$ depending smoothly on m and satisfying

$$J^2=-\operatorname{Id},\quad J^T\Omega J=\Omega,\quad J^T\Omega>0. \tag{12}$$

The second condition is, in view of $f>0$, equivalent to $J^TJ_0J=J_0$, hence equivalent to $\det J=1$. The last condition requires the inner product $g(z,z')=\big\langle\Omega z,Jz'\big\rangle=\big\langle J^T\Omega z,z'\big\rangle$, for the coordinates $z,z'\in\mathbb{R}^2$ of ξ_m, to be positive definite. It is equivalent to $\Omega^TJ>0$ and hence, since $f>0$, equivalent to

$$-J_0J>0.$$

Finally, the projection $\pi\colon\mathbb{R}^3\to\xi$ along the Reeb vector field X onto the contact planes takes the form

$$\pi_m(k)=k-\lambda_m(k)X(m)\in\xi_m \tag{13}$$

for $k\in\mathbb{R}^3$.

The positive energy cylinder $\widetilde{v} = (a, v)\colon \mathbb{R} \times S^1 \to M$ of Theorem 1.2 becomes in the local coordinates φ of Lemma 2.2 the map

$$\widetilde{u} = (a, u) = (a, \varphi^{-1} \circ v)\colon [s_0, \infty) \times S^1 \to \mathbb{R}^4$$

for some $s_0 > 0$ large. We shall use the notations

$$\begin{aligned} u(s,t) &= \big(u^1(s,t), u^2(s,t), u^3(s,t)\big) \\ &= \big(\vartheta(s,t), x(s,t), y(s,t)\big). \end{aligned}$$

Working, as usual, in the covering space $\mathbb{R}$ of $S^1 = \mathbb{R}/\mathbb{Z}$, the functions $a(s,t), x(s,t)$ and $y(s,t)$ are 1-periodic in the t variable. The function $\vartheta(s,t)$, however, represents a map from S^1 onto S^1 and satisfies $\vartheta(s, t+1) = \vartheta(s,t) + k$. Indeed, this follows from the fact, proved in [4], that for any sequence $s_n \to \infty$ there is a constant $c \in [0,1)$ such that $u(s_n, t) \to \xi(Tt + c)$ as $n \to \infty$. Here $\xi(t)$ is the T-periodic solution of $\dot{\xi} = X(\xi)$ and $T = k\tau$ with the minimal period τ. In view of $X(\vartheta, 0, 0) = \frac{1}{\tau}(1,0,0)$ we find $\xi(Tt) = (Tt/\tau, 0, 0) = (kt, 0, 0)$ and the claim follows. By construction, the functions $\widetilde{u} = (a,u)\colon [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^4$ solve the equation

$$\widetilde{u}_s + \widetilde{J}(\widetilde{u})\widetilde{u}_t = 0. \tag{14}$$

At the point $(a, m) \in \mathbb{R}^4$, the almost complex structure $\widetilde{J}$ is given by

$$\widetilde{J}(a,m)(h,k) = \big(-\lambda_m(k), j_m(\pi k) + hX(m)\big),$$

where $\pi = \pi_m$ is the projection as in (13). More explicitly we can write the equation (14) as follows:

$$a_s - \lambda(u_t) = 0 \tag{15}$$

$$(\lambda(u_s) + a_t)X(u) = 0 \tag{16}$$

$$\pi u_s + j(\pi u_t) = 0. \tag{17}$$

Note that $X(u) \neq 0$. Abbreviating the partial derivatives

$$u_s = (\vartheta_s, x_s, y_s), \quad u_t = (\vartheta_t, x_t, y_t)$$

we next express the equation (17) in the basis e_1, e_2 of the contact plane ξ_u. In view of (15) and (16) and using the formula (13) we obtain for (15)-(17) the equations

$$\begin{aligned} a_s - \lambda(u_t) &= 0 \\ a_t + \lambda(u_s) &= 0 \end{aligned} \tag{18}$$

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} + J \begin{pmatrix} x_t \\ y_t \end{pmatrix} + a_t \begin{pmatrix} X_2(u) \\ X_3(u) \end{pmatrix} - a_s J \begin{pmatrix} X_2(u) \\ X_3(u) \end{pmatrix} = 0. \tag{19}$$

Here $J = J(u)$ is the 2×2 matrix which represents the almost complex structure j_u in the basis (e_1, e_2) of ξ_u. For the derivatives of ϑ we find the additional equations

$$\begin{aligned} \vartheta_s &= -a_t X_1(u) - x(y_s + a_t)X_3(u) \\ \vartheta_t &= a_s X_1(u) - x(y_t - a_s)X_3(u) \end{aligned}$$

where $x = x(s,t)$. In view of the definition $\lambda = f \cdot \lambda_0$ we can rewrite (18) and find

$$\begin{aligned} a_s &= (\vartheta_t + xy_t)f(u) \\ a_t &= -(\vartheta_s + xy_s)f(u). \end{aligned} \tag{20}$$

We need the following lemma which will be a consequence of Proposition 2.1 and a standard bubbling-off argument as given in [4]. We shall use the notation $|\alpha| = \alpha_1 + \alpha_2$ for the partial derivatives $\alpha = (\alpha_1, \alpha_2)$.

Lemma 2.3. *As* $s \to \infty$

$$\begin{aligned} \partial^\alpha x(s,t) &\to 0 \\ \partial^\alpha y(s,t) &\to 0, \end{aligned}$$

uniformly in t*, for all* $|\alpha| \geq 0$*. Moreover,*

$$\begin{aligned} \partial^\alpha \big[\vartheta(s,t) - kt\big] &\to 0 \\ \partial^\alpha \big[a(s,t) - Ts\big] &\to 0, \end{aligned}$$

uniformly in t*, provided* $|\alpha| \geq 1$*.*

Proof. First we recall from [4] that for a finite energy plane $\widetilde{v} = (a, v)$ all the partial derivatives of v, and the partial derivatives of a satisfying $|\alpha| \geq 1$ are uniformly bounded. (Here we view M as being embedded in some $\mathbb{R}^n$). In addition, we recall from [4] that every sequence $v(s_k, t)$, with $s_k \to \infty$, possesses a subsequence converging with all its t derivatives uniformly to a T-periodic solution of X. By Proposition 2.1 we, therefore, conclude that the statement for the functions $x(s,t)$ and $y(s,t)$ hold true for all the derivatives in time, i.e. for $\alpha = (0, k)$, $k \geq 0$. That it holds true for all derivatives follows from the equations (19) and (20) together with the second statement.

In order to prove the second statement, i.e. the statement for the functions a and ϑ we argue by contradiction. If the assertion is wrong, we find a sequence (s_k, t_k) with $s_k \to \infty$ and $t_k \to t_0 \in [0,1]$ satisfying

$$\big|\partial^\alpha (a - Ts, \vartheta - kt)(s_k, t_k)\big| \geq \varepsilon, \tag{21}$$

for some $\varepsilon > 0$ and some multi-index α of order at least 1. We can always add a real constant to a and an integer to ϑ so that still the equations (18) and (19) hold. This will also not affect our assertion. Define a sequence of functions (a_k, b_k) by

$$\big(a_k(s,t), b_k(s,t)\big) = \big(a(s + s_k, t) - a(s_k, t_k), \vartheta(s + s_k, t) - \vartheta(s_k, t_k)\big).$$

Eventually taking a subsequence the above sequence has a C^∞_{loc}-convergent subsequence, whose limit we denote by $(\widehat{a}, \widehat{b})$. The map $\big(\widehat{a}, \widehat{b}\big)$ is defined on $\mathbb{R} \times \mathbb{R}$, and $\widehat{a}$ is 1-periodic in t while $\widehat{b}$ satisfies $\widehat{b}(s, t+1) = \widehat{b}(s,t) + k$. Moreover, it solves the equation

$$\begin{aligned} \widehat{a}_s &= \tau \widehat{b}_t \\ \widehat{a}_t &= -\tau \widehat{b}_s \end{aligned}$$

on $\mathbb{R} \times \mathbb{R}$. Indeed, this follows from (19) and (20) taking into account that $f(\vartheta, 0, 0) = \tau$, $X_2(\vartheta, 0, 0) = X_3(\vartheta, 0, 0) = 0$, and that $\big(x(s,t), y(s,t)\big) \to (0,0)$ as $s \to \infty$. Hence, the function $f(s + it) = \widehat{a}(s,t) + i\tau \widehat{b}(s,t)$ is holomorphic on $\mathbb{C}$. Recall that gradients of finite energy planes and finite energy cylinders are bounded (Proposition 27 and Proposition 30 in [4]). Consequently, the first derivative of f is bounded and hence the function f is linear, and necessarily of the form

$$(\widehat{a}, \widehat{b})(s,t) = (Ts + c,\ kt + d),$$

with real constants c and d. Recall that $T = k\tau$. Since $|\alpha| \geq 1$, we deduce

$$\partial^{\alpha}(a - Ts, \vartheta - kt)(s_k, t_k) \to \partial^{\alpha}(\widehat{a} - Ts, \widehat{b} - kt)(0, t_0) = (0, 0). \tag{22}$$

Clearly (22) contradicts (21). This completes the proof of the lemma. $\square$

If $X = (X_1, X_2, X_3)$ is the Reeb vector field, we next introduce

$$Y(t, x, y) = \begin{pmatrix} X_2(t, x, y) \\ X_3(t, x, y) \end{pmatrix} \in \mathbb{R}^2.$$

Since $X(t, 0, 0) = (1/\tau, 0, 0)$ we have $Y(t, 0, 0) = 0$ and, therefore, by the mean value theorem

$$Y(t, x, y) = D(t, x, y) \begin{pmatrix} x \\ y \end{pmatrix},$$

with the matrix function

$$D(t, x, y) = \int_0^1 dY(t, \tau x, \tau y) d\tau.$$

In particular,

$$D(t, 0, 0) = dY(t, 0, 0) = \frac{1}{\tau^2} \begin{pmatrix} f_{xy} & f_{yy} \\ -f_{xx} & -f_{xy} \end{pmatrix}, \tag{23}$$

where the right hand side is evaluated at $(t, 0, 0)$. Introducing

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

so that $u(s, t) = \big(\vartheta(s, t), z(s, t)\big)$ and the matrix functions along the solution $u(s, t)$,

$$\begin{aligned} J(s, t) &= J\big(u(s, t)\big) = J\big(\vartheta(s, t), z(s, t)\big) \\ S(s, t) &= \big[a_t - a_s J(s, t)\big] D\big(u(s, t)\big), \end{aligned} \tag{24}$$

we can represent the equation (19) for $z(s, t)$ in the form

$$z_s + J(s, t) z_t + S(s, t) z = 0. \tag{25}$$

Introducing the family of loops

$$z(s) : S^1 \to \mathbb{R}^2$$

by $z(s)(t) = z(s, t)$ our next aim is to show that $\big\|z(s)\big\|_{L^2(S^1)}$ converges exponentially to 0 as as $s \to \infty$.

