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THE ASYMPTOTIC DISTRIBUTION OF SERIAL COVARIANCES

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A central limit theorem is proved for the sample serial covariances of an ergodic, stationary, purely nondeterministic process whose linear innovations have their first four moments as for a sequence of independent random variables. The necessary and sufficient condition for the theorem is then that the spectra be square integrable.

1. Introduction. Let $x(n)$, $n = 1, \cdots, N$, be part of a realisation of a stationary, discrete time, vector process that is ergodic. Let the components of $x(n)$ be $x_a(n)$, $a = 1, \cdots, v$. Consider the autocovariances

$$c_{ab}(n) = N^{-1} \sum_{m=1}^{N-n} \{x_a(m) - \bar{x}_a\}\{x_b(m+n) - \bar{x}_b\}, \quad n \geqq 0 .$$

The $x(n)$ may be taken to have zero mean. If $x(n)$ is linearly purely nondeterministic with finite variances then

$$(1) \qquad x(n) = \sum_0^\infty A(j)\varepsilon(n-j), \qquad E\{\varepsilon(m)\varepsilon(n)'\} = \delta_{mn} G,$$
$$\sum_0^\infty ||A(j)||^2 < \infty, \qquad E\{\varepsilon(n) = 0\}$$

where δ_{mn} is Kronecker's delta, G is a $v \times v$ nonsingular matrix, $||\cdot||$ is a norm for $v \times v$ matrices and $\varepsilon(n)$ is the vector of errors of (one step, linear) prediction, using the infinite past. Call $\alpha_{ab}(j)$ the typical element of $A(j)$ and g_{ab} the typical element of G.

We wish to prove a central limit theorem (CLT) for the $c_{ab}(n)$ under general conditions since many distributional problems in time series analysis reduce to this problem. An example is the autoregressive moving average model. In this example the *linear* model is, presumably, being constructed for *linear* prediction and it is not unreasonable to say that the best linear predictor is the best predictor (both best in the least squares sense). Then the $\varepsilon(n)$ are martingale differences ([3]). However for a usable (i.e. reasonably neat) result more is needed for a CLT for the $c_{ab}(n)$. Let $\mathscr{F}_n$ be the sub σ-algebra (of the σ-algebra with respect to which all $x_a(n)$ are measurable) generated by $x_a(m)$, $m \leqq n$, $a = 1, \cdots, v$. It is required that

$$(2) \qquad E\{\varepsilon_a(n) \mid \mathscr{F}_{n-1}\}, \qquad E\{\varepsilon_a(n)\varepsilon_b(n) \mid \mathscr{F}_{n-1}\}, \qquad E\{\varepsilon_a(n)\varepsilon_b(n)\varepsilon_c(n) \mid \mathscr{F}_{n-1}\}$$
$$E\{\varepsilon_a(n)\varepsilon_b(n)\varepsilon_c(n)\varepsilon_d(n) \mid \mathscr{F}_{n-1}\}$$

should be constants for all a, b, c, d. Call the last of these four constants κ_{abcd}.

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Let

$$h(\omega) = \sum_0^\infty A(j)e^{ij\omega}, \qquad f(\omega) = (2\pi)^{-1}h(\omega)Gh^*(\omega),$$

where the star indicates transposition combined with conjugation. Put

$$\gamma_{ab}(n) = E\{x_a(m)x_b(m+n)\}, \qquad \tau_{ab}(n) = N^{\frac{1}{2}}\{c_{ab}(n) - \gamma_{ab}(n)\}.$$

As will be seen below the covariances of the latter in their limiting distribution are (for $\tau_{ab}(m)$ with $\tau_{cd}(n)$)

$$(3) \qquad 2\pi \int_{-\pi}^{\pi} \{f_{ac}(\omega)\overline{f_{bd}(\omega)}e^{-i(n-m)\omega} + f_{ad}(\omega)\overline{f_{bc}(\omega)}e^{i(n+m)\omega}\}\, d\omega$$
$$+ \sum\sum\sum\sum_{pqrs} \left\{\kappa_{pqrs}\frac{1}{2\pi}\int_{-\pi}^{\pi} h_{ap}(\omega)\overline{h_{bq}(\omega)}e^{im\omega} + \overline{h_{cr}(\omega)}h_{ds}(\omega)e^{-in\omega}\right\} d\omega\,.$$

Here $h_{ap}(\omega)$ is the indicated element of $h(\omega)$ and the sum over p, q, r, s is for these subscripts running from 1 to v.

