J. Math. Kyoto Univ. (JMKYAZ)
19-3 (1979) 583-599

The asymptotic distribution of eigenvalues for the Laplacian in semi-infinite domains

By

Fumioki ASAKURA

(Received Aug. 9, 1978)

§ 0. Introduction

We consider the asymptotic distribution of eigenvalues in the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in} \quad G, \quad G \subset R^2 \\ u = 0 & \text{on} \quad \partial G. \end{cases} \tag{0.1}$$

Let $N(\lambda)$ denote the number of eigenvalues not exceeding λ. Then it is a question of the asymptotic behaviour of $N(\lambda)$ as $\lambda \to \infty$. When G is a bounded domain in R^2, the spectrum of the Laplacian in (0. 1) is totally discrete and $N(\lambda)$ behaves as follows

$$N(\lambda) \sim (\lambda/4\pi) \ \text{area}(G) \quad (\text{Weyl's law}). \tag{0.2}$$

On the other hand, F. Rellich showed, in [5], there is some class of domains with infinite area where the spectrum is totally discrete. Naturally, one might be inclined to study the distribution of eigenvalues in such domains. As far as I know, H. Tamura [6] is the only work obtaining the asymptotic formula of the distribution. He considered the problem in the domain $G = \{(x, y) \in R^2 | 0 < y < b(x)\}$, where $A/(1+|x|)^{\alpha} \leq b(x) \leq B/(1+|x|)^{\alpha}$ is assumed. And he obtained the formula in the form

$$N(\lambda) \sim \sum_{n=1}^{\infty} (1/\pi) \int (\lambda - n^2\pi^2 b^{-2}(x))^{1/2} dx \tag{0.3}$$

under some additional assumptions.

In this note, we obtain (0. 3) for another class of domains in R^2 and with different methods from his. Here we must assume at least $b(x)$ is monotonic.

In § 1 and § 2, we introduce a work of F. Rellich and note such variational considerations are available in our case. In § 3, we show we might have only to consider the distant part of the domain and obtain the eigenvalue problems in the form

$$(0.4)\quad \begin{cases} \dfrac{\partial^2 u}{\partial x^2}+b^{-2}(x)\dfrac{\partial^2 u}{\partial y^2}+\lambda u=0 \quad \text{in} \quad (R,\infty)\times(0,1) \\ u=0 \quad \text{on} \quad x=R,\ y=0,1 \\ \text{or} \quad \dfrac{\partial u}{\partial x}=0 \quad \text{on} \quad x=R,\ u=0 \quad \text{on} \quad y=0,1 \end{cases}$$

which can be solved by separation of variables. So we must consider the following singular Sturm-Liouville problems.

$$\begin{cases} \Psi''+(\lambda-n^2q(x))\Psi=0 \quad \text{in} \quad (R,\infty) \\ \Psi(R)=0 \quad \text{or} \quad \Psi'(R)=0,\ q(x)=\pi^2 b^{-2}(x) \end{cases}$$

§ 4 is devoted to study the above problems, and we obtain the formula of the asymptotic distribution of eigenvalues of (0.5). As is well known, if $q(x)$ is sufficiently smooth and increases monotonocally, we have

$$(0.6)\qquad N_n(\lambda)=(1/\pi)\int_{\lambda\geq n^2 q(x)}(\lambda-n^2q(x))^{1/2}dx(1+o(1)).$$

But we must pay special attention to the remainder term. We show the estimate of the remainder term is, in a sense, uniform in n. In order to attain it, we follow the method of E. C. Titchmarsh developped in [7] (or Langer's method), where a uniform asymptotic expansion at a turning point plays a central role. The proof of the crucial lemma is put off unitil § 6. In § 5 we come back to the original problem and show the asymptotic distribution of the eigenvalues of (0.4) is equal to that of (0.1) and obtain the formula

$$N(\lambda)\sim\sum_{n=1}^{\infty}(1/\pi)\int(\lambda-n^2\pi^2 b(x)^{-2})^{1/2}dx\,.$$

In conclusion, I would like to express my gratitude to Prefessor S. Mizohata and Professor N. Shimakura for valuable advice and encouragement.

§ 1. Semi-infinite domains

In this section, following F. Rellich [5] and D. S. Jones [4], we introduce a class of domains where the spectrum of (0.1) is totally discrete.

Definition 1.1. A domain G in R^n is called a semi-infinite domain if $G_R^{(1)}=\{x\in G|\langle x,a\rangle>R\}$ goes to infinity and $G_R^{(2)}=\{x\in G|\langle x,a\rangle<R\}$ is compact for every $R>0$. Here a denotes a fixed vector in R^n and $\langle\ ,\ \rangle$ the Euclidean inner product in R^n.

When a semi-infinite domain G is given, we can consider the following four eigenvalue problems.

$$(\mathrm{I})_j \qquad \begin{cases} \Delta u+\lambda u=0 & \text{in} \quad G_R^{(j)} \\ u=0 & \text{on} \quad \partial G_R^{(j)} \cap \bar{G}_R^{(j)} \\ \dfrac{\partial u}{\partial n}=0 & \text{on} \quad \bar{G} \cap \{x \in R^n \mid \langle x, a\rangle=0\} \end{cases}$$

$$(\mathrm{II})_j \qquad \begin{cases} \Delta u+\lambda u=0 & \text{in} \quad G_R^{(j)} \\ u=0 & \text{on} \quad \partial G_R^{(j)} \quad (j=1,2) \end{cases}$$

Since $\bar{G}_R^{(2)}$ is compact, the spectrum of $(\mathrm{I})_2$ and $(\mathrm{II})_2$ totally discrete

Let G be a semi-infinite domain (we may assume $a=(0, \cdots 0, 1)$). Then we can consider the eigenvalue problem of the form

$$(1.1) \qquad \begin{cases} \Delta' v+\mu v=0 & \text{in} \quad G \cap \{x_n=R\} \\ v=0 & \text{on} \quad \partial G \cap \{x_n=R\}. \end{cases}$$

where Δ' denotes the Laplacian in R^{n-1}.

Definition 1.2. A semi-infinite domain G is said to be infinitely narrow if the first eigenvalue $\mu_1(R)$ of (1.1) goes to infinity as $R \rightarrow \infty$.

