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**THE ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES
FOR A CLASS OF MARKOV OPERATORS**

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1. Introduction. This paper is an extension of the preceding paper "Markov Operators and their Associated Semi-groups" (hereafter referred to as MO) by R. K. Getoor. Throughout this paper we will retain the terminology, notations, and all the assumptions of §2 of MO. Let G be an open subset of X with $m(\partial G) = 0$ and suppose that for each $t > 0$, $f(t, x, y)$ is in $L_2(G \times G, m \times m)$. This is condition (K) in §6 of MO. Assume further that $f(t, x, y) = f(t, y, x)$ for all t, x, y and, for simplicity, that $f(t, x, y) > 0$ for all t, x, y . These assumptions will be retained throughout this paper. It is proved in §6 of MO that under these conditions there is a non-decreasing sequence $\{\lambda_j\}$ of non-negative numbers tending to infinity and a complete orthonormal set $\{\varphi_j\}$ in $L_2(G, m)$ such that the series

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y)$$

converges absolutely. It is further proved that if $k(t, x, y)$ denotes this sum (with $k(t, x, y) = 0$ if x or y is not in G) then $K(V, G; t, x, A) = \int_A k(t, x, y) dm(y)$ for all $t > 0$, x in G , A in $\mathcal{B}(X)$.

Intuitively one can think of k as the transition density of a Markov process that is obtained from $x(t)$ by "killing" $x(t)$ at the boundary of G and upon which a "local death rate" $V(x)$ is imposed. From this interpretation one would expect $k(t, x, y)$ to behave, in some sense, like $f(t, x, y)$ at least for small t and y close enough to x , provided x is in G and V is bounded. In the terminology of Kac [4] "the boundary and death rate aren't felt for small t ". In §2 we make this statement precise by proving that if V is bounded and a certain regularity condition is imposed on f , then for all x in G , $k(t, x, y)f(t, x, y)^{-1} \rightarrow 1$ as $t \rightarrow 0$ for almost all y in a suitable neighborhood of x (Theorem 2.1). From this we are then able to show the somewhat surprising fact that $k(t, x, x)f(t, x, x)^{-1} \rightarrow 1$ as $t \rightarrow 0$ for all x in G (Theorem 2.2). Using these facts we derive the asymptotic distribution of the eigenvalues $\{\lambda_j\}$ for a wide class of processes (Theorem 2.3). In §3 we apply this theory to the symmetric stable processes on the real line and to the Ornstein-Uhlenbeck processes.

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2. **The main theorems.** Let $\{(\mathcal{L}, \mathcal{B}(\mathcal{L}), P_x)\}_{x \in X}$ be the probability spaces constructed in §2 of MO. Let $G(t) = \{x(\cdot): x(\tau) \in \bar{G}; 0 \leq \tau \leq t\}$ and let $H(t)$ be the complement in \mathcal{L} of $G(t)$, that is

$$H(t) = \{x(\cdot): x(\tau) \notin \bar{G} \text{ for some } \tau \leq t\} .$$

It was shown in MO that $G(t)$, and hence $H(t)$, are in $\mathcal{B}(\mathcal{L})$.

From the definition of k above and the orthonormality of $\{\varphi_j\}$ it follows that

$$(2.1) \quad k(t + s, x, y) = \int k(t, x, z)k(s, z, y)dm(z)$$

for all t, s, x, y , and that $k(t, x, y) = k(t, y, x)$ for all t, x, y . Since $K(V, G; t, x, A) \leq p(t, x, A)$ it follows that for each t and x , $0 \leq k(t, x, y) \leq f(t, x, y)$ a.e. (m) , and from (2.1) and the symmetry of k and f it follows that these inequalities hold for all y . From now on we will assume that V is bounded on \bar{G} . In this case we have, for x in G , $e^{-Mt}K(0, G; t, x, A) \leq K(V, G; t, x, A) \leq K(0, G; t, x, A)$ where M is any upper bound of V on \bar{G} . If, for the moment, we let k and k' denote the densities of $K(V, G; t, x, A)$ and $K(0, G; t, x, A)$ respectively, defined by the corresponding series above, then for each t and x

$$(2.2) \quad e^{-Mt}k'(t, x, y) \leq k(t, x, y) \leq k'(t, x, y) \quad \text{a.e. } (m)$$

and since k and k' each satisfy (2.1) and are symmetric these inequalities hold for all y .