2. The central limit theorem.

THEOREM. *If $x(n)$ is ergodic and is generated by* (1) *with the expressions* (2) *all constants then the necessary and sufficient condition that any finite set of the $\tau_{ab}(n)$ be jointly asymptotically normal with covariances given by* (3) *is that the $f_{aa}(\omega)$, $a = 1, \cdots, v$, be all square integrable.*

The condition is evidently necessary since otherwise (3) need not be finite. For the proof of sufficiency replace $\tau_{ab}(n)$ by

$$\tilde{\tau}_{ab}(n) = N^{-\frac{1}{2}} \sum_{m=1}^{N} \{x_a(m)x_b(m+n) - \gamma_{ab}(n)\}.$$

This replacement introduces asymptotically negligible effects since, in the first place, $N^{-\frac{1}{2}}x_a(N)$ converges in probability to zero (because $x_a(n)$ has finite fourth moment) and in the second place, as is now shown, $N^{\frac{1}{2}}\bar{x}_a{}^2$ converges in probability to zero. Indeed

$$(4) \qquad E(N^{\frac{1}{2}}\bar{x}_a{}^2) = N^{-\frac{1}{2}} \int_{-\pi}^{\pi} f_{aa}(\omega)L_N(\omega)\, d\omega\,, \qquad L_N(\omega) = \frac{1}{N}\left(\frac{\sin\frac{1}{2}N\omega}{\sin\frac{1}{2}\omega}\right)^2.$$

$L_N(\omega)$ converges uniformly to zero outside of any interval $(-\delta, \delta)$, $\delta > 0$, ([4, page 88]). Thus it is sufficient to show that the contribution to (4) from such an interval can be made arbitrarily small. However, using Schwarz inequality,

$$N^{-\frac{1}{2}} \int_{-\delta}^{\delta} f_{aa}(\omega)L_N(\omega)\, d\omega \leqq \{2\pi \int_{-\delta}^{\delta} f_{aa}^2(\omega)N^{-1}L_N(\omega)\, d\omega\}^{\frac{1}{2}} \leqq \{2\pi \int_{-\delta}^{\delta} f_{aa}^2(\omega)\, d\omega\}^{\frac{1}{2}}$$

which shows what is required since $f_{aa}(\omega) \in L_2$.

The covariance between $\tilde{\tau}_{ab}(m)$, $\tilde{\tau}_{cd}(n)$ is ([2, pages 209–211])

$$(5) \qquad \sum_{j=-N+1}^{N-1}\left(1 - \frac{|j|}{N}\right)\{\gamma_{ac}(j)\gamma_{bd}(j+n-m) + \gamma_{ad}(j+n)\gamma_{bc}(j-m)$$
$$+ \sum\sum\sum\sum_{pqrs} \kappa_{pqrs} \sum_k \alpha_{ap}(k)\alpha_{bq}(k+m)\alpha_{cr}(k+j)\alpha_{ds}(k+j+n)\}\,.$$

However $\gamma_{ac}(j)\gamma_{bd}(j+n-m)$ is $(4\pi^2)$ times the jth Fourier coefficient of the convolution of $f_{ac}(\omega)$ with $f_{bd}(\omega)\exp\{-i(n-m)\omega\}$. These two functions are in

L_2 and thus their convolution is continuous. Thus the first term in (5) is the Cesaro sum, evaluated at the origin, of the Fourier series of a continuous function and thus it converges to the first term in (3). The same argument applies to the second term in (5). The sum over k in the third term in the jth Fourier coefficient of the product of two functions, with Fourier coefficients respectively, $\alpha_{ap}(k)\alpha_{bq}(k+m)$ and $\alpha_{cr}(k)\alpha_{ds}(k+n)$. Each of these functions is the convolution of two functions in L_2 (indeed in L_4) and hence is continuous. Thus the last term again is the Cesaro sum of a continuous function and converges to the last term in (3).

