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THE ASYMPTOTIC BEHAVIOUR OF THE SAMPLE AUTOCOVARIANCE FUNCTION FOR AN AUTOREGRESSIVE INTEGRATED MOVING AVERAGE PROCESS

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We derive the first two asymptotic moments of the sample autocovariance function of a time series generated from an autoregressive d-th integrated moving average process for any positive integer d. The obtained results show that the sample autocovariance function is a random variable of order N^{2d-1}, where N is the observed length of data.

1. Introduction

Box and Jenkins [2] used an autoregressive integrated moving average model to describe a homogeneous nonstationary time series. At the first stage of their three-step procedure, i.e., identification, estimation and diagnostic checking, the sample autocovariance function of the data is computed as one of the key statistics. In this paper, asymptotic values of the mean and the covariance of the sample autocovariance function of the data generated from an autoregressive integrated moving average process are obtained.

Some authors dealt with the parameter estimation problem for once integrated processes (e.g., see White [12], Fuller [4], Rao [10], Dickey and Fuller [3]). Hasza and Fuller [5] considered this problem for an autoregressive twice integrated process. Kawashima [8] studied the least square parameter estimation of autoregressive integrated moving average processes of arbitrary orders. Hasza [6] studied the asymptotic distribution of the sample autocorrelation function for an autoregressive once integrated moving average process. Wichern [13] and Roy [11] obtained the mean and the asymptotic covariance of the sample autocovariance function for an once integrated first order moving average process. Nakano and Tagami [9] calculated higher order terms for Roy's result.

We extend Wichern and Roy's results for an autoregressive integrated moving average process of arbitrary orders. Section 2 describes definitions and basic lemmas. The mean and the covariance of the sample autocovariance function are calculated in Sections 3 and 4, respectively. Some parts of calculations in these sections are verified by means of the algebraic programming system REDUCE (Hearn [7]). Simulation results and discussions are given in Section 5.

2. Preliminaries

Let $y_0, y_1, \cdots, y_{N-1}$ be N consecutive observations generated from an autoregressive integrated moving average process of order (p, d, q), which is denoted as ARIMA (p, d, q)

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process, defined by the difference equation

$$\phi(B)(1-B)^d y_t=\theta(B)a_t \tag{2.1}$$

where

$$\phi(B)=1-\phi_1 B-\cdots-\phi_p B^p, \tag{2.2}$$

$$\theta(B)=1-\theta_1 B-\cdots-\theta_q B^q, \tag{2.3}$$

and B is a backward shift operator defined by $By_t=y_{t-1}$. It is assumed that the absolute values of the roots of the equations $\phi(z)=0$ and $\theta(z)=0$ are greater than one and $\{a_t\}$ is a sequence of uncorrelated random variables with mean zero and variance σ^2. The sample autocovariance function $\{c_k\}$ of the data $\{y_t\}$ is defined by

$$c_k=\frac{1}{N}\sum_{t=0}^{N-k-1}(y_t-\bar{y})(y_{t+k}-\bar{y}) \qquad (k=0, 1, \cdots, N-1) \tag{2.4}$$

where $\bar{y}=(y_0+\cdots+y_{N-1})/N$ is the sample mean.

Let x_t be d times backward difference of y_t, i.e., $x_t=(1-B)^d y_t$. The autocovariance function of the stationary process $\{x_t\}$ is given by

$$\sigma(k)=\mathrm{E}[x_t x_{t+k}]=\int_{-\pi}^{\pi} f(\lambda)e^{i\lambda k}d\lambda \tag{2.5}$$

where

$$f(\lambda)=\frac{\sigma^2|\theta(e^{i\lambda})|^2}{2\pi|\phi(e^{i\lambda})|^2}. \tag{2.6}$$

Kawashima [8] gave the following lemma for decomposing y_t into the summation of x_t's and remainder terms.

LEMMA 1. *For $d\leqq t\leqq N-1$, y_t has a unique representation given by*

$$y_t=\sum_{j=0}^{t-d} f_{j+1}^d x_{t-j}+\sum_{j=0}^{d-1} g_j^t (1-B)^j y_j, \tag{2.7}$$

where

$$f_{j+1}^d=\frac{1}{(d-1)!}\frac{(j+d-1)!}{j!}, \tag{2.8}$$

$$g_j^t=\frac{t!}{j!(t-j)!}. \tag{2.9}$$

In the next section, we use LEMMA 8.3.3. of Anderson [1] (p. 462) for evaluating integrals. Next lemma is a extension of Anderson's lemma and is required for the same purposes in Section 4. The proof is not difficult and is omitted.