A theorem of Rellich says

Theorem 1.3. *When G is infinitely narrow, the spectrum of* (0.1) *is totally discrete.*

By the assumption $\lim_{R \rightarrow \infty} \mu_1(R)=\infty$, we can see the first eigenvalue of $(\mathrm{I})_1$ tends to infinity as $R \rightarrow \infty$. Then we can establish Friedrichs' inequality (see Courant-Hilbert [2] Chap. 7). which shows the spectrum of (0.1) is totally discrete. From now on we shall confine ourselves to the domains in R^2 which have the following properties.

(1.2) i.) $G_R^{(1)}$ is represented as $\{(x, y) \in \boldsymbol{R}^2 \mid g(x)<y<f(x), x>R\}$ by smooth functions $f(x)$ and $g(x)$. $G_R^{(2)}$ is compact for all $R>R_0$ for some R_0.

ii.) $b(x)=f(x)-g(x)$ tends to zero as $x \rightarrow \infty$.

iii.) $\int_{R_0}^{\infty} b(x)\, dx=\infty$.

iv.) $f'(x)$ and $g'(x)$ tend to zero as $x \rightarrow \infty$.

v.) $A/x \leq -b'(x)/b(x) \leq B/x$ for some $A, B>0$.

vi.) $|b''(x)|/b(x) \leq C/x^2$, $|b'''(x)|/b(x) \leq C/x^3$ for some $C>0$.

We can see the domain G is infinitely narrow by i.) and ii.). iii.) means G has infinite area. We need iv.) and v.) in § 3 and v.) and vi.) in § 4. v.) shows $b(x)$ is monotonically decreasing, which is essential in applying the method of Langer.

Remark 1.3. i.) Considerations in this note are valid in such a domain as a finite union of those with the properties (1.2). But for simplicity, we treat the case where it has one piece.
ii.) We can also consider the domain which is a tubelar neighbourhood of a curve in R^2. If we assume, in addition to some modified version of (1.2), the curvature $k(s)$ of the curve satisfies the following conditions a.) and b.), then we can reduce the case into that of §3 by similar transformations as (3.1) and (3.2).

a.) $k(s)\to 0$ as $s\to\infty$. b.) $|k'(s)|/|k(s)|\leq C/s$.

iii.) When G has finite area, we expect to obtain Weyl's law (0.2) and the argument will be a little different.

§2. The eigenvalue problem

We consider the eigenvalue problem (0.1) in a domain with the properties (1.2). As we have mentioned, Friedrichs' inequality is valid in such a domain, we can rely on variational methods in considering (0.1). Here we recollect some basic properties for eigenvalues. In a well known manner, $D[\phi]$ denotes the Dirichlet integral $\iint_G \left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2 dxdy$ and $\|\phi\|$ the L^2-norm of ϕ. Unless otherwise stated, all the functions in this note will be real-valued and sufficiently smooth.

We may assume the eigenvalues $\{\lambda_n\}$ are arranged such that $\lambda_1\leq\lambda_2\leq\cdots$ and so on. The next proposition is so called "Courant's minimax principle".

Proposition 2.1. *The n-th eigenvalue of the Dirichlet problem is characterized as the following.*

$$\lambda_n=\max_{v_1\cdots v_{n-1}\in L^2}\Big(\min_{\|\phi\|=1,\,\phi\perp v_j,\,\phi=0\text{ on }\partial G} D[\phi]\Big) \tag{2.1}$$

Now we define $N(\lambda)$, $A_R^{(j)}$ and $B_R^{(j)}$ as follows.

Definition 2.2. $N(\lambda)=$ the number of eigenvalues for (1.1) not exceeding λ. $A_R^{(j)}(\lambda)$ and $B_R^{(j)}(\lambda)$ are similarly defined for the problems $(\mathrm{I})_j$, $(\mathrm{II})_j$ respectively.

Proposition 2.3. $N(\lambda)$, $A_R^{(j)}(\lambda)$ *and* $B_R^{(j)}(\lambda)$ *satisfy*

$$A_R^{(1)}(\lambda)+A_R^{(2)}(\lambda)\leq N(\lambda)\leq B_R^{(1)}(\lambda)+B_R^{(2)}(\lambda). \tag{2.2}$$

This is a direct consequence of Proposition 2.1. (See Courant-Hilbert [1]).

When the domain has infinite area, $A_R^{(1)}$ and $B_R^{(1)}$ seem to have the main influence on the distribution of eigenvalues. So we may try to find a function $\tilde{N}(\lambda)$ satisfying:

(2. 3) $$\lim_{\lambda\to\infty} \lambda/\tilde{N}(\lambda) = 0 .$$

If we fix R and make λ infinite, then

(2. 4) $$\varliminf_{\lambda\to\infty} A_R^{(1)}(\lambda)/\tilde{N}(\lambda) \geq 1+\varepsilon_1(R)$$
$$\varlimsup_{\lambda\to\infty} B_R^{(1)}(\lambda)/\tilde{N}(\lambda) \leq 1+\varepsilon_2(R)$$

where $\varepsilon_j(R)$ are functions of R which tend to zero as $R\to\infty$.

Proposition 2. 4. *If there exists a function* $\tilde{N}(\lambda)$ *satisfying* (2. 3) *and* (2. 4), *then* $N(\lambda)\sim\tilde{N}(\lambda)$ *i.e.,* $\lim_{\lambda\to\infty}\tilde{N}(\lambda)/N(\lambda)=0.$

Proof. By proposition 2. 3. we have

$$A_R^{(1)}(\lambda)/\tilde{N}(\lambda)+A_R^{(2)}(\lambda)/\tilde{N}(\lambda)\leq N(\lambda)/\tilde{N}(\lambda)$$
$$\leq B_R^{(1)}(\lambda)/\tilde{N}(\lambda)+B_R^{(2)}(\lambda)/\tilde{N}(\lambda).$$

We fix R and make λ infinite, then we have by (2. 3) and (2. 4)

(2. 5) $$1+\varepsilon_2(R)\leq\lim N(\lambda)/\tilde{N}(\lambda)\leq\lim N(\lambda)/\tilde{N}(\lambda)\leq 1+\varepsilon_1(R).$$

As R is chosen arbitrarily large, (2. 5) says $N(\lambda)\sim\tilde{N}(\lambda)$.

