Geometry & Topology 13 (2009) 2253–2278

The asymptotic behavior of least pseudo-Anosov dilatations

CHIA-YEN TSAI

For a surface S with n marked points and fixed genus $g \geq 2$, we prove that the logarithm of the minimal dilatation of a pseudo-Anosov homeomorphism of S is on the order of $(\log n)/n$. This is in contrast with the cases of genus zero or one where the order is $1/n$.

37E30; 57M99, 30F60

1 Introduction

Let $S = S_{g,n}$ be an orientable surface with genus g and n marked points. The *mapping class group* of S is defined to be the group of homotopy classes of orientation preserving homeomorphisms of S. We denote it by $\mathrm{Mod}(S)$. Given a pseudo-Anosov element $f \in \mathrm{Mod}(S)$, let $\lambda(f)$ denote the *dilatation* of f (see Section 2.1). We define

$$\mathcal{L}(S_{g,n}) := \{\log \lambda(f) \mid f \in \mathrm{Mod}(S_{g,n}) \text{ pseudo-Anosov}\}.$$

This is precisely the length spectrum of the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points with respect to the Teichmuller metric; see Ivanov [8]. There is a shortest closed geodesic and we denote its length by

$$l_{g,n} = \min\{\log \lambda(f) \mid f \in \mathrm{Mod}(S_{g,n}) \text{ pseudo-Anosov}\}.$$

Our main theorem is the following:

Theorem 1.1 *For any fixed $g \geq 2$, there is a constant $c_g \geq 1$ depending on g such that*

$$\frac{\log n}{c_g n} < l_{g,n} < \frac{c_g \log n}{n},$$

for all $n \geq 3$.

To contrast with known results, recall that in [13] Penner proves that for $2g-2+n > 0$,

$$l_{g,n} \geq \frac{\log 2}{12g - 12 + 4n},$$

Published: 26 May 2009 DOI: 10.2140/gt.2009.13.2253

and for closed surfaces with genus $g \geq 2$,

$$\frac{\log 2}{12g-12} \leq l_{g,0} \leq \frac{\log 11}{g}.$$

The bounds on $l_{g,0}$ have been improved by a number of authors; see Bauer [1], McMullen [10], Minakawa [11]and Hironaka and Kin [7].

In [13], Penner suggests that there may be an "analogous upper bound for $n \neq 0$". In [7], Hironaka and Kin use a concrete construction to prove that for genus $g = 0$,

$$l_{0,n} < \frac{\log(2+\sqrt{3})}{\lfloor \frac{n-2}{2} \rfloor} \leq \frac{2\log(2+\sqrt{3})}{n-3},$$

for all $n \geq 4$. The inequality is proven for even n in [7], but it follows for odd n by letting the fixed point of their example be a marked point. Combining this with Penner's lower bound, one sees for $n \geq 4$,

$$\frac{\log 2}{4n-12} \leq l_{0,n} < \frac{2\log(2+\sqrt{3})}{n-3},$$

which shows that the upper bound is on the same order as Penner's lower bound for $g = 0$. A similar situation holds for $g = 1$; see Section 5.1 of the Appendix.

Inspired by the construction of Hironaka and Kin, we tried to find examples of pseudo-Anosov $f_{g,n} \in \mathrm{Mod}(S_{g,n})$ with

$$\log \lambda(f_{g,n}) = O\left(\frac{1}{|\chi(S_{g,n})|}\right),$$

for $\chi(S_{g,n}) = 2-2g-n < 0$. However for any fixed $g \geq 2$, all attempts resulted in $f_{g,n} \in \mathrm{Mod}(S_{g,n})$ pseudo-Anosov with

$$\log \lambda(f_{g,n}) = O_g\left(\frac{\log |\chi(S_{g,n})|}{|\chi(S_{g,n})|}\right) \quad \text{and not} \quad O\left(\frac{1}{|\chi(S_{g,n})|}\right).$$

This led us to prove Theorem 1.1.

The preceding discussion suggests that the asymptotic behavior of $l_{g,n}$ while varying both g and n can be quite complicated, in general. Hence, we will focus on understanding what happens along different (g,n)–rays. In addition to the results discussed above, there are other rays in which the asymptotic behavior of $l_{g,n}$ can be understood via examples (see Section 5.2 of the Appendix) and Penner's lower bound. Table 1 summarizes these behaviors for $\chi(S_{g,n}) < 0$.

Question What are asymptotic behaviors of $l_{g,n}$ along different (g,n)–rays in the (g,n) plane?

(g,n)–rays	*The asymptotic behavior of* $l_{g,n}$
$g=0$	$1/\lvert\chi(S_{g,n})\rvert$
$g=1$ and n is even	$1/\lvert\chi(S_{g,n})\rvert$
$g=$ constant ≥ 2	$\log\left(\lvert\chi(S_{g,n})\rvert\right)/\lvert\chi(S_{g,n})\rvert$
$n=0,1,2,3,$ or 4	$1/\lvert\chi(S_{g,n})\rvert$
$n=g, g+1,$ or $g+2$	$1/\lvert\chi(S_{g,n})\rvert$
$n=g-1$ or $2(g-1)$	$1/\lvert\chi(S_{g,n})\rvert$

Table 1

1.1 Outline of the paper

We will first recall some definitions and properties in Section 2. In Section 3 we prove the lower bound of Theorem 1.1. We construct examples in Section 4 which give an upper bound for the genus 2 case, and we extend the example to arbitrary genus $g \geq 2$ to obtain the upper bound of Theorem 1.1. Finally, we construct a pseudo-Anosov element in $\mathrm{Mod}(S_{1,2n})$ and obtain an upper bound on $l_{1,2n}$ in the Appendix.

Acknowledgements The author would like to thank Christopher Leininger for key discussions and for revising an earlier draft. Kasra Rafi and A J Hildebrand offered helpful suggestions and insights. I would also like to thank MSRI for its stimulating, collaborative research environment during its fall 2007 programs.

2 Preliminaries

2.1 Homeomorphisms of a surface

We say that a homeomorphism $f\colon S \to S$ is *pseudo-Anosov* if there are transverse singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ together with transverse measures μ^s and μ^u such that for some $\lambda > 1$,

$$f(\mathcal{F}^s,\mu^s) = (\mathcal{F}^s,\lambda\mu^s),$$
$$f(\mathcal{F}^u,\mu^u) = (\mathcal{F}^u,\lambda^{-1}\mu^u).$$

The number $\lambda = \lambda(f)$ is called the *dilatation* of f. We call f *reducible* if there is a finite disjoint union U of simple essential closed curves on S such that f leaves U invariant. If there exists $k > 0$ such that f^k is the identity, then f is *periodic*.

A mapping class $[f]$ is pseudo-Anosov, reducible or periodic (respectively) if f is homotopic to a pseudo-Anosov, reducible or periodic homeomorphism (respectively). The following is proved in Fathi, Laudenbach and Poenaru [4].

Theorem 2.1 (Nielsen–Thurston) *A mapping class* $[f] \in \mathrm{Mod}(S)$ *is either periodic, reducible, or pseudo-Anosov.*

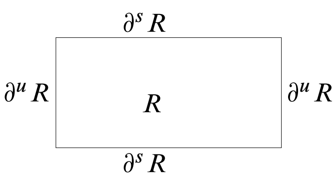
As a slight abuse of notation, we sometimes refer to a mapping class $[f]$ by one of its representatives f.

2.2 Markov partitions

Suppose $f\colon S \to S$ is pseudo-Anosov with stable and unstable measured singular foliations $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$. We define a rectangle R to be a map

$$\rho\colon I \times I \to S,$$

such that ρ is an embedding on the interior, $\rho(\text{point} \times I)$ is contained in a leaf of $\mathcal{F}^u$, and $\rho(I \times \text{point})$ is contained in a leaf of $\mathcal{F}^s$. We denote $\rho(\partial I \times I)$ by $\partial^u R$ and $\rho(I \times \partial I)$ by $\partial^s R$.



As a standard abuse of notation, we will write $R \subset S$ for the image of a rectangle map $\rho\colon I \times I \to S$.