In the remainder of this section we will assume that the density f satisfies the following condition:

(D) for every compact set A and every $\eta > 0$ there are numbers $t_0 > 0$ and $M > 0$ such that $f(\sigma, x, y)f(t, x, z)^{-1} \leq M$ for all $\sigma \leq t \leq t_0$, x in A , y and z in X with $\rho(x, y) \geq \eta$, $\rho(x, z) < \eta$. (ρ is the metric on X .)

In the applications, where X is the real line and ρ is the usual metric we will verify this condition for certain familiar process densities.

THEOREM 2.1. *For each x in G there is an open neighborhood $U \subset G$ of x such that $k(t, x, y)f(t, x, y)^{-1} \rightarrow 1$ as $t \rightarrow 0$ for almost all y in U . (Note that an assumption of MO is that the support of m is X and hence $m(U) > 0$ whenever U is open and non-empty.)*

Proof. In view of (2.2) and the remark following it we may assume $V \equiv 0$. Let $q(t, x, y) = f(t, x, y) - k(t, x, y)$. Then

$$\begin{aligned} \int_A q(t, x, y)dm(y) &= P_x[H(t) \cap \{x(\cdot): x(t) \in A\}] \\ &= Q(G; t, x, A) . \end{aligned}$$

Fix x in G and let $S_\varepsilon(x)$ be an open ε -neighborhood of x which is wholly contained in G . Let $\delta > 0$, be such that $4\delta < \varepsilon$ and $S_{2\delta}(x)$ has compact closure. Now if $\{x_k\}$ is a countable dense subset of X then for every $r_0 \geq 1$ $\{S_{1/r}(x_k); r \geq r_0, k \geq 1\}$ is a countable family of sets which generates $\mathcal{B}(X)$. Thus we can construct a sequence $\{\mathcal{N}_n\}$ of finite partitions of X into $\mathcal{B}(X)$ sets such that for every n , \mathcal{N}_{n+1} is a refinement of \mathcal{N}_n , $\mathcal{B}(X)$ is generated by the sets in these partitions, and any set in any of these partitions which intersects $S_\delta(x)$ is contained in $S_\delta(x)$. Since $Q(G; t, x, \cdot)$ is absolutely continuous with respect to $p(t, x, \cdot)$ and since $q(t, x, y)f(t, x, y)^{-1}$ is the derivative of Q with respect to p , it follows from known theorems on derivatives (see [2], pp. 343-344) that for almost all y in the sense of $p(t, x, \cdot)$ and hence for almost all y in the sense of m

$$(2.3) \quad \lim_{n \rightarrow \infty} \frac{Q(G; t, x, B_n)}{p(t, x, B_n)} = \frac{q(t, x, y)}{f(t, x, y)}$$

where B_n denotes that element of \mathcal{N}_n which contains y . (The quotients on the left are taken to be 0 whenever the denominator vanishes.)

Given any $t > 0$ let $\{T_k\}$ ($T_k = \{t_{k1} < \dots < t_{kk}\}$) with $t_{k1} = 0$ and $t_{kk} = t$ be an increasing sequence of subsets of $[0, t]$ becoming dense in $[0, t]$ as $k \rightarrow \infty$. Let

$$A_{kj} = \{x(\cdot): x(t_{kl}) \notin \bar{G}, x(t_{kl}) \in \bar{G}; l = 1, \dots, j - 1\}$$

and let $A_k = \bigcup_{j=1}^k A_{kj}$. For each k the A_{kj} 's are disjoint and $A_k \subset A_{k+1}$.