Inserting $(\vartheta, z) = (kt + c, 0)$ into the matrices $J(\vartheta, z)$ and $dY(\vartheta, z)$ we introduce the family of matrix functions

$$S_{\infty}^c(t) = -TJ(kt + c, 0) \cdot dY(kt + c, 0, 0)$$

for $c \in \mathbb{R}$. The matrix $S_{\infty}^c(t)$ is symmetric, not with respect to the Euclidean inner product but with respect to the inner product $\big\langle \cdot \, , -J_0 J(kt + c) \cdot \big\rangle$ on $\mathbb{R}^2$. To verify this, recall that $J(kt + c, 0)J_0$ is symmetric. Moreover, using $J_0^2 = -\operatorname{Id}$, we have the formula

$$S_{\infty}^c(t) = \frac{T}{\tau^2} J(kt + c, 0) J_0 \cdot \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix},$$

where the last matrix is evaluated at $(kt+c,0,0)$. We introduce the family of operators A^c_∞ in $L^2(S^1,\mathbb{R}^2)$ by

$$A^c_\infty = -J^c(t)\frac{d}{dt} - S^c_\infty(t)\colon W^{1,2}(S^1,\mathbb{R}^2)\subset L^2(S^1)\to L^2(S^1).$$

We have abbreviated

$$J^c(t) := J(kt+c,0).$$

Every operator A^c_∞ is self-adjoint with respect to the inner product

$$\langle x,y\rangle_\infty = \int_0^1 \langle x(t), -J_0 J(kt+c,0)y(t)\rangle dt.$$

Since the inclusion $W^{1,2}(S^1)\to L^2(S^1)$ is compact, the resolvent of A^c_∞ is compact. Hence the spectrum $\sigma(A^c_\infty)$ consists of isolated eigenvalues of multiplicity at most 2, which accumulate at $+\infty$ and $-\infty$. The spectrum $\sigma(A^c_\infty)$ does not depend on the value of c. Observe that A^c_∞ is a relatively compact perturbation of the self-adjoint operator $-J(kt+c,0)\frac{d}{dt}$.

Lemma 2.4. *The T-periodic solution $x(t)=(t/\tau,0,0)\in\mathbb{R}^3$ of the Reeb vector field X is nondegenerate if and only if*

$$0\notin\sigma(A^c_\infty).$$

Proof. Let φ_t denote the flow of $X=(X_1,Y)$. The derivative $d\varphi_t$ along the solution $\varphi_t(0)=x(t)=(t/\tau,0,0)$ leaves the splitting $\mathbb{R}^3=\mathbb{R}\times\mathbb{R}^2=\mathbb{R}X(m)\oplus\xi_m$, $m=x(t)$, invariant and satisfies $d\varphi_t(0)X(0)=X\big(\varphi_t(0)\big)$. Hence, with $d\varphi_t(0)|_{\mathbb{R}^2}=R(t)$, it is of the form

$$d\varphi_t(0)=\begin{pmatrix}1 & 0\\ 0 & R(t)\end{pmatrix},$$

where $R(t)$ satisfies $\frac{d}{dt}R(t)=dY\big(x(t)\big)R(t)$ and $R(0)=I$. The T-periodic orbit $x(t)$ is degenerate if and only if 1 is an eigenvalue of $R(T)$ which is equivalent to 1 is an eigenvalue of $\widetilde{R}(1)$, where $\widetilde{R}(t)=R(Tt)$. Let $e\in\mathbb{R}^2$ be a corresponding eigenvector of $\widetilde{R}(1)$. Then $w(t)=\widetilde{R}(t)e$ is 1-periodic and solves the equation

$$\frac{d}{dt}w(t)=T\,dY\big(x(Tt)\big)w(t), \tag{26}$$

or, equivalently,

$$-J^0(t)\frac{d}{dt}w(t)-S^0_\infty(t)w(t)=0 \tag{27}$$

and hence $w\in\ker(A^0_\infty)$. Conversely if $0\neq w\in\ker(A^0_\infty)$ then $w(t+1)=w(t)$ is a solution of (27) and (26). Hence $\widetilde{R}(t)\;w(0)$ and $w(t)$ are solutions of (26) having the same initial condition $w(0)$ and therefore $w(t)=\widetilde{R}(t)w(0)$. Consequently, $\widetilde{R}(1)w(0)=w(1)=w(0)$ and 1 is an eigenvalue of $\widetilde{R}(1)$ so that $x(t)$ is degenerate. Now the conclusion of the lemma follows from the fact that the spectrum $\sigma(A^c_\infty)$ does not depend on the value of c. $\square$

It will be convenient to introduce the following family of inner products and corresponding norms in $L^2(S^1,\mathbb{R}^2)$

$$\begin{aligned}\big\langle x,y\big\rangle_s &:= \int_0^1 \big\langle x(t), -J_0 J\big(\vartheta(s,t),z(s,t)\big)y(t)\big\rangle\, dt\\ \big\|x\big\|_s^2 &:= \big\langle x,x\big\rangle_s\end{aligned}$$

for $x,y\in L^2(S^1,\mathbb{R}^2)$. These norms are equivalent to the standard $L^2(S^1,\mathbb{R}^2)$ norm. In fact, there exists a constant c independent of s such that

$$\frac{1}{c}\big\|x\big\|_s \leqslant \big\|x\big\|_{L^2} \leqslant c\big\|x\big\|_s \tag{28}$$

for all $x\in L^2(S^1,\mathbb{R}^2)$.

Denote by $S^*(s,t)$ the transpose matrix of $S(s,t)$ with respect to the scalar product $\big\langle \cdot, -J_0 J(s,t)\cdot\big\rangle$ in $\mathbb{R}^2$. It is given by

$$S^* = JJ_0 S^T J_0 J.$$

where $S^T(s,t)$ means the adjoint of $S(s,t)$ with respect to the standard inner product in $\mathbb{R}^2$. Introducing the symmetric and anti-symmetric parts of $S(s,t)$,

$$\begin{aligned}B(s,t) &= \frac{1}{2}\left[S(s,t)+S^*(s,t)\right],\\ C(s,t) &= \frac{1}{2}\left[S(s,t)-S^*(s,t)\right]\end{aligned}$$

we may write the equation (25) in the form

$$z_s + J(s,t)z_t + B(s,t)z + C(s,t)z = 0. \tag{29}$$

Define the operator $A(s): W^{1,2}(S^1,\mathbb{R}^2)\subset L^2(S^1,\mathbb{R}^2)\to L^2(S^1,\mathbb{R}^2)$ by

$$A(s) := -J(s,t)\frac{d}{dt} - B(s,t). \tag{30}$$

The operator $A(s)$ is self-adjoint with respect to the inner product $\big\langle\cdot,\cdot\big\rangle_s$ for every s.

We shall use Lemma 2.4 in order to prove

Lemma 2.5. *Assume the T-periodic solution $x(t)$, with $x(Tt)=(kt,0,0)$ is non degenerate. Then there exists a constant $\eta>0$ and s_0 such that*

$$\big\|A(s)\xi\big\|_s \geq \eta\big\|\xi\big\|_s$$

for all $s\geq s_0$ and $\xi\in W^{1,2}(S^1)$.

Proof. We abbreviate the L_2-norm by $\|\cdot\|$. Since $(1/c)\big\|\xi\big\|_s \leqslant \big\|\xi\big\| \leqslant c\big\|\xi\big\|_s$ for a constant $c>0$ which is independent of s, it is sufficient to prove the estimate in the lemma for L_2-norms. Arguing by contradiction we assume the existence of sequences $s_n\to\infty$, $\varepsilon_k\to 0$ and $\xi_n\in W^{1,2}(S^1)$ satisfying

$$\big\|\xi_n\big\|_{s_n} = 1 \quad\text{and}\quad \big\|A(s_n)\xi_n\big\|_{s_n} \leqslant \varepsilon_n \tag{31}$$

for all k. From Hofer [4] we know that there exists a subsequence denoted again by s_n and a constant $c \in \mathbb{R}$ such that the solution $\big(\vartheta(s,t), z(s,t)\big)$ satisfies

$$\big(\vartheta(s_n,t), z(s_n,t)\big) \to \big(kt + c, 0\big) \tag{32}$$

as $s_n \to \infty$ uniformly in $t \in \mathbb{R}$. This will allow us to control coefficients of the differential operators $\widetilde{A}(s) = -J(s,t)\dfrac{d}{dt} - B(s,t)$. We recall the matrix

$$B(s,t) = \frac{a_t}{2}\big(D + JJ_0D^TJ_0J\big) - \frac{a_s}{2}\big(JD + JJ_0(JD)^TJ_0J\big). \tag{33}$$

Since $a_t \to 0$ as $s \to \infty$ uniformly in t, by Lemma 2.3, the first term in (33) converges to 0 as $s \to \infty$ uniformly in t. As for the second term we recall that $J^TJ_0J = J_0$ and $J_0^2 = -\operatorname{Id}$, so that $JJ_0(J_0D)^TJ_0J = -JJ_0(J_0D)^T$ and hence

$$JD + JJ_0(JD)^TJ_0J = -JJ_0\big[J_0D + (J_0D)^T\big].$$

Abbreviating $R = J_0D + (J_0D)^T$ and observing that at $z = 0$ the matrix $J_0D(\vartheta, 0)$ is symmetric we have the representation

$$R(\vartheta, z) = 2J_0D(\vartheta, 0) + \left[\int_0^1 (\partial_z R)(\vartheta, \tau z)d\tau\right] \cdot z, \tag{34}$$

where $\partial_z R$ denotes the derivative of R with respect to the second variable z. In view of Lemma 2.3, we have $a_s(s_n,t) \to T$ and $z(s_n,t) \to 0$ as $k \to \infty$ and we obtain, using (32) and (34),

$$B(s_n,t) \to TJ(kt+c,0)D(kt+c,0) \tag{35}$$

$$J(s_n,t) \to J(kt+c,0) \tag{36}$$

as $s_n \to \infty$, uniformly in $t \in \mathbb{R}$.

Since $\big\|J(s,\cdot)\xi\big\|_s = \big\|\xi\big\|_s$, there exists a constant $C_0 > 0$ such that for all $\xi \in W^{1,2}(S^1, \mathbb{R}^2)$ and all s

$$\big\|\dot{\xi}\big\| \leqslant C_0\big(\big\|A(s)\xi\big\| + \big\|B(s,\cdot)\xi\big\|\big), \tag{37}$$

from which we conclude that the sequence ξ_n in (31) is bounded in $W^{1,2}$. Since $W^{1,2}$ is compactly embedded in L^2, a subsequence converges in L^2. Consequently, in view of the assumptions (31) and using (35), (36) and (37), the subsequence (still denoted by ξ_n) is a Cauchy sequence in $W^{1,2}$ so that

$$\xi_n \to \xi \quad \text{in } W^{1,2}(S^1, \mathbb{R}^2).$$

From $\widetilde{A}(s_n)\xi_n = -J(s_n,\cdot)\dot{\xi}_n - B(s_n,\cdot)\xi_n \to 0$ in L^2 we derive in view of (35) and (36) that ξ solves the equation

$$-J(kt+c,0)\dot{\xi}(t) - S^c_\infty(t)\xi(t) = 0.$$

Recalling $\big\|\xi\big\| \neq 0$ we see by Lemma 2.4 that the periodic orbit $x(t)$ is degenerate, in contradiction to the assumption of the lemma. The proof of Lemma 2.5 is finished. □

We shall use Lemma 2.4 in order to prove

Lemma 2.6. *Assume the T-periodic solution $x(t)$, with $x(Tt) = (kt, 0, 0)$ is non degenerate. Then the solution $z(s,t) = \big(x(s,t), y(s,t)\big)$ converges exponentially to zero in L^2. There exist $r > 0$ and $s_1 > 0$ such that*

$$\big\|z(s)\big\|_{L^2} \leqslant \big\|z(s_1)\big\|_{L^2} e^{-r(s-s_1)}$$

for $s \geq s_1$. Here $z(s)(t) := z(s,t)$.