Definition 2.2 A Markov partition for $f\colon S \to S$ is a decomposition of S into a finite union of rectangles $\{R_i\}_{i=1}^k$, such that:

(1) $\mathrm{Int}(R_i) \cap \mathrm{Int}(R_j)$ is empty, when $i \neq j$,

(2) $f(\bigcup_{j=1}^k \partial^u R_j) \subset \bigcup_{j=1}^k \partial^u R_j$,

(3) $f^{-1}(\bigcup_{i=1}^k \partial^s R_i) \subset \bigcup_{i=1}^k \partial^s R_i$.

Given a pseudo-Anosov homeomorphism $f\colon S \to S$, a Markov partition is constructed in Bestvina and Handel [2] from a train track map for f. The advantage of this construction over Fathi, Laudenbach and Poenaru [4], for example, is that the number of rectangles is substantially smaller. From [2], one has the following:

Theorem 2.3 *For any pseudo-Anosov homeomorphism $f\colon S \to S$ of a surface S with at least one marked point, there exists a Markov partition for f with at most $-3\chi(S)$ rectangles.*

We say that a matrix is *positive* (respectively, *nonnegative*) if all the entries are positive (respectively, nonnegative).

We can define a *transition matrix* M associated to the Markov partition with rectangles $\{R_i\}_{i=1}^k$. The entry $m_{i,j}$ of M is the number of times that $f(R_j)$ wraps over R_i, so M is a nonnegative integral $k \times k$ matrix. In Bestvina and Handel's construction, M is the same as the transition matrix of the train track map and they show it is an integral Perron–Frobenius matrix (ie it is irreducible with nonnegative integer entries); see Gantmacher [5]. Furthermore, the Perron–Frobenius eigenvalue $\mu(M) = \lambda(f)$ is the dilatation of f. The width (respectively, height) of R_i is the i–th entry of the corresponding Perron–Frobenius eigenvector of M (respectively, M^T), where the eigenvectors are both positive by the irreducibility of M.

The following proposition will be used in proving the lower bound.

Proposition 2.4 *Let M be a $k \times k$ integral Perron–Frobenius matrix. If there is a nonzero entry on the diagonal of M, then M^{2k} is a positive matrix and its Perron–Frobenius eigenvalue $\mu(M^{2k})$ is at least k.*

Proof We construct a directed graph Γ from M with k vertices $\{i\}_{i=1}^k$ such that the number of the directed edge from i to j in Γ equals $m_{i,j}$. We observe that for any $r > 0$ the (i,j)–th entry $m_{i,j}^{(r)}$ of M^r is the number of directed edge paths from i to j of length r in Γ.

Since M is a Perron–Frobenius matrix, we know that Γ is path-connected by directed paths. Suppose M has a nonzero entry at the (l,l)–th entry, then we will see at least one corresponding loop edge at the vertex l. For any i and j in Γ, path-connectivity ensures us that there are directed edge paths of length $\leq k$ from i to l and from l to j. This tells us that there is a directed edge path P of length $\leq 2k$ from i to j passing through l. Since we can wrap around the loop edge adjacent to l to increase the length of P, there is always a directed edge path of length $2k$ from i to j. In other words, $m_{i,j}^{(2k)}$ is at least 1 for all i and j, so M^{2k} is a positive matrix.

Let v be a corresponding Perron–Frobenius eigenvector, so that we have $M^{2k}v = \mu(M^{2k})v$. This implies that if $v = [v_1 \cdots v_k]^T$, for all i,

$$\sum_{j=1}^{k} m_{i,j}^{(2k)} v_j = \mu(M^{2k}) v_i,$$

or equivalently,
$$\mu(M^{2k}) = \sum_{j=1}^{k} m_{i,j}^{(2k)} \frac{v_j}{v_i}.$$

Choosing i such that $v_i \le v_j$ for all j, we obtain
$$\mu(M^{2k}) \ge \sum_{j=1}^{k} m_{i,j}^{(2k)} \ge \sum_{j=1}^{k} 1 = k. \qquad \square$$

The following proposition will be used in proving the upper bound.

Proposition 2.5 *Let Γ be the induced directed graph of an integral Perron–Frobenius matrix M with Perron–Frobenius eigenvalue $\mu(M) = \mu$. Let $P_\Gamma(i,d)$ be the total number of paths of length d emanating from vertex i in Γ. Then, for all i,*
$$\sqrt[d]{P_\Gamma(i,d)} \longrightarrow \mu(M) \quad \text{as } d \to \infty.$$

Proof Let M be an integral $k \times k$ Perron–Frobenius matrix with Perron–Frobenius eigenvalue μ and Perron–Frobenius eigenvector v. As above
$$\sum_{j=1}^{k} m_{i,j}^{(d)} v_j = \mu(M^d) v_i = \mu^d v_i.$$

Let $v_{\max} = \max_i\{v_i\}$ and $v_{\min} = \min_i\{v_i\}$. According to the Perron–Frobenius theory, the irreducibility of M implies that $v_i > 0$ for all i. For all i we have
$$\frac{v_{\min}\left(\sum_j m_{i,j}^{(d)}\right)}{\mu^d} \le \frac{\sum_j m_{i,j}^{(d)} v_j}{\mu^d} \le \frac{v_{\max}\left(\sum_j m_{i,j}^{(d)}\right)}{\mu^d},$$
hence
$$\frac{v_i}{v_{\max}} \le \frac{\sum_j m_{i,j}^{(d)}}{\mu^d} \le \frac{v_i}{v_{\min}}.$$

We are done, since $\sum_j m_{i,j}^{(d)} = P_\Gamma(i,d)$ and for all i,
$$\sqrt[d]{\frac{v_i}{v_{\max}}} \to 1 \quad \text{and} \quad \sqrt[d]{\frac{v_i}{v_{\min}}} \to 1, \quad \text{as } d \text{ tends to } \infty. \qquad \square$$

2.3 Lefschetz numbers

We will review some definitions and properties of Lefschetz numbers. A more complete discussion can be found in Guillemin and Pollack [6] and Bott and Tu [3].

Let X be a compact oriented manifold, and $f\colon X \to X$ be a map. Define

$$\mathrm{graph}(f) = \{(x, f(x)) | x \in X\} \subset X \times X$$

and let Δ be the diagonal of $X \times X$. The algebraic intersection number $I(\Delta, \mathrm{graph}(f))$ is an invariant of the homotopy class of f, called the *(global) Lefschetz number* of f and it is denoted $L(f)$. As in [3], this can be alternatively described by

$$L(f) = \sum_{i \geq 0} (-1)^i \,\mathrm{trace}(f_*^{(i)}), \tag{1}$$

where $f_*^{(i)}$ is the matrix induced by f acting on $H_i(X) = H_i(X;\mathbb{R})$. The Euler characteristic is the self-intersection number of the diagonal Δ in $X \times X$,

$$\chi(X) = I(\Delta, \Delta) = L(\mathrm{id}).$$

As seen in [6], if f has isolated fixed points, we can compute the *local Lefschetz number* of f at a fixed point x in local coordinates as

$$L_x(f) = \deg\left(z \mapsto \frac{f(z) - z}{|f(z) - z|}\right),$$

where z is on the boundary of a small disk centered at x which contains no other fixed points. Moreover we can compute the Lefschetz number by summing the local Lefschetz numbers of fixed points,

$$L(f) = \sum_{f(x)=x} L_x(f).$$

This description of $L_x(f)$ is given for smooth f in [6], but it is equally valid for continuous f since such a map is approximated by smooth maps. We will be computing the Lefschetz number of a homeomorphism $f\colon S_{g,n} \to S_{g,n}$, *ignoring the marked points*.

Proposition 2.6 *If a homeomorphism $f\colon S_{g,n} \to S_{g,n}$ is homotopic (not necessarily fixing the marked points) to the identity or a multitwist, then*

$$L(f) = \chi(S_{g,0}) = 2 - 2g.$$

A *multitwist* is a composition of powers of Dehn twists on pairwise disjoint simple essential closed curves.

Proof If f is homotopic to the identity, the homotopy invariance of the Lefschetz number tells us $L(f) = L(\mathrm{id}) = I(\Delta, \Delta)$ which is $\chi(S_{g,0})$.

Suppose f is homotopic to a multitwist. We will use (1) to compute $L(f)$. Note that $H_i(S_{g,0})$ is 0 for $i \geq 3$, $H_0(S_{g,0}) \cong H_2(S_{g,0}) \cong \mathbb{R}$ and $f_*^{(i)}$ is the identity when $i = 0$ or 2, so this implies $L(f) = 2 - \mathrm{trace}(f_*^{(1)})$.