LEMMA 2. *If $h(\lambda,\mu)$ is bounded and is continuous at $(\lambda,\mu)=(0,0)$, if for some N_0*

$$\iint_{-\pi}^{\pi} l_N(\lambda, \mu)d\lambda d\mu=1, \qquad N=N_0, \cdots, \tag{2.10}$$

if there is a number K_1 and integer N_1 such that

$$\iint_{-\pi}^{\pi} |l_N(\lambda, \mu)|d\lambda d\mu\leqq K_1, \qquad N=N_1, \cdots, \tag{2.11}$$

if there is a number K_2 *and integer* N_2 *such that*

$$|l_N(\lambda,\mu)|\leqq K_2 f(a)\,,\qquad |\lambda|\geqq a \text{ or } |\mu|\geqq a,\ N=N_2,\cdots, \tag{2.12}$$

for every $a>0$, *and* $f(a)$ *is monotonically nonincreasing function of a for* $0<a<\pi$, *and if for some* N_3

$$|l_N(\lambda,\mu)|\leqq m_N g(a)\,,\qquad |\lambda|\geqq a \text{ and } |\mu|\geqq a,\ N=N_3,\cdots, \tag{2.13}$$

for every $a>0$, *and* $g(a)$ *is monotonically nonincreasing function of a for* $0<a<\pi$, *and* $m_N\to 0$ *as* $N\to\infty$, *then*

$$\lim_{N\to\infty}\iint_{-\pi}^{\pi} h(\lambda,\mu)l_N(\lambda,\mu)d\lambda d\mu=h(0,0)\,. \tag{2.14}$$

3. Asymptotic mean of the sample autocovariance function

In this section, we obtain the highest order term of the expectation of c_k, which is rewritten as

$$c_k=\frac{1}{N}\sum_{t=0}^{N-k-1} y_t y_{t+k}-\bar{y}^2+\bar{y}\frac{1}{N}\sum_{t=0}^{k-1}(y_t+y_{N-t-1})-\frac{k}{N}\bar{y}^2\,. \tag{3.1}$$

By the Lemma 1 of Section 2, the expectation of the first term of the right-hand side of this equation is expanded as

$$\begin{aligned}\mathrm{E}\Big[\frac{1}{N}\sum_{t=0}^{N-k-1} y_t y_{t+k}\Big]=\mathrm{E}\Big[&\frac{1}{N}\sum_{t=d}^{N-k-1}\Big\{\sum_{j=0}^{d-1} g_j^t(1-B)^j y_j\Big\}\Big\{\sum_{h=0}^{d-1} g_h^{t+k}(1-B)^h y_h\Big\}\\ &+\frac{1}{N}\sum_{t=d}^{N-k-1}\sum_{j=0}^{t-d}\sum_{h=0}^{t+k-d} f_{j+1}^d f_{h+1}^d x_{t-j}x_{t+k-h}+\frac{1}{N}\sum_{t=0}^{d-1} y_t y_{t+k}\\ &+\frac{1}{N}\sum_{t=d}^{N-k-1}\Big\{\Big(\sum_{j=0}^{t-d} f_{j+1}^d x_{t-j}\Big)\Big(\sum_{h=0}^{d-1} g_h^{t+k}(1-B)^h y_h\Big)\\ &+\Big(\sum_{h=0}^{t+k-d} f_{h+1}^d x_{t+k-h}\Big)\Big(\sum_{j=0}^{d-1} g_j^t(1-B)^j y_j\Big)\Big\}\Big]\,.\end{aligned} \tag{3.2}$$

Through this paper, the condition about the initial data,

$$\sum_{t=0}^{d-1}\mathrm{E}[y_t^2]<\infty \tag{3.3}$$

is assumed. Then we have the constant $C_1=\max_{0\leq j,h\leq d-1}|\mathrm{E}[(1-B)^j y_j(1-B)^h y_h]|$ and the first term of the right-hand side of (3.2) is shown to be

$$\begin{aligned}&\left|\mathrm{E}\Big[\frac{1}{N}\sum_{t=d}^{N-k-1}\Big\{\sum_{j=0}^{d-1} g_j^t(1-B)^j y_j\Big\}\Big\{\sum_{h=0}^{d-1} g_h^{t+k}(1-B)^h y_h\Big\}\Big]\right|\\ &\qquad\leqq\frac{C_1}{N}\sum_{t=d}^{N-k-1}\sum_{j=0}^{d-1}\sum_{h=0}^{d-1} g_j^t g_h^{t+k}\leqq C_2 N^{2d-2}\,,\end{aligned} \tag{3.4}$$

where C_2 is a constant.