Proof. We shall consider the function

$$g(s) = \frac{1}{2}\big\|z(s)\big\|_s^2 = \int_0^1 \big\langle z(s,t), -J_0 J(s,t) z(s,t)\big\rangle dt.$$

To prove the result it suffices to show

$$g''(s) \geq 4r^2 g(s) \tag{38}$$

for some positive constant r and for all $s \geq s_1$. Indeed, assuming the estimate (38) to hold true, we consider the function $f(s) = g'(s) + \delta g(s)$ where $\delta = 2r$. If $\frac{d}{ds}\big[e^{\delta s} g(s)\big] = e^{\delta s} f(s) \leqslant 0$ for all $s \geq s_1$, then

$$g(s) \leqslant e^{-\delta(s-s_1)} g(s_1)$$

which shows

$$\big\|z(s)\big\|_s \leqslant e^{-\delta(s-s_1)/2}\big\|z(s_1)\big\|_{s_1} = e^{-r(s-s_1)}\big\|z(s_1)\big\|_{s_1},$$

and so, in view of the equivalence of the s-norms and the L_2-norms in (28),

$$\big\|z(s)\big\|_{L^2} \leqslant c^2 e^{-r(s-s_1)}\big\|z(s_1)\big\|_{L^2}$$

as required. Assume that $e^{\delta s_2} f(s_2) > 0$ for some s_2. From

$$f'(s) = g''(s) + \delta g'(s) \geq \delta^2 g(s) + \delta g'(s) = \delta f(s)$$

we find $\frac{d}{ds}\big[e^{-\delta s} f(s)\big] \geq 0$ implying for $s \geq s_2$

$$e^{-\delta s} f(s) \geq e^{-\delta s_2} f(s_2)$$

which in turn gives

$$\frac{d}{ds}\big[e^{\delta s} g(s)\big] \geq e^{\delta s} e^{\delta(s-s_2)} f(s_2).$$

Integrating from s_2 to s we obtain

$$g(s) \geq e^{-\delta(s-s_2)} g(s_2) + \frac{f(s_2)}{2\delta}\big[e^{\delta(s-s_2)} - e^{-\delta(s-s_2)}\big] \to \infty$$

as $s \to \infty$ contradicting $g(s) \to 0$. Hence $e^{\delta s} f(s) \leqslant 0$ for $s \geq s_1$ and it remains to prove the estimate (38).

We shall differentiate the function g. Using the equation (29) for $z(s,t)$ and recalling that the matrix $J_0 J(s,t)$ is symmetric with respect to the Euclidean inner product $\langle \cdot\ ,\cdot \rangle$ on $\mathbb{R}^2$ and the matrix $C(s,t)$ is anti-symmetric with respect to $\langle \cdot\ , -J_0 J(s,t)\cdot \rangle$ we obtain

$$\begin{aligned} g'(s) &= \frac{1}{2}\int_0^1 \langle z_s, -J_0 J z\rangle dt + \frac{1}{2}\int_0^1 \langle z, -J_0 J z_s\rangle dt + \frac{1}{2}\int_0^1 \langle z, -J_0 J_s z\rangle dt \\ &= \langle z_s, z\rangle_s + \frac{1}{2}\int_0^1 \langle z, -J_0 J_s z\rangle dt \\ &= \langle A(s)z, z\rangle_s + \frac{1}{2}\int_0^1 \langle z, -J_0 J_s z\rangle dt. \end{aligned}$$

We differentiate the function g once more and use the fact that the operator $A(s)$ is self-adjoint with respect to the inner product $\langle \cdot\ ,\cdot \rangle_s$,

$$\begin{aligned} g''(s) &= \langle A(s)z_s, z\rangle_s + \langle A(s)z, z_s\rangle_s + \int_0^1 \langle A(s)z, -J_0 J_s z\rangle dt + \frac{1}{2}\int_0^1 \langle z_s, -J_0 J_s z\rangle dt \\ &\quad + \frac{1}{2}\int_0^1 \langle z, -J_0 J_s z_s\rangle dt + \frac{1}{2}\int_0^1 \langle z, -J_0 D J_{ss} z\rangle dt + \int_0^1 \langle -J_s z_t - B_s z, -J_0 J z\rangle dt \\ &= 2\langle A(s)z, z_s\rangle_s + \int_0^1 \langle A(s)z, -J_0 J(-JJ_s)z\rangle dt + \int_0^1 \langle z_s, -J_0 J(-JJ_s)z\rangle dt \\ &\quad + \frac{1}{2}\int_0^1 \langle z, -J_0 J(-JJ_{ss})z\rangle dt + \int_0^1 \langle -J_s z_t - B_s z, -J_0 J z\rangle dt \\ &= 2\langle A(s)z, z_s\rangle_s - \langle A(s)z, JJ_s)z\rangle_s - \langle z_s, JJ_s z\rangle_s - \frac{1}{2}\langle z, JJ_{ss} z\rangle_s \\ &\quad + \int_0^1 \langle -J_s z_t - B_s z, -J_0 D J z\rangle dt \\ &= 2\|A(s)z\|_s^2 - \langle A(s)z, Cz\rangle_s - 2\langle A(s)z, JJ_s z\rangle_s + \langle Cz, JJ_s z\rangle_s \\ &\quad - \frac{1}{2}\langle z, JJ_{ss} z\rangle_s + \int_0^1 \langle -J_s z_t - B_s z, -J_0 J z\rangle dt \end{aligned}$$

Observe that

$$\begin{aligned} \langle -J_s z_t, -J_0 J z\rangle &= \langle -J_s J[-J z_t - Bz], -J_0 J z\rangle - \langle J_s J B z, -J_0 J z\rangle \\ &= \langle -J z_t - Bz, J^T J_s^T J_0 J z\rangle - \langle J_s J B z, -J_0 J z\rangle \\ &= \langle A(s)z, J^T J_s^T J_0 J z\rangle - \langle J_s J B z, -J_0 J z\rangle \\ &= \langle A(s)z, J_0 J_s z\rangle - \langle J_s J B z, -J_0 J z\rangle \\ &= \langle A(s)z, -J_0 J(JJ_s)z\rangle - \langle J_s J B z, -J_0 J z\rangle. \end{aligned}$$

We have used that $J_s^T J_0 J = -J^T J_0 J_s$. Hence

$$\begin{aligned} \int_0^1 \langle -J_s z_t - B_s z, -J_0 J z\rangle dt &= \langle A(s)z, JJ_s z\rangle_s - \int_0^1 \langle [J_s J B + B_s]z, -J_0 J z\rangle \\ &= \langle A(s)z, JJ_s z\rangle_s - \langle [J_s J B + B_s]z, z\rangle_s \end{aligned}$$

and

$$\begin{aligned} g''(s) &= 2\big\|A(s)z\big\|_s^2 - 2\big\langle A(s)z, Cz\big\rangle_s - \big\langle A(s)z, JJ_s)z\big\rangle_s + \big\langle Cz, JJ_s z\big\rangle_s \\ &\quad - \frac{1}{2}\big\langle z, JJ_{ss}z\big\rangle_s - \big\langle\big[J_sJB + B_s\big]z, z\big\rangle_s \\ &\geq 2\big\|A(s)z\big\|_s \cdot \Big[\big\|A(s)z\big\|_s - \big\|JJ_s\big\| \cdot \big\|z\big\|_s - \big\|C\big\| \cdot \big\|z\big\|_s\Big] \\ &\quad - \Big[\big\|C\big\| \cdot \big\|JJ_s\big\| + \big\|B\big\| \cdot \big\|JJ_s\big\| + \big\|B_s\big\| + \big\|JJ_{ss}\big\|/2\Big] \cdot \big\|z\big\|_s^2. \end{aligned}$$

Since B_s, J_s and J_{ss} contain factors converging to 0 as $s \to \infty$ uniformly in t in view of Lemma 2.3 we conclude, in view of Lemma 2.5,

$$g''(s) \geq 2\eta\big[\eta - r(s) - \big\|C(s,\cdot)\big\|\big] \cdot \big\|z\big\|_s^2 - r(s)\big\|z\big\|_s^2, \tag{39}$$

where $r(s) \to 0$ as $s \to \infty$. It remains to show that $\big\|C(s,\cdot)\big\| \to 0$ as $s \to \infty$. Recall that

$$\begin{aligned} C(s,t) &= \frac{1}{2}\big[S(s,t) - S(s,t)^*\big] \\ &= \frac{a_t}{2}\big[D - JJ_0D^TJ_0J\big] - \frac{a_s}{2}\big[JD - JJ_0D^TJ_0\big], \end{aligned}$$

The first term converges to 0 since $a_t \to 0$ as $s \to \infty$ uniformly in t in view of Lemma 2.3. To estimate the second term we note that

$$JD - JJ_0D^TJ_0 = JJ_0\big[(J_0D)^T - (J_0D)\big].$$

Define $R := (J_0D)^T - (J_0D)$. Since $J_0D(\vartheta, 0)$ is symmetric we have the representation

$$R(\vartheta, z) = \left[\int_0^1 (\partial_z R)(\vartheta, \tau z) d\tau\right] \cdot z.$$

Here $\partial_z R$ means the derivative of $\widetilde{R}$ with respect to the second variable z. Now, $\widetilde{R}\big(\vartheta(s,t), z(s,t)\big) \to 0$ uniformly in t as $s \to \infty$ because $z(s,t) \to 0$ as $s \to \infty$ by Lemma 2.3. Therefore, $\big\|C(s,\cdot)\big\| \to 0$ as $s \to \infty$ and, in view of (39),

$$g''(s) \geq (\eta/4)^2 g(s)$$

for all $s \geq s_1$. So the claim (38) follows with $r = \eta/8$ and the proof of Lemma 2.6 is complete. □

We will now use the exponential L^2-estimate of the solution $z(s,t)$ in Lemma 2.6 in order to prove that

$$\begin{aligned} \vartheta(s,t) - kt - \vartheta_0 &\to 0 \\ a(s,t) - Ts - a_0 &\to 0 \end{aligned}$$

as $s \to \infty$ uniformly in t.

Lemma 2.7. *If the T-periodic solution $x(t) = (t/\tau, 0, 0)$ of X is non degenerate, then there exist constants $a_0, \vartheta_0 \in \mathbb{R}$ such that*

$$\partial^\alpha\big[a(s,t) - Ts - a_0\big] \to 0 \tag{40}$$

$$\partial^\alpha\big[\vartheta(s,t) - kt - \vartheta_0\big] \to 0 \tag{41}$$

as $s \to \infty$, uniformly in t, for all $|\alpha| \geq 0$.