§ 3. Separation of the variables

We consider the following transformations of the variables and functions.

(3. 1) $$u=x,\ v=b^{-1}(x)\,(y-g(x))$$

(3. 2) $$\psi=b^{1/2}(x)\,\phi$$

Proposition 3. 1. *By* (3. 1) *and* (3. 2) *the domain* $G_R^{(1)}$ *is transformed to* $(R,\infty)\times(0.1)$, *and*

(3. 3) $$\|\phi\|_R^2=\iint_{G_R}|\phi|^2dx\,dy=\int_R^\infty\int_0^1|\psi|^2dv\,du=\|\widetilde{\psi}\|_R^2 .$$

The last equality in the definition.

Since $dxdy=b(u)\,dudv$, proof is obvious.

Proposition 3. 2. *When we assume that* $b'(x)/b(x)=o(1)$ *and* $f'(x)$, $g'(x)=o(1)$ *as* $x\to\infty$, *we have*

(3. 4) $$(1-\varepsilon)\tilde{D}_R[\psi]-\varepsilon\|\widetilde{\psi}\|\leq D_R[\phi]\leq(1+\varepsilon)\tilde{D}_R[\psi]+\varepsilon\|\widetilde{\psi}\|$$

where $\tilde{D}_R[\psi]=\int_R^\infty\int_0^1\left(\frac{\partial\psi}{\partial u}\right)^2+b^{-2}(u)\left(\frac{\partial\psi}{\partial v}\right)^2dvdu$, ε *depends only on* R *and tends to zero as* $R\to\infty$.

Proof. Since $\frac{\partial}{\partial x}=\frac{\partial}{\partial u}-b^{-1}(b'v+g')\frac{\partial}{\partial v}$, $\frac{\partial}{\partial y}=b^{-1}\frac{\partial}{\partial v}$, we have

$$(\phi_x)^2+(\phi_y)^2=(\phi_u)^2+b^{-2}(\phi_v)^2+2b^{-1}(b'v+g')\phi_u\phi_v$$
$$+b^{-2}(b'v+g)^2(\phi_v)^2.$$

Considering $\phi_u=b^{-1/2}\psi_u+(b^{-1/2})_u\psi$

$=b^{-1/2}\psi_u-(1/2)b^{-1/2}(b'/b)\psi$, we obtain

$$(1-\varepsilon)b^{-1/2}(\psi_u^2+b^{-2}\psi_v^2)-\varepsilon\psi^2\leq\phi_x^2+\phi_y^2\leq(1+\varepsilon)b^{-1/2}(\psi_u^2+b^{-2}\psi_v^2)+\varepsilon\psi^2. \tag{3.5}$$

The proposition is a direct consequence of (3.5).

Corresponding to $(\mathrm{I})_1$ and $(\mathrm{II})_1$, we have thus the following eigenvalue problems.

$$(\widetilde{\mathrm{I}})_1\quad \begin{cases}\psi_{xx}+b^{-2}(x)\psi_{yy}+\lambda\psi=0 & \text{in}\quad (R,\infty)\times(0,1)\\ \psi_x=0 \quad \text{on}\quad x=R,\quad \psi=0\quad\text{on}\quad y=0,1\end{cases}$$

$$(\widetilde{\mathrm{II}})_1\quad \begin{cases}\psi_{xx}+b^{-2}(x)\psi_{yy}+\lambda\psi=0 & \text{in}\quad (R,\infty)\times(0,1)\\ \psi=0\quad\text{on}\quad x=R\quad\text{and}\quad y=0,1\end{cases}$$

where (x, y) stands for (u, v).

Corollary 3.3. *Let $\lambda_n^{(1)}(\tilde{\lambda}_n^{(1)})$ be the n-th eigenvalue of the problem $(\mathrm{I})_1((\widetilde{\mathrm{I}})_1$ respectively). Then we can see*

$$(1-\varepsilon)\tilde{\lambda}_n^{(1)}-\varepsilon\leq\lambda_n^{(1)}\leq(1+\varepsilon)\lambda_n^{(1)}+\varepsilon. \tag{3.6}$$

On account of Proposition 2.1. the proof is obvious and the same inequality holds between the eigenvalues $\lambda_n^{(2)}$ and $\tilde{\lambda}_n^{(2)}$.

$(\widetilde{\mathrm{I}})_1$ and $(\widetilde{\mathrm{II}})_1$ can be solved by means of separation of the variables. That is: when we set $\psi(x,y)=\Psi(x)\sin n\pi y$, $\Psi(x)$ satisfies

$$\begin{cases}\Psi''+(\lambda-n^2q(x))\Psi=0 & \text{in}\quad (R,\infty)\\ \Psi(R)=0\quad\text{or}\quad \Psi'(R)=0\end{cases} \tag{3.7}$$

where $q(x)$ denotes $\pi^2b^{-2}(x)$.

§ 4. A singular Sturm-Liouville problem

In this section we discuss the distribution of the eigenvalues of the problem (3.7).

Let $\Psi_n(x;\lambda)$ be a real-valued solution of the following problem.

(4.1)
$$\begin{cases} \Psi_n'' + (\lambda - n^2 q(x))\Psi_n = 0 \\ \lim_{x\to\infty} \Psi_n = 0 \end{cases}$$

We can see immediately Ψ_n is determined uniquely up to constant multiplication and that Ψ_n is continuous in λ. A relation between the eigenvalues of the problem and Ψ_n is the following, which can be easily verified.

Proposition 4.1. *$\lambda_n^{(m)}$ is an eigenvalue of* (3.7), *if and only if it is a zero of $\Psi_n(R;\lambda)$ (or $\Psi_n'(R;\lambda)$).*

We want to study the behaviour of Ψ_n when λ becomes large. As the interval may contain a turning point i.e. the point where $\lambda - m^2 q(x) = 0$, we make use of some special functions, namely Airy functions in order to obtain a uniform asymptotic expansion of Ψ_n at the turning point. Hereafter we shall carry out the same process as A. Erdélyi [3] in the treatment of the turning point.