Moreover $\bigcup_{k=1}^\infty A_k = H(t)$ so that for any $B \in \mathcal{B}(X)$ we have

$$Q(G, t, x, B) = \lim_{k \rightarrow \infty} \sum_{j=1}^k P_x[A_{kj} \cap \{x(\cdot): x(t) \in B\}].$$

For each x in G and A in $\mathcal{B}(X)$ define $(p(0, x, A) = I_A(x))$

$$\mu_{kj}(x, A) = \int_{\bar{G}} \dots \int_{\bar{G}} p(t_{k1}, x, dx_1) \dots p(t_{kj} - t_{k(j-1)}, x_{j-1}, A).$$

Then $\mu_{kj}(x, X - \bar{G}) = P_x[A_{kj}]$ and

$$\begin{aligned} &P_x[A_{kj} \cap \{x(\cdot): x(t) \in B\}] \\ &= \int_{X - \bar{G}} \mu_{kj}(x, dx_j) \int_B f(t - t_{kj}, x_j, y) dm(y) \end{aligned}$$

provided $t_{kj} < t$. On the other hand if $t_{kj} = t$ and $B \subset G$ the left side of this last equation is 0, so for convenience we define the right side to be 0 in this case. If B_n is in \mathcal{N}_n and $B_n \subset G$

$$(2.4) \quad \frac{Q(G; t, x, B_n)}{p(t, x, B_n)} = \lim_{k \rightarrow \infty} \frac{\sum_{j=1}^k \int_{X-\bar{G}} \mu_{kj}(x, dx_j) \int_{B_n} f(t - t_{kj}, x_j, z) dm(z)}{\int_{B_n} f(t, x, z) dm(z)}.$$

We wish to apply condition (D) with $A = \overline{S_{2\delta}(x)}$ and $\eta = 2\delta$. Let y be in $S(x)$ and let B_n be that element of \mathcal{M}_n which contains y . By construction $B_n \subset S_{2\delta}(x)$ so if z is in B_n and x_j is in $X - \bar{G}$ then $\rho(x, z) < 2\delta$ and $\rho(x_j, z) > 2\delta$. Thus for sufficiently small t the right side of (2.4) does not exceed $M \cdot P_x[H(t)]$. This estimate depends on B_n only through the fact that $B_n \subset S_{2\delta}(x)$ so combining this with (2.3) we see that

$$q(t, x, y) f(t, x, y)^{-1} \leq MP_x[H(t)]$$

for almost all y in $S_\delta(x)$ provided t is small enough (how small not depending on y). Then for almost all y in $S_\delta(x)$ we have

$$(2.5) \quad 1 \geq \frac{k(t, x, y)}{f(t, x, y)} \geq 1 - MP_x[H(t)].$$

By the right continuity of the paths $P_x[H(t)] \rightarrow 0$ as $t \rightarrow 0$ and so if we take $U = S_\delta(x)$ the proof of Theorem 2.1 is complete.

THEOREM 2.2. *For all x in G , $k(t, x, x) f(t, x, x)^{-1} \rightarrow 1$ as $t \rightarrow 0$.*

Proof. If x and δ are as in the preceding proof then

$$\begin{aligned} 1 &\geq \frac{k(2t, x, x)}{f(2t, x, x)} = \frac{\int k(t, x, y) k(t, y, x) dm(y)}{\int f(t, x, y) f(t, y, x) dm(y)} \\ &\geq \frac{\left(\int_{S_\delta(x)} k^2(t, x, y) dm(y)\right) \left(\int_{S_\delta(x)} f^2(t, x, y) dm(y)\right)^{-1}}{1 + \left(\int_{X-S_\delta(x)} f(t, x, y) p(t, x, dy)\right) \left(\int_{S_\delta(x)} f(t, x, y) p(t, x, dy)\right)^{-1}}. \end{aligned}$$

By (2.5) the expression in the numerator is not less than $(1 - MP_x[H(t)])^2$. Applying condition (D), with $A = \{x\}$ and $\eta = \delta$ to the second term in the denominator we find that for sufficiently small t it does not exceed $N \cdot p(t, x, X - S_\delta(x)) p(t, x, S_\delta(x))^{-1}$ where N is a fixed positive number. The right continuity of the paths implies that this last expression approaches 0 as $t \rightarrow 0$, and since $P_x[H(t)] \rightarrow 0$ as $t \rightarrow 0$ Theorem 2.2 is established.