Proof. Recall that a and $\vartheta\colon [s_1,\infty)\times\mathbb{R}\to\mathbb{R}$ are smooth solutions of

$$\begin{aligned} a_s &= \quad(\vartheta_t + xy_t)f(\vartheta,z) \\ a_t &= -(\vartheta_s + xy_s)f(\vartheta,z). \end{aligned} \tag{42}$$

Moreover, $f(\vartheta,0)\equiv\tau$ and we can write

$$f(\vartheta,z)=\tau+\left[\int_0^1 f_z(\vartheta,tz)dt\right]z\equiv\tau+b(\vartheta,z)z.$$

With the 1-periodic functions $\widetilde{a},\widetilde{\vartheta}\colon [s_1,\infty)\times S^1\to\mathbb{R}$, defined by

$$\begin{aligned} \widetilde{a}(s,t)&=a(s,t)-Ts \\ \widetilde{\vartheta}(s,t)&=\vartheta(s,t)-kt, \end{aligned}$$

we know from Lemma 2.3,

$$\begin{aligned} \partial^\beta\widetilde{a}(s,t)&\to 0 \\ \partial^\beta\widetilde{\vartheta}(s,t)&\to 0, \end{aligned}$$

for $|\beta|\geq 1$ as $s\to\infty$, uniformly in t. Recalling $T=k\tau$ equation (42) becomes

$$\begin{aligned} \widetilde{a}_s-\tau\widetilde{\vartheta}_t &= \tau xy_t+\big(\vartheta_t+xy_t\big)bz \\ \widetilde{\vartheta}_s+\frac{1}{\tau}\widetilde{a}_t &= -xy_s-\frac{1}{\tau}\big(\vartheta_s+xy_s\big)bz \end{aligned} \tag{43}$$

Abbreviating

$$w(s,t)=\begin{pmatrix}\widetilde{a}\\ \widetilde{\vartheta}\end{pmatrix},$$

we can write equation (43) in the form

$$w_s+\widehat{J}w_t=h,\qquad \widehat{J}=\begin{pmatrix}0 & -\tau\\ 1/\tau & 0\end{pmatrix}, \tag{44}$$

with the smooth function $h\colon [s_1,\infty)\times S^1\to\mathbb{R}$ defined by the right hand side of (43) and satisfying, in view of the exponential estimate of $\big\|z(s)\big\|$, the estimate

$$\big\|h(s)\big\|\leqslant Me^{-rs},$$

for some constant M. Introducing the mean values $\alpha(s)=\int_0^1 w(s,t)dt$ we define $\widetilde{w}(s,t)=w(s,t)-\int_0^1 w(s,t)dt=w(s,t)-\alpha(s)$. Then

$$\alpha'(s)=\int_0^1 w_s(s,t)dt=\int_0^1 h(s,t)dt$$

and so,

$$|\alpha'(s)|\leqslant\int_0^1 |h(s,t)|dt\leqslant\big\|h(s)\big\|\leqslant Me^{-rs}.$$

Hence

$$|\alpha(s_2)-\alpha(s_1)|\leqslant\frac{2M}{r}\big[e^{-rs_1}-e^{-rs_2}\big],$$

for $s_1<s_2$, showing that $\alpha(s)\to c\in\mathbb{R}^2$ as $s\to\infty$. Since $\int_0^1\widetilde{w}(s,t)dt=0$, it follows from Lemma 2.3 that

$$\big|\widetilde{w}(s,t)\big|\leqslant\sup_{\tau\in S^1}\big|\widetilde{w}_t(s,\tau)\big|=\sup_{\tau\in S^1}\big|w_t(s,\tau)\big|\to 0.$$

Consequently, $w(s,t) \to c =: (a_0, \vartheta_0)$ as $s \to 0$ and together with the estimates for the derivatives $|\alpha| \geq 1$ in Lemma 2.3 the proof is finished. $\square$

Without loss of generality we shall put for convenience $a_0 = \vartheta_0 = 0$.

Thus inserting $(\vartheta, z) = (kt, 0)$ into $J(\vartheta, z)$ and $dY(\vartheta, z)$ we abbreviate

$$
\begin{aligned}
J(t) &= J(kt, 0) \\
S_\infty(t) &= -TJ(t)dY(kt, 0)
\end{aligned} \tag{45}
$$

so that

$$
S(s,t) \to S_\infty(t) \tag{46}
$$

as $s \to \infty$ uniformly in t. The limit operator $A_\infty : W^{1,2}(S^1, \mathbb{R}^2 \subset L^2(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2)$ is defined by

$$
A_\infty = -J(t)\frac{d}{dt} - S_\infty(t). \tag{47}
$$

Summarizing the set up and the results so far we consider a finite energy cylinder $\widetilde{v} = (a, v) \colon \mathbb{R} \times S^1 \to \mathbb{R} \times M$ and assume that it is nondegenerate in the sense of Theorem 1.2 requiring the existence of a nondegenerate T-periodic solution $x(t)$ of the Reeb vector field X satisfying $v(s, \cdot) \to x(T\cdot)$ as $s \to \infty$ in $C^\infty(S^1)$. The period is $T = k\tau$, with the minimal period τ. Then there are local coordinates in a tubular neighborhood of $x(\mathbb{R}) \subset M$ in which the cylinder is represented by the map

$$
\widetilde{u} = (a, u) \colon [s_0, \infty) \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^3
$$

for some large $s_0 > 0$. The functions $\widetilde{u}(s,t) = \big(a(s,t), \vartheta(s,t), x(s,t), y(s,t)\big)$ are periodic in t except the function ϑ which satisfies $\vartheta(s, t+1) = \vartheta(s,t) + k$. The convergence to the periodic solution $x(Tt)$ becomes

$$
u(s,t) \to (kt, 0, 0) \in \mathbb{R}^3 \quad (s \to \infty)
$$

and the map (a, u) has the asymptotic properties described in Lemma 2.3. The functions solve the equations

$$
\begin{aligned}
a_s &= \quad (\vartheta_t + x y_t) f(u) \\
a_t &= -(\vartheta_s + x y_s) f(u)
\end{aligned} \tag{48}
$$

$$
z_s + J(s,t) z_t + S(s,t) z = 0, \tag{49}
$$

where $z = (x, y)$. The function $f = f(t, z) \colon \mathbb{R}^3 \to \mathbb{R}$ is smooth, 1-periodic in t, satisfies $f(t, 0) = \tau$ and $df(t, 0) = 0$. Moreover, $S(s,t) \to S_\infty(t)$ as $s \to \infty$ in C^∞. The matrix S_∞ is periodic in t, symmetric with respect to the inner product $\big\langle \cdot\, , -J_0 J(t) \cdot \big\rangle$, and satisfies $0 \notin \sigma(A_\infty)$.

Our aim is to prove the following result about the asymptotic behavior of nondegenerate finite energy planes locally near the limit periodic solution.

Theorem 2.8. (Asymptotic behavior of nondegenerate finite energy planes) *Assume the functions $(a, u) \colon [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^4$ meet the above conditions. Then*

$$
\begin{aligned}
a(s,t) &= Ts + a_0 + \widehat{a}(s,t),\\
\vartheta(s,t) &= kt + \vartheta_0 + \widehat{\vartheta}(s,t)
\end{aligned}
$$

and **either**

(i) $z(s,t) \equiv 0$ *for all* $(s,t) \in [s_0, \infty) \times \mathbb{R}$,

or

(ii)

$$z(s,t) = \mathrm{e}^{\int_0^s \gamma(\tau)d\tau}\big[e(t) + \widehat{r}(s,t)\big].$$

Here, a_0 *and* ϑ_0 *are two real constants, and*

$$\partial^\alpha \widehat{r}(s,t) \to 0 \qquad \text{as } s \to \infty$$

uniformly in $t \in \mathbb{R}$ *and for all derivatives* $\alpha = (\alpha_1, \alpha_2)$. *In addition, there are constants* $M_\alpha > 0$ *and* $d > 0$ *such that*

$$\big|\partial^\alpha \widehat{a}(s,t)\big|, \quad \big|\partial^\alpha \widehat{\vartheta}(s,t)\big| \leqslant M_\alpha e^{-ds}$$

for $s \geq 0$ *and all derivatives* α. *Moreover, the smooth function* $\gamma : [0,\infty) \to \mathbb{R}$ *converges,* $\gamma(s) \to \mu < 0$ *as* $s \to \infty$. *The limit* μ *is an eigenvalue of a self-adjoint operator* A_∞ *in* $L^2(S^1, \mathbb{R}^2)$. *The nowhere vanishing function* $e(t) = e(t+1)$ *represents an eigenvector belonging to* μ.

We emphasize, that the first alternative does not occur if the data (a,u) is the restriction of a finite energy plane. The reason is as follows. If $z(s,t) \equiv 0$ for $s \geq s_0$, then $\pi u_s(s,t) \equiv 0$ for $s \geq s_0$ since the contact plane ξ along the periodic orbit agrees with the z-plane in our coordinates. By means of the similarity principle as in the proof of Theorem 5.2 below we conclude that $\pi \circ Tu = 0$ everywhere on the plane $\mathbb{C}$. Consequently, $u^*d\lambda = |\pi u_s|^2 = 0$ by (6), contradicting the statement

$$\int_{\mathbb{C}} u^* d\lambda = T > 0$$

in Theorem 1.2.

3. Proof of the asymptotic formula (Theorem 2.8)

We shall study smooth functions $(a, \vartheta, z) \colon [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^4$ solving the equation (48) and (49) and having the asymptotic properties described in Lemma 2.3, Lemma 2.4 and Lemma 2.7. In particular,

$$
\begin{aligned}
a(s,t) - Ts - a_0 &\to 0\\
\vartheta(s,t) - kt - \vartheta_0 &\to 0\\
z(s,t) &\to 0
\end{aligned}
\tag{50}
$$

as $s \to \infty$ uniformly in t with all their derivatives. The period of the limiting periodic solution $x(t)$ of the Reeb vector field satisfies $T = k\tau$ with the minimal period τ. In order to simplify the notation in the proof we assume

$$T = \tau = k = 1$$
$$a_0 = \vartheta_0 = 0.$$

Our aim is to prove the asymptotic formula for the map $z(s,t)$ which solves the equation, (25), namely,

$$z_s + J(s,t)z_t + S(s,t)z = 0. \tag{51}$$

We claim that by the means of a coordinate transformation we can assume without loss of generality that

$$J(s,t) = J_0. \tag{52}$$

In order to prove this claim we first recall that $-J_0J(m)$ is a positive definite and symplectic matrix for every m.

Define the new coordinates $(\vartheta', z') \in \mathbb{R} \times \mathbb{R}^2$ by $\vartheta' = \vartheta$ and

$$z = T(m)z'$$
$$T(m) = \big(-J_0J(m)\big)^{-1/2},$$

$m = (\vartheta, x, y)$. Then $T = T(m)$ is symmetric and symplectic, so that $TJ_0T = J_0$, and we claim that

$$T(m)^{-1}J(m)T(m) = J_0, \qquad \text{all } m. \tag{53}$$

Indeed with T also T^{-1} is symmetric and symplectic: $T^{-1}J_0T^{-1} = J_0$. Since, by definition, $T^{-2} = -J_0J$ and $J_0^{-1} = -J_0 = J_0^T$, we have $T^{-1} = J_0T(-J_0)$ and hence $T^{-1} = J_0T(-J_0J) = J_0TT^{-2} = J_0T^{-1}$ proving the formula (53). Introduce now

$$T(s,t) \equiv T\big(u(s,t)\big),$$

and define the map ξ by

$$z(s,t) = T(s,t)\xi(s,t). \tag{54}$$

Then $\xi(s,t)$ is a solution of the equation

$$\xi_s + J_0\xi_t + \widehat{S}(t,s)\xi = 0,$$

where

$$\widehat{S}(t,s) = J_0T^{-1}T_t + T^{-1}T_s + T^{-1}ST.$$

Moreover, from $T(s,t) \to T_\infty(t)$ as $s \to \infty$ in $C^\infty(\mathbb{R})$, where

$$T_\infty(t) = \big(-J_0M(kt,0)\big)^{-1/2},$$

we find that $\widehat{S}(t,s) \to \widehat{S}_\infty(t)$ as $s \to \infty$ in $C^\infty(\mathbb{R})$, where

$$\widehat{S}_\infty(t) = -J_0[T_\infty^{-1}dYT_\infty - T_\infty^{-1}\dot{T}_\infty].$$

The dot denotes the derivative of T_∞ in t, and $dY(m)$ is the restriction of the linearized Reeb vector field $\pi_m dX(m)\pi_m$ along $m = (kt, 0)$, as introduced above. Since J_0dY along the periodic solution is symmetric, one verifies that $\widehat{S}_\infty(t)$ is symmetric, using that T_∞ is symmetric and symplectic. Hence the operator

$$\widehat{A}_\infty = -J_0\frac{d}{dt} - \widehat{S}_\infty(t)$$

is self-adjoint, and, as before, $0 \notin \sigma(\widehat{A}_\infty)$ if and only if the periodic solution of the Reeb vector field X is non degenerate.