There exists a set $\{\gamma_i\}_{i=1}^k$ of disjoint simple essential closed curves with some integers $n_i \neq 0$ such that

$$f \simeq T_{\gamma_1}^{n_1} \circ \cdots \circ T_{\gamma_k}^{n_k},$$

where $T_{\gamma_i}^{n_i}$ is the n_i–th power of a Dehn twist along γ_i.

For any curve γ,

$$T_{\gamma_i *}^{n_i}([\gamma]) = [\gamma] + n_i \langle \gamma, \gamma_i \rangle [\gamma_i],$$

where $[\gamma]$ is the homology class of γ and $\langle \gamma, \gamma_i \rangle$ is the algebraic intersection number of $[\gamma]$ and $[\gamma_i]$. If any γ_i is a separating curve, then $[\gamma_i]$ is the trivial homology class and $T_{\gamma_i *}^{n_i}$ acts trivially on $H_1(S_{g,0})$. We may therefore assume that each γ_i is nonseparating. After renaming the curves, we can assume that there is a subset $\{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ such that $\widehat{\gamma} = \bigcup_{i=1}^s \gamma_i$ is nonseparating and $\widehat{\gamma} \cup \gamma_j$ is separating for all $j > s$. Thus, for all $k \geq j > s$,

$$[\gamma_j] = \sum_{i=1}^{s} c_{ji} [\gamma_i],$$

for some constants $c_{ji} \in \mathbb{R}$. We can extend $\{[\gamma_i]\}_{i=1}^s$ to a basis of $H_1(S_{g,0})$,

$$\{\alpha_1, \alpha_2, \ldots, \alpha_g, \beta_1, \beta_2, \ldots, \beta_g\},$$

where $[\gamma_i] = \alpha_i$ for $i \leq s \leq g$ and $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$, $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$.

First suppose $s = k$, then $\langle \alpha_j, \gamma_i \rangle = \langle \alpha_j, \alpha_i \rangle = 0$ for all i and j. Therefore, for all j,

$$f_*^{(1)}(\alpha_j) = \alpha_j$$

and $$f_*^{(1)}(\beta_j) = \beta_j + \sum_{i=1}^{k} n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] = \beta_j + \sum_{i=1}^{k} n_i \langle \beta_j, \alpha_i \rangle \alpha_i = \beta_j - n_j \alpha_j.$$

So we have

$$f_*^{(1)} = \left(\begin{array}{c|c} I_{g \times g} & * \\ \hline 0 & I_{g \times g} \end{array} \right)$$

and $L(f) = 2 - \mathrm{trace}(f_*^{(1)}) = 2 - 2g$.

For $s < k$, we will have

$$\begin{aligned}
f_*^{(1)}(\alpha_j) &= \alpha_j + \sum_{i=1}^{k} n_i \langle \alpha_j, \gamma_i \rangle [\gamma_i] \\
&= \alpha_j + \sum_{i=1}^{s} n_i \langle \alpha_j, \alpha_i \rangle \alpha_i + \sum_{i=s+1}^{k} n_i \langle \alpha_j, \gamma_i \rangle [\gamma_i] \\
&= \alpha_j + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \alpha_j, \gamma_t \rangle [\gamma_t] \\
&= \alpha_j + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \alpha_j, \alpha_t \rangle \alpha_t \\
&= \alpha_j
\end{aligned}$$

and

$$\begin{aligned}
f_*^{(1)}(\beta_j) &= \beta_j + \sum_{i=1}^{k} n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] \\
&= \beta_j + \sum_{i=1}^{s} n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \beta_j, \gamma_t \rangle [\gamma_t] \\
&= \beta_j + \sum_{i=1}^{s} n_i \langle \beta_j, \alpha_i \rangle \alpha_i + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \beta_j, \alpha_t \rangle \alpha_t \\
&= \begin{cases} \beta_j, & \text{if } j > s, \\ \beta_j - n_j \alpha_j - \sum_{i=s+1}^{k} n_i c_{ij} \alpha_j, & \text{if } j \le s. \end{cases}
\end{aligned}$$

Therefore, the diagonal of the matrix $f_*^{(1)}$ is still all 1's and

$$L(f) = 2 - \operatorname{trace}(f_*^{(1)}) = 2 - 2g. \qquad \square$$

3 Bounding the dilatation from below

Lemma 3.1 *For any pseudo-Anosov element $f \in \operatorname{Mod}(S_{g,n})$ equipped with a Markov partition, if $L(f) < 0$, then there is a rectangle R of the Markov partition, such that the interiors of $f(R)$ and R intersect.*

Proof Since f is a pseudo-Anosov homeomorphism, it has isolated fixed points. Suppose x is an isolated fixed point of f such that one of the following happens:

(1) x is a nonsingular fixed point and the local transverse orientation of $\mathcal{F}^s$ is reversed.

(2) x is a singular fixed point and no separatrix of $\mathcal{F}^s$ emanating from x is fixed.

A *separatrix* of $\mathcal{F}^s$ is a maximal arc starting at a singularity and contained in a leaf of $\mathcal{F}^s$.

Claim $L_x(f) = +1$.

Let B be a small disk centered at x containing no other fixed point of f. First we show that (in local coordinates) for every $z \in \partial B$, $f(z) - z \neq \alpha z$ for all $\alpha > 0$.

It is easy to verify this in case 1 by choosing local coordinates (ξ_1, ξ_2) around x so that f is given by

$$f(\xi_1, \xi_2) = \left(-\lambda \xi_1, \frac{-1}{\lambda} \xi_2 \right).$$

In case 2, we choose local coordinates around x such that the separatrices of $\mathcal{F}^s$ emanating from x are sent to rays from 0 through the k–th roots of unity in $\mathbb{R}^2$. This means f rotates each of the sectors bounded by these rays through an angle $2\pi j/k$ for some $j = 1, \ldots, k-1$, and so for all $z \in \partial B$ $f(z) - z \neq \alpha z$ for all $\alpha > 0$.

Define a smooth map $h_0\colon \partial B \to S^1$ by $h_0(z) = (f(z) - z)/|f(z) - z|$, so $L_x(f) = \deg(h_0)$ by definition. Let $g\colon \partial B \to S^1$ be defined by $g(z) = z/|z|$ and $h_1\colon S^1 \to S^1$ be defined by $h_1(z/|z|) = (f(z) - z)/|f(z) - z|$, so that $h_0 = h_1 g$. Then

$$L_x(f) = \deg(h_0) = \deg(h_1 g) = \deg(h_1)\deg(g) = \deg(h_1)$$

since $\deg(g) = 1$. Note that h_1 has no fixed point since for all $z \in \partial B$,

$$f(z) - z \neq \alpha z,$$

for all $\alpha > 0$. Therefore $L_x(f) = \deg(h_1) = (-1)^{(1+1)} = +1$.

The assumption of $L(f) < 0$ implies that there exists a fixed point x of f which is in neither of the cases above. In other words, it falls into one of the cases in Figure 1. As seen in Figure 1, there is a rectangle R of the Markov partition such that the interiors of $f(R)$ and R intersect. □

Let $\Gamma_S(3) \lhd \mathrm{Mod}(S)$ denote the kernel of the action on $H_1(S; \mathbb{Z}/3\mathbb{Z})$, where $S = S_{g,0}$. In [9], it is shown that $\Gamma_S(3)$ consists of pure mapping classes. Setting

$$\Theta(g) = [\mathrm{Mod}(S) : \Gamma_S(3)],$$

we conclude the following.

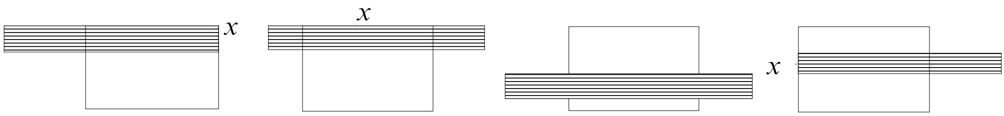


Figure 1: The intersection of $f(R)$ and R. R is the underlying rectangle and $f(R)$ is the shaded rectangle.