Let $N(\lambda)$ be the number of the eigenvalues $\{\lambda_j\}$ which do not exceed λ , that is $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$. We next prove the following theorem concerning the asymptotic behavior of $N(\lambda)$.

THEOREM 2.3. *Suppose*

$$m(G) < \infty, \int_G f(t, x, x) dm(x) < \infty$$

for all sufficiently small t , and

$$\left[\int_G f^2(t, x, x) dm(x) \right]^{1/2} \left[\int_G f(t, x, x) dm(x) \right]^{-1}$$

remains bounded as $t \rightarrow 0$. Then

$$\int_G k(t, x, x) dm(x) \left(\int_G f(t, x, x) dm(x) \right)^{-1} \rightarrow 1 \quad \text{as } t \rightarrow 0.$$

If in addition $\int_G f(t, x, x) dm(x) \sim At^{-\gamma}$ as $t \rightarrow 0$ for some A and $\gamma > 0$ then $N(\lambda) \sim A\lambda^\gamma \Gamma(1 + \gamma)^{-1}$ as $\lambda \rightarrow \infty$.

Proof. We have

$$(2.6) \quad \frac{\int_G q(t, x, x) dm(x)}{\int_G f(t, x, x) dm(x)} = \frac{\int_G \frac{q(t, x, x)}{f(t, x, x)} f(t, x, x) dm(x)}{\int_G f(t, x, x) dm(x)} \\ \leq \left(\int_G \frac{q^2(t, x, x)}{f^2(t, x, x)} dm(x) \right)^{1/2} \frac{\left(\int_G f^2(t, x, x) dm(x) \right)^{1/2}}{\int_G f(t, x, x) dm(x)}$$

$m(G)$ is finite, $q(t, x, x)f(t, x, x)^{-1}$ is bounded by 1 and by Theorem 2.2 approaches 0 as $t \rightarrow 0$ for all x in G . The second factor in the last expression in (2.6) remains bounded as $t \rightarrow 0$, so

$$\left(\int_G q(t, x, x) dm(x) \right) \left(\int_G f(t, x, x) dm(x) \right)^{-1} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

This yields the first assertion of Theorem 2.3. From the definition of k it follows that

$$\int_G k(t, x, x) dm(x) = \sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_0^{\infty} e^{-\lambda t} dN(\lambda).$$

Thus by the first part of the theorem and the additional hypothesis of the second part we have

$$\int_0^{\infty} e^{-\lambda t} dN(\lambda) \sim \int_G f(t, x, x) dm(x) \sim At^{-\gamma} \quad \text{as } t \rightarrow 0.$$

The conclusion of the theorem then follows by applying the Karamata tauberian theorem [6. p. 192].

3. Applications. In this section we apply the results of §2 to the symmetric stable processes and the Ornstein-Uhlenbeck processes on the real line. First consider the symmetric stable process of index α ($0 < \alpha \leq 2$). Here $X = R^1$, m is Lebesgue measure, and $f(t, x, y) = g(t, x - y)$ where

$$(3.1) \quad g(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{txu} e^{-t|u|^\alpha} du .$$

It is well known that the symmetric stable processes satisfy the conditions of §2 of MO and clearly f is symmetric in x and y . For each t , $f(t, x, y)$ is uniformly bounded so condition (K) in §6 of MO is satisfied if $m(G)$ is finite (in particular if \bar{G} is compact). We wish to verify condition (D) for the density f . To this end we state three lemmas.