We note that the exponential decay estimates in Lemma 2.4 still holds true after the change of variables. Without loss of generality we therefore assume in the following that $J(s,t) \equiv J_0$. Hence, using the old notation and writing $\xi = z$, $\widehat{S} = S$ and $\widehat{A}_\infty = A_\infty$ we shall study an equation of the form

$$z_s + J_0 z_t + S(s,t)z = 0 \tag{55}$$

with the standard almost complex structure J_0.

We begin with a proposition concerning the L^2-convergence of (x,y). It is of course related to our previous discussion. However the conclusion is now somewhat stronger, since we have convergence of $u(s,\cdot)$ as $s \to \infty$.

Proposition 3.1. *Assume $(a,\vartheta,z)\colon [s_0,\infty) \times \mathbb{R} \to \mathbb{R}^4$ solves the equations* (48) *and* (55) *and has the asymptotic properties of Lemma 2.3. Then $\|z(s)\|_{L^2} \to 0$ as $s \to \infty$. If*

$$\|z(s^*)\|_{L^2} = 0$$

for some $s^* \geq s_0$, then

$$z(s,t) \equiv 0$$

for all $s \geq s_0$ and $t \in \mathbb{R}$.

Proof. The first claim is an immediate consequence of Lemma 2.3. Assume that $\|z(s^*)\| = 0$, then $z(s^*,t) = 0$ for all $t \in \mathbb{R}$ and we pick a $t^* \in \mathbb{R}$ such that $z(s^*,t^*) = 0$. Since $z(s,t)$ solves the partial differential equation (55) there is an open neighborhood $D \subset \mathbb{R}^2$ of the zero (s^*,t^*) of z, on which z can be represented as

$$z(s,t) = \Phi(\zeta)h(\zeta), \quad \zeta = s + it \in \mathbb{C}.$$

Here $\Phi\colon D \to GL(\mathbb{C})$ is continuous and $h\colon D \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic function. (This is the generalized similarity principle for which we refer to [1] or [4]). Since (s^*,t^*) is a cluster point of zeroes of z we conclude that $h \equiv 0$ on D and hence $z(s,t) \equiv 0$ on D. Consequently, $z(s,t) = 0$ for all $s \geq s_0$ and $t \in \mathbb{R}$. The proof of the proposition is complete. $\square$

For the remainder of this section we shall assume that $\|z(s)\|_{L^2} \neq 0$ for all $s \geq s_0$. We know that $\|z(s)\|_{L^2} \to 0$ as $s \to \infty$. Using the assumption $0 \notin \sigma(A_\infty)$, we shall derive an exponential formula.

Lemma 3.2.

$$\|z(s)\|_{L^2} = e^{\int_{s_0}^{s} \alpha(\tau)d\tau} \cdot \|z(s_0)\|_{L^2},$$

for a smooth function $\alpha\colon [s_0,\infty) \to \mathbb{R}$ satisfying $\lim_{s\to\infty} \alpha(s) = \mu < 0$ and $\mu \in \sigma(A_\infty)$.

Proof. We abbreviate $\|\cdot\| \equiv \|\cdot\|_{L^2}$ and assume $\|z(s)\| \neq 0$. Introduce the smooth function

$$\xi(s,t) = \frac{z(s,t)}{\|z(s)\|}, \qquad \text{then} \qquad \|\xi(s)\| = 1.$$

Differentiating in s, using that z solves the equation (55) we obtain

$$\xi_s = -J\xi_t - S\xi - \frac{1}{2}\frac{\frac{d}{ds}\|z\|^2}{\|z\|^2}\xi, \tag{56}$$

abbreviating $J \equiv J_0$. Denoting by $\langle\cdot,\cdot\rangle$ the $L^2(S^1)$ scalar product we conclude from $\langle\xi,\xi\rangle = 1$ that $\langle\xi_s,\xi\rangle = 0$, and inserting the above equation we find

$$\frac{1}{2}\frac{\frac{d}{ds}\|z\|^2}{\|z\|^2} = \big\langle -J\xi_t - S\xi, \xi\big\rangle \equiv \alpha(s), \tag{57}$$

so that

$$\|z(s)\| = e^{\int_{s_0}^{s}\alpha(\tau)d\tau}\|z(s_0)\|. \tag{58}$$

We have to show that the smooth function α converges as $s \to \infty$ to a negative eigenvalue of A_∞. Dropping the subscript we recall that $A = A_\infty = -J\frac{d}{dt} - S_\infty(t)$ and write

$$-J\frac{d}{dt} - S(s,t) = A + \varepsilon \qquad \text{and} \qquad \varepsilon \equiv (S_\infty - S).$$

Then, by the previous section, $\varepsilon(s,t) \to 0$ as $s \to \infty$ in $C^\infty(\mathbb{R})$. Denoting by the prime the derivative in the s-variable we can write the equation (56) for ξ and the smooth function α in (57) as

$$\xi' = (A+\varepsilon)\xi - \alpha\xi \tag{59}$$

$$\alpha(s) = \big\langle (A+\varepsilon)\xi, \xi\big\rangle. \tag{60}$$

We differentiate this function. Using the selfadjointnes of A and the identity $\big\langle \xi, (A+\varepsilon)\xi - \alpha\xi\big\rangle = 0$, we obtain

$$\alpha' = 2\|\xi'\|^2 - \big\langle \varepsilon\xi, \xi'\big\rangle + \big\langle \varepsilon'\xi, \xi\big\rangle. \tag{61}$$

In view of $\|\xi\| = 1$, we have the estimates:

$$\big|\big\langle \varepsilon\xi, \xi'\big\rangle\big| \leqslant O(s)\|\xi'(s)\|$$
$$\big|\big\langle \varepsilon'\xi, \xi\big\rangle\big| \leqslant O(s),$$

where $O(s) \to 0$, as $s \to \infty$. Consequently,

$$\alpha'(s) \geq 2\|\xi'\|\,\big[\|\xi'\| - O(s)\big] - O(s). \tag{62}$$

Next we claim that α is bounded

$$|\alpha(s)| \leqslant C, \quad s \geq s_0 \tag{63}$$

for some $C > 0$. Arguing by contradiction we assume that α is not bounded from above. Then there is a sequence $s_n \to \infty$ such that $\alpha(s_n) \to \infty$. On the other hand if $\alpha(s) \geq \delta > 0$ for all s large, then $\|z(s)\| \to +\infty$ in view of (58) which contradicts $\|z(s)\| \to 0$, as $s \to \infty$. Hence there exists another sequence $s_n' \to \infty$ such that $\alpha(s_n') < \delta$ and so the function α has an "oscillatory" behavior as $s \to \infty$. Since $\sigma(-J\frac{d}{dt}) = 2\pi\mathbb{Z}$ it follows from Kato's perturbation theory for isolated eigenvalues of self-adjoint operators [12], that there is an $L > 0$ and an integer m, so that every interval of $\mathbb{R}$ having length equal to L contains at most m points of the spectrum $\sigma(A = -J\frac{d}{dt} - S_\infty)$ belonging to the perturbed operator A. Consequently, there are spectral gaps of fixed size: there is a sequence $r_n \to \infty$ and a constant $d > 0$ satisfying

$$[r_n - d, r_n + d] \cap \sigma(A) = \emptyset. \tag{64}$$

Hence by the oscillatory behavior of α we find a sequence $\tau_n \to \infty$ satisfying $\alpha(\tau_n) = r_n$ and $\alpha'(\tau_n) \leqslant 0$. It then follows from (62) that $\|\xi'(\tau_n)\| \to 0$ as $n \to \infty$. Since, by (59), $\xi' = A\xi - \alpha(s)\xi + \varepsilon$, we can estimate

$$\begin{aligned} \|\xi'(s)\| &\geq \|[A - \alpha(s)]\xi\| - O(s) \\ &\geq \operatorname{dist}\big(\alpha(s), \sigma(A)\big) - O(s). \end{aligned} \tag{65}$$

Here we have used, that $\|\xi\| = 1$ and that for the resolvent of a self-adjoint operator $\|(A-\gamma)^{-1}\| = \big[\operatorname{dist}\big(\gamma, \sigma(A)\big)\big]^{-1}$. Using (65) we conclude from (64) that $\|\xi'(\tau_n)\| \geq \frac{d}{2} > 0$ contradicting $\|\xi'(\tau_n)\| \to 0$ as $m \to \infty$. This contradiction shows that α is indeed bounded from above. The same argument shows that α is also bounded from below, proving the claim (63).

There exists a sequence $s_n \to \infty$ such that $\|\xi'(s_n)\| \to 0$. Indeed, otherwise, for all large s, $\|\xi'\| \geq \delta > 0$, hence $\alpha' \geq \delta^2$ in view of (62), and $\alpha(s) \geq \delta^2(s-s_0)+\alpha(s_0)$, so that $\|z(s)\| \to \infty$, in view of (58). This contradicts $\|z(s)\| \to 0$. Since α is bounded, the sequence $\alpha(s_n)$ has a convergent subsequence, $\lim_{n\to\infty} \alpha(s_n) = \mu$ and we conclude from (65) that $\mu \in \sigma(A)$. Since α is bounded, every sequence $\alpha(\tau_n)$, $\tau_n \to \infty$ possesses a convergent subsequence, $\lim_{n\to\alpha} \alpha(\tau_n) = \gamma$ and we claim that $\gamma = \mu$. Indeed, if f.e. $\gamma < \mu$, then α has again an oscillatory behavior and we can pick $\gamma < \nu < \mu$ satisfying $\nu \notin \sigma(A)$, and a sequence $s_n \to \infty$ satisfying $\alpha(s_n) = \nu$ and $\alpha'(s_n) \leqslant 0$. Consequently, in view of (62), $\|\xi'(s_n)\| \to 0$ and hence, in view of (65), $\nu \in \sigma(A)$, contradicting $\nu \notin \sigma(A)$. We have proved that $\lim_{s\to\infty} \alpha(s) = \mu$ and $\mu \in \sigma(A)$. Clearly $\mu \leqslant 0$, since otherwise $\|z(s)\| \to +\infty$. But then $\mu < 0$ in view of our assumption $0 \notin \sigma(A)$. This finishes the proof of Lemma 3.2 □

For the solution $z = z(s,t)$ of the equation (55) we have, in view of Lemma 3.2, the formula

$$z(s,t) = \|z(s_0)\| e^{\int_{s_0}^{s} \alpha(\tau)d\tau} \xi(s,t), \qquad \|\xi(s)\| = 1. \tag{66}$$

The exponential decay of z as $s \to \infty$ will be concluded from the C^∞ bounds of ξ and α in the next Lemma.