Since $f_{aa}(\omega) \in L_2$ then $h_{ab}(\omega) \in L_4$, $a, b = 1, \cdots, v$. Indeed $\operatorname{tr}(f(\omega)) = (2\pi)^{-1} \operatorname{tr}(h^*hG) \geqq K \operatorname{tr}(h^*h)$ where K is a constant determined by the smallest eigenvalue of G. Put

$$x(n) = x^{(1)}(n) + x^{(2)}(n)\,, \qquad x^{(1)}(n) = \textstyle\sum_0^M A(j)\varepsilon(n-j)\,.$$

Then $\tilde{\tau}_{ab}(n)$ is the sum of four terms $\tilde{\tau}_{ab}^{(jk)}(n)$, $j, k = 1, 2$ where these are formed as for $\tilde{\tau}$ but with $x^{(j)}(n)$, $x^{(k)}(n)$. Of course (3) still holds but with f_{ab}, h_{ap} replaced by $f_{ab}^{(jk)}$, $h_{ap}^{(j)}$ where

$$f^{(jk)}(\omega) = \frac{1}{2\pi} h^{(j)}(\omega)Gh^{(k)}(\omega)^*\,, \qquad h(\omega) = h^{(1)}(\omega) + h^2(\omega)\,,$$

$$h^{(1)}(\omega) = \textstyle\sum_0^M A(j)e^{ij\omega}\,.$$

Put $\gamma_{ab}^{(jk)}(n) = E\{x_a^{(j)}(m)x_b^{(k)}(m+n)\}$. The expression $\tilde{\tau}_{ab}^{(11)}(n)$ is composed of a finite linear combination of terms such as

$$N^{-\frac{1}{2}} \textstyle\sum_{m=1}^N \{\varepsilon_a(m)\varepsilon_b(m+n) - \delta_{0n}g_{ab}\} \tag{6}$$

plus some end terms that are asymptotically negligible. However the summands in (6) are stationary, ergodic, martingale differences with finite variance and zero mean and the central limit theorem then follows from [1]. Thus to prove the theorem it is sufficient to show that the contribution to the asymptotic variance of $\tilde{\tau}_{ab}(n)$ from the $\tilde{\tau}_{ab}^{(jk)}(n)$, j, k not both 1, may be made arbitrarily small by taking M large. Examining (3) we see that the first term contributes to the asymptotic variance of $\tilde{\tau}_{ab}^{(jk)}(n)$ the quantity

$$2\pi \textstyle\int_{-\pi}^{\pi} f_{aa}^{(jj)}(\omega)f_{bb}^{(kk)}(\omega)\,d\omega\,.$$

If, say, $j = 2$ then, again using Schwarz inequality, this expression is dominated by

$$K \textstyle\int_{-\pi}^{\pi} \{f_{aa}^{(22)}(\omega)\}^2\,d\omega\,.$$

However $f_{aa}^{(22)}(\omega)$ is a linear combination of expressions $h_{aj}^{(2)}(\omega)\overline{h_{ak}^{(2)}(\omega)}$ and thus it is only necessary to show that

$$\textstyle\int_{-\pi}^{\pi} |h_{ab}^{(2)}(\omega)|^4\,d\omega \tag{7}$$

may be made arbitrarily small by appropriate choice of M. Since $h_{ab}(\omega) \in L_4$ and the Fourier series of a function in L_4 converges in the L_4 norm ([4, page 266]) then (7) may indeed be made small. The same argument applies to the second and third terms in (3) and the result is proved.

REFERENCES

[1] BILLINGSLEY, P. (1961). The Lindberg-Lévy theorem for martingales. *Proc. Amer. Math. Soc.* **12** 788-792.

[2] HANNAN, E. J. (1970). *Multiple Time Series.* Wiley, New York.

[3] HANNAN, E. J. and HEYDE, C. C. (1972). On limit theorems for quadratic functions of discrete time series. *Ann. Math. Statist.* **43** 2058-2066.

[4] ZYGMUND, A. (1959). *Trigonometric Series.* Cambridge Univ. Press.

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