LEMMA 3.1. *For each $t > 0$, $g(t, x)$ decreases as $|x|$ increases.*

LEMMA 3.2. *Suppose φ , a real valued function defined on $[0, \infty)$, has N continuous derivatives and that $\varphi, \varphi^{(1)}, \dots, \varphi^{(N)}$ are all absolutely integrable on $[0, \infty)$. Suppose that for each $n \leq N - 1$, $\varphi^{(n)}(u) \rightarrow 0$ as $u \rightarrow \infty$. Then if $0 < \lambda < 1$ we have*

$$(3.2) \quad \int_0^\infty u^{\lambda-1} \varphi(u) \cos bu \, du = - \sum_{n=0}^{N-1} \frac{\Gamma(n + \lambda)}{n!} \cos\left(\frac{\pi}{2}(n + \lambda - 2)\right) \varphi^{(n)}(0) b^{-n-\lambda} + O(b^{-N}) \text{ as } b \rightarrow \infty .$$

LEMMA 3.3. *For each $x \neq 0$, $g(t, x)$ is an increasing function of t in the domain $0 < t < B_\alpha |x|^\alpha$ where B_α is a positive constant independent of x .*

Lemma 3.1 is reasonably well known and a proof may be found in [7, Th. 11.8, p. 32]. Lemma 3.2 is a trivial modification of a theorem of Erdélyi [3, p. 48], to which we refer the reader. Lemma 3.3 is doubtless well known, but we are unable to find an explicit reference to it in the literature and so we give a proof.

Proof of Lemma 3.3. We fix $x \neq 0$ and look at the derivative $dg/dt = -(\pi)^{-1} \int_0^\infty (\cos xu) u e^{\alpha-tu} du$. Making the change of variable $tu^\alpha = y^\alpha$ we obtain

$$(3.3) \quad \frac{dg}{dt} = -\frac{1}{\pi} t^{-1-1/\alpha} h_\alpha(b)$$

where $b = |x|t^{-1/\alpha}$ and

$$h_\alpha(b) = \int_0^\infty y^\alpha e^{-y^\alpha} \cos by \, dy .$$

If $0 < \alpha < 1$ we apply Lemma 3.2 with $N = 2$, $\lambda = \alpha$, and $\varphi(y) = ye^{-y^\alpha}$. φ clearly satisfies the assumptions of Lemma 3.2 and $\varphi(0) = 0$, $\varphi'(0) = 1$ so we obtain

$$(3.4) \quad \begin{aligned} h_\alpha(b) &= -\Gamma(1 + \alpha)\cos\left[\frac{\pi}{2}(\alpha - 1)\right]b^{-1-\alpha} + O(b^{-2}) \\ &= -A(\alpha)b^{-1-\alpha} + O(b^{-2}) \end{aligned} \quad \text{as } b \rightarrow \infty$$

where $A(\alpha) = \Gamma(1 + \alpha)\cos\left[\frac{\pi}{2}(\alpha - 1)\right] > 0$. If $1 < \alpha < 2$ we take $N = 3$, $\lambda = \alpha - 1$, and $\varphi(y) = y^2e^{-y^\alpha}$ and obtain

$$(3.5) \quad h_\alpha(b) = -A(\alpha)b^{-1-\alpha} + O(b^{-3}) \quad \text{as } b \rightarrow \infty .$$

If $0 < \alpha < 1$ then (3.4) implies that there are constants M_α and b_α such that $|h_\alpha(b) + A(\alpha)b^{-1-\alpha}| \leq M_\alpha b^{-2}$ if $b > b_\alpha$. Thus

$$\left| \frac{dg}{dt} - \frac{A(\alpha)}{\pi}|x|^{-1-\alpha} \right| \leq M'_\alpha|x|^{-2}t^{-1+1/\alpha}$$

provided $|x|t^{-1/\alpha} > b_\alpha$ or equivalently $0 < t < b'_\alpha|x|^\alpha$. Then dg/dt will be positive if $M'_\alpha|x|^{-2}t^{-1+1/\alpha} < A(\alpha)\pi^{-1}|x|^{-1-\alpha}$ or equivalently if $0 < t < M''_\alpha|x|^\alpha$. Thus if we take $B_\alpha = \min(b'_\alpha, M''_\alpha)$ Lemma 3.3 is established for $0 < \alpha < 1$. If $1 < \alpha < 2$ a similar analysis beginning with (3.5) yields the desired result. Finally $g(t, x) = \pi^{-1}t(t^2 + x^2)^{-1}$ if $\alpha = 1$ and

$$g(t, x) = (2\sqrt{\pi t})^{-1}\exp(-x^2/4t)$$

if $\alpha = 2$ and the conclusion of the lemma is easily verified in these cases.