By (2.5), the second term of the right-hand side of (3.2) is written as

$$\begin{aligned}&\mathrm{E}\Big[\frac{1}{N}\sum_{t=d}^{N-k-1}\sum_{j=0}^{t-d}\sum_{h=0}^{t+k-d} f_{j+1}^d f_{h+1}^d x_{t-j}x_{t+k-h}\Big]\\ &\quad=\int_{-\pi}^{\pi}\frac{1}{N}\sum_{t=d}^{N-k-1}\sum_{j=0}^{t-d}\sum_{h=0}^{t-d} f_{j+1}^d f_{h+1}^d e^{i\lambda(j-h)}e^{i\lambda k}f(\lambda)d\lambda\\ &\qquad+\int_{-\pi}^{\pi}\frac{1}{N}\sum_{t=d}^{N-k-1}\sum_{j=0}^{t-d}\sum_{h=t-d+1}^{t-d+k} f_{j+1}^d f_{h+1}^d e^{i\lambda(j-h+k)}f(\lambda)d\lambda\,.\end{aligned} \tag{3.5}$$

The order of the second term of the right-hand side of this equation is lower than that of the first term. To have the asymptotic value of the first term, we define

$$g_N(\lambda)=\frac{1}{N}\sum_{t=d}^{N-k-1}\sum_{j=0}^{t-d}\sum_{h=0}^{t-d}f_{j+1}^d f_{h+1}^d e^{i\lambda(j-h)}=\frac{1}{N}\sum_{t=d}^{N-k-1}\left|\sum_{j=0}^{t-d}f_{j+1}^d e^{i\lambda j}\right|^2. \tag{3.6}$$

After some algebra, we obtain

$$\int_{-\pi}^{\pi}g_N(\lambda)d\lambda=\frac{2\pi}{\{(d-1)!\}^2 2d(2d-1)}N^{2d-1}+O(N^{2d-2}), \tag{3.7}$$

where the symbol O denotes "order" (e.g., see Fuller [4], p. 179), and for some constant C_8,

$$|g_N(\lambda)|<\frac{C_8}{|1-e^{i\lambda}|^{2d}}N^{2d-2} \qquad (\lambda\neq 0). \tag{3.8}$$

These results imply that we can utilize the Lemma 8.3.3. of Anderson [1] by defining $l_N(\lambda)=g_N(\lambda)\Big/\int_{-\pi}^{\pi}g_N(\lambda)d\lambda$. Then we obtain

$$\begin{aligned}&\mathrm{E}\left[\frac{1}{N}\sum_{t=d}^{N-k-1}\sum_{j=0}^{t-d}\sum_{h=0}^{t+k-d}f_{j+1}^d f_{h+1}^d x_{t-j}x_{t+k-h}\right]\\&=\frac{2\pi f(0)}{\{(d-1)!\}^2 2d(2d-1)}N^{2d-1}+o(N^{2d-1}),\end{aligned} \tag{3.9}$$

where the symbol o denotes "smaller order".

Considering (3.4) and (3.9), orders of the remaining terms of (3.2) are shown to be less than N^{2d-1}.

In the same way, we can evaluate the expectation of the second term of the right-hand side of (3.1) as

$$\mathrm{E}[\bar{y}^2]=\frac{2\pi f(0)}{(d!)^2(2d+1)}N^{2d-1}+o(N^{2d-1}). \tag{3.10}$$

We can easily show that orders of the expectations of the remaining terms of the right-hand side of (3.1) are less than N^{2d-1}. Combining these results, we obtain the asymptotic value of the expectation of c_k as follows:

$$\mathrm{E}[c_k]=2\pi f(0)\frac{2d^2-3d+2}{2(d!)^2(2d-1)(2d+1)}N^{2d-1}+o(N^{2d-1}). \tag{3.11}$$