We set $q_n(x;\lambda) = (n^2/\lambda)q(x)$, $p_n(x;\lambda) = 1 - q_n(x;\lambda)$, then the equation becomes $\Psi'' + \lambda p_n(x;\lambda)\Psi = 0$. As p_n is monotonically decreasing in x, the turning point X_n is uniquely determined. We introduce the function $\phi_n(x;\lambda)$ as follows.

(4.3)
$$(2/3)\phi_n^{3/2} = \int_{X_n}^{x} (-p_n)^{1/2} dt \quad \text{if } x \geq X_n$$
$$(2/3)(-\phi_n)^{3/2} = \int_{x}^{X_n} p_n^{1/2} dt \quad \text{if } x \leq X_n.$$

We can see easily ϕ is C^∞, if p_n is so and that satisfies an equation

(4.4)
$$\phi_n(\phi_n')^2 = -p_n.$$

We set

(4.5)
$$A_n(x;\lambda) = \phi_n'(x;\lambda)^{-1/2} Ai(\lambda^{1/3}\phi_n(x;\lambda))$$
$$B_n(x;\lambda) = \phi_n'(x;\lambda)^{-1/2} Bi(\lambda^{1/3}\phi_n(x;\lambda))$$

where Ai and Bi are Airy functions (see A. Erdelyi [3]). Roughly speaking, Ai (Bi) is a solution of $y'' = xy$ which is decreasing (increasing) exponentialy as $x\to\infty$. We may take $A_n(x;\lambda)$ as the first approximation of $\Psi_n(x;\lambda)$.

A_n and B_n satisfy the equation of the form

(4.6)
$$A'' + p_n(x;\lambda)A + (1/2)\{\phi_n, x\}A = 0$$

where $\{\phi, x\}$ is the Schwarzian derivative of ϕ i.e., $\{\phi, x\} = \phi'''/\phi' - (3/2)(\phi''/\phi)^2$. In our case $(1/2)\{\phi_n; x\} = p_n''/4p_n - (5/16)\{p_n'/\phi_n^3 + (p_n'/p_n)^2\}$.

When we set

(4.7) $$K_n(x, t; \lambda) = -\pi\lambda^{1/3}\{A_n(x; \lambda)B_n(t; \lambda) - A_n(t; \lambda)B_n(x; \lambda)\},$$

the equation (3.7) is then formally equivalent to the integral equation of the form

(4.8) $$\Psi_n(x; \lambda) = A_n(x; \lambda) - (1/2)\int_x^\infty K_n(x, t; \lambda)\{\phi_n, t\}\Psi_n(t; \lambda)\,dt.$$

The equation (4.8) can be solved by iteration, if $|K_n(x, t; \lambda)\{\phi_n, t\}|$ is dominated by an integrable function of t.

Theorem 4.2. *Suppose* $\int_{R_0}^\infty |\{\phi_n, t\}||p_n(t; \lambda)|^{-1/2}dt = C_n < \infty$, *then* Ψ_n *and* Ψ_n' *have the following asymptotic forms.*

(4.9) $$\Psi_n(x; \lambda) = \begin{cases} A_n(x; \lambda)\{1+O(C_n\lambda^{-1/2})\} & \text{if } x \geq X_n \\ A_n(x; \lambda)\{1+O(C_n\lambda^{-1/2})\} \\ \quad + O(B_n(x; \lambda)C_n\lambda^{-1/2}) & \text{if } x \leq X_n \end{cases}$$

(4.10) $$\Psi_n'(x; \lambda) = \begin{cases} A_n'(x; \lambda)\{1+O(C_n\lambda^{-1/2})\} & \text{if } x \geq X_n \\ A_n'(x; \lambda)\{1+O(C_n\lambda^{-1/2})\} \\ \quad + O(B_n'(x; \lambda)C_n\lambda^{-1/2}) & \text{if } x \leq X_n \end{cases}$$

Since $C_n < \infty$, we can prove the theorem in the same way as in the case of a finite interval (see A. Erdelyi [3]). So we may omit the proof.

Then we have found it essential to get the estimate of the integral $\int_{R_0}^\infty |\{\phi_n, t\}||p_n(t; \lambda)|^{-1/2}dt$.

Lemma 4.3. *If we assume the following,*

(4.11) i.) $q(x) > 0$. ii.) $A/x \leq q'(x)/q(x) \leq B/x$.

iii.) $|q''(x)|/q(x) \leq C/x^2$. iv.) $|q(x)|/q(x) \leq C/x^3$.

then we obtain

(4.12) $$\int_{R_0}^\infty |\{\phi_n, t\}||p_n(t; \lambda)|^{-1/2} \leq CX_n^{-1}$$

where C is independent of λ and n.

The proof is put off until § 6.

Owing to Lemma 4.3. and theorem 4.2., we have

Theorem 4.4. *Under the assumptions in Lemma* 4.3., Ψ_n *can be represented asymptotically in the form*

(4.13) $$\Psi_n(x; \lambda) = \phi_n'(x; \lambda)^{-1/2}\{Ai(\lambda^{1/3}\phi_n(x; \lambda)) + 0(\lambda^{-1/2}X_n^{-1})\} \quad \text{if } x \leq X_n.$$

For derivatives, we have

Theorem 4.5.

(4.14) $\Psi'_n(x;\lambda)=\lambda^{1/3}\phi'_n(x;\lambda)^{1/2}\{Ai'(\lambda^{1/3}\phi_n(x;\lambda))+O(\lambda^{-1/2}X_n^{-2/3})\}$ *if* $x\leq X_n$.

Proof. By computation we get

(4.15) $$A'_n=\lambda^{1/3}(\phi'_n)^{1/2}Ai'(\lambda^{1/3}\phi_n)-2^{-1}\lambda^{-1/3}\phi''_n\phi_n^{-2}Ai(\lambda^{1/3}\phi_n).$$

We can show $|\phi''_n|/|\phi_n|^2\leq CX_n^{-2/3}$, by similar computation as in the proof of lemma 4.3. Then the theorem follows from (4.15).

Remark 4.6. When we assume, instead of ii.), iii.) and iv.), ii.') $A/x\log x\leq q'(x)/q(x)\leq B/x$. iii.') $|q''(x)|/q(x)\leq Cx^2\log x$. iv.') $|q'''(x)|/q(x)\leq C/x^3\log x$. we can get a similar result to Lemma 4.3. Here the estimate of the remainder term is of order $\lambda^{-1/2}X_n^{-1}\log(\lambda^{1/2}X_n)$.