Lemma 3.2 *Let $f \in \mathrm{Mod}(S_{g,n})$ be a pseudo-Anosov element and $\widehat{f} \in \mathrm{Mod}(S_{g,o})$ be the induced mapping class obtained by forgetting marked points. There exists a constant $1 \le \alpha \le \Theta(g)$ such that $\widehat{f}^{\alpha}$ satisfies exactly one of the following:*

(1) *$\widehat{f}^{\alpha}$ restricts to a pseudo-Anosov map on a connected subsurface.*

(2) $\widehat{f}^{\alpha} = \mathrm{Id}$.

(3) *$\widehat{f}^{\alpha}$ is a multitwist map.*

Remark For the first two cases of Lemma 3.2, one can find α bounded by a linear function of g, but in case 3, α may be exponential in g.

Theorem 3.3 *For $g \ge 2$, given any pseudo-Anosov $f \in \mathrm{Mod}(S_{g,n})$, let α be as in Lemma 3.2. Then*

$$\log \lambda(f) \ge \min\left\{\frac{\log 2}{\alpha(12g-12)}, \frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)}\right\}.$$

Proof We will deal with case 1 of Lemma 3.2 first.

If $\widehat{f}^{\alpha}$ restricts to a pseudo-Anosov homeomorphism on a connected subsurface $\sum_{g_0,n_0}$ of $S_{g,0}$ of genus g_0 with n_0 boundary components (we have $2g_0 + n_0 \le 2g$), then Penner's lower bound tells us

$$\log \lambda(\widehat{f}^{\alpha}) \ge \frac{\log 2}{12g_0 - 12 + 4n_0} \ge \frac{\log 2}{12g - 12}.$$

Hence $\log \lambda(f) \ge \log \lambda(\widehat{f}) > \log 2/\alpha(12g-12)$.

If $\widehat{f}^{\alpha}$ is homotopic to the identity or a multitwist map, from Proposition 2.6, we have $L(f^{\alpha}) = L(\widehat{f}^{\alpha}) = \chi(S_{g,0}) = 2-2g < 0$. Theorem 2.3 tells us that for any pseudo-Anosov f there is a Markov partition with k rectangles, where $k \le -3\chi(S)$. Recall that the transition matrix M obtained from the rectangles is a $k \times k$ Perron–Frobenius matrix and the Perron–Frobenius eigenvalue $\mu(M)$ equals $\lambda(f)$.

By Lemma 3.1, there is a rectangle R such that the interiors of $f^{\alpha}(R)$ and R intersect. This implies that there is a nonzero entry on the diagonal of M^{α}. Applying

Proposition 2.4, we obtain that $\mu((M^\alpha)^{2k}) = \mu(M^{2k\alpha})$ is at least k, so we have

$$(\lambda(f))^{2k\alpha} = \lambda(f^{2k\alpha}) = \mu(M^{2k\alpha}) \geq k.$$

One can easily check $(\log x)/x$ is monotone decreasing for $x \geq 3$. Since

$$3 \leq k \leq -3\chi(S) = 6g + 3n - 6,$$

hence

$$\log \lambda(f) \geq \frac{\log k}{2\alpha k} \geq \frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)}.$$

□

Remark Penner's proof in [13] does not use Lefschetz numbers which we used to conclude that $\mu(M^{2k\alpha})$ is at least k, so we obtain a sharper lower bound for $n \gg g$.

4 An example which provides an upper bound

4.1 For the genus two case

In this section, we will construct a pseudo-Anosov $f \in \mathrm{Mod}(S_{2,n})$ for all $n \geq 31$ then we compute its dilatation which gives us an upper bound for $l_{2,n}$.

Let $S_{0,m+2}$ be a genus 0 surface with $m+2$ marked points (ie a marked sphere), and recall an example of pseudo-Anosov $\phi \in \mathrm{Mod}(S_{0,m+2})$ in [7]. We view $S_{0,m+2}$ as a sphere with $s+1$ marked points X circling an unmarked point x and $t+1$ marked points Y circling an unmarked point y, and a single extra marked point z. We can also draw this as a "turnover", as in Figure 2. Note that $|X \cap Y| = 1$, $|X| = s+1$, $|Y| = t+1$ and $m = s+t$.

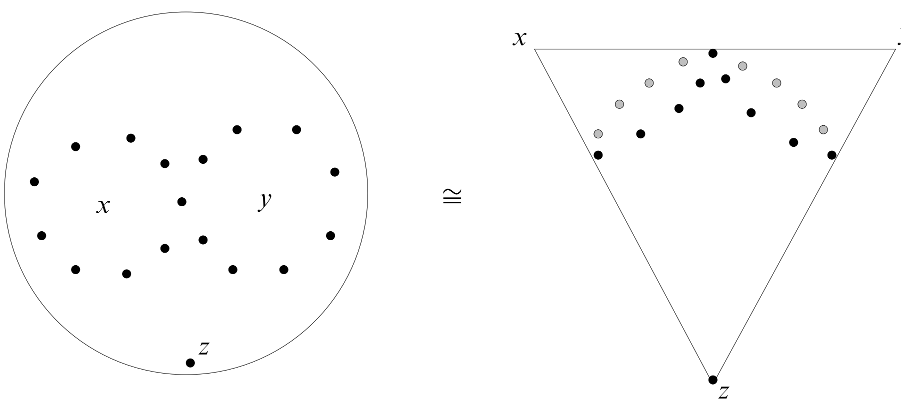


Figure 2: Two way of viewing a marked sphere. Black dots are marked points and the shaded dots on the right are marked points at the back.

We define homeomorphisms α_s, β_t: $S_{0,m+2} \to S_{0,m+2}$ such that α_s rotates the marked points of X counterclockwise around x and β_t rotates the marked points of Y clockwise around y; see Figure 3. Define $\phi_{s,t} := \beta_t\alpha_s$. In [7], it is shown that $\phi_{s,t}$

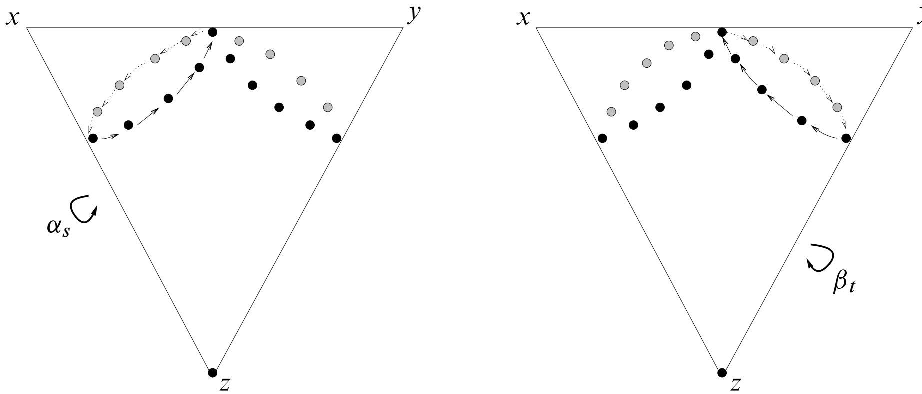


Figure 3: Homeomorphisms α_s and β_t

is pseudo-Anosov by checking it satisfies the criterion of [2]. We also note that from this one can check that x, y and z are fixed points of a pseudo-Anosov representative of $\phi_{s,t}$. Moreover, for s, $t \geq 1$ the dilatation of $\phi_{s,t}$ equals the largest root of the polynomial

$$\begin{aligned} T_{s,t}(x) &= x^{t+1}(x^s(x-1)-2) + x^{s+1}(x^{-s}(x^{-1}-1)-2) \\ &= (x-1)x^{(s+t+1)} - 2(x^{s+1}+x^{t+1}) - (x-1). \end{aligned}$$

The dilatation is minimized when $s = \lfloor m/2 \rfloor$ and $t = \lceil m/2 \rceil$. Let us define $\phi := \phi_{\lfloor m/2\rfloor,\lceil m/2\rceil}$ and its dilatation is the largest root of the polynomial

$$\begin{aligned} T_m(x) &:= T_{\lfloor m/2\rfloor,\lceil m/2\rceil}(x) \\ &= (x-1)x^{(m+1)} - 2\big(x^{\lfloor m/2\rfloor+1} + x^{\lceil m/2\rceil+1}\big) - (x-1). \end{aligned}$$

Proposition 4.1 *If $m \geq 5$, then the largest real root of $T_m(x)$ is bounded above by $m^{3/m}$.*

Proof For all m, we have $T_m(1) = -4$. It is sufficient to show that for all $x \geq m^{3/m}$, we have $T_m(x) > 0$. Dividing the inequality by $x^{(m+1)}$, it is equivalent to show

$$(x-1) + x^{-(m+1)} > 2\big(x^{\lfloor m/2\rfloor-m} + x^{\lceil m/2\rceil-m}\big) + x^{-m}.$$

For $m \geq 5$, one can verify the following inequalities hold for all $x \geq m^{3/m}$:

(1) $x-1>(3\log m)/m\geq 9/(2m)$,

(2) $x^{\lfloor m/2\rfloor-m}\leq x^{\lceil m/2\rceil-m}\leq 1/m$,

(3) $x^{-m}\leq 1/(25m)$.