4. Asymptotic covariance of the sample autocovariance function

The asymptotic covariance of the sample autocovariance function can be evaluated by the similar but more formidable calculations. In this section, we assume that $\{a_t\}$ consists of idenpendent $\mathrm{N}(0,\sigma^2)$ random variables. Considering a basic relation

$$\mathrm{Cov}[c_k, c_{k'}]=\mathrm{E}[c_k c_{k'}]-\mathrm{E}[c_k]\mathrm{E}[c_{k'}] \tag{4.1}$$

and results obtained in the previous section, we need to calculate the first term of the right-hand side of this equation. The term can be expanded as

$$\begin{aligned}\mathrm{E}[c_k c_{k'}]&=\mathrm{E}\left[\frac{1}{N^2}\left(\sum_{t=d}^{N-k-1}y_t y_{t+k}\right)\left(\sum_{t=d}^{N-k'-1}y_t y_{t+k'}\right)\right]+\mathrm{E}[\bar{y}^4]\\&-\mathrm{E}\left[\bar{y}^2\frac{1}{N}\sum_{t=d}^{N-k-1}y_t y_{t+k}\right]-\mathrm{E}\left[\bar{y}^2\frac{1}{N}\sum_{t=d}^{N-k'-1}y_t y_{t+k'}\right]\\&+(\text{lower order terms}).\end{aligned} \tag{4.2}$$

We need some calculations to prove that orders of the "lower order terms" are lower than those of other terms, but they are omitted. From Lemma 1 of Section 2, the first term of the right-hand side of (4.2) is shown to be

$$(4.3)\quad \mathrm{E}\left[\frac{1}{N^2}\left(\sum_{t=d}^{N-k-1} y_t y_{t+k}\right)\left(\sum_{t=d}^{N-k'-1} y_t y_{t+k'}\right)\right]$$
$$=\frac{1}{N^2}\sum_{t=d}^{N-k-1}\sum_{s=d}^{N-k-1}\sum_{j=0}^{t-d}\sum_{h=0}^{t-d}\sum_{m=0}^{s-d}\sum_{n=0}^{s-d} f_{j+1}^d f_{h+1}^d f_{m+1}^d f_{n+1}^d$$
$$\times \mathrm{E}[x_{t-j}x_{t+k-h}x_{s-m}x_{s+k'-n}]+(\text{lower order terms})$$
$$=\frac{1}{N^2}\sum_t\sum_s\sum_j\sum_h\sum_m\sum_n f_{j+1}^d f_{h+1}^d f_{m+1}^d f_{n+1}^d\{\sigma(t-j-s+m)\sigma(t-h-s+n+k-k')$$
$$+\sigma(t-j-s+n-k')\sigma(t-h-s+m+k)+\sigma(k-h+j)\sigma(k'-n+m)\}$$
$$+(\text{lower order terms})\,,$$

where we assume that $k\geqq k'$.

From (2.5), the first term is written as

$$(4.4)\quad \frac{1}{N^2}\sum\sum\sum\sum\sum\sum f_{j+1}^d f_{h+1}^d f_{m+1}^d f_{n+1}^d\sigma(t-j-s+m)\sigma(t-h-s+n+k-k')$$
$$=\iint_{-\pi}^{\pi}\frac{1}{N^2}\sum\sum\sum\sum\sum\sum f_{j+1}^d f_{h+1}^d f_{m+1}^d f_{n+1}^d$$
$$\times e^{i\lambda(t-j-s+m)}e^{i\mu(t-h-s+n)}e^{i\mu(k-k')}f(\lambda)f(\mu)d\lambda d\mu\,.$$

Let us define

$$(4.5)\quad g_N(\lambda,\mu)=\frac{1}{N^2}\sum\sum\sum\sum\sum\sum f_{j+1}^d f_{h+1}^d f_{m+1}^d f_{n+1}^d e^{i\lambda(t-j-s+m)}e^{i\mu(t-h-s+n)}$$
$$=\frac{1}{N^2}\left|\sum_{t=d}^{N-k-1}e^{i(\lambda+\mu)t}\left(\sum_{j=0}^{t-d}f_{j+1}^d e^{-i\lambda j}\right)\left(\sum_{h=0}^{t-d}f_{h+1}^d e^{-i\mu h}\right)\right|^2.$$