$Ai(x)$ has the following asymptotic forms.

(4.16) $$Ai(-x)=\pi^{-1/2}x^{-1/4}\{\cos((2/3)x^{3/2}-\pi/4)+O(x^{-3/2})\}$$

$$Ai'(-x)=\pi^{-1/2}x^{1/4}\{\sin((2/3)x^{3/2}-\pi/4)+O(x^{-3/2})\}.$$

Then if we set

$$Z_n(x;\lambda)=(2/3)\lambda^{1/2}(-\phi_n)^{3/2}=\int_x^{X_n}(\lambda-n^2q(t))^{1/2}dt,$$

we obtain

(4.17) $$\Psi_n(x;\lambda)=\pi^{-1/2}\phi'_n(x;\lambda)^{-1/2}Z_n(x;\lambda)^{-1/6}\{\cos(Z_n(x;\lambda)-\pi/4)+O(Z_n^{-1})\}$$

(4.18) $$\Psi'_n(x;\lambda)=\pi^{-1/2}\phi'_n(x;\lambda)^{1/2}Z_n(x;\lambda)^{1/6}\{\sin(Z_n(x;\lambda)-\pi/4)+O(Z_n^{-1})\},$$

considering $Z_n(x;\lambda)\leq\lambda^{1/2}X_n$.

From now on, we shall confine ourselves to the case where the boundary condition is $\Psi_n(R)=0$. In the other case, the discussion is the same.

Proposition 4.7. *Let $\lambda_n^{(m)}$ be the m-th eigenvalue of* (3.7), *then there exists an integer m_0 and the following asymptotic relation holds.*

(4.19) $$Z_n^{(m)}=\int_R^{X_n}(\lambda_n^{(m)}-n^2q(t))^{1/2}dt=(m+m_0+3/4)\pi+O(1/Z_n^{(m)}),$$

the first equality is the definition.

Proof. By Proposition 4.1. $\lambda_n^{(m)}$ is a zero of $\Psi_n(R;\lambda)$. We have only to consider the case $R\leq X_n$ and we may assume $R\geq R_0$ for sufficiently large R_0. If $R>X_n$, then we see $p_n<0$ and Ψ_n has no zeros. In the case $R\leq X_n$, it follows from (4.17)

$$\cos(Z_n(R;\lambda)-\pi/4)=O(1/Z_n)$$

so we can see there exists one zero in the interval $\pi/4+2m\pi<Z_n(R;\lambda)<5\pi/4+2m\pi$. If we set

$$Z_n^{(m)}=Z_n(R;\lambda)_n^{(m)})=(m+m_0+3/4)\pi+\delta_{m,n}$$

$(|\delta_{m,n}|<\pi/2)$, we can show $\delta_{m,n}=O(1/Z_n^{(m)})$.

It remains to show m_0 is independent of n. To proceed further, we need a result on the location of the zeros of Airy function. That is:

Proposition 4.8. *$Ai(-x)$ has exactly n zeros in the interval $0<x<\{(3/2)(n+1/4)\pi\}^{2/3}$, if n is sufficiently large.*

We notice $Ai(-x)=(1/3)x^{1/2}\{J_{1/3}(\zeta)+J_{-1/3}(\zeta)\}$, where $J_\nu(\zeta)$ is the Bessel function of order ν and $\zeta=(2/3)x^{3/2}$. And the location of zeros of $x^{1/2}\{J_{1/3}(x)+J_{-1/3}(x)\}$ is well studied thanks to the next lemma (in detail, see E. C. Titchmarsh [7]).

Lemma 4.9. (*G. N. Watson*) *If n is large enough and $\nu>-1$, $J_\nu(x)$ has exactly n zeros in the interval $0<x<(n+\nu/2+1/4)\pi$.*

A proof of this lemma is found in E. C. Titchmarsh [7].

Now we are able to show the asymptotic form of $N_n(\lambda)$ which is uniform in n.

Theorem 4.10. *Let $N_n(\lambda)$ be the number of eigenvalues not exceeding λ, then we have*

$$\int_R^{X_n}(\lambda-n^2q(t))^{1/2}dt=N_n(\lambda)+O(1), \tag{4.20}$$

where the remainder term is uniform in n.

Proof. First we fix the notations. $\{a_j\}$ denote the zeros of $Ai(-x)$ such that $0<a_1<a_2<\cdots$, $\{b_j\}$ the zeros of $Ai'(-x)$ such that $b_1<b_2<\cdots$. We set $A_n=\{3/2(n+1/4)\pi\}^{3/2}$. We take sufficiently large N such that $1/N<\pi/2$ and Proposition 4.8. holds, and fix it. We set $\delta=1/3\times\min\{|Ai(-b_j)|\,|0<b_j\leq A_{N+1}\}$. We can see easily there exists a series of points α_j, β_j which satisfy the following.

$$0<\alpha_1<a_1<\beta_1<\alpha_2<a_2<\beta_2<\cdots<\alpha_N<a_N<\beta_N \tag{4.21}$$

$$Ai(-\alpha_1)>2\delta,\ Ai(-\beta_1)<-2\delta,\ Ai(-\alpha_2)>2\delta,\ \cdots \tag{4.22}$$

$$|Ai'(-x)|>\delta \text{ in } \alpha_j<x<\beta_j. \tag{4.23}$$

Since $\Psi_n(R;\lambda)=\phi_n'(R;\lambda)^{1/2}\{Ai(\lambda^{1/3}\phi_n(R;\lambda))+O(\lambda^{-1/2}X_n^{-1})\}$, we can see there exists at least one zero of $\Psi_n(R; \lambda)$ in the interval $\alpha_j < -\lambda^{1/3}\phi_n(R; \lambda) < \beta_j$, provided that λ is so large that the remainder term is less than δ. As $X_n \geq R_0$ in our case, we can make the remainder term uniformly small in n. Since $-\lambda^{1/3}\phi_n(R; \lambda)$ is an increasing function of λ and α_j, β_j are chosen satisfying (4. 22) and (4. 23), we know the zero is the only one in the interval.