Therefore,

$$(x-1)+x^{-(m+1)}>x-1>\frac{9}{2m}>\frac{101}{25m}=2\left(\frac{1}{m}+\frac{1}{m}\right)+\frac{1}{25m}$$
$$\geq 2\big(x^{\lfloor m/2\rfloor-m}+x^{\lceil m/2\rceil-m}\big)+x^{-m}. \qquad \square$$

Remark Proposition 4.1 fails if we try to replace the bound with $c^{1/m}$ where c is any constant.

Remark Hironaka and Kin [7] construct two infinite families of pseudo-Anosovs in $\mathrm{Mod}(S_{0,m})$, with $\phi_{s,t}$ being one of them. Unlike $\phi_{s,t}$, the other family provides the sharp bound on $l_{0,m}$.

Next, we take a cyclic branched cover $S_{2,n}$ of $S_{0,m+2}$ with branched points x, y, and z, where $n=5(m+1)+1$ (See Figure 4.). Define $\widetilde{X}=\{\text{marked points around } \tilde{x}\}$ and $\widetilde{Y}=\{\text{marked points around } \tilde{y}\}$, so we have $|\widetilde{X}\cap\widetilde{Y}|=5$, $|\widetilde{X}|=5(s+1)$ and $|\widetilde{Y}|=5(t+1)$.

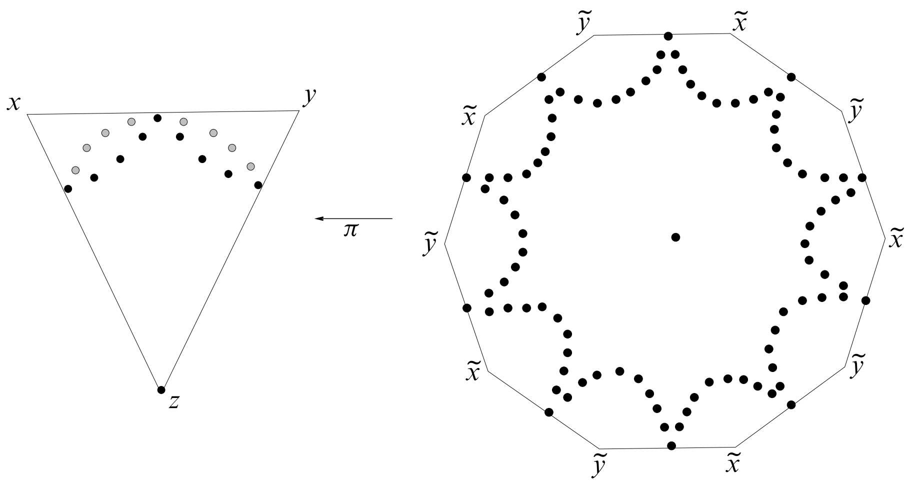


Figure 4: π is the covering map. To form $S_{2,n}$ from the decagon, identify the opposite sides. Then π is the quotient by the group generated by rotation of an angle $2\pi/5$.

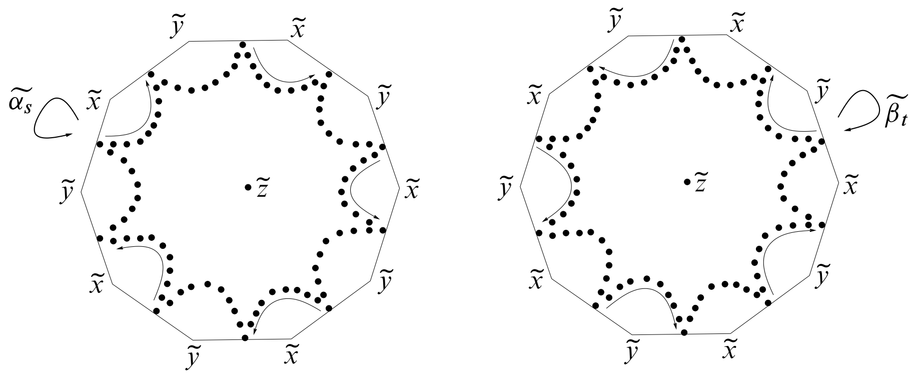


Figure 5: Homeomorphisms $\widetilde{\alpha_s}$ and $\widetilde{\beta_t}$

We lift α_s, β_t to $S_{2,n}$ and call them $\widetilde{\alpha_s}$, $\widetilde{\beta_t}$, so that $\widetilde{\alpha_s}$ rotates the marked points of $\widetilde{X}$ counterclockwise around $\widetilde{x}$ and $\widetilde{\beta_t}$ rotates the marked points of $\widetilde{Y}$ clockwise around $\widetilde{y}$; see Figure 5. We define $\psi_{s,t} := \widetilde{\beta_t}\,\widetilde{\alpha_s}$. It follows that $\psi_{s,t}$ is a lift of $\phi_{s,t}$, and so is pseudo-Anosov with $\lambda(\psi_{s,t}) = \lambda(\phi_{s,t})$. An invariant train track for $\psi_{s,t}$ is obtained by lifting the one constructed in [7], and is shown in Figure 6 for $s = t = 3$.

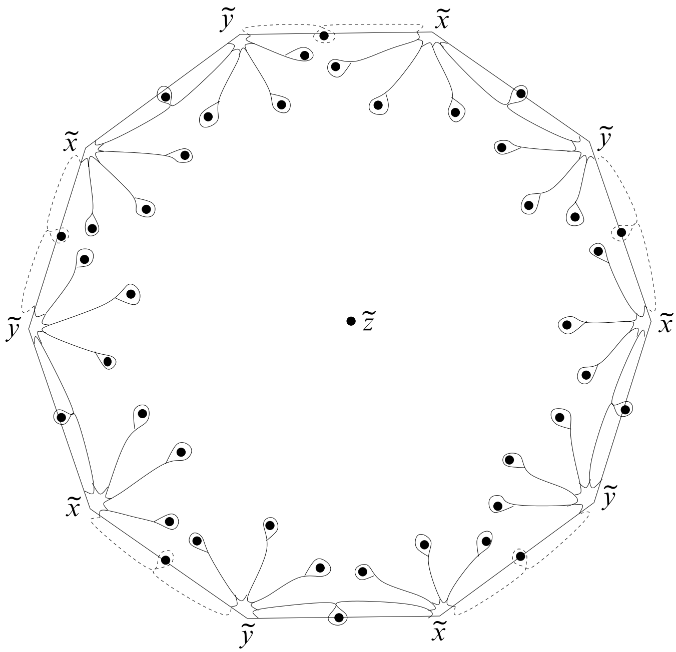


Figure 6: A train track for $\psi_{3,3}$

Hence for $n = 5(m+1)+1 \geq 31$, we have constructed a pseudo-Anosov $\psi = \psi_{\lfloor m/2 \rfloor, \lceil m/2 \rceil} \in \mathrm{Mod}(S_{2,n})$ with $\lambda(\psi) = \lambda(\phi) \leq m^{3/m}$ which implies

$$\log \lambda(\psi) \leq \frac{3 \log m}{m} = \frac{15 \log(n-6) - 15 \log 5}{n-6}.$$

We will now extend ψ so that n can be an arbitrary number ≥ 31. We add an extra marked point p_1 on $S_{2,n}$ between points in $\widetilde{X}$ or $\widetilde{Y}$ *except the places shown in Figure 7*.