After some tedious algebra, we find

$$(4.6)\quad \iint_{-\pi}^{\pi}g_N(\lambda,\mu)d\lambda d\mu=(2\pi)^2\left[\frac{4}{\{(d-1)!\}^4}\sum_{p=0}^{d-1}\sum_{q=0}^{d-1}\binom{d-1}{p}\binom{d-1}{q}\right.$$
$$\left.\times\frac{B(2d+p+q+2,\,2d-p-q-1)}{(p+d)(q+d)(2d+p+q+1)}\right]N^{4d-2}+O(N^{4d-3})\,,$$

$$(4.7)\quad |g_N(\lambda,\mu)|<\frac{C_4}{|1-e^{i\lambda}|^{2d}}N^{4d-2}\qquad(\lambda\neq0,\ \mu=0)\,,$$

and

$$(4.8)\quad |g_N(\lambda,\mu)|\leqq\frac{C_5}{|1-e^{i\lambda}|^{2d}|1-e^{i\mu}|^{2d}}N^{4d-4}\qquad(\lambda\neq0,\ \mu\neq0)$$

where $\binom{n}{r}$ is the binomial coefficient defined by $\binom{n}{r}=\frac{n!}{r!(n-r)!}$, $B(p,q)$ is the beta function defined by $B(p,q)=\frac{(p-1)!(q-1)!}{(p+q-1)!}$, and C_4 and C_5 are constants.

Therefore we define $l_N(\lambda,\mu)=g_N(\lambda,\mu)/\iint_{-\pi}^{\pi}g_N(\lambda,\mu)d\lambda d\mu$ and apply Lemma 2 of Section 2 to (4.4), we obtain

$$\frac{1}{N^2}\sum\sum\sum\sum\sum\sum\sum f^d_{j+1}f^d_{h+1}f^d_{m+1}f^d_{n+1}\sigma(t-j-s+m)\sigma(t-h-s+n+k-k')$$
$$=\{2\pi f(0)\}^2\left[\frac{4}{\{(d-1)!\}^4}\sum_{p=0}^{d-1}\sum_{q=0}^{d-1}\binom{d-1}{p}\binom{d-1}{q}\times\frac{B(2d+p+q+2,2d-p-q-1)}{(p+d)(q+d)(2d+p+q+1)}\right]N^{4d-2}+o(N^{4d-2})\,. \tag{4.9}$$

The highest order term of the second term of (4.3) is easily shown to be same as that of the first term. The third term is evaluated in the same way, and the result is

$$\frac{1}{N^2}\sum\sum\sum\sum\sum\sum\sum f^d_{j+1}f^d_{h+1}f^d_{m+1}f^d_{n+1}\sigma(k-h+j)\sigma(k'-n+m)=\frac{\{2\pi f(0)\}^2}{\{(d-1)!\}^4(2d-1)^2 4d^2}N^{4d-2}+o(N^{4d-2})\,. \tag{4.10}$$

After same kind of calculations, we have the results for the remaining terms of the right-hand side of (4.2) as follows:

$$\mathrm{E}[\bar{y}^4]=\{2\pi f(0)\}^2\frac{3}{(2d+1)^2(d!)^4}N^{4d-2}+o(N^{4d-2}) \tag{4.11}$$

and

$$\mathrm{E}\left[\bar{y}^2\frac{1}{N}\sum_{t=d}^{N-k-1}y_t y_{t+k}\right]=\{2\pi f(0)\}^2\left[\frac{2}{d^2\{(d-1)!\}^4}\sum_{p=0}^{d}\sum_{q=0}^{d}\binom{d}{p}\binom{d}{q}\frac{B(2d+p+q+1,2d-p-q+1)}{(p+d)(q+d)}+\frac{1}{(2d+1)(d!)^2\{(d-1)!\}^2(2d-1)2d}\right]N^{4d-2}+o(N^{4d-2})\,. \tag{4.12}$$

From the results above, we have the final evaluation, i.e.,

$$\mathrm{Cov}[c_k,c_{k'}]=\{2\pi f(0)\}^2\left[\frac{4}{\{(d-1)!\}^4}\sum_{p=0}^{d-1}\sum_{q=0}^{d-1}\binom{d-1}{p}\binom{d-1}{q}\frac{B(2d+p+q+2,2d-p-q-1)}{(p+d)(q+d)(2d+p+q+1)}-\frac{4}{d^2\{(d-1)!\}^4}\sum_{p=0}^{d}\sum_{q=0}^{d}\binom{d}{p}\binom{d}{q}\frac{B(2d+p+q+1,2d-p-q+1)}{(p+d)(q+d)}+\frac{4}{(d!)^4(2d+1)^2}\right]N^{4d-2}+o(N^{4d-2})\,. \tag{4.13}$$

5. Discussion

In this section, we assume the normality of $\{a_t\}$. For a special case of ARIMA (0, 1, 1) model

$$y_t-y_{t-1}=a_t-\theta a_{t-1}\,, \tag{5.1}$$

(3.11) and (4.13) are reduced to

$$\mathrm{E}[c_k]=\frac{1}{6}N\lambda^2\sigma^2+o(N)\,, \tag{5.2}$$

$$\mathrm{Var}[c_k]=\frac{1}{45}N^2\lambda^4\sigma^4+o(N^2)\,, \tag{5.3}$$

respectively, where $\lambda=1-\theta$. They naturally coincide with the results of Wichern [13] and Roy [11].