Thus we have proved $\Psi_n(R; \lambda)$ has exactly N zeros in the interval $0 < -\lambda^{1/3}\phi_n(R; \lambda) < A_N$ namely in $0 < \int_R^{X_n} (\lambda - n^2 q(t))^{1/2} dt < (N+1/4)\pi$. As we may assume

$$(4.26) \qquad (N-3/4)\pi < \int_R^{X_n} (\lambda_n^{(N)} - n^2 q(t))^{1/2} dt < (N+1/4)\pi\,,$$

we can see Proposition 4. 7. is compatible with (4. 26) if and only if $m_0 = -1$. Then we obtain

$$(4.25) \qquad (m-1/4)\pi + O(1/Z_n^{(m)}) = \int_R^{X_n} (\lambda_n^{(m)} - n^2 q(t))^{1/2} dt\,.$$

the theorem is a direct consequence of (4. 25).

Using theorem 4. 10., we have the distribution of eigenvalues of $(\widetilde{\mathrm{I}})_1$ and $(\widetilde{\mathrm{II}})_1$ in § 3.

Theorem 4. 11. *Let $N_I(\lambda)$ be the number of eigenvalues of $(\widetilde{\mathrm{I}})_1$ not exceeding λ, then we have*

$$N_I(\lambda) = (1/\pi) \sum_{n=1}^{\infty} \int_R^{X_n} (\lambda - n^2 q(t))^{1/2} dt + O(\lambda^{1/2})\,.$$

Proof. As we have only to consider the case $\lambda \geq n^2 q(R)$, the summation is, in fact, finite and of order $\lambda^{1/2}$.

Hence
$$N_I(\lambda) = \sum_{n=1}^{\infty} N_n(\lambda) = \sum_{n=1}^{(\lambda/q(R))^{1/2}} N_n(\lambda)$$
$$= (1/\pi) \sum_{n=1}^{\infty} \int_R^{X_n} (\lambda - n^2 q(t))^{1/2} dt + O(\lambda^{1/2})\,.$$

Remark 4. 12. When we define $N_{II}(\lambda)$ for $(\widetilde{\mathrm{II}})_1$, the same conclusion as in the theorem 4. 11 holds for $N_{II}(\lambda)$.

§ 5. Distribution of eigenvalues

We come back to the original eigenvalue problem. First, summing up the results of § 4, we have

Theorem 5. 1. *Suppose $b(x)$ has the properties* v.) *and* vi.) *in* (1. 2), *then we have the asymptotic formula for $N_I(\lambda)$ and $N_{II}(\lambda)$ in the form*

(5. 1) $$N_{*}(\lambda)=(1/\pi)\sum_{n=1}^{\infty}\int_{R}^{X_n}(\lambda-n^2\pi^2b^{-2}(x))^{1/2}dx+O(\lambda^{1/2})\quad (*=\mathrm{I}\ or\ \mathrm{II}).$$

Proof. When we put $q(x)=\pi^2 b(x)^{-2}$, $q(x)$ satisfies the conditions of Lemma 4. 3. Then theorem 4. 11. holds.

Corollary 5. 2. *If we fix* R_0, *we have*

(5. 2) $$N_{*}(\lambda)=(1/\pi)\sum_{n=1}^{\infty}\int_{R_0}^{X_n}(\lambda-n^2\pi^2b(x)^{-2})^{1/2}dx+O(\lambda)\quad (*=\mathrm{I}\ or\ \mathrm{II}).$$

Proof. Since

$$\sum_{n=1}^{\infty}\int_{R_0}^{X_n}(\lambda-n^2\pi^2b(x)^{-2})^{1/2}dx=\sum_{n=1}^{\infty}\int_{R}^{X_n}+\int_{R_0}^{R}=\sum_{n=1}^{\infty}\int_{R}^{X_n}+O(\lambda),$$

we have the corollary.

We set $\tilde{N}(\lambda)=(1/\pi)\sum_{n=1}^{\infty}\int_{R_0}^{X_n}(\lambda-n^2\pi^2b(x)^{-2})^{1/2}dx$. We want to show $\tilde{N}(\lambda)$ has the required properties (2. 3) and (2. 4).

Proposition 5. 3.

(2. 3) $$\lim_{\lambda\to\infty}\lambda/\tilde{N}(\lambda)=0.$$

Proof. We recall X_n denotes the solution of the equation in x, $n^2\pi^2b(x)^2=\lambda$.

By exchanging the order of integration and summation, we have

$$\begin{aligned}\tilde{N}(\lambda)&=\int_{X_2}^{X_1}(\lambda-\pi^2b(x)^{-2})^{1/2}dx+\int_{X_3}^{X_2}\sum_{j=1}^{2}(\lambda-j^2\pi^2b(x)^{-2})^{1/2}dx\\&\quad+\int_{X_{n+1}}^{X_n}\sum_{j=1}^{n}(\lambda-j^2\pi^2b(x)^{-2})^{1/2}dx+\cdots\cdots\\&=(\lambda/\pi^2)\sum_{n=1}^{\infty}\int_{X_{n+1}}^{X_n}b(x)\sum_{j=1}^{n}(\pi/b(x)\lambda^{1/2})\{1-(j\pi/\lambda^{1/2}b(x))^2\}^{1/2}dx.\end{aligned}$$

Since $\sum_{j=1}^{n}(\pi/b(x)\lambda^{1/2})\{1-(j\pi/\lambda^{1/2}b(x))^2\}^{1/2}$ is a Riemannian sum for the area of the quarter of the unit disk, we observe

$$\lambda/16\pi\int_{R_0}^{X_2}b(x)dx\leq\tilde{N}(\lambda)\leq\lambda/4\pi\int_{R_0}^{X_1}b(x)dx.$$

As we have assumed $\int_{R_0}^{\infty}b(x)dx=\infty$, we get the proposition.

Proposition 5. 4.