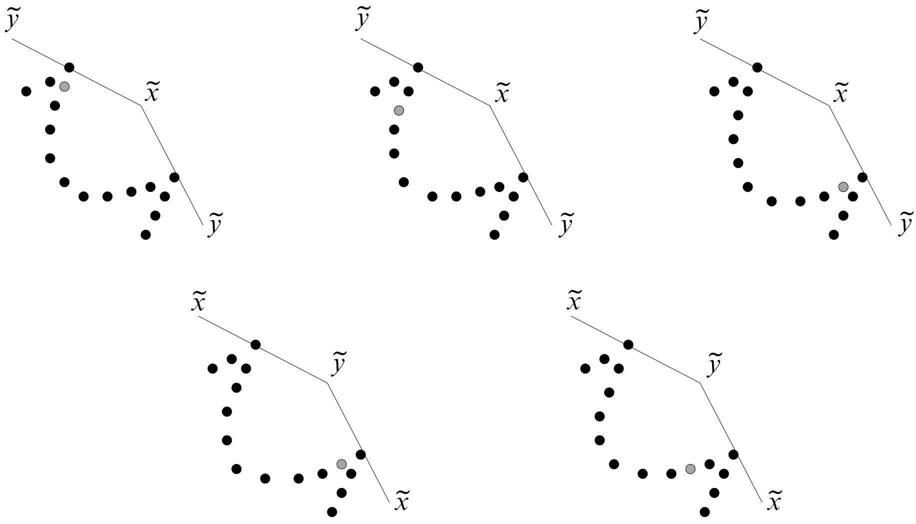


Figure 7: We are *not* allowed to add p_1 in the places indicated by a shaded point.

Without loss of generality we assume p_1 is added in $\widetilde{X}$ to obtain $S_{2,n+1}$ and we define $\psi_1 := \widetilde{\beta_t}\, \widetilde{\alpha_s}' \in \mathrm{Mod}(S_{2,n+1})$ where $\widetilde{\alpha_s}'$ is extended from $\widetilde{\alpha_s}$ in the obvious way; see Figure 8. One can check that ψ_1 is pseudo-Anosov via the techniques of [2]. An invariant train track for ψ_1 is shown in Figure 9 and is obtained by modifying the invariant train track for ψ shown in Figure 6.

Next, we will show $\lambda(\psi_1) \leq \lambda(\psi)$. Let H (respectively, H_1) be the associated transition matrix of the train track map for ψ (respectively, ψ_1), and let Γ (respectively, Γ_1) be the induced directed graph as constructed in Section 2.2.

From the construction above (ie adding p_1), the directed graph Γ_1 is obtained by adding a vertex on the edge going out from some vertex i in Γ (that is, subdividing the edge going out from i) where i has exactly one edge coming in and exactly one

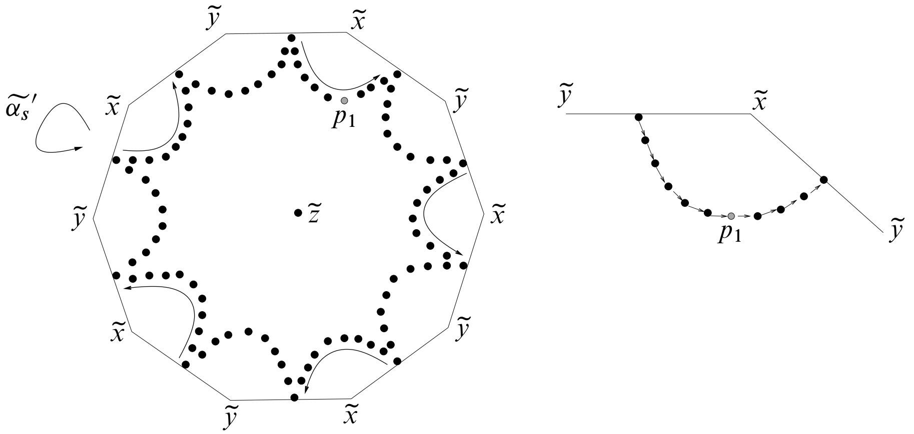


Figure 8: The homeomorphism $\widetilde{\alpha_s}'$. The figure on the right is a local picture near the added point p_1.

edge going out. This implies $P_{\Gamma_1}(i, k+1) = P_\Gamma(i,k)$ and

$$\sqrt[k+1]{P_{\Gamma_1}(i, k+1)} \leq \sqrt[k]{P_{\Gamma_1}(i, k+1)} = \sqrt[k]{P_\Gamma(i,k)}$$

for all k. Since H and H_1 are Perron–Frobenius matrices with Perron–Frobenius eigenvalues corresponding to the dilatations of ψ and ψ_1, and Proposition 2.5 tells us $\mu(H_1) \leq \mu(H)$, we have $\lambda(\psi_1) = \mu(H_1)$ is no greater than $\lambda(\psi) = \mu(H)$.

We can obtain ψ_2, ψ_3 and ψ_4 by repeating the construction above of adding more marked points without increasing dilatations (ie $\lambda(\psi_c) \leq \lambda(\psi)$ for $c = 1, 2, 3, 4$). Since $(\log m)/m \geq (\log(m+1))/(m+1)$, we need not consider the cases with $c \geq 5$. Therefore, set $f\colon S_{2,n} \to S_{2,n}$ to be ψ_c, where $n = 5(m+1)+1+c$ with $c < 5$, and where $\psi_0 = \psi$. For $n \geq 31$, we have

$$\log \lambda(f) \leq \log \lambda(\psi) < \frac{3 \log m}{m} < \frac{3 \log\left(\frac{n-11}{5}\right)}{\left(\frac{n-11}{5}\right)},$$

where $m = \lfloor (n-6)/5 \rfloor$.

Theorem 4.2 *There exists $\kappa_2 > 0$ such that*

$$l_{2,n} < \frac{\kappa_2 \log n}{n},$$

for all $n \geq 3$.

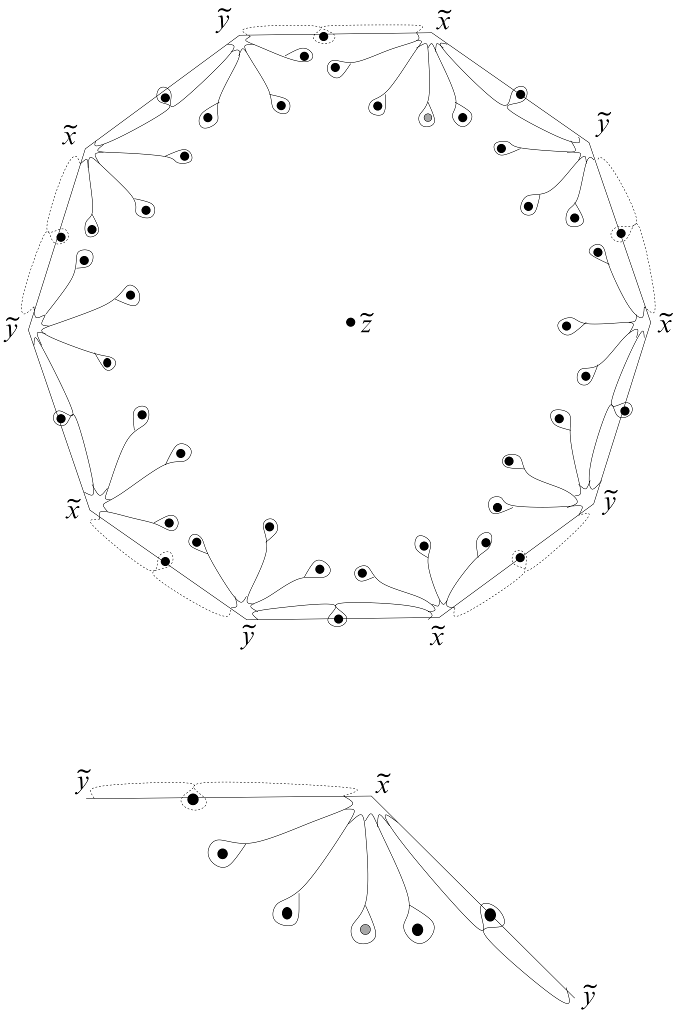


Figure 9: A train track for ψ_1. The figure on the bottom is a local picture.