Some simulations were done to investigate the adequacy of the asymptotic results. Five thousands sets of observations of length $N=100$ were generated from the ARIMA $(0, d, 1)$ model

(5.4) $$(1-B)^d y_t=(1+0.8B)a_t$$

for $d=1, 2, 3$ and $\sigma^2=1$. For each set, sample autocovariance of lag 0 divided by N^{2d-1} were computed, then means and sample variances of them were calculated. Repeating these experiments fifty times, means and standard deviations of both of them were obtained. Table 1 gives the results, together with asymptotic values calculated by the first terms of the right-hand sides of (3.11) and (4.13). For expectations and variances of c_0/N^{2d-1}, three numbers in a group denote the asymptotic value, the sample mean and the sample standard deviation. As expected, it is found that asymptotic values and sample means give close values.

Table 1. Values of the asymptotic value, the sample mean and the sample standard deviation for expectations and variances of the sample autocovariances at lag 0 of time series of length 100 generated from ARIMA(0, d, 1) processes.

d	1	2	3
	0.540	0.108	0.141×10^{-1}
$E[c_0/N^{2d-1}]$	0.530	0.110	0.150×10^{-1}
	0.021	0.007	0.009×10^{-1}
	0.233	0.215×10^{-1}	0.391×10^{-3}
$\mathrm{Var}[c_0/N^{2d-1}]$	0.231	0.219×10^{-1}	0.442×10^{-3}
	0.030	0.034×10^{-1}	0.070×10^{-3}

From (3.11) and (4.13), it follows that

(5.5) $$E[c_k]=O(N^{2d-1}),$$

(5.6) $$\mathrm{Var}[c_k]=O(N^{4d-2}),$$

therefore

(5.7) $$c_k=O_p(N^{2d-1}),$$

where O_p denotes "order in probability" (e.g., see Fuller [4], p. 181). This property is very different from that of stationary processes in which c_k approaches to autocovariance function as N becomes large. So, as was suggested by Roy [11], it may be useful to compute the sample autocovariance function of the first N observations for various increasing values of N in order to detect the integration order of the obtained time series at the model identification stage. Some examples are shown in Figure 1. For the model (5.4) and $d=1, 2, 3$, three sets of observations of length $N=400$ were generated. Then sample autocovariances of lag 0 were calculated for first 50, 100, 150, $\cdots$, 400 observations and plotted in common logarithmic forms. Equation (5.7) suggests that these points fall about the straight line whose slope is $2d-1$. Figure 1 shows this tendency and we can distinguish the difference of values of d.

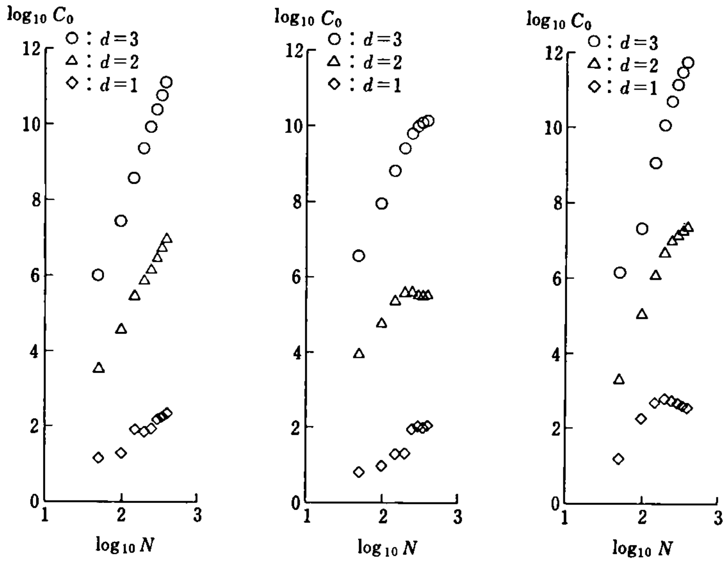

Fig. 1. Examples of plots of the sample autocovariances at lag 0 of time series generated from ARIMA $(0, d, 1)$ processes for various increasing values of N.

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