Lemma 3.3. *Define, as in Lemma 3.2,*

$$\xi(s,t) = \frac{z(s,t)}{\|z(s)\|}$$
$$\alpha(s) = \langle A_\infty \xi, \xi\rangle + \langle (S_\infty - S)\xi, \xi\rangle.$$

Then for every $j \in \mathbb{N} = \{0, 1, 2, \dots\}$ and every multi index $\beta \in \mathbb{N} \times \mathbb{N}$

$$\sup_{s,t} |\partial^\beta \xi(s,t)| < \infty \quad \textit{and} \quad \sup_{s} |\partial^j \alpha| < \infty.$$

Proof. The scalar product and the norm $\|\cdot\|$ above refer to the L^2-space in the t variable, $0 \leqslant t \leqslant 1$. By (55) the smooth function $\xi = \xi(s,t) \colon [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ is 1-periodic in $t \in \mathbb{R}$ and solves the equation

$$\bar{\partial}\xi = -S(s,t)\xi - \alpha(s)\xi, \quad \bar{\partial} := \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t}, \tag{67}$$

with smooth functions S and α having the bounds

$$\sup\left|\partial^\beta S\right| < \infty \quad \text{and} \quad \sup|\alpha| < \infty$$

for all multi indices $\beta \in \mathbb{N} \times \mathbb{N}$. The supremum is taken over all $s \geq s_0$ and $t \in \mathbb{R}$, the function S is 1-periodic in t and the constant matrix J_0 satisfies $J_0^2 = -Id$. In order to derive uniform $W^{k,p}_{\text{loc}}$ bounds for ξ we pick $\delta_0 > 0$ and $s^* > s_0$ and define the sequence $\delta_j \searrow \delta_0/2$ by $\delta_j = \delta_0(1 + 2^{-j})/2$. We choose smooth bump functions $\beta_j \colon \mathbb{R} \to [0,1]$ vanishing outside of $(s^* - \delta_{j-1}, s^* + \delta_{j-1})$ and equal to 1 on $[s^* - \delta_j, s^* + \delta_j]$. Introducing the nested intervals $I_j = [s^* - \delta_j, s^* + \delta_j] \subset \mathbb{R}$ and $Q_j = I_j \times [0,1] \subset \mathbb{R}^2$ we claim that for every $N \geq 1$ and every $2 < p < \infty$ there exists a constant $C_{N,p} > 0$ such that

$$\begin{aligned} \left\|\xi\right\|_{W^{N,p}(Q_N)} &\leqslant C_{N,p} \\ \left\|\alpha\right\|_{W^{N,p}(I_N)} &\leqslant C_{N,p}, \end{aligned} \tag{68}$$

where the constants $C_{N,p}$ are independent of s^*. Lemma 3.3 is an immediate consequence of the local uniform estimates (68) in view of the Sobolev embedding theorem.

In order to prove (68) we proceed inductively making use of the well known a-priori estimate for the $\bar{\partial}$-operator:

$$\left\|\xi\right\|_{W^{j,p}(Q_j)} \leqslant M_p \left\|\bar{\partial}(\beta_j \xi)\right\|_{W^{j-1,p}(Q_{j-1})}, \tag{69}$$

where M_p only depends on $\bar{\partial}$. Starting with $j = 1$ we first show that ξ is uniformly bounded. Recalling (67) we deduce from (69), setting $p = 2$, the estimate $\left\|\xi\right\|_{W^{1,2}(Q_1)} \leqslant c\left\|\xi\right\|_{L^2(Q_0)}$. The constant $c > 0$ depends on the C^1-norm of β_1, on $\sup|S|$ and $\sup|\alpha|$ but not on s^*. Since $\left\|\xi(s)\right\|_{L^2(S^1)} = 1$ we have $\left\|\xi\right\|^2_{L^2(Q_0)} = 2\delta_0$ so that $\left\|\xi\right\|_{W^{1,2}(Q_1)} \leqslant c_1$ for a constant c_1 independent of s^*. Therefore, using the Sobolev embedding theorem, $\left\|\xi\right\|_{L^p(Q_1)} \leqslant c'_p$, again independent of s^*, for every $1 < p < \infty$. In view of this local uniform L^p-estimate for ξ we deduce from (50) for $p > 1$ the estimate $\left\|\xi\right\|_{W^{1,p}(Q_1)} \leqslant c_p$, the constant being independent of s^*. Hence choosing $p > 2$ we conclude $\sup|\xi| < \infty$ by means of the Sobolev embedding theorem.

Recall now equation (61) for α, namely

$$\alpha'(s) = 2\left\|\xi'(s)\right\|^2 - \langle \varepsilon\xi, \xi'\rangle + \langle \varepsilon'\xi, \xi\rangle,$$

where prime denotes the partial derivative in the s-variable and where the smooth function $\varepsilon = \varepsilon(s,t)$ and all its partial derivatives are uniformly bounded. From the above equation we deduce $|\alpha'(s)| \leqslant c_1\left\|\xi'(s)\right\|^2 + c_2$. Integrating and using Hölder's inequality we find the local L^p-estimate $\left\|\alpha'\right\|^p_{L^p(I_1)} \leqslant c_3\left\|\xi\right\|_{W^{1,2p}(Q_1)} + c_4 \leqslant c_p$ independent of s^*.

We have verified (68) for $N = 1$ and all $p > 2$. Proceeding now inductively, using (67), (69) then (61) and the Sobolev embedding theorems, the desired estimates (68) are verified for all N and the lemma is proved. $\square$

Recall that $\alpha(s) \to \mu$ as $s \to \infty$ and $\mu < 0$. We deduce from Lemma 3.3 and the formula (66) for the map $z(s,t)$ the following Corollary.

Corollary 3.4. *Let $0 < r < |\mu|$, then*

$$\left|\partial^\beta z(s,t)\right| \leqslant Me^{-rs},$$

for all derivatives β, with constants $M = M_\beta$.

Proposition 3.5. *Recall that $\alpha(s) \to \mu$ as $s \to \infty$ with $\mu < 0$ and $\mu \in \sigma(A_\infty)$. There exists an eigenvector $e(t+1) = e(t)$ of $A_\infty e = \mu e$ satisfying $\|e\|_{L^2(S^1)} = 1$ and*

$$\xi(s,t) \to e(t) \qquad in \qquad C^\infty(\mathbb{R}) \qquad as \qquad s \to \infty.$$

Proof. In view of the previous Lemma it is sufficient to prove the convergence in $W^{1,2}(S^1)$. We start with

Lemma 3.6. *Let $E \subset L^2(S^1)$ be the eigenspace of A_∞ belonging to $\mu \in \sigma(A_\infty)$. Then*

$$\operatorname{dist}(\xi(s), E) \to 0 \qquad as \qquad s \to \infty,$$

where the distance is taken in the $W^{1,2}(S^1)$-norm.

Proof of Lemma 3.6 Arguing by contradiction we assume that $\operatorname{dist}(\xi(s_n), E) \geq \varepsilon$ for some $\varepsilon > 0$ and for a sequence $s_n \to \infty$. Since, by Lemma 3.3, the derivatives of $\xi(s,t)$ are uniformly bounded, there is a constant $c > 0$, such that

$$\|\xi(s) - \xi(s')\|_{W^{1,2}} \leqslant c|s - s'|.$$

Therefore, we find intervals around s_n, $I_n = [s_n - d, s_n + d]$, with some $d > 0$, such that

$$\operatorname{dist}(\xi(s), E) \geq \frac{\varepsilon}{2}, \qquad s \in I_n. \tag{70}$$

We claim that there exists a sequence $\tau_k \in I_{n_k}$ such that

$$\|A_\infty \xi(\tau_k) - \mu\xi(\tau_k)\| \to 0 \tag{71}$$

as $k \to \infty$. Indeed, we recall, using (65) and $\|\xi\| = 1$, that

$$\|\xi'(s)\| \geq \|[A - \alpha(s)]\xi\| - O(s) \geq \|[A - \mu]\xi\| - |\mu - \alpha(s)| - O(s).$$

Therefore, it is sufficient to prove that $\|\xi'(\tau_k)\| \to 0$. If not, then we have $\|\xi'(s)\| \geq \delta$ for $s \in I_n$, all n, with some $\delta > 0$. Hence $\alpha'(s) \geq \delta^2$ in view of the estimate (62), $s \in I_n$ and hence, by the mean value theorem $|\alpha(s_n + d) - \alpha(s_n - d)| \geq 2d\delta^2 > 0$ for all n, which contradicts the convergence $\alpha(s) \to \mu$ as $s \to \infty$ and proves the claim (71).

Now, by Lemma 3.3 we know that $\|\xi(\tau_k)\|_{W^{2.2}(S^1)} \leqslant C$ and hence, since $W^{2,2}(S^1)$ is compactly embedded in $W^{1,2}(S^1)$, we find a subsequence which converges in $W^{1,2}$ such that $\xi(\tau_k) \to e \in W^{1,2}(S^1)$. It follows from $\|\xi(s)\| = 1$ that $\|e\|_{L^2} = 1$ and, in view of (71), that $A_\infty e = \mu e$, so that $e \in E$. This contradicts (70) and hence Lemma 3.6 is proved. $\square$

Lemma 3.7. *There exists $e \in E$, i.e. $A_\infty e = \mu e$ such that $\|e\|_{L^2(S^1)} = 1$ and*

$$\xi(s) \to e \quad \text{in } W^{1,2}(S^1) \text{ as } s \to \infty.$$

Proof. Let P denote the orthogonal projection of $L^2(S^1)$ onto the eigenspace E of A_∞, and define

$$\zeta(s) = P\xi(s).$$

Recall that, by (59), ξ solves the equation $\xi' = A_\infty \xi + \varepsilon\xi - \alpha\xi$. Using $A_\infty(P\xi) = PA_\infty\xi = \mu(P\xi)$ we find for $\zeta(s)$ the equation

$$\zeta' = [\mu - \alpha(s)]\zeta + P\varepsilon\xi. \tag{72}$$

From Lemma 3.6 we conclude

$$\|\zeta(s) - \xi(s)\| \to 0 \qquad \text{and} \qquad \|\zeta(s)\| \to 1 \tag{73}$$

as $s \to \infty$. Therefore, $\|\zeta(s)\| \geq \frac{1}{2}$ for large s and we define the smooth function η by

$$\eta(s,t) = \frac{\zeta(s,t)}{\|\zeta(s)\|}, \qquad \text{then} \qquad \|\eta(s)\| = 1.$$

This function satisfies the differential equation

$$\eta' = \frac{\zeta'}{\|\zeta\|} - \frac{1}{2}\frac{\frac{d}{ds}\|\zeta\|^2}{\|\zeta\|^2} \cdot \frac{\zeta}{\|\zeta\|}.$$

Using that $\langle \eta', \eta \rangle = 0$ we find, inserting the equation (72) for ζ' that

$$\eta' = \frac{P\varepsilon\xi}{\|\zeta\|} - \frac{\langle P\varepsilon\xi, \eta \rangle}{\|\zeta\|} \cdot \eta.$$