Proof From the discussion above, for $n \geq 31$,

$$l_{2,n} < \frac{3\log\left(\frac{n-11}{5}\right)}{\left(\frac{n-11}{5}\right)} < \frac{\kappa_2' \log n}{n},$$

for some κ_2'. For $3 \leq n \leq 30$, let $\kappa_2'' = \max\{l_{2,3}, l_{2,4}, \dots, l_{2,30}\}$ then

$$l_{2,n} \leq \kappa_2'' = \left(\kappa_2'' \frac{31}{\log 31}\right) \frac{\log 31}{31} < \left(\kappa_2'' \frac{31}{\log 31}\right) \frac{\log n}{n}.$$

Let $\kappa_2 := \max\{\kappa_2', \kappa_2''(31/\log 31)\}$. □

4.2 Higher genus cases

We can generalize our construction and extend to any genus $g > 2$. For any fixed $g > 2$, we define ψ to be a homeomorphism of $S_{g,n}$ in the same fashion with $n = (2g+1)(m+1)+1$ by taking an appropriate branched cover over $S_{0,m+2}$, and we can again extend to arbitrary n by adding c extra marked points and constructing ψ_c. Define $f\colon S_{g,n} \to S_{g,n}$ to be ψ_c where $n = (2g+1)(m+1)+1+c$. If $n \geq 6(2g+1)+1$, then

$$\begin{aligned}\log\lambda(f) &< \frac{3\log m}{m}, \quad \text{where } m = \left\lfloor \frac{n-1}{2g+1} \right\rfloor - 1\\ &< \frac{3\log\left(\frac{n-4g-3}{2g+1}\right)}{\left(\frac{n-4g-3}{2g+1}\right)}.\end{aligned}$$

Theorem 4.3 *For any fixed $g \geq 2$, there exists $\kappa_g > 0$ such that*

$$l_{g,n} < \frac{\kappa_g \log n}{n},$$

for all $n \geq 3$.

Proof This is similar to the proof of Theorem 4.2, where κ_g is defined to be

$$\kappa_g := \max\left\{\kappa'_g, \kappa''_g \frac{12g+7}{\log(12g+7)}\right\}. \qquad \square$$

Proof of Theorem 1.1 We only need to prove that the lower bounds on $\log\lambda(f)$ of Theorem 3.3 are bounded below by $(\log n)/(\omega_g n)$ for some ω_g depending only on g, then let $c_g = \max\{\kappa_g, \omega_g\}$. We use the monotone decreasing property of $(\log n)/n$ for $n \geq 3$. Let

$$\omega'_g(\alpha) := \frac{\alpha(12g-12)}{\log 2}\frac{\log 3}{3} \geq \frac{\alpha(12g-12)}{\log 2}\frac{\log n}{n}$$

and so

$$\frac{\log 2}{\alpha(12g-12)} \geq \frac{\log n}{\omega'_g(\alpha) n}.$$

For $n \geq g-1$,

$$\frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)} \geq \frac{\log 9n}{2\alpha 9n} > \frac{1}{18\alpha}\frac{\log n}{n}.$$

For $3 \le n < g-1$,

$$\frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)} > \frac{\log(9(g-1))}{2\alpha 9(g-1)} > \frac{\log g}{18\alpha g}\frac{3}{\log 3}\frac{\log n}{n}.$$

Let $\omega_g := \max\{\omega'_g(\alpha), 18\alpha, (6\alpha g \log 3)/\log g\}$, where $0 \le \alpha \le \Theta(g)$. □

5 Appendix

5.1 Torus with marked points

We will construct an example to prove that $l_{1,2n}$ has an upper bound of the same order as Penner's lower bound in [13], ie $l_{1,2n} = O(1/n)$. The construction is analogous to the one given by Penner for $S_{g,0}$ in [13].

Let $S_{1,2n}$ be a marked torus of $2n$ marked points. Let a and b be essential simple closed curves as in Figure 10. Let T_a^{-1} be the left Dehn twist along a and T_b be the

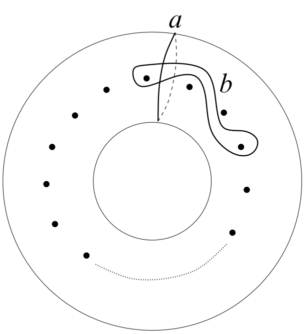


Figure 10: Essential simple closed curves a and b on a marked torus

right Dehn twist along b, then we define

$$f := \rho \circ T_b \circ T_a^{-1} \in \mathrm{Mod}(S_{1,2n})$$

where ρ rotates the torus clockwise by an angle of $2\pi/n$, so it sends each marked point to the one which is two to the right. As in [12], f^n is shown to be pseudo-Anosov, and thus so is f. Figure 11 shows a bigon track for f^n.

We obtain the $2n \times 2n$ transition matrix M^n associated to the train track map of f^n where M^n is an integral Perron–Frobenius matrix and the Perron–Frobenius

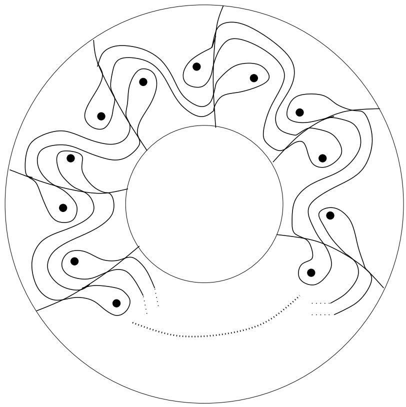


Figure 11: A bigon track for f^n

eigenvalues $\mu(M^n)$ is the dilatation $\lambda(f^n)$ of f^n. For $n \geq 5$, we have $M^n = N$, where

$$N = \begin{pmatrix} A_1 & B_1 & 0 & 0 & \cdots & 0 & 0 & D_1 \\ A_2 & B_2 & B_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & B_3 & B_2 & B_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & B_3 & B_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & B_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_2 & B_1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_3 & B_2 & D_2 \\ A_3 & C & 0 & 0 & \cdots & 0 & B_3 & D_3 \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

For $n \geq 5$, the greatest column sum of M^n is 9 and the greatest row sum of M^n is 11. One can verify that both the greatest column sum and the greatest row sum are ≤ 11 for $0 < n \leq 4$. Therefore, for $n \geq 1$,

$$\begin{aligned} 11 \geq \mu(M^n) = \lambda(f^n) = (\lambda(f))^n \\ \Rightarrow l_{1,2n} \leq \log \lambda(f) \leq \frac{\log 11}{n}. \end{aligned}$$

5.2 Higher genus with marked points

In all of the following examples we obtain a mapping class $\tilde{f} \in \mathrm{Mod}(S_{g,n})$ from $f \in \mathrm{Mod}(S_{g,0})$ by adding marked points on the closed surface $S_{g,0}$, where f is a composition of Dehn twists along some set $\mathcal{T}$ of closed geodesics. We can add one marked point in each of the complementary disks of the curves in $\mathcal{T}$ without creating essential reducing curves. By [12, Theorem 3.1], the induced mapping class $\tilde{f} \in \mathrm{Mod}(S_{g,n})$ is pseudo-Anosov with dilatation $\lambda(\tilde{f}) = \lambda(f)$.

Example 1 Penner [13] constructed a pseudo-Anosov mapping class $f \in \mathrm{Mod}(S_{g,0})$ with dilatation $\lambda(f) \leq (\log 11)/g$ for $g \geq 2$, where

$$f := \rho \circ T_c \circ T_a^{-1} \circ T_b.$$

and T_α is the Dehn twist along α. Here $\mathcal{T} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ with

$$\mathcal{A} = \bigsqcup_{i=1}^{g} a_i, \quad \mathcal{B} = \bigsqcup_{i=1}^{g} b_i \quad \text{and} \quad \mathcal{C} = \bigsqcup_{i=1}^{g} c_i.$$

We can add g marked points as in the Figure 12 so that $\tilde{f} \in \mathrm{Mod}(S_{g,g})$ is pseudo-Anosov. Therefore,

$$l_{g,g} \leq \log \lambda(\tilde{f}) \leq \frac{\log 11}{g}.$$

We can also add extra marked points at the fixed points of the rotation. For $g \geq 2$, we will have for $c = 0, 1$ and 2,

$$l_{g,g+c} \leq \log \lambda(\tilde{f}) \leq \frac{\log 11}{g},$$

where $\tilde{f} \in \mathrm{Mod}(S_{g,g+c})$.