Now to verify condition (D) let A , a compact subset of R^1 , and $\eta > 0$ be given. If $t_0 < B_\alpha\eta^\alpha$ where B_α is the constant of Lemma 3.3, if $|x - y| > \eta$ and if $0 < \sigma < t < t_0$ then $f(\sigma, x, y) = g(\sigma, x - y) \leq g(t, x - y)$, and if $|x - z| \leq \eta$ then

$$\frac{f(\sigma, x, y)}{f(t, x, z)} = \frac{g(\sigma, x - y)}{g(t, x - z)} \leq \frac{g(t, x - y)}{g(t, x - z)} \leq 1$$

the last inequality being a consequence of Lemma 3.1. In this case these estimates do not depend on x being in A .

Since

$$f(t, x, x) = g(t, 0) = (\pi)^{-1}\int_0^\infty e^{-tu^\alpha} du = (\alpha\pi)^{-1}t^{-1/\alpha}\Gamma(1/\alpha) ,$$

if $m(G) < \infty$ then the conditions of Theorem (2.3) are satisfied and we have

$$(3.6) \quad N(\lambda) \sim \frac{\lambda^{1/\alpha}}{\pi} m(G).$$

This is the asymptotic distribution of the eigenvalues for the symmetric stable process of index α on an open set G of finite Lebesgue measure with V bounded. This should be compared with the results of Kac [5]. (Kac's V is different from ours. His $V \equiv 1$ yields our results with our $V \equiv 0$.)

Next we turn to the Ornstein-Uhlenbeck processes. It is well known [1] that these processes satisfy the conditions of §2 of MO (in fact the paths can be taken to be continuous.) The transition density relative to Lebesgue measure of the $0 - U$ process with parameter $\beta > 0$ is given by

$$(3.7) \quad \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp \left[-\frac{1}{2} \frac{(y - \rho x)^2}{1 - \rho^2} \right]$$

where $\rho = \rho(t) = e^{-\beta t}$, $\beta > 0$, $t > 0$. This density is not symmetric, but if we introduce the measure m defined by $dm(y) = e^{-y^2/2} dy$ then the transition density relative to m is

$$(3.8) \quad f(t, x, y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp \left[-\frac{1}{2} \frac{\rho^2 x^2 - 2\rho xy + y^2}{1 - \rho^2} \right]$$

which is symmetric. We now verify condition (D) for this density. Let the compact set A and number η be given. Then

$$(3.9) \quad \frac{f(\sigma, x, y)}{f(t, x, z)} = \frac{(1-\rho^2)^{-1/2} \exp \left[-\frac{1}{2} \frac{(y-x)^2}{1-\rho^2} \right] \exp \left[-\frac{xy}{1+\rho} \right] \exp \left[\frac{x^2}{2} \right] \exp \left[\frac{y^2}{2} \right]}{(1-\theta^2)^{-1/2} \exp \left[-\frac{1}{2} \frac{(z-x)^2}{1-\theta^2} \right] \exp \left[-\frac{xz}{1+\theta} \right] \exp \left[\frac{x^2}{2} \right] \exp \left[\frac{z^2}{2} \right]}$$

where $\rho = e^{-\beta\sigma}$ and $\theta = e^{-\beta t}$. The fourth factors in the numerator and denominator cancel. If we consider only x, y , and z such that x is in A , $|x - z| < \eta$ and $|y - x| \geq \eta$ then the third and fifth factors in the denominator are bounded away from 0 and the second factor is no smaller than $\exp \left[-\frac{1}{2} \frac{\eta^2}{1-\theta^2} \right]$. Thus there exists a positive constant N_1 such that