(2. 4) $$\varliminf_{\lambda\to\infty}A_R^{(1)}(\lambda)/\tilde{N}(\lambda)\geq 1+\varepsilon_1(R)$$

$$\overline{\lim_{\lambda\to\infty}} B_R^{(1)}(\lambda)/\tilde{N}(\lambda)\leq 1+\varepsilon_2(R).$$

Proof. By corollary 3. 3, we observe $(1-\varepsilon)\tilde{\lambda}_n^{(1)}\leq\lambda_n^{(1)}\leq(1+\varepsilon)\tilde{\lambda}_n^{(1)}$ when n is large. So we can see

$$\tilde{N}(\lambda/1+\varepsilon(R))=\tilde{N}((1-\varepsilon'(R))\lambda)\leq A_R^{(1)}(\lambda),\ B_R^{(1)}(\lambda)\leq$$
$$\tilde{N}(\lambda/1-\varepsilon(R))=\tilde{N}((1+\varepsilon'(R))\lambda).$$

A direct computation shows $\tilde{N}((1\pm\varepsilon'(R))\lambda)=(1\pm\tilde{\varepsilon}(R))\tilde{N}(\lambda)$. Hence we have the proposition.

Now we can show the asymptotic formula of the distribution of the eigenvalues of the original problem.

Theorem 5. 5. *Assume the domain G has the properties* (1. 2), *then we get the asymptotic formula*

$$N(\lambda)\sim(1/\pi)\sum_{n=1}^{\infty}\int_{\lambda\geq n^2\pi^2b(x)^{-2}}(\lambda-n^2\pi^2b(x)^{-2})^{1/2}dx. \tag{5. 3}$$

Proof. As $\tilde{N}(\lambda)$ has the required properties, we can see $N(\lambda)\sim\tilde{N}(\lambda)$. Hence $\sum_{n=1}^{\infty}\int_{R_0}^{X_n}(\lambda-n^2\pi^2b(x)^{-2})^{1/2}dx$

$$=\sum_{n=1}^{\infty}\int_{\lambda\geq n^2\pi^2b(x)^{-2}}(\lambda-n^2\pi^2b(x)^{-2})^{1/2}dx+O(\lambda),$$

we obtain the formula.

§ 6. A proof of Lemma 4. 3

To begin with, we note some properties of $q_n(x)$ under the assumptions v.) and vi.) in (1. 2). Since $q_n(x)=(n^2/\lambda)q(x)$, $q_n(x)$ itself has the following properties.

(6. 1) i.) $q_n(x)>0$. ii.) $Ax^{-1}\leq q_n'(x)/q_n(x)\leq Bx^{-1}$.

iii.) $|q_n''(x)|\leq Cx^{-2}q_n(x)$, $|q_n'''(x)|\leq Cx^{-3}q_n(x)$.

Integrating the both sides of ii.) from x to αx $(x>0,\alpha>1)$, we obtain

$$\alpha^A\leq q_n(\alpha x)/q_n(x)\leq\alpha^B. \tag{6. 2}$$

By ii.) and iii.), we can see

$$|q_n''(x)|\leq ACx^{-1}q_n'(x)\ \text{and}\ |q_n'''(x)|\leq ACx^{-2}q_n'(x). \tag{6. 3}$$

Now we estimate the integral

(6. 4) $\int_{R_0}^{\infty}|\{\phi_n, t\}||p_n(t)|^{-1/2}dt$, dividing it into four parts.

That is: we choose a positive number $\alpha\,(\alpha>1$, later we will make α arbitrarily close to 1) and set

$$\int_{R_0}^{\infty}=\int_{\alpha X_n}^{\infty}+\int_{X_n}^{\alpha X_n}+\int_{\alpha^{-1}X_n}^{X_n}+\int_{R_0}^{\alpha^{-1}X_n}$$
$$=I_1+I_2+I_3+I_4\,.$$

From now on we assume C stands for any constant.

I.) Estimate of $I_1\,(x\geq\alpha X_n)$.

As we have noticed

(6. 5) $$\{\phi_n, t\}=p_n''/4p_n-(5/16)\{p_n/\phi_n^3+(p_n'/p_n)^2\},$$

(6. 6) $$(2/3)\phi_n^{3/2}=\int_{X_n}^{x}(q_n(t)-q_n(X_n))^{1/2}dt\,,$$

we estimate (6. 5) term by term.

i.)

(6. 7) $$\begin{aligned}&\int_{\alpha X_n}^{\infty}|p_n''/p_n||p_n|^{-1/2}dt\\&=\int_{\alpha X_n}^{\infty}q_n''(t)\,(q_n(t)-q_n(X_n))^{-3/2}dt\\&\leq CX_n^{-1}\int_{\alpha X_n}^{\infty}q_n'(t)\,(q_n(t)-q_n(X_n))^{-3/2}dt\\&\leq CX_n^{-1}(q_n(\alpha X_n)-q_n(X_n))^{-1/2}\\&\leq CX_n^{-3/2}|q_n'(\xi)|^{-1/2}.\quad(X_n\leq\xi\leq\alpha X_n)\end{aligned}$$

Hence $q_n'(\xi)^{-1}\leq A^{-1}\xi q_n(\xi)^{-1}\leq A^{-1}\alpha X_n q_n(X_n)^{-1}$,

we have $\int_{\alpha X_n}^{\infty}\leq CX_n^{-1}$.

ii.)

(6. 8) $$\int_{\alpha X_n}^{\infty}|p_n/\phi_n^3||p_n|^{-1/2}dt=\int_{\alpha X_n}^{\infty}(p_n)^{-1/2}\phi_n^{-3}dt\,.$$

When we put $\zeta_n=(2/3)\phi_n^{3/2}$, we observe (6. 8) is equivalent to $\int_{\alpha X_n}^{\infty}\zeta_n'\zeta_n^{-2}dt=\zeta_n(\alpha X_n)^{-1}$. Considering

$$\begin{aligned}\zeta_n(\alpha X_n)&\geq\inf_{X_n\leq\xi\leq\alpha X_n}(q_n'(\xi))^{1/2}\int_{X_n}^{\alpha X_n}(t-X_n)^{1/2}dt\\&\geq C\inf|\xi^{-1}q_n(\xi)|^{1/2}X_n^{3/2}\geq CX_n\,,\end{aligned}$$

we obtain $\int_{\alpha X_n}^{\infty}dt\leq CX_n^{-1}$.

iii.)