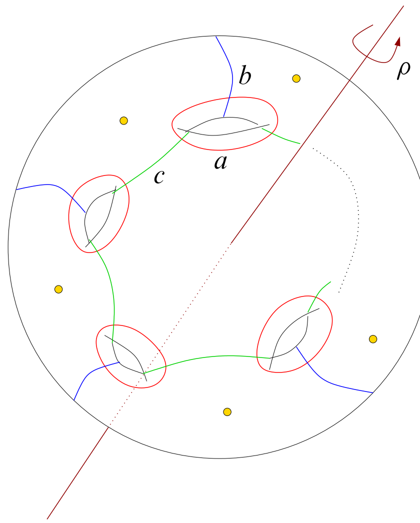


Figure 12: A pseudo-Anosov $\widetilde{f} \in \mathrm{Mod}(S_{g,g})$

Example 2 For all $g \geq 3$, define $f\colon S_{g,0} \to S_{g,0}$ to be

$$f := \rho \circ T_{b_1} \circ T_{a_1}^{-1},$$

where

$$\rho(a_1) = a_{g+1}, \qquad \rho(b_1) = b_{g+1}$$

and $$\rho(a_i) = a_{i-1}, \qquad \rho(b_i) = b_{i-1}, \qquad i = 2, \ldots, g+1.$$

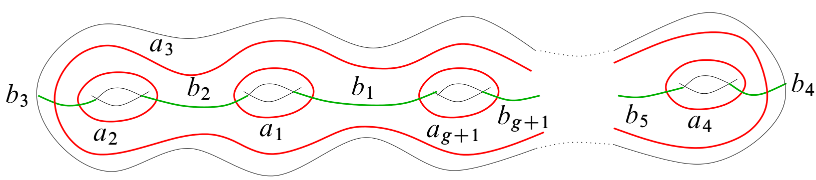


Figure 13: A pseudo-Anosov $f \in \mathrm{Mod}(S_{g,0})$

We construct the $(2g+2) \times (2g+2)$ transition matrix $M^{(g+1)}$ with respect to the spanning vectors associated with geodesics in $\mathcal{T}$. We will get $M^{(g+1)} = N$ for $g \geq 3$, where the matrices are the same as in the Appendix (Section 5.1). Therefore for $g \geq 3$ we have

$$\log \lambda(f) \leq \frac{\log 9}{g+1}.$$

Here $\mathcal{T} = \mathcal{A} \cup \mathcal{B}$ with

$$\mathcal{A} = \bigsqcup_{i=1}^{g} a_i \quad \text{and} \quad \mathcal{B} = \bigsqcup_{i=1}^{g} b_i.$$

For $g \geq 3$ and $c = 0, 1, 2, 3, 4$, we have

$$l_{g,c} \leq \log \lambda(\tilde{f}) \leq \frac{\log 9}{g+1},$$

where $\tilde{f} \in \mathrm{Mod}(S_{g,c})$.

Example 3 For $g \geq 5$, define $f\colon S_{g,0} \to S_{g,0}$ by

$$f := \rho \circ T_{d_1} \circ T_{c_1}^{-1} \circ T_{b_1} \circ T_{a_1},$$

where

$$\rho(a_1) = a_{g-1},\ \rho(b_1) = b_{g-1},\ \rho(c_1) = c_{g-1},\ \rho(d_1) = d_{g-1}$$

and $\rho(a_i) = a_{i-1},\ \ \rho(b_i) = b_{i-1},\ \ \rho(c_i) = c_{i-1},\ \ \rho(d_i) = d_{i-1},\ i = 2, \dots, g-1.$

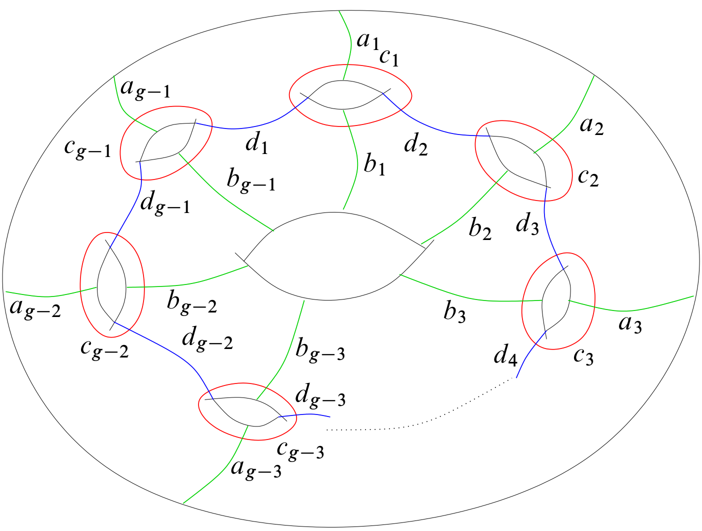


Figure 14: A pseudo-Anosov $f \in \mathrm{Mod}(S_{g,0})$

Similarly, we have the $(4g-4) \times (4g-4)$ transition matrix $M^{(g-1)}$ with respect to the spanning vectors associated with the geodesics in $\mathcal{T}$. For $g \geq 5$ we have $M^{(g-1)} = N$

where

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 3 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 4 & 3 \end{pmatrix}.$$

For $g \geq 5$, the greatest column sum of $M^{(g-1)}$ is 17 and the greatest row sum of $M^{(g-1)}$ is 21, hence

$$\log \lambda(f) \leq \frac{\log 17}{g-1}.$$

Here $\mathcal{T} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ with

$$\mathcal{A} = \bigsqcup_{i=1}^{g} a_i, \quad \mathcal{B} = \bigsqcup_{i=1}^{g} b_i, \quad \mathcal{C} = \bigsqcup_{i=1}^{g} c_i \quad \text{and} \quad \mathcal{D} = \bigsqcup_{i=1}^{g} d_i.$$

For $c = 1$ and 2, we can induce $\widetilde{f} \in \mathrm{Mod}(S_{g,c(g-1)})$ with

$$l_{g,c(g-1)} \leq \log \lambda(\widetilde{f}) \leq \frac{\log 17}{g-1},$$

when $g \geq 5$.

References

[1] **M Bauer**, *An upper bound for the least dilatation*, Trans. Amer. Math. Soc. 330 (1992) 361–370 MR1094556

[2] **M Bestvina**, **M Handel**, *Train-tracks for surface homeomorphisms*, Topology 34 (1995) 109–140 MR1308491

[3] **R Bott**, **L W Tu**, *Differential forms in algebraic topology*, Graduate Texts in Math. 82, Springer, New York (1982) MR658304

[4] **A Fathi**, **F Laudenbach**, **V Poenaru**, **editors**, *Travaux de Thurston sur les surfaces*, second edition, Astérisque 66, Soc. Math. France, Paris (1991) MR568308

[5] **F R Gantmacher**, *The theory of matrices. Vols. 1, 2*, Translated by K. A. Hirsch, Chelsea Publ. Co., New York (1959) MR0107649

[6] **V Guillemin**, **A Pollack**, *Differential topology*, Prentice-Hall, Englewood Cliffs, N.J. (1974) MR0348781

[7] **E Hironaka**, **E Kin**, *A family of pseudo-Anosov braids with small dilatation*, Algebr. Geom. Topol. 6 (2006) 699–738 MR2240913

[8] **N V Ivanov**, *Coefficients of expansion of pseudo-Anosov homeomorphisms*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167 (1988) 111–116, 191 MR964259

[9] **N V Ivanov**, *Subgroups of Teichmüller modular groups*, Transl. of Math. Monogr. 115, Amer. Math. Soc. (1992) MR1195787 Translated from the Russian by E J F Primrose and revised by the author

[10] **C T McMullen**, *Polynomial invariants for fibered 3–manifolds and Teichmüller geodesics for foliations*, Ann. Sci. École Norm. Sup. (4) 33 (2000) 519–560 MR1832823

[11] **H Minakawa**, *Examples of pseudo-Anosov homeomorphisms with small dilatations*, J. Math. Sci. Univ. Tokyo 13 (2006) 95–111 MR2277516

[12] **R C Penner**, *A construction of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc. 310 (1988) 179–197 MR930079

[13] **R C Penner**, *Bounds on least dilatations*, Proc. Amer. Math. Soc. 113 (1991) 443–450 MR1068128

Department of Mathematics, The University of Illinois at Urbana-Champaign
1409 West Green Street, Urbana, IL 61801, USA

ctsai6@math.uiuc.edu

http://www.math.uiuc.edu/~ctsai6/

Proposed: Joan Birman
Seconded: Danny Calegari, Walter Neumann

Received: 8 October 2008
Revised: 6 May 2009