Since, in the L^2-norms, $\|\xi\| = \|\eta\| = 1$ and $\|\zeta\| \geq \frac{1}{2}$ we find the estimate

$$\|\eta'(s)\| \leqslant 4\|\varepsilon(s)\|,$$

where $\varepsilon(s,t) = S(s,t) - S_\infty(t)$. By definition, $S(s,t) = N\big(\vartheta(s,t), z(s,t)\big)$ and $S_\infty(t) = N(t,0)$ for a smooth matrix function N. Therefore, we can estimate:

$$\|\varepsilon(s)\| \leqslant C(\|z(s)\| + \|\widetilde{\vartheta}(s)\|_{L^2}),$$

with the t-periodic function $\widetilde{\vartheta}(s,t) = \vartheta(s,t) - t$. We shall prove below that $\|\widetilde{\vartheta}(s)\|_{L^2} \leqslant Ce^{-rs}$ for some $r > 0$. Consequently, we find together with the exponential estimate (66) for z, that

$$\|\eta'(s)\| \leqslant Ce^{-rs} \tag{74}$$

for some $r > 0$. Take any sequence $s_n \to \infty$. Since $\xi(s_n)$ is, by Lemma 3.3, bounded in $W^{2,2}(S^1)$ it possesses a subsequence converging in $W^{1,2}(S^1)$, such that $\xi(s_n) \to e \in W^{1,2}(S^1)$. From Lemma 3.6 we conclude that $e \in E$ and it remains to prove the uniqueness of this limit. Assume $\xi(s_n) \to e$ and $\xi(\tau_n) \to e'$ in $W^{1,2}(S^1)$, then, by (73), $\eta(s_n) \to e$ and $\eta(\tau_n) \to e'$ in $L^2(S^1)$. Using (74) we can estimate in L^2

$$\|\eta(s_n) - \eta(\tau_n)\| \leqslant \left| \int_{s_n}^{\tau_n} \|\eta'(s)\| ds \right| \leqslant C\left| \int_{s_n}^{\tau_n} e^{-rs} ds \right| \to 0$$

for $n \to \infty$. Hence $e = e'$ and the proof of Lemma 3.7 and, therefore, also the proof of Proposition 3.5 is finished. $\square$

As a consequence of Lemma 3.7 and the C^∞ bounds of Lemma 3.6 we have $\xi(s,t) \to e(t)$ as $s \to \infty$ in $C^\infty(\mathbb{R})$. Now define $r(s,t) = \xi(s,t) - e(t)$. Using the equation (59) for the derivative of ξ in the s-variable, the convergence $\alpha(s) \to \mu$, and $(A-\mu)e = e$, we deduce inductively that $\partial^\beta r(s,t) \to 0$ as $s \to \infty$, uniformly in t, for all derivatives. Recalling formula (66), we have established the asymptotic formula for the function $z(s,t)$ in Theorem 2.8. It remains to demonstrate the exponential decay of the functions $a(s,t) - Ts$ and $\vartheta(s,t) - kt$. Again, for simplicity of the notation, we assume $T = k = \tau = 1$.

4. End of the proof

We shall now use the exponential estimate of z, Corollary 3.4, in order to derive the desired exponential estimate for the functions a and ϑ. Recall that a and ϑ: $[s_0, \infty) \times \mathbb{R} \to \mathbb{R}$ are smooth solutions of

$$\begin{aligned} a_s &= \quad (\vartheta_t + xy_t) f(\vartheta, z) \\ a_t &= -(\vartheta_s + xy_s) f(\vartheta, z). \end{aligned} \tag{75}$$

Moreover $f(t,0) \equiv 1$ and we can write

$$f(\vartheta, z) = 1 + \int_0^1 f_z(\vartheta, \tau z) d\tau z \equiv 1 + k(s,t) z.$$

Introduce the 1-periodic functions $\widetilde{a}, \widetilde{b}$: $[s_0, \infty) \times S^1 \to \mathbb{R}$ by

$$\begin{aligned} \widetilde{a}(s,t) &= a(s,t) - s \\ \widetilde{\vartheta}(s,t) &= \vartheta(s,t) - t. \end{aligned}$$

By Lemma 2.7,

$$\begin{aligned} \partial^\beta \widetilde{a}(s,t) &\to 0 \qquad \text{for } |\beta| \geq 0 \\ \partial^\beta \widetilde{\vartheta}(s,t) &\to 0 \qquad \text{for } |\beta| \geq 0. \end{aligned}$$

as $s \to \infty$, uniformly in t. The equation (75) becomes

$$\begin{aligned} \widetilde{a}_s &= \quad \widetilde{\vartheta}_t + xy_t + (\vartheta_t + xy_t) kz \\ \widetilde{\vartheta}_s &= -\widetilde{a}_t - xy_s - (\vartheta_s + xy_s) kz. \end{aligned} \tag{76}$$

Hence, abbreviating

$$w(s,t) = \begin{pmatrix} \widetilde{a} \\ \widetilde{\vartheta} \end{pmatrix},$$

we can write the equation (76) in the form

$$w_s + J_0 w_t = h, \qquad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{77}$$

with a smooth function h: $[s_0, \infty) \times S^1 \to \mathbb{R}$ satisfying, in view of Corollary 3.4, the exponential estimates

$$\left| \partial^\beta h(s,t) \right| \leqslant M e^{-rs}, \qquad |\beta| \geq 0 \tag{78}$$

for constants $M = M_\beta$, and $0 < r < |\mu|$. Our aim is to deduce similar estimates for w. We start with a simple observation.

Lemma 4.1. *Assume $v(s,t)$ and $h(s,t)$ are smooth, 1-periodic in t and solve the partial differential equation*

$$v_s + J_0 v_t = h \quad \text{on } [s_0, \infty] \times S^1. \tag{79}$$

Assume, in addition, that

$$(i) \qquad \int_0^1 v(s,t)dt = 0$$

$$(ii) \qquad \|\partial_s h(s,\cdot)\|^2 + \|\partial_t h(s,\cdot)\|^2 \leqslant f(s)$$

$$(iii) \qquad \|v(s,\cdot)\| \to 0, \;\; f(s) \to 0 \quad \text{as } s \to \infty$$

for a smooth function f satisfying $f''(s) = r^2 f(s)$, with some constant $r > 0$. Then, choosing $0 < \nu < r$ and $\nu \leqslant 1$ we have for all $s \in [s_0, \infty)$

$$\frac{1}{2}\|v(s)\|^2 \leqslant \beta(s_0) e^{-\nu(s-s_0)},$$

where

$$\beta(s_0) = \frac{1}{2}\|v(s_0)\|^2 + bf(s_0).$$

with a positive constant b independent of s.

Proof. In terms of the operator A, defined by

$$A = J_0 \frac{d}{dt},$$

the equation (79) looks as follows,

$$v' + Av = h, \tag{80}$$

where during the proof prime denotes ∂_s and dot denotes ∂_t. Define

$$\alpha(s) = \frac{1}{2}\|v(s)\|^2.$$

Differentiation in s gives in view of the equation (80) for w,

$$\alpha' = \langle w, w' \rangle = \langle w, h - Aw \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2. The operator A is self-adjoint and

$$\|Aw\|^2 = \|\dot{w}\|^2$$

since J is an isometry. Due to the assumption $\int_0^1 w(s,t)dt = 0$,

$$\|\dot{w}\|^2 \geq \|w\|^2.$$

Together with $\|Ah\| = \|\dot{h}\|$, we obtain for the second derivative of α the estimate

$$\begin{aligned}
\alpha''(s) &= \|h - Aw\|^2 + \|Aw\|^2 + \langle w, h' \rangle - \langle w, Ah \rangle \\
&\geq \|\dot{w}\|^2 - \|w\| \cdot \|h'\| - \|w\| \cdot \|\dot{h}\| \\
&\geq \|w\|^2 - \lambda \|w\|^2 - \frac{1}{2\lambda}\{ \; \|h'\|^2 + \|\dot{h}\|^2 \; \} \\
&\geq (2 - 2\lambda)\frac{1}{2}\|w\|^2 - \frac{1}{2\lambda}\{ \; \|h'\|^2 + \|\dot{h}\|^2 \; \}
\end{aligned}$$

for every $\lambda > 0$. Choosing $\lambda = 1/2$ one concludes, using assumption (ii), the estimate

$$\alpha''(s) \geq \nu^2 \alpha(s) - af(s) \tag{81}$$

for every $\nu \leqslant 1$, with a constant $a > 0$. Choose $0 < \nu < r$ and define

$$\beta(s) = \alpha(s) + bf(s), \quad b = \frac{a}{r^2 - \nu^2}. \tag{82}$$

Then

$$\alpha(s) \leqslant \beta(s). \tag{83}$$

From (81), (82) and $f''(s) = r^2 f(s)$ we deduce

$$\beta''(s) \geq \nu^2 \beta(s).$$

This leads to the desired estimate. Indeed, we consider the function $g(s) = \beta'(s) + \nu\beta(s)$. If $g(s) \leqslant 0$ for all $s \geq s_0$, then $\frac{d}{ds}\left[e^{\nu s}\beta(s)\right] = e^{\nu s}\beta(s) \leqslant 0$ for all $s \geq s_0$ which implies

$$\alpha(s) \leqslant \beta(s) \leqslant e^{-\nu(s-s_0)}\beta(s_0)$$

for all $s \geq s_0$ as required. Hence it suffices to show that $g(s) \leqslant 0$ for $s \geq s_0$. Arguing by contradiction assume that $g(s_1) > 0$ for some $s_1 > s_0$. From

$$g'(s) = \beta''(s) + \nu\beta'(s) \geq \nu^2\beta(s) + \nu\beta'(s) = \nu g(s)$$

we find $\frac{d}{ds}\left[e^{-\nu s} g(s)\right] \geq 0$ implying for $s \geq s_1$

$$e^{-\nu s} g(s) \geq e^{-\nu s_1} g(s_1)$$

which in turn gives

$$\frac{d}{ds}\left[e^{\nu s}\beta(s)\right] \geq e^{\nu s} e^{\nu(s-s_q)} g(s_1).$$

Integrating from s_1 to s we obtain

$$\beta(s) \geq e^{-\nu(s-s_1)}\beta(s_1) + \frac{g(s_1)}{\nu}\sinh[\nu(s-s_1)] \to \infty$$

as $s \to \infty$ contradicting $\beta(s) \to 0$. Hence $g(s) \leqslant 0$ for $s \geq s_0$. The proof of Lemma 4.1 is complete. □

As a consequence we have

Lemma 4.2. *If w is a solution of* (77) *and $0 < \nu < r$ and $\nu \leqslant 1$, then there exist constants M_β such that for all derivatives*

$$\partial^\beta = \partial_t^{\beta_1}\partial_s^{\beta_2}, \quad \beta_1 \geq 1,\ \beta_2 \geq 0$$

the following estimate holds

$$\left\|\partial^\beta w(s,\cdot)\right\| \leqslant Me^{-\nu s},$$

for all $s \in (s_0, \infty)$.