$$(3.10) \quad \frac{f(\sigma, x, y)}{f(t, x, z)} \leq N_1 \frac{(1-\rho^2)^{-1/2} \exp \left[-\frac{1}{2} \frac{(y-x)^2}{1-\rho^2} \right] \exp \left[-\frac{xy}{1+\rho} \right] \exp \left[\frac{y^2}{2} \right]}{(1-\theta^2)^{-1/2} \exp \left[-\frac{1}{2} \frac{\eta^2}{1-\theta^2} \right]}$$

The product of the exponentials in the numerator is precisely

$$\exp\left[-\frac{1}{2} \frac{(\rho y - x)^2}{1 - \rho^2}\right].$$

If $|y| > 2\max_{x \in A} |x| + 2\eta$ and $\rho > 1/2$ then $(\rho y - x)^2 > \eta^2$. But for any other y such that $|x - y| \geq \eta$, the second and third exponentials in the numerator of (3.10) are uniformly bounded while the first exponential does not exceed $\exp\left[-\frac{1}{2} \frac{\eta^2}{1 - \rho^2}\right]$. Thus if t_0 is such that $e^{-\beta t_0} > 1/2$, then for $\sigma < t \leq t_0$, x in A , $|x - y| \geq \eta$, and $|x - z| < \eta$ we have

$$(3.11) \quad \frac{f(\sigma, x, y)}{f(t, x, y)} < N_2 \frac{(1 - \rho^2)^{-1/2} \exp\left[-\frac{1}{2} \frac{\eta^2}{1 - \rho^2}\right]}{(1 - \theta^2)^{-1/2} \exp\left[-\frac{1}{2} \frac{\eta^2}{1 - \theta^2}\right]}$$

where N_2 is a positive constant. The right side of (3.11) is easily seen to be uniformly bounded for $0 < \sigma < t \leq t_0$ and thus condition (D) is verified.

For this density $f(t, x, x) = b(t) \exp(\rho x^2 / (1 + \rho))$ where $b(t) = [2\pi(1 - \rho^2)]^{-1/2}$ and $\rho = \rho(t) = e^{-\beta t}$. One verifies easily that if $\mu(G) < \infty$, where μ denotes Lebesgue measure, then condition (K) as well as all the hypotheses of Theorem 2.3 are satisfied. In particular since ρ increases to 1 as, $t \rightarrow 0$ we have

$$\begin{aligned} \int_G f(t, x, x) dm(x) &= b(t) \int_G e^{(\rho x^2 / (1 + \rho))} e^{-(x^2 / 2)} dx \\ &\sim b(t) \mu(G) \sim \frac{\mu(G)}{2\sqrt{\beta\pi}} t^{-1/2} \quad \text{as } t \rightarrow 0. \end{aligned}$$

So applying Theorem 2.3 we obtain for the $0-U$ process with parameter β

$$(3.12) \quad N(\lambda) \sim \frac{\mu(G)\lambda^{1/2}}{\pi\sqrt{\beta}}.$$

If G is the open interval (a, b) then the infinitesimal generator Ω'_a is given by the differential operator $\Omega'_a \varphi = \beta[\varphi'' + (x\varphi)'] - V\varphi$ on an appropriate domain in $L_2[G, m]$ subject to the boundary conditions $\varphi(a) = \varphi(b) = 0$. If $\beta = 1$ notice that (3.12) reduces to (3.6) with $\alpha = 2$. If $\alpha = 2$ in (3.6) the corresponding infinitesimal generator is given by $\varphi'' - V\varphi$ on an appropriate domain with the same boundary conditions. Thus the term $(x\varphi)'$ does not affect the asymptotic distribution of the eigenvalues, which is certainly what one would expect. The λ_j are the eigenvalues of $-\Omega'_a$ in each case. See Theorem 6.3 of MO.

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