$$\int_{\alpha X_n}^{\infty} (p_n'/p_n)^2 |p_n|^{1/2} dt \tag{6.9}$$

$$= \int_{\alpha X_n}^{\infty} q_n'(t)^2 (q_n(t) - q_n(X_n))^{-5/2} dt$$

$$\leq C X_n^{-1} \int_{\alpha X_n}^{\infty} q_n'(t) q_n(t) (q_n(t) - q_n(X_n))^{-5/2} dt$$

$$\leq C X_n^{-1} \int_{\alpha X_n}^{\infty} q_n'(t) (q_n(t) - q_n(X_n))^{-3/2} dt$$

$$+ C X_n^{-1} \int_{\alpha X_n}^{\infty} q_n'(t) (q_n(t) - q_n(X_n))^{-5/2} dt,$$

then (6.9) is less than

$$C X_n^{-1} (q_n(\alpha X_n) - q_n(X_n))^{-1/2} + C X_n^{-1} (q_n(\alpha X_n) - q_n(X_n))^{-3/2} \leq C X_n^{-1}.$$

Thus we have shown $I_1 \leq C X_n^{-1}$. The estimate of I_4 goes in the same way, even easier, so we may omit the proof.

II.) Estimate of I_2 ($\alpha X_n \geq x \geq X_n$).

Integrating (6.6) by parts twice, we obtain

$$\phi_n^{3/2} = -p_n^{3/2} (p_n')^{-1} \{1 + (2/5) p_n'' p_n (p_n')^{-2} + S\},$$

where $S = (2/5) p_n' p_n^{-3/2} \int_{X_n}^{x} (-p_n)^{5/2} \{p_n''' (p_n')^{-3} - 3 (p_n'')^2 (p_n')^{-4}\} dt.$

First we observe we make $p_n'' p_n (p_n')^{-2}$ and $p_n' p_n^{-3/2} S$ arbitrarily small, if we choose α sufficiently close to 1.

i.) $|p_n'' p_n (p_n')^{-2}| = |p_n''(x)| (q_n(x)) (p_n'(x))^{-2}$

$$\leq C x^{-1} (q'(\xi)/q'(x)) (x - X_n)$$

$$\leq C \xi^{-1} (q(\xi)/q(x)) (x - X_n) \quad (X_n \leq \xi \leq \alpha X_n).$$

By (6.2), we can see

$$|p_n'' p_n (p_n')^{-2}| \leq C(\alpha - 1) \tag{6.10}$$

ii.) $|S| \leq C q_n'(x) (q_n(x) - q_n(X_n))^{3/2} \int_{X_n}^{x} (q_n(t) - q_n(X_n))^{5/2}$

$$\times \{q_n'''(t) q_n'(t)^{-2} - 3 q_n''(t)^2 q_n'(t)^{-4}\} dt.$$

Considering (6.3), we obtain

$$|S| \leq C q_n'(x) (q_n(x) - q_n(X_n))^{-3/2} \int_{X_n}^{x} (q_n(t) - q_n(X_n))^{5/2} q_n'(t)^{-2} dt$$

$$\leq C\{\sup q_n'(\xi)^{5/2}/\inf q_n'(\xi)^{3/2}\} q_n'(x)(x-X_n)^{-3/2}$$

$$\times \int_{X_n}^{x} (t-X_n)^{5/2} t^{-2} q_n'(t)^{-2} dt.$$

Hence we have

$$|S| \leq C X_n^{-2}(x-X_n)^2 \leq C(\alpha-1)^2. \tag{6.11}$$

By (6.10) and (6.8), we can see

$$\phi_n^{3/2} = -(-p_n)^{3/2}(p_n')^{-2}\{1+\theta\},$$

where we can make θ arbitrarily small as $\alpha \to 1$. Hence $p_n/\phi_n^3 = -(p_n'/p_n)^2 \{1-(4/5)p_n''p_n/(p_n')^2 + O(S) + O((p_n''p_n)^2(p_n')^{-4})\}$, we get

$$p_n/\phi_n^3 + (p_n'/p_n)^2 - (4/5)(p_n''/p_n) \quad \text{(see (6.5))} \tag{6.12}$$

$$= O(S(p_n')^2(p_n)^{-2}) + O((p_n'')^2(p_n')^{-2}).$$

Thus we have only to estimate $S(p_n'/p_n)^2$ and $(p_n''/p_n')^2$.

iii.) $(p_n''/p_n')^2 = (q''/q')^2 \leq C X_n^{-1}$.

iv.) $|S|(p_n'/p_n)^2 \leq |S| q_n'(x)^2 (q_n(x)-q_n(X_n))^{-2}$

$$\leq |S|(q'(x)^2/q'(\xi)^2)|x-X_n|^{-2}.$$

Since $|q'(x)/q'(\xi)| \leq C\xi/x \leq q(x)/q(\xi) \leq C$, we can see $|S||p_n'/p_n|^2 \leq C X_n^{-1}$ considering together (6.11).

Thus we have shown

$$(1/2)|\{\phi_n, t\}| \leq C X_n^{-1} \text{ in } X_n \leq x \leq \alpha X_n,$$

then $I_2 \leq C X_n^{-1} \int_{X_n}^{\alpha X_n} (q_n(t)-q_n(X_n))^{-1/2} dt \leq C X_n^{-1}$.

The estimate of I_3 can be carried out just in the same manner. Thus we have obtained the lemma.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] R. Courant and D. Hilbert; Methoden der Mathematischen Physik. Band I. Springer (1931).
[2] R. Courant and D. Hilbert; ibid. Band II, Springer (1937).
[3] A. S. Erdélyi; Asymptotic Expansions. Dover (1959).
[4] D. S. Jones; Proc. Camb. Phil. Soc. 49, 668-684 (1953).
[5] F. Rellich; Studies and Essays Presented to R. Courant. 329-344 (1948).
[6] H. Tamura; Nagoya, Math. J. Vol. 60, 7-33 (1976).
[7] E. C. Titchmarsh; Eigenfunction Expansions. Vol. I, 2nd edition, Oxford (1962).
[8] E. C. Titchmarsh; ibid. Vol. II, Oxford (1958).

Added in proof: We should emphasize the fact that the remainder terms in theorem 4.2, 4.4, and 4.5 are valid uniformly in n.