Proof. The function w is a solution of (77). Abbreviate

$$v = \partial^\beta w = \partial_t^{\beta_1}\partial_s^{\beta_2} w.$$

Since $\beta_1 \geq 1$, the mean values over a period vanish and v solves the equation

$$v_s + J_0 v_t = \partial^\beta h =: g.$$

In view of Lemma 2.7,

$$\left\|\partial_s g(s,\cdot)\right\|^2 + \left\|\partial_t g(s,\cdot)\right\|^2 \leqslant Ce^{-rs}$$

for some constant C. Choosing $f(s)$ equal to the right hand side, the assumptions of Lemma 4.1 are met. The proof of Lemma 4.2 is complete. □

Having proved the estimates for the derivatives we finally turn to the estimates of the functions. Recall from Lemma 2.7 that

$$|w(s,t) - c| \to 0$$

as $s \to \infty$, uniformly in t. Introduce $\widetilde{w}(s,t) = w(s,t) - \int_0^1 w(s,t)dt$ and the mean values $\alpha(s) = \int_0^1 w(s,t)dt$. In view of (77),

$$\widetilde{w}_s + J_0 \widetilde{w}_t = h - \int_0^1 h(s,t)dt =: f.$$

In view of (78) the function f satisfies $|f(s,t)| \leqslant Me^{-rs}$ and since $\int_0^1 \widetilde{w}(s,t)dt = 0$ we conclude from Lemma 4.2 that

$$\left|\widetilde{w}(s,t)\right| \leqslant \left\|\widetilde{w}_t(s,\cdot)\right\| = \left\|w_t(s,\cdot)\right\| \leqslant Me^{-\nu s}. \tag{84}$$

Next consider the function $\alpha(s) = \int_0^1 w(s,t)dt$. In view of (77),

$$\alpha'(s) = \int_0^1 w_s(s,t)dt = \int_0^1 h(s,t)dt.$$

Consequently, recalling the definition of the function $h(s,t)$,

$$\begin{aligned}
|\alpha(s') - \alpha(s)| &= \int_s^{s'} \alpha'(\tau)\ d\tau = \int_s^{s'} \left[\int_0^1 h(\tau,t)dt\right] d\tau \\
&\leqslant \int_s^{s'} \left[\int_0^1 |h(\tau,t)|^2 dt\right]^{1/2} d\tau \leqslant M \int_s^{s'} \left\|z(\tau)\right\| d\tau \\
&\leqslant \frac{M}{r}\left(e^{-rs} - e^{-rs'}\right)
\end{aligned}$$

for $s' > s$. Using $\alpha(s') \to c$, as $s' \to \infty$, we obtain

$$|c - \alpha(s)| \leqslant \frac{M}{r} e^{-rs}.$$

Together with (84) and $0 < \nu < r$, we see that

$$\left|w(s,t) - c\right| \leqslant \left|\widetilde{w}(s,t)\right| + \left|\alpha(s) - c\right| \leqslant M_0 e^{-\nu s}.$$

By means of the Sobolev embedding theorem one concludes from the above considerations and from Lemma 4.2 the following pointwise estimates.

Proposition 4.3. *Assume (a, ϑ, z) meets the assumptions of Theorem 2.8. Then there exist constants a_0 and $\vartheta_0 \in \mathbb{R}$ such that for $0 < \nu < r$ and $\nu \leqslant 1$,*

$$\left|\partial^\beta (a(s,t) - Ts - a_0)\right| \leqslant Me^{-\nu s}$$
$$\left|\partial^\beta (\vartheta(s,t) - kt - \vartheta_0)\right| \leqslant Me^{-\nu s},$$

for all $|\beta| \geq 0$ and all $(s,t) \in [s_0, \infty) \times S^1$, with constants $M = M_\beta$.

We should mention that during the proof we have set, in order to simplify the notation, $T = k = 1$ and also $a_0 = 0 = \vartheta_0$.

This completes the proof of Theorem 2.8 about the asymptotics of nondegenerate finite energy planes. We shall use next the asymptotic formula in order to derive some global properties of nondegenerate finite energy planes.

5. Intersections of the finite energy plane with its asymptotic limit

If $(a, u)\colon \mathbb{C} \to \mathbb{R} \times M$ is a finite energy plane, which is nondegenerate as in Theorem 1.2, then $u\big(Re^{2\pi i t}\big) \to p(Tt)$ as $R \to \infty$. Here $p(t)$ is a periodic solution of the Reeb vector field $\dot{x} = X(x)$ associated to the contact structure λ on M. The period T is positive and we assume that $T = 1$ (for notational convenience). It turns out that outside of a large disc the energy plane does not hit the "limit" periodic solution p. We shall abbreviate $P = \{p(t) \mid t \in \mathbb{R}\} \subset M$.

Theorem 5.1. *If $(a, u)\colon \mathbb{C} \to \mathbb{R} \times M$ is a nondegenerate finite energy plane as described in Theorem 1.2, then there exists an $R > 0$, such that*

$$\begin{array}{lll} (i) & u(z) \notin P & \text{if } |z| \geq R \\ (ii) & \pi u_s(z) \neq 0 & \text{if } |z| \geq R, \end{array}$$

where $\pi\colon T_m M = \mathbb{R}\, X(m) \oplus \xi_m \to \xi_m$ is the projection onto the contact plane.

Proof. We recall that the first alternative in Theorem 2.8 does not occur for finite energy planes. The proof is an immediate application of the asymptotic formula in Theorem 2.8. We argue by contradiction and assume, in the cylinder variables $(s,t) \in \mathbb{R} \times S^1$, that $u(s_n, t_n) \in P$ for a sequence $s_n \to \infty$. We may assume that $t_n \to t^* \in S^1$. In the local coordinates near P, we then have $u(s_n, t_n) = \big(\vartheta(s_n, t_n), z(s_n, t_n)\big) \in \mathbb{R}^3$ and $z(s_n, t_n) = 0$. From

$$z(s,t) = e^{\int_{s_0}^{s} \alpha(\tau) d\tau} \big[e(t) + r(s,t)\big]$$

we deduce

$$e(t_n) + r(s_n, t_n) = 0.$$

Since $r(s,t) \to 0$ as $s \to \infty$ we conclude $e(t^*) = 0$. This contradicts the fact, that the eigenfunction $e(t)$ does not vanish and proves statement (i).

Similarly one proves the second statement. We assume that $\pi\big(u_s(s_n, t_n)\big) = 0$ for a sequence $s_n \to \infty$. Hence for s_n large $u(s_n, t_n)$ is in our local coordinate neighborhood of the periodic solution. We can write $u(t, s) = (\vartheta, z)$ and $u_s(s,t) = (\vartheta_s, z_s)$. Since $\pi u_s = u_s - \lambda(u_s) X(u)$, with the Reeb vector field X, we have

$$u_s(s_n, t_n) = \lambda(u_s) X(u),$$

and hence, at (s_n, t_n),

$$\begin{pmatrix} \vartheta_s \\ z_s \end{pmatrix} = \lambda(u_s) \begin{pmatrix} X_1(u) \\ X_2(u) \end{pmatrix} \in \mathbb{R}^3$$

Since $X_2(\vartheta, 0) = 0$, we can write $X_2(\vartheta, z) = Rz$, with a matrix function $R = R(\vartheta, z)$, so that, at (s_n, t_n)

$$z_s = f(u)(\vartheta_s + xy_s)Rz.$$

Inserting the asymptotic formula from Theorem 2.8 the exponential terms cancel, and we find

$$\alpha(s_n)\big[e(t_n) + r(s_n, t_n)\big] + r_s(s_n, t_n) = f(u)(\vartheta_s + xy_s)R\big[e(t_n) + r(s_n, t_n)\big].$$

Recall now that $\alpha(s) \to \mu < 0$, and $r(s,t)$, $r_s(s,t)$, $\vartheta_s(s,t)$, $x(s,t) \to 0$ as $s \to \infty$. We conclude $\mu e(t^*) = 0$, contradicting again $e(t) \neq 0$. This finishes the proof of Theorem 5.1. $\square$

Using the generalized similarity principle we shall deduce from Theorem 5.1 the

Theorem 5.2. *The sets*

$$\big\{z \in \mathbb{C} \mid u(z) \in P\big\}$$
$$\big\{z \in \mathbb{C} \mid \pi_{u(z)}\big(u_s(z)\big) = 0\big\}$$

consist of finitely many points.

Proof. In order to prove the first statement we argue by contradiction and assume that there is an infinite sequence $z_n \in \mathbb{C}$ such that $u(z_n) \in P$. By Theorem 5.1 we can assume that $z_n \to z^* \in \mathbb{C}$ and $u(z^*) \in P$. By Darboux's theorem there is an open neighborhood of $u(z^*) \in M$ on which we find coordinates $(\vartheta, x, y) = (\vartheta, z) \in \mathbb{R}^3$, in which the contact form λ is represented as

$$\lambda = d\vartheta + xdy,$$

and in which $u(z^*)$ corresponds to the origin 0 in $\mathbb{R}^3$. Using the cylinder coordinates $(s,t) \in \mathbb{R} \times S^1$, the map $u(s,t) = \big(\vartheta(s,t), z(s,t)\big) \in \mathbb{R} \times \mathbb{R}^2$ satisfies, in our local coordinates, the equations

$$z_s + J(s,t)z_t = 0, \tag{85}$$

where $J(s,t)^2 = -1$. This is proved as in Section 3; this time $f = 1$ and $X(\vartheta, z) = (1, 0, 0)$. By assumption, we know that

$$z(s_n, t_n) = z(s^*, t^*) = 0$$

for a sequence $(s_n, t_n) \to (s^*, t^*)$. Consequently, by the generalized similarity principle [11], there is an open neighborhood D of (s^*, t^*) on which the solution z of the equation is represented by $z(s,t) = \Phi(z)h(z)$, where $z = s + it$, $\Phi\colon D \to GL(\mathbb{R}^2)$ is continuous, and $h\colon D \to \mathbb{R}^2 \cong \mathbb{C}$ is holomorphic. By assumption, $z^* = s^* + it^*$ is a cluster point of zeroes of the holomorphic function h. Therefore, $h \equiv 0$ and hence $z \equiv 0$ on D. Consequently, $u(s,t) \in P$ for all (s,t) in the open set D. We have proved, in particular, that the set of points $z = (s,t)$ which are cluster points of z_j satisfying $u(z_j) \in P$ is an open set in $\mathbb{R} \times S^1$. It is clearly also a closed set and hence agrees with $\mathbb{R} \times S^1$ so that $u(s,t) \in P$ for all $(s,t) \in \mathbb{R} \times S^1$. This contradicts Theorem 5.1.

The second statement is proved similarly. Note that in the above local coordinates the Reeb vector field X is constant, $X(\vartheta, z) = (1, 0, 0)$. Hence, the condition $0 = \pi u_s = u_s - \lambda(u_s)X(u)$ becomes, in our local coordinates $(u = (\vartheta, z)), z_s = 0$. Introducing $\zeta = z_s$ we find, by differentiating (85) in the s-variable, that ζ solves the equations

$$\zeta_s + J(s,t)\zeta_t + A(s,t)\zeta = 0.$$

Moreover, $\zeta(s_n, t_n) = \zeta(s^*, t^*) = 0$ for a sequence $(s_n, t_n) \to (s^*, t^*)$. Hence, by the generalized similarity principle [11], $\zeta \equiv 0$ in an open neighborhood of $(s^*, t^*) = 0$. Consequently, $\pi u_s(z) = 0$ in an open neighborhood of $z^* \in \mathbb{C}$.

Arguing as before this leads to a contradiction to the second statement in Theorem 5.1. The proof of the Theorem 5.2 is complete. □

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